Research Article

The Sufficient and Necessary Conditions for the Poisson Distribution Series to Be in Some Subclasses of Analytic Functions

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In this paper, we introduce new subclasses of analytic functions in the open unit disc. Furthermore, the necessary and sufficient conditions for the Poisson distribution series to be in these new subclasses are found.

1. Introduction

Let \( \mathcal{A} \) be the class of all analytic functions \( f \) in the open unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by \( f(0) = 0 \) and \( f'(0) = 1 \). A function \( f \in \mathcal{A} \) has the Taylor series expansion of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k.
\]  
(1)

We denote by \( \mathcal{S} \) the subclass of \( \mathcal{A} \) consisting of normalized functions of the form (1) which are univalent in \( U \). Further, we denote by \( \mathcal{T} \) the subclass of \( \mathcal{S} \) consisting of functions with negative coefficients of the form

\[
f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0.
\]  
(2)

If \( f, g \in \mathcal{A} \) such that \( f \) is given by (1) and \( g \) is given by

\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k,
\]
then, the Hadamard product \( (f \ast g)(z) \) is defined by

\[
(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.
\]  
(3)

In 1837, the French mathematician Siméon Denis Poisson created the Poisson distribution which is a popular distribution expresses the probability of a given number of events occurring in a fixed interval of time or space. In [1], Porwal introduced a power series such that its coefficients are probabilities of the Poisson distribution

\[
N(m, z) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} e^{-m} z^k, \quad m > 0, \quad z \in U.
\]  
(4)

In addition, he introduced the series

\[
R(m, z) = 2z - N(m, z) = z - \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} e^{-m} z^k, \quad m > 0, \quad z \in U.
\]  
(5)

In [2], Porwal and Kumar introduced a new linear operator defined by

\[
N(m, z) \ast f(z) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} e^{-m} a_k z^k, \quad m > 0, \quad z \in U.
\]  
(6)

In [3, 4], El-Ashwah and Kota presented the functions
where the class \( H(\lambda, \beta) \) was studied by Obradovic and Joshi in [7], and the class \( H(\lambda, 0) \) was introduced and studied by Ramesha et al. in [8]. We define the class \( H(\lambda, \beta) \cap \mathcal{T} = H(\lambda, \beta) \) which was studied by Lashin [9].

\[
\mathcal{A} \left( \frac{z}{1-z}, \frac{z}{1-z}^2, z, \lambda, \beta \right) = \mathcal{A}(\lambda, \beta),
\]

where the class \( \mathcal{A}(\lambda, \beta) \) was studied by Ding et al. [10]. We define the class \( \mathcal{A}(\lambda, \beta) \cap \mathcal{T} = \mathcal{A}(\lambda, \beta) \) which was introduced by Hassan [11].

\[
\mathcal{A} \left( \frac{z}{1-z}, \frac{z}{1-z}^2, z, \lambda, \beta \right) = \mathcal{A}(\lambda, \beta),
\]

where the classes \( \mathcal{A}(\lambda, \beta) \) and \( \mathcal{A}(\lambda, \beta) \cap \mathcal{T} \equiv \mathcal{A}(\lambda, \beta) \) were introduced by Lashin et al. in [12].

In this paper, we find the necessary and sufficient conditions for the Poisson distribution series to be in the classes \( \mathcal{A}(\lambda, \beta), H(\lambda, \beta), \) and \( \mathcal{A}(\lambda, \beta) \).

## 2. Coefficient Inequalities and Related Properties

We first derive the sufficient and necessary conditions for the function \( f \) to be in the aforementioned classes.

**Theorem 2.** The sufficient condition for \( f \) to be in the class \( \mathcal{A}(\phi, \psi, \zeta, \lambda, \beta) \) is

\[
\mathcal{R} \left\{ (1-\lambda)(f * \phi)(z) + \lambda(f * \psi)(z) \right\} \beta > 0, \lambda, \beta < 1.
\]

**Proof.** We need to show that

\[
\mathcal{R} \left\{ (1-\lambda)(f * \phi)(z) + \lambda(f * \psi)(z) \right\} > \beta.
\]

Then, we have

\[
\frac{(1-\lambda)(f * \phi)(z) + \lambda(f * \psi)(z)}{f(z)} = \frac{\sum_{k=0}^{\infty} (f_k - \lambda \mu_k Y_k + \lambda \mu_k Y_k) \alpha_k z^{k-1}}{1 + \sum_{k=1}^{\infty} \alpha_k z^{k-1}}.
\]

where

\[
\sum_{k=0}^{\infty} (f_k - \lambda \mu_k Y_k + \lambda \mu_k Y_k) \alpha_k z^{k-1} \leq 1 - \beta.
\]
if condition (17) holds. This implies that \( f \in \mathcal{A}(\phi, \psi, \xi, \lambda, \beta) \) which completes the proof.

**Theorem 3.** Let \( F(k) = y_k - \rho_k + \lambda(\mu_k - y_k) \) be an increasing function of \( k \) and \( \lambda \geq \max \{0, (\rho_2 - y_1)/(\mu_2 - y_2)\} \). Then, the necessary and sufficient condition for the function \( f \) to be in the class \( \mathcal{T}(\phi, \psi, \xi, \lambda, \beta) \) is

\[
\sum_{k=2}^{\infty} \{y_k - \rho_k + \lambda(\mu_k - y_k) + (1 - \beta)\rho_k\} a_k \leq 1 - \beta. \tag{20}
\]

**Proof.** In view of Theorem 2, it suffices to show the necessary condition only. Assume that \( f \in \mathcal{T}(\phi, \psi, \xi, \lambda, \beta) \). Then

\[
\Im \left\{ \frac{1 - \sum_{k=2}^{\infty} (y_k + \lambda(\mu_k - y_k)) a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} \rho_k a_k z^{k-1}} \right\} > \beta \tag{21}
\]

which is equivalent to

\[
\Im \left\{ \frac{1 - \sum_{k=2}^{\infty} (y_k + \lambda(\mu_k - y_k)) a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} \rho_k a_k z^{k-1}} \right\} > \beta \tag{22}
\]

Choosing \( z \) on the real axis, then

\[
\left\{ \frac{1 - \sum_{k=2}^{\infty} (y_k + \lambda(\mu_k - y_k)) a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} \rho_k a_k z^{k-1}} \right\}
\]

is also real. Let \( z \rightarrow 1^{-} \) through real values, we get

\[
1 - \sum_{k=2}^{\infty} (y_k + \lambda(\mu_k - y_k)) a_k \geq \beta \left( 1 - \sum_{k=2}^{\infty} \rho_k a_k \right), \tag{24}
\]

which is equivalent to (20), and this completes the proof. \( \square \)

Putting \( \phi = z/(1 - z)^2, \psi = (z + z^2)/(1 - z)^3, \) and \( \xi = z/(1 - z)^2 \) in Theorems 3, we get the following corollary due to Lashin [9].

**Corollary 4** (see [9]). A function \( f \in \mathcal{T} \) is in the class \( \mathcal{H}(\lambda, \beta) \) if and only if

\[
\sum_{k=2}^{\infty} [(k - 1)(\lambda k + 1) + (1 - \beta)] a_k \leq 1 - \beta, 0 \leq \lambda, \beta < 1. \tag{25}
\]

Putting \( \phi = z/(1 - z), \psi = z/(1 - z)^2, \) and \( \xi = z \) in Theorem 3, we get the following corollary due to Hassan [11].

**Corollary 5** (see [11]). A function \( f \in \mathcal{T} \) is in the class \( \mathcal{F}^+(\lambda, \beta) \) if and only if

\[
\sum_{k=2}^{\infty} \lambda(k - 1) + 1 \leq 1 - \beta, 0 \leq \lambda, \beta < 1. \tag{26}
\]

Putting \( \phi = z/(1 - z), \psi = (z + z^2)/(1 - z)^3, \) and \( \xi = z/(1 - z)^2 \) in Theorem 3, we get the following corollary.

**Corollary 6** (see [12]). Let \( 1/3 \leq \lambda < 1 \). A function \( f \in \mathcal{T} \) is in the class \( \mathcal{H}^+(\lambda, \beta) \) if and only if

\[
\sum_{k=2}^{\infty} [(k - 1)(\lambda k + 1) + (1 - \beta)] a_k \leq 1 - \beta, 0 \leq \beta < 1. \tag{27}
\]

Making use of the techniques and methodology given by Porwal [1] (see also [13–18]), we get the following theorems.

**Theorem 7.** The sufficient condition for \( H_u(m, z) \) to be in the class \( \mathcal{H}(\lambda, \beta) \) is

\[
\lambda m^3 + (4\nu\lambda + \nu + \lambda)m^2 + [(1 + 2\lambda)(1 + \nu) + \nu(1 - \beta)]m \leq (1 - \beta)e^{-m}. \tag{28}
\]

Also, condition (28) is necessary and sufficient for \( H_u(m, z) \) to be in the class \( \mathcal{H}(\lambda, \beta) \).

**Proof.** According to Theorem 2, we need to show that

\[
\sum_{k=2}^{\infty} [(k - 1)(\lambda k + 1) + (1 - \beta)] [1 + \nu(k - 1)] m^{k-1} [(k - 1) \Gamma(k - 1)] e^{-m} \leq 1 - \beta. \tag{29}
\]

Thus,

\[
\sum_{k=2}^{\infty} [(k - 1)(\lambda k + 1) + (1 - \beta)] [1 + \nu(k - 1)] m^{k-1} [(k - 1) \Gamma(k - 1)] e^{-m}
= e^{-m} \sum_{k=2}^{\infty} [(k - 1)(\lambda k - 2) + (1 + 2\lambda)] (1 - \beta) m^{k-1} (k - 1) \Gamma(k - 1)
+ e^{-m} \sum_{k=2}^{\infty} \nu(\lambda(k - 1)(k - 2)(k - 3) + 2\nu(\lambda(k - 1)(k - 2)) m^{k-1} (k - 1) \Gamma(k - 1)
+ e^{-m} \sum_{k=2}^{\infty} \nu(1 + 2\lambda)(k)(k - 1) m^{k-1} (k - 1) \Gamma(k - 1)
+ e^{-m} \sum_{k=2}^{\infty} (1 + \nu)(k - 1) m^{k-1} (k - 1) \Gamma(k - 1)
+ (4\nu\lambda + \nu + \lambda)e^{-m} \sum_{k=2}^{\infty} m^{k-1} (k - 3) \Gamma(k - 3)
+ [(1 + 2\lambda)(1 + \nu) + \nu(1 - \beta)] e^{-m} \sum_{k=2}^{\infty} m^{k-1} (k - 2) m^{k-1} (k - 1) \Gamma(k - 1)
+ (1 - \beta)e^{-m} \sum_{k=2}^{\infty} m^{k-1} (k - 1) \Gamma(k - 1)
+ [(1 + 2\lambda)(1 + \nu) + \nu(1 - \beta)] m + (1 - \beta)(1 - e^{-m}) \leq (1 - \beta). \tag{30}
\]
if condition (28) holds. Then, from Theorem 3, it follows that the condition (28) is necessary and sufficient for \( H^*_c(m, z) \in \mathcal{H}(\lambda, \beta) \). Hence, the proof is completed.

In Theorem 7, if we put \( v = 0 \), then, we get the following corollary which was obtained by Murugusundaramoorthy et al. [19].

**Corollary 8.** The sufficient condition for \( N(m, z) \) to be in the class \( H(\lambda, \beta) \) is

\[
\lambda m^2 + (1 + 2\lambda)m \leq (1 - \beta)e^{-m}. \tag{31}
\]

Also, condition (31) is necessary and sufficient for \( R(m, z) \) to be in the class \( H(\lambda, \beta) \).

**Theorem 9.** The sufficient condition for \( H_s(m, z) \) to be in the class \( \mathcal{J}(\lambda, \beta) \) is

\[
\lambda m^2 + (\lambda + \nu(1 + \lambda))m + 1 - e^{-m} \leq 1 - \beta. \tag{32}
\]

Also, condition (32) is necessary and sufficient for \( H^*_c(m, z) \) to be in the class \( \mathcal{J}^*(\lambda, \beta) \).

**Proof.** According to Theorem 2, we need to show that

\[
\sum_{k=2}^{\infty} \left(1 + \lambda(k-1)(1 + \nu(k-1))\right) \frac{m^{k-1}}{(k-1)!} e^{-m} \leq 1 - \beta. \tag{33}
\]

Thus,

\[
\sum_{k=2}^{\infty} (1 + \lambda(k-1))(1 + \nu(k-1)) \frac{m^{k-1}}{(k-1)!} e^{-m} = e^{-m} \sum_{k=2}^{\infty} \left(\lambda(k-1)(k-2) + (\nu + \lambda + \nu)(k-1) + 1\right) \frac{m^{k-1}}{(k-1)!},
\]

\[
= e^{-m} \left(\lambda \sum_{k=3}^{\infty} \frac{m^{k-2}}{(k-2)!} + [\nu + 1 + \lambda] \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!}\right),
\]

\[
= \lambda m^2 + (\lambda + \nu(1 + \lambda))m + 1 - e^{-m} \leq 1 - \beta. \tag{34}
\]

if condition (32) holds. Then, from Theorem 3, it follows that condition (32) is necessary and sufficient for \( H^*_c(m, z) \in \mathcal{J}^*(\lambda, \beta) \). Hence, the proof is completed.

In Theorem 9, if we put \( v = 0 \), then, we get the following corollary which was obtained by Frasin [20].

**Corollary 10.** The sufficient condition for \( N(m, z) \) to be in the class \( \mathcal{J}(\lambda, \beta) \) is

\[
\lambda m^2 + (1 - e^{-m}) \leq 1 - \beta. \tag{35}
\]

Also, condition (35) is necessary and sufficient for \( R(m, z) \) to be in the class \( \mathcal{J}^*(\lambda, \beta) \).

**Theorem 11.** The sufficient condition for \( N(m, z) \) to be in the class \( \mathcal{X}(\lambda, \beta) \) is

\[
\lambda m^2 + (3\lambda - \beta)m \leq (1 - \beta)e^{-m}, \tag{36}
\]

where \( 1/3 \leq \lambda < 1 \). Also, condition (36) is necessary and sufficient for \( R(m, z) \) to be in the class \( \mathcal{X}^*(\lambda, \beta) \).

**Proof.** According to Theorem 2, we need to show that

\[
\sum_{k=2}^{\infty} ((k-1)|\lambda(k+1) - 1| + (1 - \beta)k) \frac{m^{k-1}}{(k-1)!} e^{-m} \leq 1 - \beta. \tag{37}
\]

Thus,

\[
\sum_{k=2}^{\infty} ((k-1)|\lambda(k+1) - 1| + (1 - \beta)k) \frac{m^{k-1}}{(k-1)!} e^{-m} = e^{-m} \sum_{k=2}^{\infty} (\lambda(k-1)(k-2) + (3\lambda - \beta)(k-1) + (1 - \beta)) \frac{m^{k-1}}{(k-1)!},
\]

\[
= e^{-m} \left\{ \lambda \sum_{k=3}^{\infty} \frac{m^{k-2}}{(k-2)!} + (3\lambda - \beta) \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} + (1 - \beta) \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!}\right\},
\]

\[
= \lambda m^2 + (3\lambda - \beta)m + (1 - \beta)(1 - e^{-m}) \leq (1 - \beta). \tag{38}
\]

if condition (36) holds. Then, from Theorem 3, it follows that the condition (36) is necessary and sufficient for \( R(m, z) \in \mathcal{X}^*(\lambda, \beta) \). Hence, the proof is completed.

In Theorem 9, if we put \( v = 0 \), then, we get the following corollary which was obtained by Frasin [20].

**Theorem 12.** The sufficient condition for \( H_s(m, z) \) to be in the class \( \mathcal{X}(\lambda, \beta) \) is

\[
\lambda m^3 + [\lambda + \nu(5\lambda - \beta)]m^2 + [(3\lambda - \beta)(\nu + 1) + \nu(1 - \beta)]m \leq (1 - \beta)e^{-m}, \tag{39}
\]

where \( 1/3 \leq \lambda < 1 \). Also, condition (39) is necessary and sufficient for \( N(m, z) \in \mathcal{X}^*(\lambda, \beta) \).
Proof. According to Theorem 2, we need to show that
\[
\sum_{k=2}^{\infty} \left( (k-1)(\lambda(k+1) - 1) + (1 - \beta)k \right) \frac{m^{k-1}}{(k-1)!} e^{-m} \leq 1 - \beta.
\]
(40)

Thus,
\[
\sum_{k=2}^{\infty} \left( (k-1)(\lambda(k+1) - 1) + (1 - \beta)k \right) \frac{m^{k-1}}{(k-1)!} e^{-m} = e^{-m} \sum_{k=2}^{\infty} \left( \lambda k(k+1) - 1 + (1 - \beta)k \right) \frac{m^{k-1}}{(k-1)!} e^{-m}
\]
\[
= e^{-m} \left( \lambda m^2 + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} + (1 - \beta) \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} \right),
\]
\[
= \lambda m^2 + \lambda m \left( 5 - \beta \right) m + (1 - \beta) (1 - e^{-m}) \leq 1 - \beta.
\]
(41)

if condition (39) holds. Then, from Theorem 3, it follows that the condition (39) is necessary and sufficient for \( H^*_{\alpha}(m, z) \in \mathcal{A}^* (\lambda, \beta) \). Hence, the proof is completed. \( \square \)

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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