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Research Article

Applications of the Bell Numbers on Univalent Functions Associated with Subordination

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The motivation of the present paper is to define a new subclass of univalent functions associated with the q-analogue of the exponential function and the well-known Bell numbers based on subordination structure. Furthermore, we estimate the coefficient bound and extreme points. Also, geometric properties such as convexity and convolution preserving concept are investigated.

1. Introduction

For a fixed nonnegative integer k, the Bell numbers B_k is the number of equivalent relations on a set with k elements or equivalently the number of possible disjoint partitions of a set with k elements into nonempty subsets. The function

$$Q(z) = e^{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}, \tag{1}$$

involving the Bell numbers was considered by Kumar et al. [1], see also [2, 3].

Let $\mathscr{A}$ denote the class of all functions F which are analytic in the open unit disk,

$$\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}, \tag{2}$$

and normalized by conditions:

$$F(0) = F'(0) - 1 = 0. \tag{3}$$

Hence, $F \in \mathscr{A}$ has a Taylor–Maclaurin series representation:

$$F(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{D}). \tag{4}$$

Also, $\mathscr{S}$ is the subclass of $\mathscr{A}$ consisting of all well-known univalent functions in $\mathbb{D}$.

Furthermore, we denote by $\mathscr{N}$ a subclass of $\mathscr{A}$ consisting of functions with negative coefficients of the type:

$$F(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{D}, a_k \geq 0). \tag{5}$$

Since $(zF'(z)/F(z))$ maps $\mathbb{D}$ onto the right half-plane of $\mathbb{C}$, so $\mathrm{Re}(zF'(z)/F(z)) > 0$, and it is a usual subclass of normalized univalent function class $\mathscr{S}$, which are star-like functions, see [4]. Thus, $Q(z)$, given by (1), is star-like with respect to 1, and its coefficients are the Bell numbers.

The theory of q-calculus (or quantum calculus) operators is used in various areas of science and also in the geometric function theory. Also, the theory of q-derivative operators has played an important role in differential equations, physics, mechanics, and so on. The application of q-calculus was initiated by Jackson [5, 6]. He was the first to develop q-integral and q-derivative in a systematic way. Q-calculus is equivalent to classical calculus without the notion of limits. A comprehensive study on applications of q-calculus and q-analogue of well-known operators in theory of univalent functions may be found in [7–12].

The q-analogue of the exponential function e^z is given by

$$e_q^z = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(k+1)}, \quad (6)$$

where $q \in (0,1)$ and $\Gamma_q(k+1)$ is the q-gamma function defined by

$$\Gamma_q(k+1) = [k]_q \Gamma_q(k), \quad \Gamma_q(1) = 1, \quad (7)$$

and q-number $[k]_q$ is introduced by

$$[k]_q = \begin{cases} \dfrac{1-q^k}{1-q}, & k \in \mathbb{C}, \\ \sum_{n=0}^{k-1} q^n = 1 + q + q^2 + \cdots + q^{k-1}, & k \in \mathbb{N}, \end{cases} \quad (8)$$

see [13–15].

The Hadamard product (convolution) for function $F(z)$, given by (5) and $G(z) = z - \sum_{k=2}^{\infty} b_k z^k$ denoted by $F * G$, is defined by

$$(F * G)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (G * F)(z), \quad (z \in \mathbb{D}). \quad (9)$$

Let F and G be analytic in $\mathbb{D}$; then, F is said to be subordinate to G, written $F \prec G$, if there exists a function W analytic in $\mathbb{D}$, with $W(0) = 0$ and $|W(z)| < 1$, such that

$$F(z) = G(W(z)). \quad (10)$$

If G is univalent, then $F \prec G$ if and only if $F(0) = G(0)$ and $F(\mathbb{D}) \subset G(\mathbb{D})$, see [16].

Definition 1. A function H is said to belong to the class $\mathscr{W}^t(\alpha, \beta, \gamma)$ if

$$\frac{zH'(z)}{F_t(z)} \prec \frac{1 + (\gamma + \alpha(1-\beta))z}{1 + \gamma z}, \quad (11)$$

where $0 < \beta < 1$, $-1 \le \gamma \le 1$, $-1 \le \alpha \le 1$, $0 \le t \le 1$, $F_t(z) = (1-t)z + tF(z)$, $F(z) \in \mathscr{N}$, and

$$H(z) = \left[\left(1 + 2z - e_q^z\right) * (2z + 1 - Q(z))\right] * F(z). \quad (12)$$

$Q(z)$, $F(z)$, and e_q^z are given by (1), (5), and (6), respectively.

2. Main Results

In this section, we obtain the coefficient bounds and extreme points of $\mathscr{W}^t(\alpha, \beta, \gamma)$.

Theorem 1. *Let $H(z)$ be analytic in $\mathbb{D}$. Then, $H \in \mathscr{W}^t(\alpha, \beta, \gamma)$ if and only if*

$$\sum_{k=2}^{\infty} \left[\left(\frac{B_k}{(k-1)!\Gamma_q(k+1)} - t\right)(1-\gamma) + t\alpha(1-\beta)\right] a_k \le \alpha(1-\beta). \quad (13)$$

Proof. The γ subordination relation (11) is equivalent to

$$\left| \frac{(zH'(z)/F_t(z)) - 1}{\gamma + \alpha(1-\beta) - \gamma(zH'(z)/F_t(z))} \right| < 1. \quad (14)$$

Suppose that (13) holds true. We must show that (11) or equivalently (14) holds. However, we have

$$\begin{aligned} &\left|zH'(z) - F_t(z)\right| - \left|(\gamma + \alpha(1-\beta))F_t(z) - \gamma zH'(z)\right| \\ &= \left| z - \sum_{k=2}^{\infty} \frac{B_k}{(k-1)!\Gamma_q(k+1)} a_k z^k - z + \sum_{k=2}^{\infty} t a_k z^k \right| \\ &\quad - \left| \alpha(1-\beta)z - \sum_{k=2}^{\infty} \left[t(\gamma + \alpha(1-\beta)) - \frac{\gamma B_k}{(k-1)!\Gamma_q(k+1)} \right] a_k z^k \right| \\ &\le \left| \sum_{k=2}^{\infty} \left[\left(\frac{B_k}{(k-1)!\Gamma_q(k+1)} - t\right)(1-\gamma) + t\alpha(1-\beta)\right] a_k - \alpha(1-\beta) \right|. \end{aligned} \quad (15)$$

By (13) and letting $|z| = 1$, the above expression is less than or equal to zero, so (14) holds true.

To prove the converse, let $H(z) \in \mathscr{W}^t(\alpha, \beta, \gamma)$; thus,

$$\begin{aligned}&\left|\frac{(zH'(z)/F_t(z)) - 1}{\gamma + \alpha(1-\beta) - \gamma z(H'(z)/F_t(z))}\right| \\ &= \frac{\left|z\left(1 - \sum_{k=2}^{\infty}\left(B_k/(k-1)!\Gamma_q(k+1)\right)a_k z^{k-1}\right) - (1-t)z - t\left(z - \sum_{k=2}^{\infty} a_k z^k\right)\right|}{\left|(\gamma + \alpha(1-\beta))\left((1-t)z + t\left(z - \sum_{k=2}^{\infty} a_k z^k\right)\right) - \gamma z\left(1 - \sum_{k=2}^{\infty}\left(B_k/(k-1)!\Gamma_q(k+1)\right)a_k z^{k-1}\right)\right|} < 1,\end{aligned} \tag{16}$$

for all $z \in \mathbb{D}$. Since $\mathrm{Re}(z) \leq |z|$, we have

$$\mathrm{Re}\left\{\frac{\sum_{k=2}^{\infty}\left(B_k/(k-1)!\Gamma_q(k+1) - t\right)a_k z^k}{\alpha(1-\beta)z - \sum_{k=2}^{\infty}\left[t(\gamma + \alpha(1-\beta)) - \left(\gamma B_k/(k-1)!\Gamma_q(k+1)\right)\right]a_k z^k}\right\} < 1. \tag{17}$$

By letting $z \longrightarrow 1$ through positive values and choosing the values of z such that $(zH'(z)/F_t(z))$ is real, we have

$$\sum_{k=2}^{\infty}\left[\left(\frac{B_k}{(k-1)!\Gamma_q(k+1)} - t\right)(1-\gamma) + t\alpha(1-\beta)\right]a_k \leq \alpha(1-\beta), \tag{18}$$

and this completes the proof. □

Remark 1. We note that the function,

$$V(z) = z - \frac{\alpha(1-\beta)}{\left(\left(B_2/\Gamma_q(3)\right) - t\right)(1-\gamma) + t\alpha(1-\beta)}z^2, \tag{19}$$

shows that inequality (13) is sharp.

Theorem 2. *Let $H_1(z) = z$ and*

$$H_k(z) = z - \frac{\alpha(1-\beta)}{\left(\left(B_k/(k-1)!\Gamma_q(k+1)\right) - t\right)(1-\gamma) + t\alpha(1-\beta)}z^k, \tag{20}$$

where $k = 2, 3, \ldots$. Then, $H \in \mathscr{W}^t(\alpha, \beta, \gamma)$ if and only if it can be expressed in the form:

$$H(z) = \sum_{k=1}^{\infty}\lambda_k H_k(z), \tag{21}$$

where $\lambda_k \geq 0$ and $\sum_{k=1}^{\infty}\lambda_k = 1$. In particular, the extreme points of $\mathscr{W}^t(\alpha, \beta, \gamma)$ are the functions $H_k(z)$, where $k = 1, 2, 3, \ldots$.

Proof. Let H be expressed by (21). This means that we can write

$$\begin{aligned}H(z) &= \sum_{k=1}^{\infty}\lambda_k H_k(z) = \lambda_1 H_1(z) + \sum_{k=2}^{\infty}\lambda_k H_k(z) \\ &= \left(\sum_{k=1}^{\infty}\lambda_k\right)z - \sum_{k=2}^{\infty}\frac{\alpha(1-\beta)\lambda_k}{\left(\left(B_k/(k-1)!\Gamma_q(k+1)\right) - t\right)(1-\gamma) + t\alpha(1-\beta)}z^k.\end{aligned} \tag{22}$$

Since $\sum_{k=1}^{\infty} \lambda_k = 1$ and

$$\sum_{k=2}^{\infty}\left[\left(\frac{B_k}{(k-1)!\Gamma_q(k+1)}-t\right)(1-\gamma)+t\alpha(1-\beta)\right]\left[\frac{\alpha(1-\beta)\lambda_k}{\left(\left(B_k/(k-1)!\Gamma_q(k+1)\right)-t\right)(1-\gamma)+t\alpha(1-\beta)}\right]$$
$$=\sum_{k=2}^{\infty}\alpha(1-\beta)\lambda_k=\alpha(1-\beta)\sum_{k=2}^{\infty}\lambda_k=\alpha(1-\beta)(1-\lambda_1)<\alpha(1-\beta), \tag{23}$$

so, by Theorem 1, we conclude that $H \in \mathscr{W}^t(\alpha,\beta,\gamma)$.

Conversely, suppose that $H \in \mathscr{W}^t(\alpha,\beta,\gamma)$. Then, by (13), we have

$$a_k<\frac{\alpha(1-\beta)}{\left(\left(B_k/(k-1)!\Gamma_q(k+1)\right)-t\right)(1-\gamma)+t\alpha(1-\beta)}, \quad (k=2,3,\ldots). \tag{24}$$

By setting

$$\lambda_k=\frac{\left(\left(B_k/(k-1)!\Gamma_q(k+1)\right)-t\right)(1-\gamma)+t\alpha(1-\beta)}{\alpha(1-\beta)}a_k, \quad (k\geq 2), \tag{25}$$

and $\lambda_1 = 1-\sum_{k=2}^{\infty}\lambda_k$, we obtain the required result. So, the proof is complete. □

3. Geometric Properties of $\mathscr{W}^t(\alpha,\beta,\gamma)$

In this section, we show convexity of $\mathscr{W}^t(\alpha,\beta,\gamma)$. Also, we obtain convolution preserving property.

Theorem 3. *$\mathscr{W}^t(\alpha,\beta,\gamma)$ is a convex set.*

Proof. We must show that if $H_j(z)$, for $j=1,2,\ldots,m$, belong to $\mathscr{W}^t(\alpha,\beta,\gamma)$, then the function,

$$H(z)=\sum_{j=1}^{m}\sigma_j H_j(z), \tag{26}$$

is also in the same class, where $0<\sigma_j<1$ and $\sum_{j=1}^{m}\sigma_j=1$.

Since $H_j(z) \in \mathscr{W}^t(\alpha,\beta,\gamma)$, we have

$$\sum_{k=2}^{\infty}\left[\left(\frac{B_k}{(k-1)!\Gamma_q(k+1)}-t\right)(1-\gamma)+t\alpha(1-\beta)\right]a_{k,j}\leq\alpha(1-\beta), \quad (j=1,2,\ldots,m). \tag{27}$$

However,

$$H(z)=\sum_{j=1}^{m}\sigma_j H_j(z)$$
$$=\sum_{j=1}^{m}\sigma_j\left(z-\sum_{k=2}^{\infty}\frac{B_k}{\Gamma_q(k+1)k!}a_{k,j}z^k\right) \tag{28}$$
$$=z-\sum_{k=2}^{\infty}\left(\sum_{j=1}^{m}\sigma_j a_{k,j}\right)\frac{B_k}{\Gamma_q(k+1)k!}z^k.$$

It is enough to verify inequality (13) for $H(z)$. However,

$$\sum_{k=2}^{\infty}\left[\left(\frac{B_k}{(k-1)!\Gamma_q(k+1)}-t\right)(1-\gamma)+t\alpha(1-\beta)\right]\left(\sum_{j=1}^{m}\sigma_j a_{k,j}\right)$$
$$=\sum_{j=1}^{m}\sigma_j\left\{\sum_{k=2}^{\infty}\left[\left(\frac{B_k}{(k-1)!\Gamma_q(k+1)}-t\right)(1-\gamma)+t\alpha(1-\beta)\right]\right\}$$
$$<\left(\sum_{j=1}^{m}\sigma_j\right)\alpha(1-\beta)=\alpha(1-\beta). \tag{29}$$

This inequality by (13) shows that $H \in \mathscr{W}^t(\alpha,\beta,\gamma)$, and the proof is complete. □

Theorem 4. *Let the functions $H_j(z)$, $j = 1, 2$, be in the class $\mathscr{W}^t(\alpha, \beta, \gamma)$; then, $(H_1 * H_2)(z)$ belongs to $\mathscr{W}^t(\alpha, \beta^*, \gamma)$, where $\beta^* \leq 1 - X$, and*

$$X = \frac{\alpha\left(\left(B_k/(k-1)!\Gamma_q(k+1)\right) - t\right)(1-\gamma)}{\left[\left(\left(B_k/(k-1)!\Gamma_q(k+1)\right) - t\right)(1-\gamma) + t\alpha(1-\beta)\right)/(1-\beta)\right] - t\alpha^2}. \tag{30}$$

Proof. Since $H_j(z) \in \mathscr{W}^t(\alpha, \beta, \gamma)$, so

$$H_j(z) = z - \sum_{k=2}^{\infty} \frac{B_k}{\Gamma_q(k+1)k!} a_{k,j} z^k. \tag{31}$$

It is sufficient to show that

$$\sum_{k=2}^{\infty}\left[\left(\frac{B_k}{(k-1)!\Gamma_q(k+1)} - t\right)\left(\frac{1-\gamma}{\alpha(1-\beta)}\right) + t\right] a_{k,1}a_{k,2} \leq 1. \tag{32}$$

By using Cauchy–Schwarz inequality, from (13), we obtain

$$\sum_{k=2}^{\infty}\left[\left(\frac{B_k}{(k-1)!\Gamma_q(k+1)} - t\right)\left(\frac{1-\gamma}{\alpha(1-\beta)}\right) + t\right] \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{33}$$

Hence, we find β^* such that

$$\begin{aligned}&\sum_{k=2}^{\infty}\left[\left(\frac{B_k}{(k-1)!\Gamma_q(k+1)} - t\right)\left(\frac{1-\gamma}{\alpha(1-\beta^*)}\right) + t\right] a_{k,1}a_{k,2} \leq \\ &\sum_{k=2}^{\infty}\left[\left(\frac{B_k}{(k-1)!\Gamma_q(k+1)} - t\right)\left(\frac{1-\gamma}{\alpha(1-\beta)}\right) + t\right] \sqrt{a_{k,1}a_{k,2}} \leq 1,\end{aligned} \tag{34}$$

or equivalently

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{\left(\left(B_k/(k-1)!\Gamma_q(k+1)\right) - t\right)((1-\gamma)/\alpha(1-\beta)) + t}{\left(\left(B_k/(k-1)!\Gamma_q(k+1)\right) - t\right)((1-\gamma)/\alpha(1-\beta^*)) + t}. \tag{35}$$

This inequality holds if

$$\frac{\alpha(1-\beta)}{\left(\left(B_k/(k-1)!\Gamma_q(k+1)\right) - t\right)(1-\gamma) + t\alpha(1-\beta)} \leq \frac{\left(\left(B_k/(k-1)!\Gamma_q(k+1)\right) - t\right)((1-\gamma)/\alpha(1-\beta)) + t}{\left(\left(B_k/(k-1)!\Gamma_q(k+1)\right) - t\right)((1-\gamma)/\alpha(1-\beta^*)) + t}, \tag{36}$$

or equivalently $\beta^* \leq 1 - X$, where X is given in (30). So, the proof is complete. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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