

Research Article

Hardy-Leindler-Type Inequalities via Conformable Delta Fractional Calculus

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In this article, some fractional Hardy-Leindler-type inequalities will be illustrated by utilizing the chain law, Hölder's inequality, and integration by parts on fractional time scales. As a result of this, some classical integral inequalities will be obtained. Also, we would have a variety of well-known dynamic inequalities as special cases from our outcomes when $\alpha = 1$.

1. Introduction

The Hardy discrete inequality is known as (see [1])

$$\sum_{r=1}^{\infty} \left(\frac{1}{r} \sum_{j=1}^r l(j) \right)^{\mu} \leq \left(\frac{\mu}{\mu-1} \right)^{\mu} \sum_{r=1}^{\infty} l^{\mu}(r), \quad \mu > 1. \quad (1)$$

where $l(r) > 0$ for all $r \geq 1$.

In [2], Hardy employed the calculus of variations and exemplified the continuous version for (1) as follows:

$$\int_0^{\infty} \left(\frac{1}{y} \int_0^y g(s) ds \right)^{\mu} dy \leq \left(\frac{\mu}{\mu-1} \right)^{\mu} \int_0^{\infty} g^{\mu}(y) dy, \quad \mu > 1, \quad (2)$$

where $\mu \geq 0$ is integrable over any finite interval $(0, y)$, g^{μ} is

convergent and integrable over $(0, \infty)$, and $(\mu/(\mu-1))^{\mu}$ is a sharp constant in (1) and (2).

Leindler in [3] exemplified that if $\mu > 1$ and $\lambda(r), f(r) > 0$, then

$$\sum_{r=1}^{\infty} \lambda(r) \left(\sum_{s=1}^r f(s) \right)^{\mu} \leq \mu^{\mu} \left(\sum_{r=1}^{\infty} \lambda^{1-\mu}(r) \left(\sum_{s=r}^{\infty} \lambda(s) \right)^{\mu} f^{\mu}(r) \right), \quad (3)$$

$$\sum_{r=1}^{\infty} \lambda(r) \left(\sum_{k=r}^{\infty} f(k) \right)^{\mu} \leq \mu^{\mu} \left(\sum_{r=1}^{\infty} \lambda^{1-\mu}(r) \left(\sum_{s=1}^r \lambda(s) \right)^{\mu} f^{\mu}(r) \right). \quad (4)$$

The converses of (3) and (4) are exemplified by Leindler in [4]. Precisely, he established that if $0 < \mu \leq 1$, then

$$\sum_{r=1}^{\infty} \lambda(r) \left(\sum_{s=1}^r f(s) \right)^{\mu} \geq \mu^{\mu} \left(\sum_{r=1}^{\infty} \lambda^{1-\mu}(r) \left(\sum_{s=r}^{\infty} \lambda(s) \right)^{\mu} f^{\mu}(r) \right), \quad (5)$$

$$\sum_{r=1}^{\infty} \lambda(r) \left(\sum_{k=r}^{\infty} f(k) \right)^{\mu} \geq \mu^{\mu} \left(\sum_{r=1}^{\infty} \lambda^{1-\mu}(r) \left(\sum_{s=1}^r \lambda(s) \right)^{\mu} f^{\mu}(r) \right). \quad (6)$$

Saker [5] exemplified the time scale version of (3) and (4), respectively, as follows: suppose that \mathbb{T} be a time scale and $\mu > 1$. If $\Lambda(\tau) := \int_{\tau}^{\infty} \lambda(s) \Delta s$, $\Phi(\tau) := \int_1^{\tau} f(s) \Delta s$, for any $\tau \in I, \infty)_{\mathbb{T}}$, then

$$\int_I^{\infty} \lambda(\tau) (\Phi^{\sigma}(\tau))^{\mu} \Delta \tau \leq \mu^{\mu} \left(\int_I^{\infty} \lambda^{1-\mu}(\tau) \Lambda^{\mu}(\tau) f^{\mu}(\tau) \Delta \tau \right). \quad (7)$$

Also, if $\bar{\Lambda}(\tau) := \int_1^{\tau} \lambda(s) \Delta s$ and $\bar{\Phi}(\tau) := \int_{\tau}^{\infty} f(s) \Delta s$, for any $\tau \in I, \infty)_{\mathbb{T}}$, then

$$\int_I^{\infty} \lambda(\tau) (\bar{\Phi}(\tau))^{\mu} \Delta \tau \leq \mu^{\mu} \left(\int_I^{\infty} \lambda^{1-\mu}(\tau) (\bar{\Lambda}^{\sigma}(\tau))^{\mu} f^{\mu}(\tau) \Delta \tau \right). \quad (8)$$

The converses of (7) and (8) are established by Saker [5]. Precisely, he exemplified that, if \mathbb{T} is a time scale, $0 < \mu \leq 1$, $\Omega(\tau) = \int_{\tau}^{\infty} \lambda(s) \Delta s$, and $\Psi(\tau) = \int_1^{\tau} f(s) \Delta s$, for any $\tau \in I, \infty)_{\mathbb{T}}$, then

$$\int_I^{\infty} \lambda(\tau) (\Psi^{\sigma}(\tau))^{\mu} \Delta \tau \geq \mu^{\mu} \left(\int_I^{\infty} f^{\mu}(\tau) \Omega^{\mu}(\tau) \lambda^{1-\mu}(\tau) \Delta \tau \right). \quad (9)$$

Also, if $\bar{\Omega}(\tau) = \int_1^{\tau} \lambda(s) \Delta s$ and $\bar{\Psi}(\tau) = \int_{\tau}^{\infty} f(s) \Delta s$, for any $\tau \in I, \infty)_{\mathbb{T}}$, then

$$\int_I^{\infty} \lambda(\tau) (\bar{\Psi}(\tau))^{\mu} \Delta \tau \geq \mu^{\mu} \left(\int_I^{\infty} f^{\mu}(\tau) (\bar{\Omega}^{\sigma}(\tau))^{\mu} \lambda^{1-\mu}(\tau) \Delta \tau \right), \quad (10)$$

which are the time scale version for (5) and (6), respectively. For developing dynamic inequalities, see the papers ([6–11]).

Our target in this article is proving some fractional dynamic inequalities for Hardy-Leindler's type, and it is reversed with employing conformable calculus on time scales. This article is structured as follows: In Section 2, we discuss the preliminaries of conformable fractional on time scale calculus which will be required in proving our main outcomes. In Section 3, we will exemplify the major consequences.

2. Basic Concepts

In this part, we introduce the essentials of conformable fractional integral and derivative of order $\alpha \in [0, 1]$ on time scales that will be used in this article (see [12–15]). For a time scale \mathbb{T} , we define the operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$, as

$$\sigma(\tau) := \inf \{s \in \mathbb{T} : s > \tau\}. \quad (11)$$

Also, we define the function $\mu : \mathbb{T} \rightarrow [0, \infty)$ by

$$\mu(\tau) := \sigma(\tau) - \tau. \quad (12)$$

Finally, for any $\tau \in \mathbb{T}$, we refer to the notation $\xi^{\sigma}(\tau)$ by $\xi(\sigma(\tau))$, i.e., $\xi^{\sigma} = \xi \circ \sigma$. In the following, we define conformable α -fractional derivative and α -fractional integral on \mathbb{T} .

Definition 1 (see [16], Definition 3.1). Suppose that $\xi : \mathbb{T} \rightarrow \mathbb{R}$ and $\alpha \in (0, 1]$. Then, for $\tau > 0$, we define $D_{\alpha}(\xi^{\Delta})(\tau)$ to be the number with the property that, for any $\varepsilon > 0$, there is a neighborhood V of τ s.t. $\forall \tau \in V$, we have

$$\left| (\xi^{\sigma}(\tau) - \xi(s)) \sigma^{1-\alpha}(\tau) - D_{\alpha}(\xi^{\Delta})(\tau) (\sigma(\tau) - s) \right| \leq \varepsilon |\sigma(\tau) - s|. \quad (13)$$

The conformable α -fractional derivative on \mathbb{T} at 0 is

$$D_{\alpha}(\xi^{\Delta})(0) = \lim_{\tau \rightarrow 0} D_{\alpha}(\xi^{\Delta})(\tau). \quad (14)$$

Theorem 2 (see [16], Theorem 3.6). Assume $0 < \alpha \leq 1$ and $\nu, \xi : \mathbb{T} \rightarrow \mathbb{R}$ are conformable α -fractional derivatives at $\tau \in \mathbb{T}^k$. Then, we have the following.

(i) The sum $\nu + \xi$ is a conformable α -fractional derivative and

$$D_{\alpha}((\nu + \xi)^{\Delta})(\tau) = D_{\alpha}(\nu^{\Delta})(\tau) + D_{\alpha}(\xi^{\Delta})(\tau) \quad (15)$$

(ii) The product $\nu \xi : \mathbb{T} \rightarrow \mathbb{R}$ is a conformable α -fractional derivative with

$$\begin{aligned} D_{\alpha}((\nu \xi)^{\Delta})(\tau) &= D_{\alpha}(\nu^{\Delta})(\tau) \xi(\tau) + \nu(\sigma(\tau)) D_{\alpha}(\xi^{\Delta})(\tau) \\ &= \nu(\tau) D_{\alpha}(\xi^{\Delta})(\tau) + D_{\alpha}(\nu^{\Delta})(\tau) \xi(\sigma(\tau)) \end{aligned} \quad (16)$$

(iii) If $\xi(\tau) \xi(\sigma(\tau)) \neq 0$, then ν/ξ is a conformable α -fractional derivative with

$$D_{\alpha} \left(\left(\frac{\nu}{\xi} \right)^{\Delta} \right) (\tau) = \frac{D_{\alpha}(\nu^{\Delta})(\tau) \xi(\tau) - \nu(\tau) D_{\alpha}(\xi^{\Delta})(\tau)}{\xi(\tau) \xi(\sigma(\tau))} \quad (17)$$

Lemma 3 (Chain rule). Suppose that $\xi : \mathbb{T} \rightarrow \mathbb{R}$ is continuous and α -fractional differentiable at $\tau \in \mathbb{T}$, for $\alpha \in (0, 1]$ and $\nu : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable. Then, $(\nu \circ \xi) : \mathbb{T} \rightarrow \mathbb{R}$ is α -fractional differentiable and

$$D_\alpha \left((v \circ \zeta)^\Delta \right) (\tau) = v'(\xi(d)) D_\alpha \left(\xi^\Delta \right) (\tau), \text{ where } d \in [\tau, \sigma(\tau)]. \tag{18}$$

Definition 4 (see [16], Definition 4.1). For $0 < \alpha \leq 1$, then the α -conformable Δ fractional integral of ξ is defined as

$$I_\alpha \left(\xi^\Delta \right) (s) = \int \xi(s) \Delta_\alpha s = \int \xi(s) \sigma^{\alpha-1}(s) \Delta s, \text{ for all } s \in \mathbb{T}^k. \tag{19}$$

Theorem 5 (see [16], Theorem 4.3). Let $l, m, n \in \mathbb{T}, \beta \in \mathbb{R}, \alpha \in (0, 1]$, and $v, \xi : \mathbb{T} \rightarrow \mathbb{R}$ be rd-continuous functions. Then,

- (i) $\int_l^m [v(s) + \xi(s)] \Delta_\alpha s = \int_l^m v(s) \Delta_\alpha s + \int_l^m \xi(s) \Delta_\alpha s$
- (ii) $\int_l^m \beta v(s) \Delta_\alpha s = \beta \int_l^m v(s) \Delta_\alpha s$
- (iii) $\int_l^m v(s) \Delta_\alpha s = - \int_m^l v(s) \Delta_\alpha s$
- (iv) $\int_l^m v(s) \Delta_\alpha s = \int_l^n v(s) \Delta_\alpha s + \int_n^m v(s) \Delta_\alpha s$
- (v) $\int_l^l v(s) \Delta_\alpha s = 0$

Lemma 6 (Integration by parts formula [16], Theorem 4.3). Suppose that $l, m \in \mathbb{T}$ where $m > l$. If v, ξ are rd-continuous functions and $\alpha \in (0, 1]$, then

$$(i) \int_l^m v(s) D_\alpha \left(\xi^\Delta \right) (s) \Delta_\alpha s = [v(s) \xi(s)]_l^m - \int_l^m \xi^\sigma(s) D_\alpha (v^\Delta) (s) \Delta_\alpha s. \tag{20}$$

$$(ii) \int_l^m v^\sigma(s) D_\alpha \left(\xi^\Delta \right) (s) \Delta_\alpha s = [v(s) \xi(s)]_l^m - \int_l^m \xi(s) D_\alpha (v^\Delta) (s) \Delta_\alpha s. \tag{21}$$

Lemma 7 (Hölder's inequality). Let $l, m \in \mathbb{T}$ where $m > l$. If $\alpha \in (0, 1]$ and $F, G : \mathbb{T} \rightarrow \mathbb{R}$, then

$$\int_l^m |F(s)G(s)| \Delta_\alpha s \leq \left(\int_l^m |F(s)|^\beta \Delta_\alpha s \right)^{1/\beta} \left(\int_l^m |G(s)|^\mu \Delta_\alpha s \right)^{1/\mu}, \tag{22}$$

where $\beta > 1$ and $1/\beta + 1/\mu = 1$.

Through our paper, we will consider the integrals are given exist (are finite, i.e., convergent).

3. Main Results

Here, we will exemplify our major results in this article. In the pursuing theorem, we will exemplify Leindler's inequality (7) for fractional time scales as follows.

Theorem 8. Suppose that \mathbb{T} be a time scale and $0 < \alpha \leq 1$. If $\mu > 1, \Lambda(\tau) := \int_\tau^\infty \lambda(s) \Delta_\alpha s$ and $\Phi(\tau) := \int_l^\tau f(s) \Delta_\alpha s$, for any $\tau \in [l, \infty)_\mathbb{T}$, then

$$\begin{aligned} & \int_l^\infty \lambda(\tau) (\Phi^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau \\ & \leq (\mu - \alpha + 1) \left(\int_l^\infty \lambda^{1-\mu}(\tau) \Lambda^\mu(\tau) f^\mu(\tau) \Delta_\alpha \tau \right)^{1/\mu} \\ & \quad \times \left(\int_l^\infty \lambda(\tau) (\Phi^\sigma(\tau))^{\mu(\mu-\alpha)/\mu-1} \Delta_\alpha \tau \right)^{(\mu-1)/\mu}. \end{aligned} \tag{23}$$

Proof. By utilizing (20) on

$$\int_l^\infty \lambda(\tau) (\Phi^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau, \tag{24}$$

with $\zeta^\sigma(\tau) = (\Phi^\sigma(\tau))^{\mu-\alpha+1}$ and $D_\alpha(v^\Delta)(\tau) = \lambda(\tau)$, we have

$$\begin{aligned} & \int_l^\infty \lambda(\tau) (\Phi^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau \\ & = v(\tau) \Phi^{\mu-\alpha+1}(\tau) \Big|_l^\infty \\ & \quad + \int_l^\infty (-v(\tau)) D_\alpha \left((\Phi^\Delta)^{\mu-\alpha+1} \right) (\tau) \Delta_\alpha \tau, \end{aligned} \tag{25}$$

where

$$v(\tau) = - \int_\tau^\infty \lambda(s) \Delta_\alpha s = -\Lambda(\tau). \tag{26}$$

Substituting (26) into (25), we get

$$\begin{aligned} & \int_l^\infty \lambda(\tau) (\Phi^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau \\ & = v(\tau) \Phi^{\mu-\alpha+1}(\tau) \Big|_l^\infty + \int_l^\infty \Lambda(\tau) D_\alpha \left((\Phi^\Delta)^{\mu-\alpha+1} \right) (\tau) \Delta_\alpha \tau. \end{aligned} \tag{27}$$

Using $\Phi(l) = 0$ and $\Lambda(\infty) = 0$ in (27), we have

$$\int_l^\infty \lambda(\tau) (\Phi^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau = \int_l^\infty \Lambda(\tau) D_\alpha \left((\Phi^\Delta)^{\mu-\alpha+1} \right) (\tau) \Delta_\alpha \tau. \tag{28}$$

Utilizing the chain rule (18), we get

$$D_\alpha \left((\Phi^\Delta)^{\mu-\alpha+1} \right) (\tau) = (\mu - \alpha + 1) \Phi^{\mu-\alpha}(d) D_\alpha (\Phi^\Delta) (\tau) \tag{29}$$

$$\leq (\mu - \alpha + 1) D_\alpha (\Phi^\Delta) (\tau) (\Phi^\sigma(\tau))^{\mu-\alpha}.$$

Since $D_\alpha (\Phi^\Delta) (\tau) = f(\tau)$, we get

$$D_\alpha \left((\Phi^\Delta)^{\mu-\alpha+1} \right) (\tau) \leq (\mu - \alpha + 1) f(\tau) (\Phi^\sigma(\tau))^{\mu-\alpha}. \tag{30}$$

Substituting (30) into (28) yields

$$\int_l^\infty \lambda(\tau) (\Phi^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau \leq (\mu - \alpha + 1) \int_l^\infty \Lambda(\tau) f(\tau) (\Phi^\sigma(\tau))^{\mu-\alpha} \Delta_\alpha \tau. \tag{31}$$

Inequality (31) can be written as

$$\int_l^\infty \lambda(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau \leq (\mu - \alpha + 1) \int_l^\infty \frac{\Lambda(\tau)f(\tau)}{\lambda^{(\mu-1)/\mu}(\tau)} \lambda^{(\mu-1)/\mu}(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha} \Delta_\alpha \tau. \quad (32)$$

Implementing Hölder's inequality on the R.H.S of (32) with indices $\mu, \mu/(\mu - 1)$, we get

$$\begin{aligned} & \int_l^\infty \frac{\Lambda(\tau)f(\tau)}{\lambda^{(\mu-1)/\mu}(\tau)} \lambda^{(\mu-1)/\mu}(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha} \Delta_\alpha \tau \\ & \leq \left(\int_l^\infty \left(\frac{\Lambda(\tau)f(\tau)}{\lambda^{(\mu-1)/\mu}(\tau)} \right)^\mu \Delta_\alpha \tau \right)^{1/\mu} \left(\int_l^\infty (\lambda^{(\mu-1)/\mu}(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha})^{\mu/(\mu-1)} \Delta_\alpha \tau \right)^{(\mu-1)/\mu} \\ & = \left(\int_l^\infty \frac{\Lambda^\mu(\tau)f^\mu(\tau)}{\lambda^{\mu-1}(\tau)} \Delta_\alpha \tau \right)^{1/\mu} \left(\int_l^\infty \lambda(\tau)(\Phi^\sigma(\tau))^{(\mu(\mu-\alpha))/(\mu-1)} \Delta_\alpha \tau \right)^{(\mu-1)/\mu}. \end{aligned} \quad (33)$$

By substituting (33) into (32), we get

$$\begin{aligned} \int_l^\infty \lambda(\tau)(\Phi^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau & \leq (\mu - \alpha + 1) \left(\int_l^\infty \lambda^{1-\mu}(\tau) \Lambda^\mu(\tau) f^\mu(\tau) \Delta_\alpha \tau \right)^{1/\mu} \\ & \quad \times \left(\int_l^\infty \lambda(\tau)(\Phi^\sigma(\tau))^{\mu(\mu-\alpha)/(\mu-1)} \Delta_\alpha \tau \right)^{(\mu-1)/\mu}, \end{aligned} \quad (34)$$

which is (23). \square

Corollary 9. At $\alpha = 1$ in Theorem 8, then

$$\begin{aligned} & \int_l^\infty \lambda(\tau)(\Phi^\sigma(\tau))^\mu \Delta \tau \\ & \leq \mu \left(\int_l^\infty \lambda^{1-\mu}(\tau) \Lambda^\mu(\tau) f^\mu(\tau) \Delta \tau \right)^{1/\mu} \\ & \quad \times \left(\int_l^\infty \lambda(\tau)(\Phi^\sigma(\tau))^\mu \Delta \tau \right)^{(\mu-1)/\mu}, \end{aligned} \quad (35)$$

where $>1, \Lambda(\tau) := \int_\tau^\infty \lambda(s) \Delta s$, and $\Phi(\tau) := \int_l^\tau f(s) \Delta s$, for any $\tau \in l, \infty)_\mathbb{T}$.

Remark 10. In Corollary 9, if we divide both sides of (35) by the factor

$$\left(\int_l^\infty \lambda(\tau)(\Phi^\sigma(\tau))^\mu \Delta \tau \right)^{(\mu-1)/\mu}, \quad (36)$$

and using the fact that $1 - (\mu - 1)/\mu = 1/\mu$, then

$$\left(\int_l^\infty \lambda(\tau)(\Phi^\sigma(\tau))^\mu \Delta \tau \right)^{1/\mu} \leq \mu \left(\int_l^\infty \lambda^{1-\mu}(\tau) \Lambda^\mu(\tau) f^\mu(\tau) \Delta \tau \right)^{1/\mu}. \quad (37)$$

Elevating the last inequality to the μ th power, we get

$$\int_l^\infty \lambda(\tau)(\Phi^\sigma(\tau))^\mu \Delta \tau \leq \mu^\mu \int_l^\infty \lambda^{1-\mu}(\tau) \Lambda^\mu(\tau) f^\mu(\tau) \Delta \tau, \quad (38)$$

which is (7) in Introduction.

Remark 11. If we put $\mathbb{T} = \mathbb{R}$ (i.e., $\sigma(\tau) = \tau$) in Theorem 5, then

$$\begin{aligned} \int_l^\infty \lambda(\tau) \Phi^{\mu-\alpha+1}(\tau) \tau^{\alpha-1} d\tau & \leq (\mu - \alpha + 1) \left(\int_l^\infty \lambda^{1-\mu}(\tau) \Lambda^\mu(\tau) f^\mu(\tau) \tau^{\alpha-1} d\tau \right)^{1/\mu} \\ & \quad \times \left(\int_l^\infty \lambda(\tau)(\Phi(\tau))^{\mu(\mu-\alpha)/(\mu-1)} \tau^{\alpha-1} d\tau \right)^{(\mu-1)/\mu}, \end{aligned} \quad (39)$$

where $\alpha \in (0, 1], \mu > 1, \Lambda(\tau) := \int_\tau^\infty \lambda(s) s^{\alpha-1} ds$, and $\Phi(\tau) := \int_l^\tau f(s) s^{\alpha-1} ds$, for any $\tau \in [l, \infty)$.

Remark 12. Clearly, for $\alpha = 1$ and $l = 1$, Remark 12 coincides with Remark 10 in [5].

Remark 13. When $\mathbb{T} = \mathbb{Z}$ (i.e., $\sigma(\tau) = \tau + 1$), $\mu > 1$, and $l = 1$ in (23), then we get

$$\begin{aligned} & \sum_{r=1}^\infty \lambda(r) \left(\sum_{s=1}^{\sigma(r)-1} f(s)(s+1)^{\alpha-1} \right)^{\mu-\alpha+1} (r+1)^{\alpha-1} \\ & \leq (\mu - \alpha + 1) \left(\sum_{r=1}^\infty \lambda^{1-\mu}(r) \left(\sum_{s=r}^\infty \lambda(s)(s+1)^{\alpha-1} \right)^\mu f^\mu(r)(r+1)^{\alpha-1} \right)^{1/\mu} \\ & \quad \times \left(\sum_{r=1}^\infty \lambda(r) \left(\sum_{s=1}^{\sigma(r)-1} f(s)(s+1)^{\alpha-1} \right)^{(\mu(\mu-\alpha))/(\mu-1)} (r+1)^{\alpha-1} \right)^{(\mu-1)/\mu}. \end{aligned} \quad (40)$$

If $\alpha = 1$, then (40) becomes

$$\begin{aligned} & \sum_{r=1}^\infty \lambda(r) \left(\sum_{s=1}^{\sigma(r)-1} f(s) \right)^\mu \\ & \leq \mu \left(\sum_{r=1}^\infty \lambda^{1-\mu}(r) \left(\sum_{s=r}^\infty \lambda(s) \right)^\mu f^\mu(r) \right)^{1/\mu} \\ & \quad \times \left(\sum_{r=1}^\infty \lambda(r) \left(\sum_{s=1}^{\sigma(r)-1} f(s) \right)^\mu \right)^{\mu-1/\mu}, \end{aligned} \quad (41)$$

which is Remark 11 in [5].

In the pursuing theorem, we will exemplify Leindler's inequality (8) on fractional time scales as follows.

Theorem 14. Suppose that \mathbb{T} be a time scale and $0 < \alpha \leq 1$. If $>1, \bar{\Lambda}(\tau) := \int_l^\tau \lambda(s) \Delta_\alpha s$, and $\Psi(\tau) := \int_\tau^\infty f(s) \Delta_\alpha s$, for any $\tau \in l, \infty)_\mathbb{T}$, then

$$\begin{aligned} \int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau & \leq (\mu - \alpha + 1) \left(\int_l^\infty \lambda^{1-\mu}(\tau) (\bar{\Lambda}^\sigma(\tau))^\mu f^\mu(\tau) \Delta_\alpha \tau \right)^{1/\mu} \\ & \quad \times \left(\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu(\mu-\alpha)/(\mu-1)} \Delta_\alpha \tau \right)^{(\mu-1)/\mu}. \end{aligned} \quad (42)$$

Proof. By utilizing (20) on

$$\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau, \tag{43}$$

with $\nu(\tau) = (\Psi(\tau))^{\mu-\alpha+1}$ and $D_\alpha(\zeta^\Delta)(\tau) = \lambda(\tau)$, we have

$$\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau = \zeta(\tau)(\Psi(\tau))^{\mu-\alpha+1} \Big|_l^\infty + \int_l^\infty \zeta^\sigma(\tau) \left(-D_\alpha((\Psi^\Delta)^{\mu-\alpha+1})(\tau)\right) \Delta_\alpha \tau, \tag{44}$$

where

$$\zeta(\tau) = \int_l^\tau \lambda(s) \Delta_\alpha s = \bar{\Lambda}(\tau). \tag{45}$$

Substituting (45) into (44), we get

$$\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau = \bar{\Lambda}(\tau)(\Psi(\tau))^{\mu-\alpha+1} \Big|_l^\infty + \int_l^\infty (\bar{\Lambda}(\tau))^\sigma \left(-D_\alpha((\Psi^\Delta)^{\mu-\alpha+1})(\tau)\right) \Delta_\alpha \tau. \tag{46}$$

Using the fact that $\Psi(\infty) = 0$ and $\bar{\Lambda}(l) = 0$, (46) became

$$\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau = \int_l^\infty (\bar{\Lambda}(\tau))^\sigma \left(-D_\alpha((\Psi^\Delta)^{\mu-\alpha+1})(\tau)\right) \Delta_\alpha \tau. \tag{47}$$

Utilizing chain rule (18), we get

$$\begin{aligned} -D_\alpha((\Psi^\Delta)^{\mu-\alpha+1})(\tau) &= -(\mu - \alpha + 1)\Psi^{\mu-\alpha}(d)D_\alpha(\Psi^\Delta)(\tau) \\ &= -(\mu - \alpha + 1)(-f(\tau))\Psi^{\mu-\alpha}(d) \\ &\leq (\mu - \alpha + 1)f(\tau)\Psi^{\mu-\alpha}(\tau). \end{aligned} \tag{48}$$

By substituting (48) into (47), we get

$$\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau \leq (\mu - \alpha + 1) \int_l^\infty (\bar{\Lambda}(\tau))^\sigma f(\tau)\Psi^{\mu-\alpha}(\tau) \Delta_\alpha \tau. \tag{49}$$

Inequality (49) can be written as

$$\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau \leq (\mu - \alpha + 1) \int_l^\infty \frac{(\bar{\Lambda}(\tau))^\sigma f(\tau)}{\lambda^{(\mu-1)/\mu}(\tau)} \lambda^{(\mu-1)/\mu}(\tau)\Psi^{\mu-\alpha}(\tau) \Delta_\alpha \tau. \tag{50}$$

Implementing Hölder’s inequality on the R.H.S of (50)

with indices $\mu, \mu/(\mu - 1)$, we get

$$\begin{aligned} \int_l^\infty \frac{(\bar{\Lambda}(\tau))^\sigma f(\tau)}{\lambda^{(\mu-1)/\mu}(\tau)} \lambda^{(\mu-1)/\mu}(\tau)\Psi^{\mu-\alpha}(\tau) \Delta_\alpha \tau \\ \leq \left(\int_l^\infty \left(\frac{(\bar{\Lambda}(\tau))^\sigma f(\tau)}{\lambda^{(\mu-1)/\mu}(\tau)} \right)^\mu \Delta_\alpha \tau \right)^{1/\mu} \left(\int_l^\infty \left(\lambda^{(\mu-1)/\mu}(\tau)\Psi^{\mu-\alpha}(\tau) \right)^{\mu/(\mu-1)} \Delta_\alpha \tau \right)^{(\mu-1)/\mu} \\ = \left(\int_l^\infty \frac{(\bar{\Lambda}^\sigma(\tau))^\mu f^\mu(\tau)}{\lambda^{\mu-1}(\tau)} \Delta_\alpha \tau \right)^{1/\mu} \left(\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu(\mu-\alpha)/(\mu-1)} \Delta_\alpha \tau \right)^{(\mu-1)/\mu}. \end{aligned} \tag{51}$$

By substituting (51) into (50), we get

$$\begin{aligned} \int_l^\infty \lambda(\tau)\Psi^{\mu-\alpha+1}(\tau) \Delta_\alpha \tau \leq (\mu - \alpha + 1) \left(\int_l^\infty \lambda^{1-\mu}(\tau)(\bar{\Lambda}^\sigma(\tau))^\mu f^\mu(\tau) \Delta_\alpha \tau \right)^{1/\mu} \\ \times \left(\int_l^\infty \lambda(\tau)(\Psi(\tau))^{\mu(\mu-\alpha)/(\mu-1)} \Delta_\alpha \tau \right)^{(\mu-1)/\mu}, \end{aligned} \tag{52}$$

which is (42). □

Corollary 15. *At $\alpha = 1$ in Theorem 14, then*

$$\begin{aligned} \int_l^\infty \lambda(\tau)(\Psi(\tau))^\mu \Delta \tau \leq \mu \left(\int_l^\infty \lambda^{1-\mu}(\tau)(\bar{\Lambda}^\sigma(\tau))^\mu f^\mu(\tau) \Delta \tau \right)^{1/\mu} \\ \times \left(\int_l^\infty \lambda(\tau)(\bar{\Phi}(\tau))^\mu \Delta \tau \right)^{(\mu-1)/\mu}, \end{aligned} \tag{53}$$

where $\mu > 1, \bar{\Lambda}(\tau) := \int_l^\tau \lambda(s) \Delta s$, and $\Psi(\tau) := \int_\tau^\infty f(s) \Delta s$, for any $\tau \in l, \infty)_{\mathbb{T}}$.

Remark 16. In Corollary 15, if we divide both sides of (53) by the factor

$$\left(\int_l^\infty \lambda(\tau)(\Psi(\tau))^\mu \Delta \tau \right)^{(\mu-1)/\mu}, \tag{54}$$

and using the fact that $1 - (\mu - 1)/\mu = 1/\mu$, then

$$\left(\int_l^\infty \lambda(\tau)(\Psi(\tau))^\mu \Delta \tau \right)^{1/\mu} \leq \mu \left(\int_l^\infty \lambda^{1-\mu}(\tau)(\bar{\Lambda}^\sigma(\tau))^\mu f^\mu(\tau) \Delta \tau \right)^{1/\mu}. \tag{55}$$

Elevating the last inequality to the μ th power, we get

$$\int_l^\infty \lambda(\tau)(\Psi(\tau))^\mu \Delta \tau \leq \mu^\mu \int_l^\infty \lambda^{1-\mu}(\tau)(\bar{\Lambda}^\sigma(\tau))^\mu f^\mu(\tau) \Delta \tau, \tag{56}$$

which is (8) in Introduction.

Remark 17. As a result, if $\mathbb{T} = \mathbb{R}$ (i.e., $\sigma(\tau) = \tau$) in Theorem 14, then

$$\int_l^\infty \lambda(\tau) \Psi^{\mu-\alpha+1}(\tau) \tau^{\alpha-1} d\tau \leq (\mu - \alpha + 1) \left(\int_l^\infty \lambda^{1-\mu}(\tau) (\bar{\Lambda}(\tau))^\mu f^\mu(\tau) \tau^{\alpha-1} d\tau \right)^{1/\mu} \times \left(\int_l^\infty \lambda(\tau) (\Psi(\tau))^{\mu(\mu-\alpha)/(\mu-1)} \tau^{\alpha-1} d\tau \right)^{(\mu-1)/\mu}, \quad (57)$$

where $\alpha \in (0, 1]$, $\mu > 1$, $\bar{\Lambda}(\tau) := \int_l^\tau \lambda(s) s^{\alpha-1} ds$, and $\Psi(\tau) := \int_\tau^\infty f(s) s^{\alpha-1} ds$, for any $\tau \in [l, \infty)$.

Remark 18. Clearly, for $\alpha = 1$ and $l = 1$, Remark 17 coincides with Remark 12 in [5].

Remark 19. When $\mathbb{T} = \mathbb{Z}$ (i.e., $\sigma(\tau) = \tau + 1$), $\mu > 1$, and $l = 1$ in (42), we get

$$\sum_{r=1}^\infty \lambda(r) \left(\sum_{s=r}^\infty f(s) (s+1)^{\alpha-1} \right)^{\mu-\alpha+1} (r+1)^{\alpha-1} \leq (\mu - \alpha + 1) \left(\sum_{r=1}^\infty \lambda^{1-\mu}(r) \left(\sum_{k=1}^{r-1} \lambda(k) (k+1)^{\alpha-1} \right)^\mu f^\mu(r) (r+1)^{\alpha-1} \right)^{1/\mu} \times \left(\sum_{r=1}^\infty \lambda(r) \left(\sum_{s=r}^\infty f(s) (s+1)^{\alpha-1} \right)^{\mu(\mu-\alpha)/(\mu-1)} (r+1)^{\alpha-1} \right)^{(\mu-1)/\mu}. \quad (58)$$

If $\alpha = 1$, then (58) becomes

$$\sum_{r=1}^\infty \lambda(r) \left(\sum_{s=r}^\infty f(s) \right)^\mu \leq \mu \left(\sum_{r=1}^\infty \lambda^{1-\mu}(r) \left(\sum_{k=1}^r \lambda(k) \right)^\mu f^\mu(r) \right)^{1/\mu} \times \left(\sum_{r=1}^\infty \lambda(r) \left(\sum_{s=r}^\infty f(s) \right)^\mu \right)^{(\mu-1)/\mu}, \quad (59)$$

which is Remark 13 in [5].

In the pursuing theorem, we will exemplify Leindler's inequality (9) for fractional time scales as follows.

Theorem 20. Suppose that \mathbb{T} be a time scale and $\alpha \in (0, 1]$. If $0 < \mu \leq 1$, $\Omega(\tau) = \int_\tau^\infty \lambda(s) \Delta_\alpha s$ and $F(\tau) = \int_l^\tau f(s) \Delta_\alpha s$, for any $\tau \in l, \infty)_{\mathbb{T}}$, then

$$\left(\int_l^\infty \lambda(\tau) (F^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau \right)^\mu \geq (\mu - \alpha + 1)^\mu \left(\int_l^\infty f^\mu(\tau) \Omega^\mu(\tau) \lambda^{1-\mu}(\tau) \Delta_\alpha \tau \right) \times \left(\int_l^\infty \lambda(\tau) (F^\sigma(\tau))^{\mu(\alpha-\mu)/(1-\mu)} \Delta_\alpha \tau \right)^{\mu-1}. \quad (60)$$

Proof. By applying (20) on

$$\int_l^\infty \lambda(\tau) (F^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau, \quad (61)$$

with $\zeta^\sigma(\tau) = (F^\sigma(\tau))^{\mu-\alpha+1}$ and $D_\alpha(\nu^\Delta)(\tau) = \lambda(\tau)$, we have

$$\int_l^\infty \lambda(\tau) (F^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau = \nu(\tau) F^{\mu-\alpha+1}(\tau) \Big|_\infty^l + \int_l^\infty (-\nu(\tau)) D_\alpha \left((F^\Delta)^{\mu-\alpha+1} \right) (\tau) \Delta_\alpha \tau, \quad (62)$$

where

$$\nu(\tau) = - \int_\tau^\infty \lambda(s) \Delta_\alpha s = -\Omega(\tau). \quad (63)$$

Substituting (63) into (62) yields

$$\int_l^\infty \lambda(\tau) (F^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau = -\Omega(\tau) F^{\mu-\alpha+1}(\tau) \Big|_\infty^l + \int_l^\infty \Omega(\tau) D_\alpha \left((F^\Delta)^{\mu-\alpha+1} \right) (\tau) \Delta_\alpha \tau. \quad (64)$$

Using the fact that $\nu(\infty) = 0$ and $F(l) = 0$, (64) became

$$\int_l^\infty \lambda(\tau) (F^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau = \int_l^\infty \Omega(\tau) D_\alpha \left((F^\Delta)^{\mu-\alpha+1} \right) (\tau) \Delta_\alpha \tau. \quad (65)$$

Utilizing chain rule (18), we get

$$D_\alpha \left((F^\Delta)^{\mu-\alpha+1} \right) (\tau) = (\mu - \alpha + 1) F^{\mu-\alpha}(d) D_\alpha (F^\Delta) (\tau) \geq (\mu - \alpha + 1) D_\alpha (F^\Delta) (\tau) (F^\sigma(\tau))^{\mu-\alpha}. \quad (66)$$

Since $D_\alpha (F^\Delta) (\tau) = f(\tau)$, we obtain

$$D_\alpha \left((F^\Delta)^{\mu-\alpha+1} \right) (\tau) \geq (\mu - \alpha + 1) f(\tau) (F^\sigma(\tau))^{\mu-\alpha}. \quad (67)$$

By substituting (67) into (65), we have

$$\int_l^\infty \lambda(\tau) (F^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau \geq (\mu - \alpha + 1) \int_l^\infty \Omega(\tau) f(\tau) (F^\sigma(\tau))^{\mu-\alpha} \Delta_\alpha \tau = (\mu - \alpha + 1) \int_l^\infty \left[\Omega^\mu(\tau) f^\mu(\tau) (F^\sigma(\tau))^{\mu(\mu-\alpha)} \right]^{1/\mu} \Delta_\alpha \tau. \quad (68)$$

Raises (68) to the factor μ , we get

$$\begin{aligned} & \left(\int_l^\infty \lambda(\tau)(F^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau \right)^\mu \\ & \geq (\mu - \alpha + 1)^\mu \left(\int_l^\infty \lambda \left[\Omega^\mu(\tau) f^\mu(\tau) (F^\sigma(\tau))^{\mu(\mu-\alpha)} \right]^{1/\mu} \Delta_\alpha \tau \right)^\mu. \end{aligned} \tag{69}$$

By applying Hölder’s inequality on

$$\left(\int_l^\infty \left[\Omega^\mu(\tau) f^\mu(\tau) (F^\sigma(\tau))^{\mu(\mu-\alpha)} \right]^{1/\mu} \Delta_\alpha \tau \right)^\mu, \tag{70}$$

with indices $1/\mu$, $1/(1 - \mu)$, and

$$\begin{aligned} F(\tau) &= \frac{f^\mu(\tau) \Omega^\mu(\tau)}{(F^\sigma(\tau))^{\mu(\alpha-\mu)}}, \\ G(\tau) &= \lambda^{1-\mu}(\tau) (F^\sigma(\tau))^{\mu(\alpha-\mu)}, \end{aligned} \tag{71}$$

we see that

$$\begin{aligned} & \left(\int_l^\infty F^{1/\mu}(\tau) \Delta_\alpha \tau \right)^\mu \\ &= \left(\int_l^\infty \left(\frac{f^\mu(\tau) \Omega^\mu(\tau)}{(F^\sigma(\tau))^{\mu(\alpha-\mu)}} \right)^{1/\mu} \Delta_\alpha \tau \right)^\mu \\ &\geq \frac{\int_l^\infty F(\tau) G(\tau) \Delta_\alpha \tau}{\left(\int_l^\infty G^{1/(1-\mu)}(\tau) \Delta_\alpha \tau \right)^{1-\mu}} \\ &= \left(\int_l^\infty \frac{f^\mu(\tau) \Omega^\mu(\tau) \lambda^{1-\mu}(\tau) (F^\sigma(\tau))^{\mu(\alpha-\mu)}}{(F^\sigma(\tau))^{\mu(\alpha-\mu)}} \Delta_\alpha \tau \right) \\ &\quad \times \left(\int_l^\infty \left(\lambda^{1-\mu}(\tau) (F^\sigma(\tau))^{\mu(\alpha-\mu)} \right)^{1/(1-\mu)} \Delta_\alpha \tau \right)^{\mu-1} \\ &= \left(\int_l^\infty f^\mu(\tau) \Omega^\mu(\tau) \lambda^{1-\mu}(\tau) \Delta_\alpha \tau \right) \\ &\quad \cdot \left(\int_l^\infty \lambda(\tau) (F^\sigma(\tau))^{\mu(\alpha-\mu)/(1-\mu)} \Delta_\alpha \tau \right)^{\mu-1}. \end{aligned} \tag{72}$$

This implies that

$$\begin{aligned} & \left(\int_l^\infty \Omega(\tau) f(\tau) (F^\sigma(\tau))^{\mu-\alpha} \Delta_\alpha \tau \right)^\mu \\ & \geq \left(\int_l^\infty f^\mu(\tau) \Omega^\mu(\tau) \lambda^{1-\mu}(\tau) \Delta_\alpha \tau \right) \\ & \quad \times \left(\int_l^\infty \lambda(\tau) (F^\sigma(\tau))^{\mu(\alpha-\mu)/(1-\mu)} \Delta_\alpha \tau \right)^{\mu-1}. \end{aligned} \tag{73}$$

By substituting (73) into (69), we get

$$\begin{aligned} & \left(\int_l^\infty \lambda(\tau)(F^\sigma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau \right)^\mu \\ & \geq (\mu - \alpha + 1)^\mu \left(\int_l^\infty f^\mu(\tau) \Omega^\mu(\tau) \lambda^{1-\mu}(\tau) \Delta_\alpha \tau \right) \\ & \quad \times \left(\int_l^\infty \lambda(\tau) (F^\sigma(\tau))^{\mu(\alpha-\mu)/(1-\mu)} \Delta_\alpha \tau \right)^{\mu-1}, \end{aligned} \tag{74}$$

which is (60). □

Corollary 21. *At $\alpha = 1$ in Theorem 20, then*

$$\begin{aligned} & \left(\int_l^\infty \lambda(\tau)(F^\sigma(\tau))^\mu \Delta \tau \right)^\mu \\ & \geq \mu^\mu \left(\int_l^\infty f^\mu(\tau) \Omega^\mu(\tau) \lambda^{1-\mu}(\tau) \Delta \tau \right) \\ & \quad \times \left(\int_l^\infty \lambda(\tau) (F^\sigma(\tau))^\mu \Delta \tau \right)^{\mu-1}, \end{aligned} \tag{75}$$

where $0 < \mu \leq 1, \Omega(\tau) = \int_\tau^\infty \lambda(s) \Delta s$, and $F(\tau) = \int_l^\tau f(s) \Delta s$, for any $\tau \in [l, \infty)$.

Remark 22. In Corollary 21, if we divide both sides of (75) by the factor

$$\left(\int_l^\infty \lambda(\tau) (F^\sigma(\tau))^\mu \Delta \tau \right)^{\mu-1}, \tag{76}$$

then (75) can be written as

$$\int_l^\infty \lambda(\tau) (F^\sigma(\tau))^\mu \Delta \tau \geq \mu^\mu \left(\int_l^\infty f^\mu(\tau) \Omega^\mu(\tau) \lambda^{1-\mu}(\tau) \Delta \tau \right), \tag{77}$$

which is (9) in Introduction.

Remark 23. As a result, if $\mathbb{T} = \mathbb{R}$ (i.e., $\sigma(\tau) = \tau$) in Theorem 20, then

$$\begin{aligned} & \left(\int_l^\infty \lambda(\tau) (F(\tau))^{\mu-\alpha+1} \tau^{\alpha-1} d\tau \right)^\mu \\ & \geq (\mu - \alpha + 1)^\mu \left(\int_l^\infty f^\mu(\tau) \Omega^\mu(\tau) \lambda^{1-\mu}(\tau) \tau^{\alpha-1} d\tau \right) \\ & \quad \times \left(\int_l^\infty \lambda(\tau) (F(\tau))^{\mu(\alpha-\mu)/(1-\mu)} \tau^{\alpha-1} d\tau \right)^{\mu-1}, \end{aligned} \tag{78}$$

where $\alpha \in (0, 1], 0 < \mu \leq 1, \Omega(\tau) = \int_\tau^\infty \lambda(s) s^{\alpha-1} ds$, and $F(\tau) = \int_l^\tau f(s) s^{\alpha-1} ds$, for any $\tau \in [l, \infty)$.

Remark 24. Clearly, for $\alpha = 1$ and $l = 1$, Remark 23 coincides with Remark 16 in [5].

Remark 25. When $\mathbb{T} = \mathbb{Z}$ (i.e., $\sigma(\tau) = \tau + 1$), $\mu \leq 1$, and $l = 1$ in (60), then we get

$$\begin{aligned} & \left(\sum_{r=1}^{\infty} \lambda(r) \left(\sum_{s=1}^{\sigma(r)-1} f(s)(s+1)^{\alpha-1} \right)^{\mu-\alpha+1} (r+1)^{\alpha-1} \right)^{\mu} \\ & \geq (\mu - \alpha + 1)^{\mu} \left(\sum_{r=1}^{\infty} f^{\mu}(r) \left(\sum_{s=r}^{\infty} \lambda(s)(s+1)^{\alpha-1} \right)^{\mu} \lambda^{1-\mu}(r)(r+1)^{\alpha-1} \right) \\ & \times \left(\sum_{r=1}^{\infty} \lambda(r) \left(\sum_{s=1}^{\sigma(r)-1} f(s)(s+1)^{\alpha-1} \right)^{(\mu-\alpha+1)/(1-\mu)} (r+1)^{\alpha-1} \right)^{\mu-1}. \end{aligned} \quad (79)$$

If $\alpha = 1$, then (79) becomes

$$\begin{aligned} & \left(\sum_{r=1}^{\infty} \lambda(r) \left(\sum_{s=1}^{\sigma(r)-1} f(s) \right)^{\mu} \right)^{\mu} \geq \mu^{\mu} \left(\sum_{r=1}^{\infty} f^{\mu}(r) \left(\sum_{s=r}^{\infty} \lambda(s) \right)^{\mu} \lambda^{1-\mu}(r) \right) \\ & \times \left(\sum_{r=1}^{\infty} \lambda(r) \left(\sum_{s=1}^{\sigma(r)-1} f(s) \right)^{\mu} \right)^{\mu-1}, \end{aligned} \quad (80)$$

which is Remark 17 in [5].

In the pursuing theorem, we will exemplify Leindler's inequality (10) for fractional time scales as follows.

Theorem 26. Suppose that \mathbb{T} be a time scale and $\alpha \in (0, 1]$. If $0 < \mu \leq 1$, $\bar{\Omega}(\tau) = \int_l^{\tau} \lambda(s) \Delta_{\alpha} s$, and $\Gamma(\tau) = \int_{\tau}^{\infty} f(s) \Delta_{\alpha} s$, for any $\tau \in [l, \infty)_{\mathbb{T}}$, then

$$\begin{aligned} & \left(\int_l^{\infty} \lambda(\tau) (\Gamma(\tau))^{\mu-\alpha+1} \Delta_{\alpha} \tau \right)^{\mu} \\ & \geq (\mu - \alpha + 1)^{\mu} \left(\int_l^{\infty} f^{\mu}(\tau) (\bar{\Omega}^{\sigma}(\tau))^{\mu} \lambda^{1-\mu}(\tau) \Delta_{\alpha} \tau \right) \\ & \times \left(\int_l^{\infty} \lambda(\tau) (\Gamma(\tau))^{\mu(\alpha-1)/1-\mu} \Delta_{\alpha} \tau \right)^{\mu-1}. \end{aligned} \quad (81)$$

Proof. By applying (20) on

$$\int_l^{\infty} \lambda(\tau) (\Gamma(\tau))^{\mu-\alpha+1} \Delta_{\alpha} \tau, \quad (82)$$

with $\nu(\tau) = (\Gamma(\tau))^{\mu-\alpha+1}$ and $D_{\alpha}(\zeta^{\Delta})(\tau) = \lambda(\tau)$, we have

$$\begin{aligned} & \int_l^{\infty} \lambda(\tau) (\Gamma(\tau))^{\mu-\alpha+1} \Delta_{\alpha} \tau \\ & = \zeta(\tau) (\Gamma(\tau))^{\mu-\alpha+1} \Big|_l^{\infty} + \int_l^{\infty} \zeta^{\sigma}(\tau) \left(-D_{\alpha} \left((\Gamma^{\Delta})^{\mu-\alpha+1} \right) (\tau) \right) \Delta_{\alpha} \tau, \end{aligned} \quad (83)$$

where

$$\zeta(\tau) = \int_l^{\tau} \lambda(s) \Delta_{\alpha} s = \bar{\Omega}(\tau). \quad (84)$$

Substituting (84) into (83), we get

$$\begin{aligned} & \int_l^{\infty} \lambda(\tau) (\Gamma(\tau))^{\mu-\alpha+1} \Delta_{\alpha} \tau \\ & = \bar{\Omega}(\tau) (\Gamma(\tau))^{\mu-\alpha+1} \Big|_l^{\infty} + \int_l^{\infty} \bar{\Omega}^{\sigma}(\tau) \left(-D_{\alpha} \left((\Gamma^{\Delta})^{\mu-\alpha+1} \right) (\tau) \right) \Delta_{\alpha} \tau. \end{aligned} \quad (85)$$

Using the fact that $\Gamma(\infty) = 0$ and $\bar{\Omega}(l) = 0$, (85) became

$$\int_l^{\infty} \lambda(\tau) (\Gamma(\tau))^{\mu-\alpha+1} \Delta_{\alpha} \tau = \int_l^{\infty} \bar{\Omega}^{\sigma}(\tau) \left(-D_{\alpha} \left((\Gamma^{\Delta})^{\mu-\alpha+1} \right) (\tau) \right) \Delta_{\alpha} \tau. \quad (86)$$

Utilizing chain rule (18), we have

$$\begin{aligned} & -D_{\alpha} \left((\Gamma^{\Delta})^{\mu-\alpha+1} \right) (\tau) \\ & = -(\mu - \alpha + 1) \Gamma^{\mu-\alpha}(d) D_{\alpha}(\Gamma^{\Delta})(\tau) \\ & \geq -(\mu - \alpha + 1) \Gamma^{\mu-\alpha}(\tau) D_{\alpha}(\Gamma^{\Delta})(\tau). \end{aligned} \quad (87)$$

Since $D_{\alpha}(\Gamma^{\Delta})(\tau) = -f(\tau)$, we get

$$-D_{\alpha} \left((\Gamma^{\Delta})^{\mu-\alpha+1} \right) (\tau) \geq (\mu - \alpha + 1) f(\tau) \Gamma^{\mu-\alpha}(\tau). \quad (88)$$

By substituting (88) into (86), we get

$$\begin{aligned} & \int_l^{\infty} \lambda(\tau) (\Gamma(\tau))^{\mu-\alpha+1} \Delta_{\alpha} \tau \\ & \geq (\mu - \alpha + 1) \int_l^{\infty} \bar{\Omega}^{\sigma}(\tau) f(\tau) \Gamma^{\mu-\alpha}(\tau) \Delta_{\alpha} \tau, \\ & = \int_l^{\infty} \left((\bar{\Omega}^{\sigma}(\tau))^{\mu} f^{\mu}(\tau) \Gamma^{\mu(\mu-\alpha)}(\tau) \right)^{1/\mu} \Delta_{\alpha} \tau. \end{aligned} \quad (89)$$

Raises (89) to the factor μ , we have

$$\begin{aligned} & \left(\int_l^{\infty} \lambda(\tau) (\Gamma(\tau))^{\mu-\alpha+1} \Delta_{\alpha} \tau \right)^{\mu} \\ & \geq (\mu - \alpha + 1)^{\mu} \left(\int_l^{\infty} \left((\bar{\Omega}^{\sigma}(\tau))^{\mu} f^{\mu}(\tau) \Gamma^{\mu(\mu-\alpha)}(\tau) \right)^{1/\mu} \Delta_{\alpha} \tau \right)^{\mu}. \end{aligned} \quad (90)$$

By applying Hölder's inequality on

$$\left(\int_l^{\infty} \left((\bar{\Omega}^{\sigma}(\tau))^{\mu} f^{\mu}(\tau) \Gamma^{\mu(\mu-\alpha)}(\tau) \right)^{1/\mu} \Delta_{\alpha} \tau \right)^{\mu}, \quad (91)$$

with indices $1/\mu, 1/(1-\mu)$, and

$$\begin{aligned} F(\tau) &= \frac{f^{\mu}(\tau) (\bar{\Omega}^{\sigma}(\tau))^{\mu}}{\Gamma^{\mu(\alpha-\mu)}(\tau)}, \\ G(\tau) &= \lambda^{1-\mu}(\tau) (\Gamma(\tau))^{\mu(\alpha-\mu)}, \end{aligned} \quad (92)$$

we see that

$$\begin{aligned} & \left(\int_l^\infty F^{1/\mu}(\tau) \Delta_\alpha \tau \right)^\mu \\ &= \left(\int_l^\infty \left(\frac{f^\mu(\tau) (\bar{\Omega}^\sigma(\tau))^\mu}{\Gamma^{\mu(\alpha-\mu)}(\tau)} \right)^{1/\mu} \Delta_\alpha \tau \right)^\mu \\ &\geq \frac{\int_l^\infty F(\tau) G(\tau) \Delta_\alpha \tau}{\left(\int_l^\infty G^{1/(1-\mu)}(\tau) \Delta_\alpha \tau \right)^{1-\mu}} \\ &= \left(\int_l^\infty \frac{f^\mu(\tau) \lambda^{1-\mu}(\tau) (\bar{\Omega}^\sigma(\tau))^\mu (\Gamma(\tau))^{\mu(\alpha-\mu)}}{\Gamma^{\mu(\alpha-\mu)}(\tau)} \Delta_\alpha \tau \right)^\mu \quad (93) \\ &\quad \times \left(\int_l^\infty \left(\lambda^{1-\mu}(\tau) (\Gamma(\tau))^{\mu(\alpha-\mu)} \right)^{1/(1-\mu)} \Delta_\alpha \tau \right)^{\mu-1} \\ &= \left(\int_l^\infty f^\mu(\tau) (\bar{\Omega}^\sigma(\tau))^\mu \lambda^{1-\mu}(\tau) \Delta_\alpha \tau \right) \\ &\quad \times \left(\int_l^\infty \lambda(\tau) (\Gamma(\tau))^{\mu(\alpha-\mu)/(1-\mu)} \Delta_\alpha \tau \right)^{\mu-1}. \end{aligned}$$

This implies that

$$\begin{aligned} \left(\int_l^\infty \left((\bar{\Omega}^\sigma(\tau))^\mu f^\mu(\tau) \Gamma^{\mu(\alpha-\mu)}(\tau) \right)^{1/\mu} \Delta_\alpha \tau \right)^\mu &\geq \left(\int_l^\infty f^\mu(\tau) (\bar{\Omega}^\sigma(\tau))^\mu \lambda^{1-\mu}(\tau) \Delta_\alpha \tau \right) \\ &\quad \times \left(\int_l^\infty \lambda(\tau) (\Gamma(\tau))^{\mu(\alpha-\mu)/(1-\mu)} \Delta_\alpha \tau \right)^{\mu-1}. \quad (94) \end{aligned}$$

By substituting (94) into (90), we get

$$\begin{aligned} & \left(\int_l^\infty \lambda(\tau) (\Gamma(\tau))^{\mu-\alpha+1} \Delta_\alpha \tau \right)^\mu \\ &\geq (\mu - \alpha + 1)^\mu \left(\int_l^\infty f^\mu(\tau) (\bar{\Omega}^\sigma(\tau))^\mu \lambda^{1-\mu}(\tau) \Delta_\alpha \tau \right) \quad (95) \\ &\quad \times \left(\int_l^\infty \lambda(\tau) (\Gamma(\tau))^{\mu(\alpha-\mu)/(1-\mu)} \Delta_\alpha \tau \right)^{\mu-1}, \end{aligned}$$

which is (81). □

Corollary 27. At $\alpha = 1$ in Theorem 26, then

$$\begin{aligned} & \left(\int_l^\infty \lambda(\tau) (\Gamma(\tau))^\mu \Delta \tau \right)^\mu \\ &\geq \mu^\mu \left(\int_l^\infty f^\mu(\tau) (\bar{\Omega}^\sigma(\tau))^\mu \lambda^{1-\mu}(\tau) \Delta \tau \right) \quad (96) \\ &\quad \times \left(\int_l^\infty \lambda(\tau) (\Gamma(\tau))^\mu \Delta \tau \right)^{\mu-1}, \end{aligned}$$

where $0 < \mu \leq 1, \bar{\Omega}(\tau) = \int_l^\tau \lambda(s) \Delta s$, and $\Gamma(\tau) = \int_\tau^\infty f(s) \Delta s$, for any $\tau \in [l, \infty)_\mathbb{T}$.

Remark 28. In Corollary 27, if we divide both sides of (96) by the factor

$$\left(\int_l^\infty \lambda(\tau) (\Gamma(\tau))^\mu \Delta \tau \right)^{\mu-1}, \quad (97)$$

then (96) can be written as

$$\int_l^\infty \lambda(\tau) (\Gamma(\tau))^\mu \Delta \tau \geq \mu^\mu \left(\int_l^\infty f^\mu(\tau) (\bar{\Omega}^\sigma(\tau))^\mu \lambda^{1-\mu}(\tau) \Delta \tau \right), \quad (98)$$

which is (10) in Introduction.

Remark 29. As a result, if $\mathbb{T} = \mathbb{R}$ (i.e., $\sigma(\tau) = \tau$) in Theorem 26, then

$$\begin{aligned} & \left(\int_l^\infty \lambda(\tau) (\Gamma(\tau))^{\mu-\alpha+1} \tau^{\alpha-1} d\tau \right)^\mu \\ &\geq (\mu - \alpha + 1)^\mu \left(\int_l^\infty f^\mu(\tau) (\bar{\Omega}(\tau))^\mu \lambda^{1-\mu}(\tau) \tau^{\alpha-1} d\tau \right) \quad (99) \\ &\quad \times \left(\int_l^\infty \lambda(\tau) (\Gamma(\tau))^{\mu(\alpha-\mu)/(1-\mu)} \tau^{\alpha-1} d\tau \right)^{\mu-1}, \end{aligned}$$

where $\alpha \in (0, 1], 0 < \mu \leq 1, \bar{\Omega}(\tau) = \int_l^\tau \lambda(s) s^{\alpha-1} ds$, and $\Gamma(\tau) = \int_\tau^\infty f(s) s^{\alpha-1} ds$, for any $\tau \in [l, \infty)$.

Remark 30. Clearly, for $\alpha = 1$ and $l = 1$, Remark 29 coincides with Remark 18 in [5].

Remark 31. When $\mathbb{T} = \mathbb{Z}$ (i.e., $\sigma(\tau) = \tau + 1$), $\mu \leq 1$, and $l = 1$ in (81), then we get

$$\begin{aligned} & \left(\sum_{r=1}^\infty \lambda(r) \left(\sum_{k=r}^\infty f(k) (k+1)^{\alpha-1} \right)^{\mu-\alpha+1} (r+1)^{\alpha-1} \right)^\mu \\ &\geq (\mu - \alpha + 1)^\mu \left(\sum_{r=1}^\infty \lambda^{1-\mu}(r) \left(\sum_{k=1}^{r-1} \lambda(k) (k+1)^{\alpha-1} \right)^\mu f^\mu(r) (r+1)^{\alpha-1} \right) \\ &\quad \times \left(\sum_{r=1}^\infty \lambda(r) \left(\sum_{k=r}^\infty f(k) (k+1)^{\alpha-1} \right)^{\mu(\alpha-\mu)/(1-\mu)} (r+1)^{\alpha-1} \right)^{\mu-1}. \quad (100) \end{aligned}$$

If $\alpha = 1$, then (100) becomes

$$\begin{aligned} & \left(\sum_{r=1}^\infty \lambda(r) \left(\sum_{k=r}^\infty f(k) \right)^\mu \right)^\mu \geq (\mu)^\mu \left(\sum_{r=1}^\infty \lambda^{1-\mu}(r) \left(\sum_{k=1}^{r-1} \lambda(k) \right)^\mu f^\mu(r) \right) \\ &\quad \times \left(\sum_{r=1}^\infty \lambda(r) \left(\sum_{k=r}^\infty f(k) \right)^\mu \right)^{\mu-1}, \quad (101) \end{aligned}$$

which is Remark 19 in [5].

4. Conclusions and Future Work

In this article, we explore new generalizations of the integral Hardy-Leindler-type inequalities by the utilization of the delta conformable calculus on time scales which are used in various problems involving symmetry. We generalize a number of those inequalities to a general time scale measure space. In addition to this, in order to obtain some new inequalities as special cases, we also extend our inequalities to a discrete and continuous calculus. In future work, we will continue to generalize more fractional dynamic inequalities by using Specht's ratio, Kantorovich's ratio, and n-tuple fractional integral.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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