Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2350193, 7 pages
https://doi.org/10.1155/2022/2350193

Research Article

Some New Fractional Inequalities Involving Convex Functions and Generalized Fractional Integral Operator

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Received 20 January 2022; Accepted 28 February 2022; Published 21 March 2022

Academic Editor: Humberto Rafeiro

In this manuscript, we are getting some novel inequalities for convex functions by a new generalized fractional integral operator setting. Our results are established by merging the (k, s)-Riemann-Liouville fractional integral operator with the generalized Katugampola fractional integral operator. Certain special instances of our main results are considered. The detailed results extend and generalize some of the present results by applying some special values to the parameters.

1. Introduction

Chebyshev [1] presented the celebrated functional described by

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \tag{1}$$

where the functions f and g are integrable on $[a, b]$. If f and g are synchronous, i.e.,

$$(f(x_1)-f(x_2))(g(x_1)-g(x_2))\geq 0, \tag{2}$$

for $x_1, x_2 \in [a, b]$, then, $T(f, g) \geq 0$.

The relation (1) has stood out for some researchers because of the different applications in numerical quadrature, statistical problems, probability, and transform theory. Among those applications, the relation (1) was utilized to yield various integral inequalities (see, e.g., [2–6], for extremely late work, see additionally [7–10]).

There is one more appealing and useful inequality, namely, the Pólya-Szegö inequality [11], which establishes the essential key of motivation in our study, which we can indicate as follows

$$\frac{\int_a^b f^2(x)dx\int_a^b g^2(x)dx}{\left(\int_a^b f(x)g(x)dx\right)^2}\leq\frac{1}{4}\left(\sqrt{\frac{MN}{mn}}+\sqrt{\frac{mn}{MN}}\right)^2, \tag{3}$$

where $m \leq f(x) \leq M$ and $n \leq g(x) \leq N$, for some $m, M, n, N \in \mathbb{R}$ and for each $x \in \mathbb{J} := [a, b]$.

In [12], Dragomir and Diamond introduced the following Grüss type integral inequality

$$|T(f,g)|\leq\frac{(M-m)(N-n)}{4(b-a)^2\sqrt{MmNn}}\int_a^b f(x)dx\int_a^b g(x)dx, \tag{4}$$

by the Pólya-Szegö result, here, $0 < m \leq f(x) \leq M < \infty$ and $0 < n \leq g(x) \leq N < \infty$, for $x \in \mathbb{J}$.

The next integral inequalities

$$\int_0^1 \nu^{\mu-1}(x)dx \geq \int_0^1 x^{\mu}\nu(x)dx, \quad (5)$$

$$\int_0^1 \nu^{\mu-1}(x)dx \geq \int_0^1 x\nu^{\sigma}(x)dx, \quad (6)$$

have been proved by Ngo [13], where $x > 0$ and ν are a positive continuous on $[0, 1]$ such that

$$\int_h^1 \nu(x)dx \geq \int_h^1 xdx, h \in 0, 1]. \quad (7)$$

In this regard, Liu et al. [14] introduced the subsequent inequality

$$\int_a^b \nu^{\mu+\nu}(x)dx \geq \int_a^b (x-a)^{\mu}\nu^{\nu}(x)dx, \quad (8)$$

with

$$\int_a^b \nu^{\varsigma}(x)dx \geq \int_a^b (x-a)^{\varsigma}dx, \quad (9)$$

where $\mu, \nu > 0$ and ν are a positive continuous on $\mathbb{J}$, and $\varsigma = \min(1, \nu)$, for $x \in \mathbb{J}$.

Since one of the primary inspiration points of fractional analysis is getting more general and valuable fundamental integral operators, the generalized fractional integral operator is a decent tool to sum up numerous past investigations and results, see [15–18]. Essentially, in inequality theory, scientists utilize such broad operators to generalize and extend their inequalities.

Lately, these fractional integral operators have been considered and used to broaden particularly Grüss, Chebychev-Grüss, Pólya-Szegö, Gronwall, Minkowski, and Hermite-Hadamard type inequalities. For additional subtleties, Agarwal [19] proved some fractional integral inequalities with Hadamard's fractional integral operators. Ntouyas et al. [20] established certain Chebyshev type inequalities involving Hadamard's fractional integral operators. Some Grüss type inequalities under k-Riemann-Liouville fractional integral operators have been investigated by Set et al. [21]. In [22], the authors introduced some Pólya-Szegö type inequalities by Hadamard k-fractional integral operators. A new version for the Gronwall type inequality involving generalized proportional fractional integral operators was presented by Alzabut et al. [23]. Rahman et al. [24] established reverse Minkowski inequalities with generalized proportional fractional integral operators. Many Chebyshev type inequalities with generalized conformable fractional integral operators have been discussed by Nisar et al. [25].

In this regard, Dahmani [26] established some new inequalities for convex functions involving Riemann-Liouville fractional integral operator. Jleli et al. [27] obtained new Hermite-Hadamard type inequalities for convex functions via generalized fractional integral operators. Some Hermite-Hadamard type inequalities for (k, s)-Riemann-Liouville fractional integral operators were obtained by Agarwal et al. [28]. The authors in [29] established certain Polya-Szego type inequalities involving generalized Katugampola fractional integral operator.

Motivated by the above works and discussions, in this manuscript, we establish certain novel inequalities for convex functions under a new generalized fractional integral operator which integrates the two proposed fractional integral operators in [30, 31]. Moreover, we consider certain special cases of our main results. The results obtained extend and generalize some of the existing results by substituting some parameters.

The manuscript is marshaled as follows: Section 2 presents some main definitions and results. The acquired results are presented in Section 3. The last section concludes the manuscript.

2. Preliminaries

Let us first present the essential definitions and properties of fractional analysis that will be frequently used in this study.

Definition 1 (see [15]). The right and left-sided Riemann-Liouville fractional integral operators of order $\alpha > 0$ of $u \in L^1[a, b]$ are defined by

$$({}_a\mathbb{I}^{\alpha}u)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}u(y)dy, \quad a < x, \quad (10)$$

$$(\mathbb{I}_b^{\alpha}u)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (y-x)^{\alpha-1}u(y)dy, \quad x < b. \quad (11)$$

Definition 2 (see [30]). For $>0, k > 0$, and $s \in \mathbb{R} \setminus \{-1\}$, the left-sided (k, s)-Riemann-Liouville fractional integral operator is defined as

$$\mathbb{I}_{a,s,k}^{\alpha}u(x) = \frac{(s+1)^{1-\alpha/k(\alpha/k)}}{k\Gamma_k(\alpha)}\int_a^x \left(x^{s+1} - y^{s+1}\right)^{\frac{\alpha}{k}-1}y^s u(y)dy, \quad a < x. \quad (12)$$

Definition 3 (see [31]). For $\alpha, \rho > 0$, and $\beta, \eta, r \in \mathbb{R}$, the left-sided generalized Katugampola fractional integral operator is defined as

$${}^{\rho}\mathbb{I}_{a,\eta,r}^{\alpha,\beta}u(x) = \frac{\rho^{1-\beta}x^r}{\Gamma(\alpha)}\int_a^x (x^{\rho} - y^{\rho})^{\alpha-1}y^{\rho(\eta+1)-1}u(y)dy, \quad a < x. \quad (13)$$

The subsequent important results have been established by Liu et al. [32].

Theorem 4 (see [32]). *Let u and ν be two positive continuous functions with $u \leq \nu$ on $\mathbb{J}$. Assume that the functions u/ν and* *u are decreasing and increasing, respectively. If the function Ω is a convex with $\Omega(0) = 0$, then*

$$\frac{\int_a^b u(x)dx}{\int_a^b v(x)dx} \geq \frac{\int_a^b \Omega(u(x))dx}{\int_a^b \Omega(v(x))dx}. \tag{14}$$

Theorem 5 (see [32]). *Let u, z, and v be positive continuous functions with $u \leq v$ on $\mathbb{J}$. Let also u, z are increasing functions and u/v is a decreasing function. If the function Ω is a convex with $\Omega(0) = 0$, then*

$$\frac{\int_a^b u(x)dx}{\int_a^b v(x)dx} \geq \frac{\int_a^b \Omega(u(x))z(x)dx}{\int_a^b \Omega(v(x))z(x)dx}. \tag{15}$$

3. Main Results

In this section, we will establish some new tolerances for the convex functions under a new fractional integral operator which combine together the two operators proposed in [30, 31].

Definition 6. Let $\alpha > 0, k > 0, s > -1$, and $\beta, \eta, r \in \mathbb{R}$. Then, the generalized fractional integral operator of order α for a continuous function u is defined as

$$ {}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta} u(x) = \frac{(s+1)^{1-\beta/k(\beta/k)} x^r}{k\Gamma_k(\alpha)} \int_a^x \left(x^{s+1} - y^{s+1}\right)^{\alpha/k\frac{\alpha}{k}-1} y^{(s+1)\eta+s} u(y)dy, a < x. \tag{16}$$

Remark 7.

(1) Setting $k = 1, s = \rho - 1$ in Eq. (16), the fractional integral operator Eq. (16) reduces to the generalized Katugampola fractional integral operator defined by Eq. (13)

(2) Setting $\alpha = \beta, r = 0$, and $\eta = 0$ in Eq. (16), the fractional integral operator Eq. (16) reduces to the generalized fractional integral operator defined by Eq. (12)

(3) Setting $k = 1, \alpha = \beta, r = 0$, $\eta = 0$, and $s \longrightarrow -1$ in Eq. (16), with L'Hôpital's rule, the fractional integral operator Eq. (16) reduces to the Hadamard fractional integral operator, namely,

$$ {}^H\mathbb{I}_a^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{y}\right)^{\alpha-1} \frac{u(y)}{y} dy, a < x. \tag{17}$$

(4) Setting $k = 1, \beta = 0, s = \rho - 1$, and $r = -\rho(\eta + \alpha)$, in Eq. (16), the fractional integral operator Eq. (16) reduces to the Erdélyi-Kober fractional integral operator (see [15])

(5) Setting $\alpha = \beta, r = 0$, $\eta = 0$, and $s = 0$ in Eq. (16), the fractional integral operator Eq. (16) reduces to the k-Riemann-Liouville fractional integral operator, i.e.,

$$ {}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta} u(x) \longrightarrow \mathbb{I}_{a,k}^\alpha u(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-y)^{\alpha/k\frac{\alpha}{k}-1} u(y)dy, \quad a < x. \tag{18}$$

(6) Setting $k = 1, \alpha = \beta, r = 0$, $\eta = 0$, and $s = 0$ in Eq. (16), the fractional integral operator Eq. (16) reduces to the Riemann-Liouville fractional integral operator defined by

$$ \mathbb{I}_a^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} u(y)dy, a < x. \tag{19}$$

Now, we are ready to provide the inequalities for convex functions by using the considered fractional integral operator defined by Eq. (16).

Theorem 8. *Let u and v be positive continuous functions with $u \leq v$ on $\mathbb{J}$. If u and u/v are increasing and decreasing on $\mathbb{J}$, respectively, then for any convex function Ω with $\Omega(0) = 0$, we have*

$$\frac{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}[u(x)]}{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}[v(x)]} \geq \frac{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}[\Omega(u(x))]}{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}[\Omega(v(x))]}, \tag{20}$$

where $\alpha, \beta, a, \eta, r, s, k$ are as in Definition 6.

Proof. By the hypotheses of theorem, Ω is convex with $\Omega(0) = 0$. Then the function $\Omega(x)/x$ is increasing. Since u is an increasing function, thus, $\Omega(u(x))/u(x)$ is an increasing function too.

Obviously, $u(x)/v(x)$ is a decreasing function. Therefore, for each $\zeta, \xi \in \mathbb{J}$, we have

$$\left(\frac{\Omega(u(\zeta))}{u(\zeta)} - \frac{\Omega(u(\xi))}{u(\xi)}\right)\left(\frac{u(\xi)}{v(\xi)} - \frac{u(\zeta)}{v(\zeta)}\right) \geq 0. \tag{21}$$

It follows that

$$\frac{\Omega(u(\zeta))}{u(\zeta)}\frac{u(\xi)}{v(\xi)} + \frac{\Omega(u(\xi))}{u(\xi)}\frac{u(\zeta)}{v(\zeta)} - \frac{\Omega(u(\xi))}{u(\xi)}\frac{u(\xi)}{v(\xi)} - \frac{\Omega(u(\zeta))}{u(\zeta)}\frac{u(\zeta)}{v(\zeta)} \geq 0. \tag{22}$$

Multiplying Eq. (22) by $\nu(\zeta)\nu(\xi)$, we obtain

$$\frac{\Omega(u(\zeta))}{u(\zeta)}u(\xi)\nu(\zeta)+\frac{\Omega(u(\xi))}{u(\xi)}u(\zeta)\nu(\xi)-\frac{\Omega(u(\xi))}{u(\xi)}u(\xi)\nu(\zeta)-\frac{\Omega(u(\zeta))}{u(\zeta)}u(\zeta)\nu(\xi)\geq 0. \tag{23}$$

Multiplying Eq. (23) by $(s+1)^{1-\beta/k}x^r/k\Gamma_k(\alpha)(x^{s+1}-\zeta^{s+1})^{\alpha/k-1}\zeta^{(s+1)\eta+s}$ and integrating Eq. (23) with respect to ζ over $[a,x]$,$a<x\leq b$, we get

$$\begin{aligned}&\frac{(s+1)^{1-\beta/k}x^r}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-\zeta^{s+1}\right)^{\alpha/k-1}\zeta^{(s+1)\eta+s}\frac{\Omega(u(\zeta))}{u(\zeta)}u(\xi)\nu(\zeta)d\zeta\\&+\frac{(s+1)^{1-\beta/k}x^r}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-\zeta^{s+1}\right)^{\alpha/k-1}\zeta^{(s+1)\eta+s}\frac{\Omega(u(\xi))}{u(\xi)}u(\zeta)\nu(\xi)d\zeta\\&-\frac{(s+1)^{1-\beta/k}x^r}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-\zeta^{s+1}\right)^{\alpha/k-1}\zeta^{(s+1)\eta+s}\frac{\Omega(u(\xi))}{u(\xi)}u(\xi)\nu(\zeta)d\zeta\\&-\frac{(s+1)^{1-\beta/k}x^r}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-\zeta^{s+1}\right)^{\alpha/k-1}\zeta^{(s+1)\eta+s}\frac{\Omega(u(\zeta))}{u(\zeta)}u(\zeta)\nu(\xi)d\zeta\geq 0.\end{aligned} \tag{24}$$

Hence

$$\begin{aligned}&u(\xi){}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}\nu(x)\right)+\left(\frac{\Omega(u(\xi))}{u(\xi)}\nu(\xi)\right){}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}(u(x))\\&-\left(\frac{\Omega(u(\xi))}{u(\xi)}u(\xi)\right){}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}(\nu(x))-\nu(\xi){}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}u(x)\right)\geq 0.\end{aligned} \tag{25}$$

Again, multiplying Eq. (25) by $(s+1)^{1-\beta/k}x^r/k\Gamma_k(\alpha)(x^{s+1}-\xi^{s+1})^{\alpha/k-1}\xi^{(s+1)\eta+s}$ and integrating Eq. (25) with respect to ξ over $[a,x]$,$a<x\leq b$, we obtain

$$\begin{aligned}&{}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}u(x){}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}\nu(x)\right)+{}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}\nu(x)\right){}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}(u(x))\\&\geq{}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}(\Omega(u(x))){}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}\nu(x)+{}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}\nu(x){}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}(\Omega(u(x))).\end{aligned} \tag{26}$$

Consequently, we have

$$\frac{{}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}u(x)}{{}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}\nu(x)}\geq\frac{{}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}(\Omega(u(x)))}{{}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}((\Omega(u(x))/u(x))\nu(x))}. \tag{27}$$

Since $u(x)\leq\nu(x)$ for all $x\in\mathbb{J}$ and the function $\Omega(x)/x$ is an increasing, thus, for $\zeta\in a,x]$,$a<x\leq b$, we have

$$\frac{\Omega(u(\zeta))}{u(\zeta)}\leq\frac{\Omega(\nu(\zeta))}{\nu(\zeta)}. \tag{28}$$

Multiplying both sides of Eq. (28) by $[(s+1)^{1-\beta/k}x^r/k\Gamma_k(\alpha)(x^{s+1}-\zeta^{s+1})^{\alpha/k-1}\zeta^{(s+1)\eta+s}]\nu(\zeta)$ then integrating with respect to ζ over $[a,x]$,$a<x\leq b$, we get

$$\begin{aligned}&\frac{(s+1)^{1-\beta/k}x^r}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-\zeta^{s+1}\right)^{\alpha/k-1}\zeta^{(s+1)\eta+s}\frac{\Omega(u(\zeta))}{u(\zeta)}\nu(\zeta)d\zeta\\&\leq\frac{(s+1)^{1-\beta/k}x^r}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-\zeta^{s+1}\right)^{\alpha/k-1}\zeta^{(s+1)\eta+s}\Omega(\nu(\zeta))d\zeta.\end{aligned} \tag{29}$$

As per Eq. (16) can be written Eq. (29) as follows

$${}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}\nu(x)\right)\leq{}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}\Omega(\nu(x)). \tag{30}$$

Hence, from Eq. (27) and Eq. (30), we obtain the desired result Eq. (20). □

Remark 9.

(i) When = 1, $\alpha=\beta$,$r=0$, $\eta=0$, and $s=0$ in Theorem 8, we get the result (Theorem 3.1) proved by Dahmani [26].

(ii) When $\alpha=\beta=1$,$k=1$,$r=\eta=s=0$, and $x=b$ in Theorem 8, we recapture Theorem 4

(iii) In Theorem 8, if we replace the operator ${}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}$ with the generalized proportional fractional integral operator, then, we obtain the result (Theorem 3.1) proved by Neamah and Ibrahim [33].

Theorem 10. *Let u and ν be positive continuous functions with $u\leq\nu$ on $\mathbb{J}$. If u is increasing and u/ν is decreasing on $\mathbb{J}$, then for any convex function Ω with $\Omega(0)=0$, we have*

$$\frac{{}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}[u(x)]{}_k^s\mathbb{J}_{a,\eta,r}^{\gamma,\beta}[\Omega(\nu(x))]+{}_k^s\mathbb{J}_{a,\eta,r}^{\gamma,\beta}[u(x)]{}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}[\Omega(\nu(x))]}{{}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}[\nu(x)]{}_k^s\mathbb{J}_{a,\eta,r}^{\gamma,\beta}[\Omega(u(x))]+{}_k^s\mathbb{J}_{a,\eta,r}^{\gamma,\beta}[\nu(x)]{}_k^s\mathbb{J}_{a,\eta,r}^{\alpha,\beta}[\Omega(u(x))]}\geq 1, \tag{31}$$

where $\gamma>0$ and $\alpha,\beta,a,\eta,r,s,k$ are as in Definition 6.

Proof. Thanks to the hypotheses of theorem, Ω is convex with $\Omega(0)=0$. Thus, $\Omega(x)/x$ is an increasing function. Furthermore, since u is increasing function, the function $\Omega u(x)/u(x)$ is increasing. Distinctly, $u(x)/\nu(x)$ is decreasing function.

Thus, by multiplying Eq. (25) by $(s+1)^{1-\beta/k}x^r/k\Gamma_k(\gamma)(x^{s+1}-\xi^{s+1})^{\gamma/k-1}\xi^{(s+1)\eta+s}$ and integrating the nascent identity with respect to ξ over $[a,x]$,$a<x\le b$, we obtain

$$\begin{aligned}&{}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}u(x){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}v(x)\right)\\&\quad+{}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}\left(\frac{\Omega(u(x))}{u(x)}v(x)\right){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(u(x))\\&\quad\ge{}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}\left(\frac{\Omega(u(x))}{u(x)}u(x)\right){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(v(x))\\&\quad+{}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}v(x){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}u(x)\right).\end{aligned}\tag{32}$$

Consequently, the inequalities Eq. (30) and Eq. (32) give the inequality Eq. (31). □

Remark 11.

(i) Applying Theorem 10 for $\alpha=\gamma$, we get Theorem 8

(ii) Applying Theorem 10 for $=1$, $\alpha=\beta$,$r=0$, $\eta=0$, and $s=0$, we get the result (Theorem 8) proved by Dahmani [26].

(iii) Applying Theorem 10 for $\alpha=\beta=\gamma=1$,$k=1$, $r=\eta=s=0$, and $x=b$, we get Theorem 4

Theorem 12. *Let u, z, and v be three positive continuous functions and $u\le v$ on $\mathbb{J}$. If u/v is decreasing, u and z are increasing, and Ω is convex function with $\Omega(0)=0$. Then*

$$\frac{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}[u(x)]}{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}[v(x)]}\ge\frac{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}[\Omega(u(x))z(x)]}{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}[\Omega(v(x))z(x)]}.\tag{33}$$

Proof. In view of conditions of theorem, Ω is convex with $\Omega(0)=0$. Thus, $\Omega(x)/x$ is increasing. Besides, from the increasing of u,$\Omega(u(x))/u(x)$ is increasing. Obviously, the function $u(x)/v(x)$ is decreasing. So, for all $\zeta,\xi\in a,x]$ and $a<x\le b$, we have

$$\left(\frac{\Omega(u(\zeta))}{u(\zeta)}z(\zeta)-\frac{\Omega(u(\xi))}{u(\xi)}z(\xi)\right)(u(\xi)v(\zeta)-u(\zeta)v(\xi))\ge 0.\tag{34}$$

Then,

$$\begin{aligned}&\frac{\Omega(u(\zeta))z(\zeta)}{u(\zeta)}u(\xi)v(\zeta)+\frac{\Omega(u(\xi))z(\xi)}{u(\xi)}u(\zeta)v(\xi)\\&\quad-\frac{\Omega(u(\xi))z(\xi)}{u(\xi)}u(\xi)v(\zeta)-\frac{\Omega(u(\zeta))z(\zeta)}{u(\zeta)}u(\zeta)v(\xi)\ge 0.\end{aligned}\tag{35}$$

Multiplying Eq. (35) by $(s+1)^{1-\beta/k}x^r/k\Gamma_k(\alpha)(x^{s+1}-\zeta_1^{s+1})^{\alpha/k-1}\zeta^{(s+1)\eta+s}$ then integrating the resulting inequality with respect to ζ over $[a,x]$,$a<x\le b$, we get

$$\begin{aligned}&\frac{(s+1)^{1-\beta/k}x^r}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-\zeta^{s+1}\right)^{\alpha/k-1}\zeta^{(s+1)\eta+s}\frac{\Omega(u(\zeta))}{u(\zeta)}u(\xi)v(\zeta)z(\zeta)d\zeta\\&\quad+\frac{(s+1)^{1-\beta/k}x^r}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-\zeta^{s+1}\right)^{\alpha/k-1}\zeta^{(s+1)\eta+s}\frac{\Omega(u(\xi))}{u(\xi)}u(\zeta)v(\xi)z(\xi)d\zeta\\&\quad-\frac{(s+1)^{1-\beta/k}x^r}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-\zeta^{s+1}\right)^{\alpha/k-1}\zeta^{(s+1)\eta+s}\frac{\Omega(u(\xi))}{u(\xi)}u(\xi)v(\zeta)z(\xi)d\zeta\\&\quad-\frac{(s+1)^{1-\beta/k}x^r}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-\zeta^{s+1}\right)_1^{\alpha/k-1}\zeta^{(s+1)\eta+s}\frac{\Omega(u(\zeta))}{u(\zeta)}u(\zeta)v(\xi)z(\zeta)d\zeta\ge 0.\end{aligned}\tag{36}$$

Hence,

$$\begin{aligned}&u(\xi){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}v(x)z(x)\right)+\left(\frac{\Omega(u(\xi))}{u(\xi)}v(\xi)z(\xi)\right){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(u(x))\\&\quad-\left(\frac{\Omega(u(\xi))}{u(\xi)}u(\xi)z(\xi)\right){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(v(x))-v(\xi){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}u(x)z(x)\right)\ge 0.\end{aligned}\tag{37}$$

With the same arguments as above for Eq. (37), we obtain

$$\begin{aligned}&{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(u(x)){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}v(x)z(x)\right)\\&\quad+{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}v(x)z(x)\right){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(u(x))\\&\quad\ge{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(\Omega(u(x))z(x)){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(v(x))\\&\quad+{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(v(x)){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(\Omega(u(x))z(x)).\end{aligned}\tag{38}$$

It follows that

$$\frac{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}u(x)}{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}v(x)}\ge\frac{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(\Omega(u(x)z(x))}{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}((\Omega(u(x))/u(x))v(x)z(x))}.\tag{39}$$

Further, since $u\le v$ on $\mathbb{J}$ then using fact that the function $\Omega(x)/x$ is increasing, we can write

$$\frac{\Omega(u(\zeta))}{u(\zeta)}\le\frac{\Omega(v(\zeta))}{v(\zeta)},\ \text{for}\ \zeta\in a,x].\tag{40}$$

By some previously repeated procedure, the inequality Eq. (40) leads to

$$\begin{aligned}&\frac{(s+1)^{1-\beta/k}x^r}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-\zeta^{s+1}\right)^{\alpha/k-1}\zeta^{(s+1)\eta+s}\frac{\Omega(u(\zeta))}{u(\zeta)}v(\zeta)z(\zeta)d\zeta\\&\quad\le\frac{(s+1)^{1-\beta/k}x^r}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-\zeta^{s+1}\right)^{\alpha/k-1}\zeta^{(s+1)\eta+s}\Omega(v(\zeta))z(\zeta)d\zeta.\end{aligned}\tag{41}$$

Given Eq. (16), the inequality Eq. (41) can be written as

$$ {}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}v(x)z(x)\right) \le {}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(\Omega(v(x))z(x)). \quad (42) $$

Therefore, from Eq. (42) and Eq. (39), we get Eq. (33), which completes the proof. □

Remark 13.

(1) Applying Theorem 12 for $k = 1, \alpha = \beta$, and $r = \eta = s = 0$, we get the result (Theorem 10) proved by Dahmani [26]

(2) It is noteworthy that Theorem 5 is a special case of Theorem 12 when $\alpha = \beta = 1, k = 1, r = \eta = s = 0$, and $x = b$

Theorem 14. *Let u, z, and v be three positive continuous functions and $u \le v$ on $\mathbb{J}$. If u/v is decreasing, u and z are increasing on $\mathbb{J}$, and Ω is convex function such that $\Omega(0) = 0$. Then, we have*

$$ \frac{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}[u(x)]{}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}[\Omega(v(x))z(x)] + {}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}[u(x)]{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}[\Omega(v(x))z(x)]}{{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}[v(x)]{}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}[\Omega(u(x))z(x)] + {}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}[v(x)]{}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}[\Omega(u(x))z(x)]} \ge 1. \quad (43) $$

Proof. Applying $(s+1)^{1-\beta/k}x^r/k\Gamma_k(\gamma)(x^{s+1}-\xi^{s+1})^{\gamma/k-1}\xi^{(s+1)\eta+s}$ on both sides of Eq. (37), then integrating the resulting inequality with respect to ξ over $[a,x], a < x \le b$, we get

$$ \begin{aligned} & {}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}u(x){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}v(x)z(x)\right) \\ & + {}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}\left(\frac{\Omega(u(x))}{u(x)}v(x)z(x)\right){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(u(x)) \\ & \ge {}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}\left(\frac{\Omega(u(x))}{u(x)}u(x)z(x)\right){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(v(x)) \\ & + {}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}v(x){}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}u(x)z(x)\right). \end{aligned} \quad (44) $$

Since $u \le v$ on $\mathbb{J}$, then using fact that the function $\Omega(x)/x$ is increasing, we have

$$ \frac{\Omega(u(\zeta))}{u(\zeta)} \le \frac{\Omega(v(\zeta))}{v(\zeta)}, \text{for } \zeta \in a, \xi] \text{ and } \xi \in \mathbb{J}. \quad (45) $$

Multiplying Eq. (45) by

$$ \left[\frac{(s+1)^{1-\beta/k}x^r}{k\Gamma_k(\alpha)}\left(x^{s+1}-\zeta^{s+1}\right)^{\alpha/k-1}\zeta^{(s+1)\eta+s}\right]v(\zeta)z(\zeta), \quad (46) $$

then integration with respect to ζ over $[a,x], a < x \le b$, we obtain

$$ {}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}\left(\frac{\Omega(u(x))}{u(x)}v(x)z(x)\right) \le {}_k^s\mathbb{I}_{a,\eta,r}^{\alpha,\beta}(\Omega(v(x))z(x)). \quad (47) $$

Following similar arguments as mentioned earlier, we conclude that

$$ {}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}\left(\frac{\Omega(u(x))}{u(x)}v(x)z(x)\right) \le {}_k^s\mathbb{I}_{a,\eta,r}^{\gamma,\beta}z(\Omega(v(x))z(x)). \quad (48) $$

Hence, by virtue of Eq. (44), Eq. (47), and Eq. (48), we obtain Eq. (43). Thus, the proof is completed. □

Remark 15.

(i) Applying Theorem 14 for $\alpha = \gamma$, we obtain Theorem 12

(ii) Applying Theorem 14 for $k = 1, \alpha = \beta = \gamma$, and $r = \eta = s = 0$, we obtain Theorem 12 proved by Dahmani [26]

4. Conclusions

In this work, we have established certain Pólya-Szegö inequalities by using convex functions under a new generalized fractional integral operator. More precisely, some new results have been established by merging the (k,s)-Riemann-Liouville fractional integral operator with the generalized Katugampola fractional integral operator. Moreover, we have introduced several new special results that cover many classical fractional integral operators.

In future work, it will be very interesting to study the inequalities considered in this work under a more general fractional integral operator in terms of another function ψ, precisely, we hint to $\psi(x) = x^{s+1}$, and this is what we will think about in the next work.

Data Availability

Data are available upon request.

Conflicts of Interest

No conflicts of interest are related to this work.

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