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Research Article

Coefficient Estimates for Some Classes of Biunivalent Function Associated with Jackson q-Difference Operator

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Using the Jackson q-difference operator, we present two new subclasses of biunivalent functions. Furthermore, we estimate the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions belonging to these new subclasses. Our results generalize some of the previously related works of several authors.

1. Introduction

Let A denote the normalized analytical function family f of the formula

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U. If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function $w = \sum_{k=1}^{\infty} w_k z^k$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)), z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence (cf., e.g., [1, 2])

$$f \prec g \Leftrightarrow f(0) = g(0), \quad f(U) \subset g(U). \tag{2}$$

q-calculus plays an important role in the theory of hypergeometric series and quantum theory, number theory, statistical mechanics, etc. In the early 1900s, studies on q-difference equations were intensified by Jackson [3, 4], Carmichael [5], Mason [6], and Trjitzinsky [7]. It was Ismail et al. [8] who introduced geometric function theory and q-theory together for the first time. Following the same idea, the q-difference operator has been extensively investigated in the field of geometric function theory by many authors; for some recent works related to this operator on the classes of analytic functions, we refer to [9–11]. For any nonnegative integer k the q-number (or basic number) $[k]_q$is defined by

$$[k]_q = \frac{1 - q^k}{1 - q} . [0]_q = 0. \tag{3}$$

For nonnegative integer k, the q-factorial is defined by

$$[k]_q ! = [1]_q [2]_q [3]_q \cdots . [k]_q \quad \left([0]_q ! = 1\right). \tag{4}$$

We note that when $q \longrightarrow 1, [k]!$ reduces to classical definition of factorial. Throughout in this paper, we will assume q to be a fixed number between 0 and 1.

For $f(z) \in A$, the q-derivative operator or q-difference operator is defined as

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z} & (z \neq 0), \\ f'(z) & (z = 0), \end{cases} \tag{5}$$

and $D_q^2(f(z)) = D_q(D_q(f(z)))$. From (5), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}. \tag{6}$$

Recently, Govindaraj and Sivasubramanian [12] defined Salagean q-derivative operator as follows:

$$\begin{aligned} S_q^0 f(z) &= f(z), \\ S_q^1 f(z) &= z D_q f(z), \\ S_q^n f(z) &= z D_q \left(S_q^{n-1}(f(z)) \right). \end{aligned} \tag{7}$$

A simple calculation implies

$$S_q^n f(z) = \left(f * G_q^n \right)(z) \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, \cdots\}), \tag{8}$$

where

$$G_q^n(z) = z + \sum_{k=2}^{\infty} [k]_q^n z^k \quad (n \in \mathbb{N}_0, z \in U). \tag{9}$$

Making use of (8) and (9), the power series of $S_q^n f(z)$ for f of the form (1) is given by

$$S_q^n f(z) = z + \sum_{k=2}^{\infty} [k]_q^n a_k z^k, \tag{10}$$

and $\lim_{q \longrightarrow 1^-} S_q^n f(z) = D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$, which is the familiar Salagean derivative [13]. Shams et al. [14] introduced and investigated the class $US^*(k;\beta)$ of parabolic starlike functions and the class $UC(k;\beta)$ of parabolic convex functions of order $\beta\,(0 \leq \beta < 1)$ as

$$US(k;\beta) = \left\{ f \in A : \Re\left(\frac{zf'(z)}{f(z)} - \beta \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\}, k \geq 0, \tag{11}$$

$$UC(k;\beta) = \left\{ f \in A : \Re\left(1 + \frac{zf''(z)}{f'(z)} - \beta \right) > k \left| \frac{zf''(z)}{f'(z)} \right|, k \geq 0 \right\}. \tag{12}$$

Since $\mathrm{Re}\, w > \alpha|w-1| + \gamma$ if and only if $\mathrm{Re}\,\{w(1 + \alpha e^{i\theta}) - \alpha e^{i\theta}\} > \gamma$ (see [15]), then the conditions (11) and (12) can be written as

$$\begin{aligned} US(k;\beta) &= \left\{ f \in A : \Re\left[\left(1 + ke^{i\theta}\right) \frac{zf'(z)}{f(z)} - ke^{i\theta} \right] > \beta \right\}, k \geq 0, \\ US(k;\beta) &= \left\{ f \in A : \Re\left[\left(1 + ke^{i\theta}\right) \left(1 + \frac{zf''(z)}{f'(z)}\right) - \beta e^{i\theta} \right] > \beta \right\}, k \geq 0. \end{aligned} \tag{13}$$

Let φ be analytic function with positive real part and normalized by the conditions $\varphi(0) = 1, \varphi'(0) > 0$ and φ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots, B_1 > 0. \tag{14}$$

Definition 1. A function $f(z) \in A$ is said to be in the class $S(\gamma, \beta, \lambda, \phi)$ $(\beta \geq 0, 0 \leq \lambda \leq 1, -\pi \leq \gamma < \pi)$ if it satisfies

$$\left(1 + \beta e^{i\gamma}\right) \left[(1-\lambda) \frac{zD_q f(z)}{f(z)} + \lambda \frac{D_q\left(zD_q f(z)\right)}{D_q f(z)} \right] - \beta e^{i\gamma} \prec \varphi(z)\ (z \in U). \tag{15}$$

Remark 2. Taking $\beta = 0$, $\varphi(z) = (1+z)/(1-z)$, and $q \longrightarrow 1^-$ in the class $S(\gamma, \beta, \lambda, \phi)$, we get the well-known class of λ-convex functions which was studied by [16].

Definition 3. A function $f(z) \in A$ is said to be in the class $B(n, \gamma, \beta, \lambda, \phi)$ $(\beta \geq 0; n \in \mathbb{N}, 0 \leq \lambda \leq 1, -\pi \leq \gamma < \pi)$ if it satisfies

$$\left[\left(1 + \beta e^{i\gamma}\right) \frac{(1-\lambda) S_q^{n+1} f(z) + \lambda S_q^{n+2} f(z)}{(1-\lambda) S_q^n f(z) + \lambda S_q^{n+1} f(z)} - \beta e^{i\gamma} \right] \prec \varphi(z) \quad (z \in U). \tag{16}$$

In Definition 3, if we set $F(z) \coloneqq (1-\lambda) S_q^n f(z) + \lambda S_q^{n+1} f(z)$, we obtain a new class $US(\lambda, \gamma, \varphi)$ given below.

Example 1. A function $F(z) \in A$ is said to be in the class $US(\lambda, \gamma, \varphi)$ $(0 \leq \lambda \leq 1, -\pi \leq \gamma < \pi)$ if it satisfies

$$\left(1 + \beta e^{i\gamma}\right) \frac{zD_q F(z)}{F(z)} - \beta e^{i\gamma} \prec \varphi(z) \quad (z \in U). \tag{17}$$

Remark 4. Taking $\beta = 0$ and $\varphi(z) = (1 + (1-2\alpha)z)/(1-z)$ in the class $B(\lambda, \gamma, \varphi)$, we get the class $S_q^*(\alpha)$ of q-starlike functions of order $\alpha(0 \leq \alpha < 1)$ which was introduced by Seoudy and Aouf [17].

The well-known Koebe one-quarter theorem [18] ensures the range of every function of the class S contains the disc $\{w : |w| < 1/4\}$. Thus, every univalent function $f \in$

S has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in U),$$
$$f(f^{-1}(\omega)) = \omega \quad \left(|\omega| < \frac{1}{4}\right), \tag{18}$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \cdots. \tag{19}$$

If a function f and its inverse f^{-1} are both univalent in U, then a member f of A is called biunivalent in U. We symbolize by $\sum$ the family of biunivalent functions in U given by (1). Lewin [19] examined the family $\sum$ and proved that $|a_2| < 1.51$ for elements of the family $\sum$. Later, Brannan et al. [20] claimed that $|a_2| < \sqrt{2}$ for $f \in \sum$. Subsequently, Tan [21] obtained some initial coefficient estimates of functions belonging to the class $\sum$. Brannan and Taha in [22] proposed biconvex and bistarlike functions, which are similar to well-known subfamilies of S. The research trend in the last decade was the study of subfamilies of $\sum$. Generally, interest was shown to obtain the initial coefficient bounds for certain subfamilies of $\sum$. In 2010, Srivastava et al. [23] introduced two interesting subfamilies of the function family and found bounds for $|a_2|$ and $|a_3|$ of functions belonging to these subfamilies. Subsequently, other writers explored related problems in this direction (see [9, 10, 24–30]).

Definition 5. A function $f \in \sum$ given by (1) is said to be in the class $S_{\sum}(\gamma, \beta, \lambda, \phi)$ if both f and its inverse map $g = f^{-1}$ are in $S(\gamma, \beta, \lambda, \phi)$.

Remark 6. Note the following:

(1) If $\beta = 0$ and $\phi = (1 + \tau^2 z^2)/(1 - \tau z - \tau^2 z^2)$, then the class $S_{\sum}(\gamma, 0, \lambda, (1 + \tau^2 z^2)/(1 - \tau z - \tau^2 z^2))$ is equivalent to the class $\mathrm{SLM}_{\sum}(q, \alpha, 0)$ introduced by [10]

(2) If $q \longrightarrow 1^-$, then the class $S_{\sum}(\gamma, \beta, \lambda, \phi)$ is equivalent to the class the class obtained by Attiya et al. [32]

(3) $\lim_{q \longrightarrow 1^-} S_{\sum}(\gamma, 1, 0, \phi) \equiv \mathfrak{R}_\sigma(\phi)$ (see Darwish et al. [33])

(4) $\lim_{q \longrightarrow 1^-} S_{\sum}(\gamma, 0, 0, (1 + Az)/(1 + Bz)) \equiv S[A, B]$ (see Hamidi and Jahangiri [34])

(5) $\lim_{q \longrightarrow 1^-} S_{\sum}(\gamma, 0, \lambda, \phi) \equiv M_{\sum}^{q}(\lambda, \phi)(\varphi(z) = 1)$ (see Goyal and Kumar [35] and also Zireh et al. [36])

Definition 7. A function $f \in \sum$ given by (1) is said to be in the class $B_{\sum}(n, \gamma, \beta, \lambda, \phi)$ if both f and its inverse map $g = f^{-1}$ are in $B(n, \gamma, \beta, \lambda, \phi)$.

In Definition 7, if we set $F(z) \coloneqq (1 - \lambda)S_q^n f(z) + \lambda S_q^{n+1} f(z)$ and $G(z) \coloneqq (1 - \lambda)S_q^n g(z) + \lambda S_q^{n+1} g(z)$, we obtain a new class $\mathrm{US}_{\sum}(\lambda, \gamma, \varphi)$ given below.

Example 2. A function $F \in \sum$ given by (1) is said to be in the class $\mathrm{US}_{\sum}(\lambda, \gamma, \varphi)$ if

$$\left(1 + \beta e^{i\gamma}\right)\frac{zD_q F(z)}{F(z)} - \beta e^{i\gamma} \prec \varphi(z) \quad (z \in U),$$
$$\left(1 + \beta e^{i\gamma}\right)\frac{wG'(w)}{G(w)} - \beta e^{i\gamma} = \phi(w) \quad (w \in U). \tag{20}$$

Remark 8. Note the following:

(1) If $\beta = 0$, then the class $B_{\sum}(n, \gamma, 0, \lambda, \phi)$ is equivalent to the class $P\sum_q^k \lambda, \phi)$ introduced by Murugusundaramoorthy et al. [31]

(2) If $q \longrightarrow 1^-$, then the class $B_{\sum}(0, \gamma, \beta, \lambda, \phi)$ is equivalent to the class introduced by Attiya et al. [32]

(3) $\lim_{q \longrightarrow 1^-} B_{\sum}(0, \gamma, 1, 0, \phi) \equiv \mathfrak{R}_\sigma(\phi)$ (see Darwish et al. [33])

(4) $\lim_{q \longrightarrow 1^-} B_{\sum}(0, \gamma, 0, 0, (1 + Az)/(1 + Bz)) = S[A, B]$ (see Hamidi and Jahangiri [34])

(5) $\lim_{q \longrightarrow 1^-} B_{\sum}(0, \gamma, 0, \lambda, \phi) \equiv M_{\sum}^{q}(\lambda, \phi)(\varphi(z) = 1)$ (see Goyal and Kumar [35] and also Zireh et al. [36])

In order to prove our main results, we will need the following result.

Lemma 9 (see [37]). *If $h \in P$, then $|c_k| \le 2$ for each k, where P is the family of all functions h analytic in U for which* $\mathrm{Re}\,(h(z)) > 0$, *$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ for $z \in U$.*

The aim of this paper is to find bounds on first two coefficients in the Taylor-Maclaurin expansion functional problem for functions belonging to the classes $S_{\sum}(\gamma, \beta, \lambda, \phi)$ and $B_{\sum}(n, \gamma, \beta, \lambda, \phi)$. We also indicate interesting cases of the main results.

2. Coefficient Estimates

Unless otherwise mentioned, we shall assume throughout the remainder of this paper that $\beta \ge 0$, $0 \le \lambda \le 1$, $-\pi \le \gamma < \pi$, $n \in \mathbb{N}_0$, and $z \in U$.

Theorem 10. *Let the function f given by (1) be in the class $S_{\sum}(\gamma, \beta, \lambda, \phi)$. Then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{q\sqrt{\left|\left(1 + \beta e^{i\gamma}\right)\left\{\left(1 - \lambda + \lambda 2]_q\right)B_1^2 + \left(1 + \beta e^{i\gamma}\right)(1 + \lambda q)^2 (B_1 - B_2)\right\}\right|}}, \tag{21}$$

$$|a_3| \le \frac{B_1}{q[2]_q|1+\beta e^{i\gamma}|\left(1+\lambda q[2]_q\right)} + \frac{B_1^2}{2q^2\left|(1+\beta e^{i\gamma})^2\right|(1+\lambda q)^2}. \tag{22}$$

Proof. Let $f \in S_\sum(\gamma, \beta, \lambda, ,\phi)$; then f and its inverse map $g = f^{-1}$ are in the class $S(\gamma, \beta, \lambda, \phi)$; there exist two analytic functions $u; v : U \longrightarrow U$ with $u(0) = v(0) = 0$, $|u(z)| < 1$ and $|v(z)| < 1$, such that

$$\left(1+\beta e^{i\gamma}\right)\left[(1-\lambda)\frac{zD_q f(z)}{f(z)} + \lambda \frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\right] - \beta e^{i\gamma} = \varphi(u(z)), \tag{23}$$

$$\left(1+\beta e^{i\gamma}\right)\left[(1-\lambda)\frac{wD_q g(w)}{g(w)} + \lambda \frac{D_q\left(wD_q g(w)\right)}{D_q g(w)}\right] - \beta e^{i\gamma} = \varphi(v(w)). \tag{24}$$

Define the functions p and q by

$$\begin{aligned} p(z) &= \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots, \\ q(w) &= \frac{1+v(w)}{1-v(w)} = 1 + q_1 w + q_2 w^2 + \cdots, \end{aligned} \tag{25}$$

or equivalently,

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \cdots\right), \tag{26}$$

$$v(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}\left(q_1 w + \left(q_2 - \frac{q_1^2}{2}\right)w^2 + \cdots\right). \tag{27}$$

We observe that $p, q \in P$, and in view of Lemma 9, we have that $|p_n| \le 2$ and $|q_n| \le 2$, for $n \ge 2$. Further, using (26) and (27) together with (14), it is evident that

$$\varphi(u(z)) = 1 + \frac{1}{2}B_1 p_1 z + \left(\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right)z^2 + \cdots, \tag{28}$$

$$\varphi(v(w)) = 1 + \frac{1}{2}B_1 q_1 w + \left(\frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2\right)w^2 + \cdots. \tag{29}$$

Therefore, in view of (23), (24), (28), and (29), we have

$$\begin{aligned} &\left(1+\beta e^{i\gamma}\right)\left[(1-\lambda)\frac{zD_q f(z)}{f(z)} + \lambda \frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\right] - \beta e^{i\gamma} \\ &\quad = 1 + \frac{1}{2}B_1 p_1 z + \left(\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right)z^2 + \cdots, \end{aligned} \tag{30}$$

$$\begin{aligned} &\left(1+\beta e^{i\gamma}\right)\left[(1-\lambda)\frac{wD_q g(w)}{g(w)} + \lambda \frac{D_q\left(wD_q g(w)\right)}{D_q g(w)}\right] - \beta e^{i\gamma} \\ &\quad = 1 + \frac{1}{2}B_1 q_1 w + \left(\frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2\right)w^2 + \cdots. \end{aligned} \tag{31}$$

Since $f \in \sum$ has the Taylor series expansion (1) and $g = f^{-1}$ the series (19), we have

$$\begin{aligned} &\left(1+\beta e^{i\gamma}\right)\left[(1-\lambda)\frac{zD_q f(z)}{f(z)} + \lambda \frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\right] - \beta e^{i\gamma} \\ &\quad = 1 + q\left(1+\beta e^{i\gamma}\right)\left\{(1+\lambda q)a_2 z + \left[\left(1+\lambda q[2]_q\right)[2]_q a_3 \right.\right. \\ &\qquad \left.\left. - \left(1+\lambda q\left([2]_q+1\right)\right)a_2^2\right]z^2 + \cdots\right\}, \end{aligned} \tag{32}$$

$$\begin{aligned} &\left(1+\beta e^{i\gamma}\right)\left[(1-\lambda)\frac{wD_q g(w)}{g(w)} + \lambda \frac{D_q\left(wD_q g(w)\right)}{D_q g(w)}\right] - \beta e^{i\gamma} \\ &\quad = 1 + q\left(1+\beta e^{i\gamma}\right)\left\{-(1+\lambda q)a_2 z + \left[-\left(1+\lambda q[2]_q\right)[2]_q a_3 \right.\right. \\ &\qquad \left.\left. + \left\{2[2]_q\left(1+\lambda q[2]_q\right) - \left(1+\lambda q\left([2]_q+1\right)\right)\right\}a_2^2\right]z^2 + \cdots\right\}. \end{aligned} \tag{33}$$

Comparing the corresponding coefficients of (30) and (32) yields

$$q\left(1+\beta e^{i\gamma}\right)(1+\lambda q)a_2 = \frac{1}{2}B_1 p, \tag{34}$$

$$\begin{aligned} &q\left(1+\beta e^{i\gamma}\right)\left\{\left(1+\lambda q[2]_q\right)[2]_q a_3 - \left(1+\lambda q\left([2]_q+1\right)\right)a_2^2\right\} \\ &\quad = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2. \end{aligned} \tag{35}$$

Similarly, from (31) and (33), we have

$$-q\left(1+\beta e^{i\gamma}\right)(1+\lambda q)a_2 = \frac{1}{2}B_1 q_1, \tag{36}$$

$$\begin{aligned} &q\left(1+\beta e^{i\gamma}\right)\left[-\left(1+\lambda q[2]_q\right)[2]_q a_3 + \left\{2[2]_q\left(1+\lambda q[2]_q\right)\right.\right. \\ &\quad \left.\left. - \left(1+\lambda q\left([2]_q+1\right)\right)\right\}a_2^2\right] = \frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2. \end{aligned} \tag{37}$$

From (34) and (36), it follows that

$$p_1 = -q_1, \tag{38}$$

$$2q^2\left(1+\beta e^{i\gamma}\right)^2(1+\lambda q)^2 a_2^2 = \frac{1}{4}B_1^2\left(p_1^2 + q_1^2\right). \tag{39}$$

Adding (35) and (37) yields

$$2q^2\left(1+\beta e^{i\gamma}\right)\left[1-\lambda+\lambda 2\right]_q a_2^2 = \frac{1}{2}B_1(p_2+q_2)+\frac{1}{4}(B_2-B_1)\left(p_1^2+q_1^2\right). \quad (40)$$

From (39) and (40), we get

$$a_2^2 = \frac{B_1^3(p_2+q_2)}{4q^2(1+\beta e^{i\gamma})\left\{\left(1-\lambda+\lambda 2\right]_q\right)B_1^2+(1+\beta e^{i\gamma})(1+\lambda q)^2(B_1-B_2)\right\}}. \quad (41)$$

Applying Lemma 9 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \le \frac{B_1\sqrt{B_1}}{q\sqrt{\left|(1+\beta e^{i\gamma})\left\{\left(1-\lambda+\lambda 2\right]_q\right)B_1^2+(1+\beta e^{i\gamma})(1+\lambda q)^2(B_1-B_2)\right\}\right|}}. \quad (42)$$

This gives the bound on $|a_2|$ as asserted in (21). Next, in order to find the bound on $|a_3|$, by subtracting (37) from (35), we get

$$a_3 = a_2^2 + \frac{B_1(p_2-q_2)}{4q[2]_q(1+\beta e^{i\gamma})\left(1+\lambda q[2]_q\right)}. \quad (43)$$

Upon substituting the value of a_2^2 from (39), we obtain

$$a_3 = \frac{B_1(p_2-q_2)}{4q[2]_q(1+\beta e^{i\gamma})\left(1+\lambda q[2]_q\right)} + \frac{B_1^2\left(p_1^2+q_1^2\right)}{8q^2(1+\beta e^{i\gamma})^2(1+\lambda q)^2}. \quad (44)$$

Applying Lemma 9 for the coefficients $p_1, p_{2,}q_1$, and q_2, we get

$$|a_3| \le \frac{B_1}{q[2]_q|1+\beta e^{i\gamma}|\left(1+\lambda q[2]_q\right)} + \frac{B_1^2}{2q^2\left|(1+\beta e^{i\gamma})^2\right|(1+\lambda q)^2}, \quad (45)$$

which yield the estimate given by (22), and so the proof of Theorem 10 is completed. □

If we set

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\eta} = 1+2\zeta z+2\zeta^2 z^2+\cdots (0<\zeta\le 1, z\in U), \quad (46)$$

in Definition 5, of the biunivalent function class $S_{\Sigma}(\gamma,\beta,\lambda,\phi)$, we obtain a new class $S_{\Sigma}^{\eta}(\gamma,\beta,\lambda)$ given by Definition 11.

Definition 11. For $0<\zeta\le 1$, a function $f(z)\in A$ is said to be in the class $S_{\Sigma}^{\eta}(\gamma,\beta,\lambda)$ if it satisfies the following conditions:

$$\left(1+\beta e^{i\gamma}\right)\left[(1-\lambda)\frac{zD_q f(z)}{f(z)}+\lambda\frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\right]-\beta e^{i\gamma} \prec \left(\frac{1+z}{1-z}\right)^{\eta} (z\in U),$$
$$\left(1+\beta e^{i\gamma}\right)\left[(1-\lambda)\frac{wD_q g(w)}{g(w)}+\lambda\frac{D_q\left(wD_q g(w)\right)}{D_q g(w)}\right]-\beta e^{i\gamma} \prec \left(\frac{1+w}{1-w}\right)^{\eta} (w\in U), \quad (47)$$

where $g=f^{-1}$.

Using the parameter setting of Definition 11, in Theorem 10, we get the following corollary.

Corollary 12. *For $0<\zeta\le 1$, let the function $f(z)\in A$ be in the class $S_{\Sigma}^{\eta}(\gamma,\beta,\lambda)$. Then*

$$|a_2| \le \frac{2\zeta}{q\sqrt{\left|(1+\beta e^{i\gamma})\left\{\left[1-\lambda+\lambda[2]_q\right](2\zeta)-(1-\zeta)(1+\beta e^{i\gamma})(1+\lambda q)^2\right|}},$$
$$|a_3| \le \frac{2\zeta}{q[2]_q|1+\beta e^{i\gamma}|\left(1+\lambda[2]_q\right)} + \frac{2\zeta^4}{q^2\left|(1+\beta e^{i\gamma})^2\right|(1+\lambda q)^2}. \quad (48)$$

Let

$$\varphi(z) = \frac{1+(1-2\vartheta)z}{1-z} = 1+2(1-\vartheta)z+2(1-\vartheta)z^2+\cdots (0<\vartheta\le 1, z\in U), \quad (49)$$

in Definition 5, of the biunivalent function class $S_{\Sigma}(\gamma,\beta,\lambda,\phi)$, we obtain a new class $S_{\vartheta}^{\Sigma}(\gamma,\beta,\lambda)$ given by Definition 13.

Definition 13. For $0<\vartheta\le 1$, a function $f(z)\in A$ is said to be in the class $S_{\vartheta}^{\Sigma}(\gamma,\beta,\lambda)$ if it satisfies the following conditions:

$$\left(1+\beta e^{i\gamma}\right)\left[(1-\lambda)\frac{zD_q f(z)}{f(z)}+\lambda\frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\right]-\beta e^{i\gamma} \prec \frac{1+(1-2\vartheta)z}{1-z} (z\in U),$$
$$\left(1+\beta e^{i\gamma}\right)\left[(1-\lambda)\frac{wD_q g(w)}{g(w)}+\lambda\frac{D_q\left(wD_q g(w)\right)}{D_q g(w)}\right]-\beta e^{i\gamma} \prec \frac{1+(1-2\vartheta)w}{1-w} (w\in U), \quad (50)$$

where $g=f^{-1}$.

Using the parameter setting of Definition 13 in Theorem 10, we get the following corollary.

Corollary 14. *For $0 < \vartheta \leq 1$, let the function $f(z) \in A$ be in the class $S_{\vartheta}^{\Sigma}(\gamma, \beta, \lambda)$. Then*

$$|a_2| \leq \frac{1}{q}\sqrt{\frac{2(1-\vartheta)}{|1+\beta e^{i\gamma}|\left[1-\lambda+\lambda[2]_q\right]}},$$

$$|a_3| \leq \frac{2(1-\vartheta)}{q[2]_q|1+\beta e^{i\gamma}|\left[1+q\lambda 2\right]_q} + \frac{2(1-\vartheta)^2}{q^2(1+\lambda q)^2\left|(1+\beta e^{i\gamma})^2\right|}. \tag{51}$$

Theorem 15. *Let the function f given by (1) be in the class $B_{\sum}(n, \gamma, \beta, \lambda, \phi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|q(1+\beta e^{i\gamma})\left[[2]_q[3]_q^n\left(1+\lambda q[2]_q\right)-[2]_q^{2n}(1+\lambda q)^2\right]B_1^2+(B_1-B_2)(1+\beta e^{i\gamma})^2 q^2[2]_q^{2n}(1+\lambda q)^2\right|}}, \tag{52}$$

$$|a_3| \leq \frac{B_1}{q[2]_q[3]_q^n(1+\beta e^{i\gamma})\left(1+\lambda q[2]_q\right)} + \frac{B_1^2}{q^2[2]_q^{2n}(1+\beta e^{i\gamma})^2(1+\lambda q)^2}. \tag{53}$$

Proof. Let $f \in B_{\sum}(n, \gamma, \beta, \lambda, \phi)$; there are two Schwarz functions u and v defined by (26) and (27), respectively, such that

$$\left(1+\beta e^{i\gamma}\right)\left[\frac{(1-\lambda)S_q^{n+1}f(z)+\lambda S_q^{n+2}f(z)}{(1-\lambda)S_q^{n}f(z)+\lambda S_q^{n+1}f(z)}\right]-\beta e^{i\gamma}=\varphi(u(z)),$$
$$\left(1+\beta e^{i\gamma}\right)\left[\frac{(1-\lambda)S_q^{n+1}g(w)+\lambda S_q^{n+2}g(w)}{(1-\lambda)S_q^{n}g(w)+\lambda S_q^{n+1}g(w)}\right]-\beta e^{i\gamma}=\varphi(v(w)). \tag{54}$$

Since

$$\begin{aligned}&\left(1+\beta e^{i\gamma}\right)\left[\frac{(1-\lambda)S_q^{n+1}f(z)+\lambda S_q^{n+2}f(z)}{(1-\lambda)S_q^{n}f(z)+\lambda S_q^{n+1}f(z)}\right]-\beta e^{i\gamma}\\ &\quad=1+q\left(1+\beta e^{i\gamma}\right)\Big\{[2]_q^n(1+\lambda q)a_2 z\\ &\qquad+\left[[2]_q[3]_q^n\left(1+\lambda q[2]_q\right)a_3-[2]_q^{2n}(1+\lambda q)^2a_2^2\right]z^2+\cdots\Big\},\end{aligned} \tag{55}$$

$$\begin{aligned}&\left(1+\beta e^{i\gamma}\right)\left[(1-\lambda)\frac{wD_qg(w)}{g(w)}+\lambda\frac{D_q\left(wD_qg(w)\right)}{D_qg(w)}\right]-\beta e^{i\gamma}\\ &\quad=1+q\left(1+\beta e^{i\gamma}\right)\Big\{-[2]_q^n(1+\lambda q)a_2 z\\ &\qquad+\Big\{\left[2[2]_q[3]_q^n\left(1+\lambda q[2]_q\right)-[2]_q^{2n}(1+\lambda q)^2\right)\Big]a_2^2\\ &\qquad-[2]_q[3]_q^n\left(1+\lambda q[2]_q\right)a_3\Big\}z^2+\cdots\Big\}.\end{aligned} \tag{56}$$

Now, upon equating the coefficients in (30) and (55) and in (31) and (56), we get

$$q[2]_q^n\left(1+\beta e^{i\gamma}\right)(1+\lambda q)a_2=\frac{1}{2}B_1p_1, \tag{57}$$

$$\begin{aligned}&q\left(1+\beta e^{i\gamma}\right)\left[[2]_q[3]_q^n\left(1+\lambda q[2]_q\right)a_3-[2]_q^{2n}(1+\lambda q)^2a_2^2\right]\\ &\quad=\frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2,\end{aligned} \tag{58}$$

$$-\left(1+\beta e^{i\gamma}\right)q[2]_q^n(1+\lambda q)a_2=\frac{1}{2}B_1q_1, \tag{59}$$

$$\begin{aligned}&q\left(1+\beta e^{i\gamma}\right)\Big\{\Big[2[2]_q[3]_q^n\left(1+\lambda q[2]_q\right)\\ &\qquad-[2]_q^{2n}(1+\lambda q)^2\Big)\Big]a_2^2-[2]_q[3]_q^n\left(1+\lambda q[2]_q\right)a_3\Big\}\\ &\quad=\frac{1}{2}B_1\left(q_2-\frac{q_1^2}{2}\right)+\frac{1}{4}B_2q_1^2.\end{aligned} \tag{60}$$

From (57) and (59), it follows that

$$p_1=-q_1, \tag{61}$$

and after a few additional measurements using (58)–(60), we find

$$a_2^2 = \frac{B_1^3(p_2+q_2)}{4\left\{ q\left(1+\beta e^{i\gamma}\right)\left[[2]_q[3]_q^n\left(1+\lambda q[2]_q\right) - [2]_q^{2n}(1+\lambda q)^2\right]B_1^2 + (B_1 - B_2)\left(1+\beta e^{i\gamma}\right)^2 q^2[2]_q^{2n}(1+\lambda q)^2 \right\}},$$
$$a_3 = \frac{B_1(p_2-q_2)}{4q[2]_q[3]_q^n(1+\beta e^{i\gamma})\left(1+\lambda q[2]_q\right)} + \frac{B_1^2(p_1^2+q_1^2)}{8q^2[2]_q^{2n}(1+\beta e^{i\gamma})^2(1+\lambda q)^2}. \quad (62)$$

Applying Lemma 9 is followed by the estimates in (52) and (53). □

If we set $\varphi(z) = ((1+z)/(1-z))^\eta$ in Definition 7, of the biunivalent function class $B_\Sigma(n,\gamma,\beta,\lambda,\phi)$, we obtain a new class $B_\Sigma^\eta(n,\gamma,\beta,\lambda)$ given by Definition 16.

Definition 16. For $0<\zeta\le 1$, a function $f(z)\in A$ is said to be in the class $B_\Sigma^\eta(n,\gamma,\beta,\lambda)$ if it satisfies the following conditions:

$$\left(1+\beta e^{i\gamma}\right)\left[\frac{(1-\lambda)S_q^{n+1}f(z)+\lambda S_q^{n+2}f(z)}{(1-\lambda)S_q^n f(z)+\lambda S_q^{n+1}f(z)}\right] - \beta e^{i\gamma} \prec \left(\frac{1+z}{1-z}\right)^\eta (z\in U),$$
$$\left(1+\beta e^{i\gamma}\right)\left[\frac{(1-\lambda)S_q^{n+1}g(w)+\lambda S_q^{n+2}g(w)}{(1-\lambda)S_q^n g(w)+\lambda S_q^{n+1}g(w)}\right] - \beta e^{i\gamma} \prec \left(\frac{1+w}{1-w}\right)^\eta (w\in U), \quad (63)$$

where $g=f^{-1}$.

Using the parameter setting of Definition 16, in Theorem 15, we get the following corollary.

Corollary 17. *For $0<\eta\le 1$, let the function $f(z)\in A$ be in the class $B_\Sigma^\eta(n,\gamma,\beta,\lambda)$. Then*

$$|a_2| \le \frac{2\zeta}{\sqrt{\left| q\left(1+\beta e^{i\gamma}\right)\left[[2]_q[3]_q^n\left(1+\lambda q[2]_q\right)-[2]_q^{2n}(1+\lambda q)^2\right](2\zeta) + q^2[2]_q^{2n}(1-\zeta)\left(1+\beta e^{i\gamma}\right)^2(1+\lambda q)^2 \right|}},$$
$$|a_3| \le \frac{2\zeta}{q[2]_q[3]_q^n|(1+\beta e^{i\gamma})|\left(1+\lambda q[2]_q\right)} + \frac{4\zeta^2}{q^2[2]_q^{2n}\left|(1+\beta e^{i\gamma})^2\right|(1+\lambda q)^2}. \quad (64)$$

If we set $\varphi(z) = (1+(1-2\vartheta)z)/(1-z)$ in Definition 7, of the biunivalent function class $B_\Sigma(n,\gamma,\beta,\lambda,\phi)$, we obtain a new class $B_\Sigma^\vartheta(n,\gamma,\beta,\lambda)$ given by Definition 18.

Definition 18. For $0<\vartheta\le 1$, a function $f(z)\in A$ is said to be in the class $B_\Sigma^\vartheta(n,\gamma,\beta,\lambda)$ if it satisfies the following conditions:

$$\left(1+\beta e^{i\gamma}\right)\left[\frac{(1-\lambda)S_q^{n+1}f(z)+\lambda S_q^{n+2}f(z)}{(1-\lambda)S_q^n f(z)+\lambda S_q^{n+1}f(z)}\right] - \beta e^{i\gamma} \prec \frac{1+(1-2\vartheta)z}{1-z} (z\in U),$$
$$\left(1+\beta e^{i\gamma}\right)\left[\frac{(1-\lambda)S_q^{n+1}g(w)+\lambda S_q^{n+2}g(w)}{(1-\lambda)S_q^n g(w)+\lambda S_q^{n+1}g(w)}\right] - \beta e^{i\gamma} \prec \frac{1+(1-2\vartheta)w}{1-w} (w\in U), \quad (65)$$

where $g=f^{-1}$.

Using the parameter setting of Definition 18 in Theorem 15, we get the following corollary.

Corollary 19. *For $0<\vartheta\le 1$, let the function $f(z)\in A$ be in the class $B_\Sigma^\vartheta(n,\gamma,\beta,\lambda)$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\vartheta)}{q\left|(1+\beta e^{i\gamma})\left[[2]_q[3]_q^n\left(1+\lambda q[2]_q\right)-[2]_q^{2n}(1+\lambda q)^2\right]\right|}},$$
$$|a_3| \le \frac{2(1-\vartheta)}{q[2]_q[3]_q^n|(1+\beta e^{i\gamma})|\left(1+\lambda q[2]_q\right)} + \frac{4(1-\vartheta)^2}{q^2[2]_q^{2n}\left|(1+\beta e^{i\gamma})^2\right|(1+\lambda q)^2}. \quad (66)$$

3. Conclusions

This study introduces two new subclasses of biunivalent functions associated with the Jackson q-difference operator in the open unit disc. We have determined upper bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions belonging to these new subclasses. Our results generalize some of the earlier work of several authors.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

The authors contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

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