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Research Article

An Approximation Theorem and Generic Convergence of Solutions of Inverse Quasivariational Inequality Problems

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In this paper, we mainly obtain an approximation theorem and generic convergence of solutions for inverse quasivariational inequality problems. First, we define the concept of the approximate solution to inverse quasivariational inequality problems under bounded rationality theory. Afterward, an approximation theorem that satisfies fairly mild assumptions is proved. Moreover, we establish a function space and discuss the convergence properties of solutions for inverse quasivariational inequality problems by the method of set-valued analysis. Finally, we prove that most of inverse quasivariational inequality problems are stable in the case of perturbation of the objective function. These results are new, which improve the corresponding outcomes of the recent literatures.

1. Introduction

Inverse quasivariational inequality (briefly, IQVI) was first proposed by Aussel et al. [1] in 2013. The specific format of this inequality is as follows:

Finding a vector $s \in R^n$ such that $\psi(s) \in H(s)$, we have

$$\langle \varphi(s), h - \psi(s) \rangle \geq 0, \text{for all } h \in H(S), \quad (1)$$

where $\varphi, \psi : R^n \longrightarrow R^n$ represents two continuous mappings, respectively. $H : R^n \longrightarrow 2^{R^n}$ denotes a set-valued mapping. For all $s \in R^n$, $H(s)$ is a nonempty closed convex set in R^n. In addition, $\langle \cdot, \cdot \rangle$ represents the inner products and $\|\cdot\|$ denotes the norms in R^n. When the mapping ψ is the identity mapping or $H(s)$ represents a constant on R^n, then the IQVI problem converts to a classical quasivariational inequality (briefly, QVI) problem or an inverse variational inequality (briefly, IVI) problem. In recent years, the IQVI model has attracted much attention. Many related results have been extensively investigated as follows:

(i) For the IQVI problems, Aussel et al. obtained the global/local error bounds of problems by using different gap (merit) functions, namely the regularized, residual, and D-gap function [1]. In addition, Han et al. proved the existence theorem for the solution of the IQVI problems [2]. Dey and Vetrivel first defined an approximate solution to IQVI problems. And based on the existence theorem of the IQVI problems in literature [2], they obtained the existence theorem of the approximate solution for the IQVI problems in a locally convex Hausdorff topological vector space [3]

(ii) For the inverse mixed quasivariational inequality problems, scholars explored several properties of problems in Hilbert space, such as generalized f-projection operator, error bounds, existence and uniqueness outcomes, and gap functions [4, 5]

(iii) For the differential IQVI problems, Li et al. obtained some existence theorems for Carathéodory weak solutions of the problems. Besides, the convergence consequent on the Euler time-dependent scheme was proved [6]

(iv) For the mixed set-valued vector IQVI and the vector inverse mixed quasivariational inequality problems, three gap functions were provided, respectively. Using generalized f-projection operator and three gap functions, scholars obtained error bounds of the generalized vector IQVI and the vector inverse mixed quasivariational inequality problems under the Lipschitz continuity and strong monotonicity of the underlying mappings [7–9]

It can be seen that the research on the model of the IQVI problem is relatively active and has a wide application. However, the uniqueness of the IQVI problem is still few. Hence, this is one of the motivations why we study the uniqueness of the IQVI problem.

On the other hand, the bounded rationality theory was proposed by Simon, and the core of this theory is the principle of satisfaction. The decision-maker is to seek the satisfactory solution rather than the optimal solution [10]. In 2001, Anderlini and Canning [11] established an abstract model for the study of bounded rationality. The model is a type of general games with abstract rational functions, which reflects the approximation of bounded rationality to full rationality. In 2007, Khanh and Luu [12] obtained semi(upper)continuous of the approximate solution sets and solution sets of parametric multivalued QVI in topological vector spaces. Chen et al. [13] first obtained a scalarization result of the ε-weakly efficient solution for a class of vector equilibrium problems under the Hausdorff topological vector space. Then, they proved the connectedness of the ε -efficient solution sets and ε-weakly efficient solution sets for this problem by applying the scalarization result. Subsequently, Han and Huang [14] provided the scalarization results and density theorems for the efficient and weakly efficient approximate solution sets of generalized vector equilibrium problems and established their connectedness. And these researchers obtained the upper (lower) semicontinuous of efficient and weakly efficient approximate solution mappings for parametric generalized vector equilibrium problems in which both the feasible regions and the objective mappings were simultaneously perturbed. Research on the bounded rationality theory has been increasing in recent years. In 2018, Qiu et al. [15] discussed an approximation theorem for equilibrium problems. At the same time, on the meaning of the Baire category, they obtained the generic convergence of the solution for the monotone equilibrium problem. Moreover, Qiu et al. discussed the applications of such approximation theorems to saddle point problems, optimization problems, and variational inequality problems. Especially, in 2020, under certain assumptions, Jia et al. [16] proved an approximation theorem and obtained several corollaries for the vector equilibrium problem. In addition, based on the meaning of the Baire category, Jia et al. obtained the generic convergence theorem for the solution of the strictly quasimonotone scalar equilibrium problems and applied a series of results to Nash equilibrium problems with multiobjective games, vector optimization problems, and vector variational inequality problems.

At the moment, the model for the IQVI problem has not been considered from the perspective of bounded rationality ideas. Therefore, this is another motivation why we study the approximation theorem for the IQVI problem.

Motivated by the aforementioned works, we consider the approximation theorem and the generic convergence of the IQVI problem. In this paper, we first define the concept of the approximate solution of IQVI problems under bounded rationality theory [10]. Then, the approximation theorem of the IQVI problem is proved, which reflects the approximation of bounded rationality to full rationality. Moreover, we establish the function space and discuss the generic convergence of IQVI problems. Finally, based on the meaning of the Baire category, we prove the results that the IQVI problem has generic convergence in the case of perturbation of the objective function. These new results generalize those of some previous literature.

This paper consists of four parts. First, in Section 2, we introduce some indispensable lemmas and definitions for later use. Next, we define the concept of approximate solutions of IQVI problems and propose approximation theorems of solutions for IQVI problems in Section 3. Then, we construct the function space and discuss the generic convergence for IQVI problems in Section 4. Section 5 summarizes this paper.

2. Preliminaries

Definition 1 (see [17, 18]). Let W and D be two metric spaces, the set-valued mapping be expressed as $\aleph : W \longrightarrow P_0(D)$, where $P_0(D)$ is the nonempty set, $\aleph$ is said to be

(1) lower semicontinuous (l.s.c.) at $w \in W$ if for any open set $\mho$ in D, $\aleph(w) \cap \mho \neq \varnothing$, there exists $O(w)$ of w, where $O(w)$ is an open neighbourhood such that $\aleph(w') \cap \mho \neq \varnothing$, for every $w' \in O(w)$

(2) upper semi-continuous (u.s.c.) at $w \in W$ if for any open set $\mho$ in D, $\aleph(w) \subset \mho$, there exists $O(w)$ of w, where $O(w)$ is an open neighbourhood such that $\aleph(w') \subset \mho$, for every $w' \in O(w)$

(3) a usco mapping on W if for any $w \in W$, $\aleph : W \longrightarrow P_0(D)$ is u.s.c and nonempty compact values

(4) continuous at $w \in W$ if $\aleph$ is both u.s.c and l.s.c. at w

Lemma 2 (see [19]). *Let $\{A_m\}$ be a sequence of nonempty bounded subset of R^n and A represents a nonempty bounded subset of R^n, where for each $m = 1, 2, 3, \cdots$. If $A_m \longrightarrow A$ and $t_m \in A_m$, then there exists a subsequence $\{t_{m_k}\}$ of $\{t_m\}$ such that $t_{m_k} \longrightarrow t_* \in A\,(m \longrightarrow \infty)$.*

Lemma 3 (see [19]). *Let $\{A_m\}$ denote a sequence of nonempty bounded subset of R^n and A represents a nonempty bounded subset of R^n, where for each $m = 1, 2, 3, \cdots$. Denote by G an open set of R^n, where $G \bigcap A \neq \varnothing$. If $A_m \longrightarrow A$, thus* *there exists a positive integer N such that for each* $m \geq N$, $G \bigcap A_m \neq \varnothing (m \longrightarrow \infty)$.

Definition 4 (see [1]). Let $\xi : R^n \longrightarrow R^n$ represent a mapping, then ξ is monotonic on R^n if for any $\varsigma, \sigma \in R^n$, one has

$$\langle \xi(\varsigma) - \xi(\sigma), \varsigma - \sigma \rangle \geq 0. \tag{2}$$

Definition 5 (see [1]). Assume that two mappings are denoted as $\xi, \phi : R^n \longrightarrow R^n$, respectively. If there exists a constant $\delta > 0$ such that for every $\varsigma, \sigma \in R^n$, one has

$$\langle \xi(\varsigma) - \xi(\sigma), \phi(\varsigma) - \phi(\sigma) \rangle \geq \delta \|\varsigma - \sigma\|^2, \tag{3}$$

then (ξ, ϕ) is said to be δ-strongly monotone couple on R^n.

Lemma 6 (see [17]). *Let M and F be two topological spaces, where F be a compact space. If the set-valued mapping* $\aleph : M \longrightarrow P_0(F)$ *is closed, then* $\aleph$ *is a usco mapping on M.*

Definition 7 (see [20]). Let $\xi : E \longrightarrow R$ and E denotes a nonempty convex set of R^n, then

(1) ξ is quasiconvex on E, if for all $e_1, e_2 \in E$ and $\rho \in (0, 1)$, we have

$$\xi(\varrho e_1 + (1 - \varrho)e_2) \leq \max \{\xi(e_1), \xi(e_2)\} \tag{4}$$

(2) ξ is quasiconcave on E, if for all $e_1, e_2 \in E$ and $\rho \in (0, 1)$, we have

$$\xi(\varrho e_1 + (1 - \varrho)e_2) \geq \min \{\xi(e_1), \xi(e_2)\} \tag{5}$$

Lemma 8 (see [21], Fort Lemma). *Let M be a Hausdorff topological space and W be a metric space. The set-valued mapping* $\aleph : M \longrightarrow P_0(W)$ *is a usco mapping. Then, there exists a residual subset Q of M such that* $\aleph$ *is l.s.c. on Q.*

3. Approximation Theorem of Inverse Quasivariational Inequality

First, we introduce the concept of an approximate solution of IQVI problems.

Definition 9. Let S be a nonempty compact subset of R^n and denote by $T(S)$ a set of all nonempty compact subsets in S. Let $\varphi, \psi : S \longrightarrow S$ represent two continuous mappings. A set-valued mapping represents as $H : S \longrightarrow T(S)$ such that for any $s \in S$, $H(s)$ is a nonempty convex compact set. For a real number $\varepsilon > 0$, finding a vector $s^* \in S$ such that $\psi(s^*) \in H(s^*)$, we have

$$\langle \varphi(s^*), h - \psi(s^*) \rangle + \varepsilon \geq 0, \text{for all } h \in H(s^*). \tag{6}$$

Then, s^* is said to be an ε-approximate solution of IQVI problems.

Theorem 10. *Let S be a nonempty compact subset of* R^n *and satisfy the following assumptions:*

(i) For every $m = 1, 2, \cdots$, *the two function sequences* $\varphi_m, \psi_m : S \longrightarrow S$ *and a set-valued mapping sequence* $H_m : S \longrightarrow T(S)$ *are satisfied by*

$$\begin{aligned} &\sup_{s\in S} \|\varphi_m(s) - \varphi(s)\| \longrightarrow 0\,(m \longrightarrow \infty), \\ &\sup_{s\in S} \|\psi_m(s) - \psi(s)\| \longrightarrow 0\,(m \longrightarrow \infty), \\ &\sup_{s\in S} h(H_m(s), H(s)) \longrightarrow 0\,(m \longrightarrow \infty), \end{aligned} \tag{7}$$

where h denotes the Hausdorff distance defined on S. $\varphi, \psi : S \longrightarrow S$ *and* $H : S \longrightarrow T(S)$ *are continuous and for all* $s \in S$, $H(s)$ *is a nonempty convex compact set*

(ii) For every $m = 1, 2, \cdots$, A_m *is a nonempty subset sequence of S and*

$$h(A_m, A) \longrightarrow 0\,(m \longrightarrow \infty), \tag{8}$$

where A is a nonempty compact set of S

(iii) For every $m = 1, 2, \cdots$, $s_m \in S$ *is satisfied with* $d(s_m, A_m) \longrightarrow 0$, $\psi_m(s_m) \in H_m(s_m)$,*we have*

$$\langle \varphi_m(s_m), h - \psi_m(s_m) \rangle + \varepsilon_m \geq 0, \text{for all } h \in H_m(s_m), \tag{9}$$

where $\varepsilon_m \geq 0$ *and* $\varepsilon_m \longrightarrow 0\,(m \longrightarrow \infty)$

(1) Then there exists a convergent subsequence $\{s_{m_k}\}$ *of* $\{s_m\}$ *which converges to some* $s^* \in A(m \longrightarrow \infty)$

(2) For all $h \in H(s^*)$, *we have*

$$\langle \varphi(s^*), h - \psi(s^*) \rangle \geq 0 \tag{10}$$

(3) If the solution of IQVI problems is a singleton set, there must be $s_m \longrightarrow s^*$

Proof.

(1) Since $(s_m, A_m) \longrightarrow 0(m \longrightarrow \infty)$, we can see that there exists $s_{m'} \in A_m$ such that $d(s_m, s_{m'}) \longrightarrow 0$. Because $h(A_m, A) \longrightarrow 0$ and A is a compact set by Lemma 2, for any sequence $\{s_{m'}\}$ in S, there must be a subsequence $\{s_{m'_k}\}$ such that $s_{m'_k} \longrightarrow s^* \in A$. Therefore, there exists a subsequence $\{s_{m_k}\}$ of the sequence $\{s_m\}$ such that $s_{m_k} \longrightarrow s^* \in A$

(2) According to conclusion (1), it may be assumed that $s_m \longrightarrow s^* \in A$. By contradiction, we assume that conclusion (2) does not hold. Thus, there exists $h \in H(s^*)$ such that

$$\langle \varphi(s^*), h - \psi(s^*) \rangle < 0 \tag{11}$$

First, since $\varphi, \psi : S \longrightarrow S$ are continuous at s and $\langle \varphi(s), h - \psi(s) \rangle$ is continuous at variables s and h, there exists a real number $\gamma_0 > 0$ and two open neighbourhoods $V(s^*)$ and $V(h^0)$ of s^* and h^0 such that for any $s' \in V(s^*)$, $h' \in V(h^0)$, we have

$$\left\langle \varphi\left(s'\right), h' - \psi\left(s'\right) \right\rangle < -\gamma_0. \tag{12}$$

According to the Cauchy-Swartz inequality, we can obtain

$$\begin{aligned} &\langle \varphi_m(s) - \varphi(s), h - (\psi_m(s) - \psi(s)) \rangle \\ &\quad \le \|\varphi_m(s) - \varphi(s)\| \|h - (\psi_m(s) - \psi(s))\| \\ &\quad \le \|\varphi_m(s) - \varphi(s)\| [\|h\| + \|\psi_m(s) - \psi(s)\|], \end{aligned} \tag{13}$$

that is,

$$\begin{aligned} &\sup_{s \in S} \|\varphi_m(s) - \varphi(s)\| \longrightarrow 0 (m \longrightarrow \infty), \\ &\sup_{s \in S} \|\psi_m(s) - \psi(s)\| \longrightarrow 0 (m \longrightarrow \infty). \end{aligned} \tag{14}$$

This means that

$$\sup_{s \in S, h \in H(s)} \langle \varphi_m(s) - \varphi(s), h - (\psi_m(s) - \psi(s)) \rangle \longrightarrow 0 (m \longrightarrow \infty). \tag{15}$$

Again, because $\varepsilon_m \longrightarrow 0 (m \longrightarrow \infty)$, there exists a positive integer N_1 such that for every $m \ge N_1$, we have

$$\sup_{s \in S, h \in H(s)} \langle \varphi_m(s) - \varphi(s), h - (\psi_m(s) - \psi(s)) \rangle < \frac{\gamma_0}{2}, \varepsilon_m < \frac{\gamma_0}{2}. \tag{16}$$

Finally, since $s_m \longrightarrow s^* \in A$, $h(H_m, H) \longrightarrow 0 (m \longrightarrow \infty)$, and $h^0 \in H(s)$, according to Lemma 3, we know that there exists a positive integer N_2. It may be assumed that $N_2 \ge N_1$ such that for all $m \ge N_2$, we have $s_m \in V(s^*)$ and $V(h^0) \bigcap H_m(s_m) \ne \varnothing$. Let $h_m \in V(h^0) \bigcap H_m(s_m)$, thus, we have

$$\begin{aligned} &\langle \varphi(s_m), h_m - \psi(s_m) \rangle < -\gamma_0, \\ &\langle \varphi_m(s_m) - \varphi(s_m), h_m - (\psi_m(s_m) - \psi(s_m)) \rangle \\ &\quad = \langle \varphi_m(s_m), h_m - \psi_m(s_m) \rangle + \langle \varphi_m(s_m), \psi(s_m) \rangle \\ &\qquad - \langle \varphi(s_m), h_m - \psi_m(s_m) \rangle - \langle \varphi(s_m), \psi(s_m) \rangle \\ &\quad = \langle \varphi_m(s_m), h_m - \psi_m(s_m) \rangle - \langle \varphi(s_m), h_m - \psi(s_m) \rangle \\ &\qquad + \langle \varphi_m(s_m) - \varphi(s_m), \psi(s_m) \rangle - \langle \varphi(s_m), \psi(s_m) - \psi_m(s_m) \rangle \\ &\quad < \frac{\gamma_0}{2}. \end{aligned} \tag{17}$$

Then,

$$\begin{aligned} &\langle \varphi_m(s_m), h_m - \psi_m(s_m) \rangle < \langle \varphi(s_m), h_m - \psi(s_m) \rangle \\ &\quad - \langle \varphi_m(s_m) - \varphi(s_m), \psi(s_m) \rangle + \langle \varphi(s_m), \psi(s_m) - \psi_m(s_m) \rangle \\ &\quad + \frac{\gamma_0}{2} \le \langle \varphi(s_m), h_m - \psi(s_m) \rangle + \|\varphi(s_m) - \varphi_m(s_m)\| \|\psi(s_m)\| \\ &\quad + \|\varphi(s_m)\| \|\psi(s_m) - \psi_m(s_m)\| + \frac{\gamma_0}{2} \le \langle \varphi(s_m), h_m - \psi(s_m) \rangle \\ &\quad + \frac{\gamma_0}{2} < -\frac{\gamma_0}{2} < -\varepsilon_m. \end{aligned} \tag{18}$$

Then, this conflicts with condition (iii). Hence, for all $h \in H(s^*)$, we have $\langle \varphi(s^*), h - \psi(s^*) \rangle \ge 0$.

(3) First, using the contradiction method, we assume that conclusion (3) does not hold. In other words, if the solution set of IQVI problems is a singleton set, then$s_m \nrightarrow s^*$. Hence, there exists $\gamma > 0$ and a subsequence $\{s_{m_k}\}$ of $\{s_m\}$ such that $\|s^* - s_{m_k}\| \ge \gamma$. Next, by conclusion (1), it can be seen that sequence $\{s_{m_k}\}$ must have subsequences. Let $s_{m_k} \longrightarrow \bar{s} \in A$, that is, $\|\bar{s} - s_{m_k}\| \longrightarrow 0$, according to conclusion (2), we can obtain $\langle \varphi(\bar{s}), h - \psi(\bar{s}) \rangle \ge 0$. Finally, because the solution of IQVI problems is singleton set, therefore $\bar{s} = s^*$, that is, $\|\bar{s} - s_{m_k}\| \ge \gamma$, which conflicts with $\|\bar{s} - s_{m_k}\| \longrightarrow 0$. Therefore, we can obtain $s_m \longrightarrow s^*$. The proof is completed.

□

Remark 11. According to Theorem 10, although the objective function, feasible solution set are all approximated (that is, $\varphi_m \longrightarrow \varphi$, $\psi_m \longrightarrow \psi$, $H_m \longrightarrow H$, and $A_m \longrightarrow A$), we can obtain an approximation sequence $\{s_m\}$, which must have convergent subsequences $\{s_{m_k}\}$, that is, $s_{m_k} \longrightarrow s^* \in A$ and s^* must be the solution of IQVI problems. If the ε_m -approximate solution s_m of IQVI problems is regarded as a "satisfactory solution" under bounded rationality, and the solution s^* of IQVI problems is regarded as an"exact solution" under full rationality. Theorem 10 implies the approximate of bounded rationality to full rationality, that is, full rationality can be approximated by a series of approximate solutions of bounded rationality, which verifies Simon's bounded rationality theory from a certain perspective.

Remark 12. Theorem 10 shows that the limit points and convergent subsequences existing for sequence $\{s_m\}$ are equivalent. It can be seen from the above proof that the sequence $\{s_m\}$ must have limit points. Each limit point belongs to compact set A and is the solution of IQVI problems. If the solution of IQVI problems is a singleton set, there is a stronger convergence result as follows: $s_m \longrightarrow s$.

In Theorem 10, if $s_m \in A_m$ $(m = 1, 2, \cdots)$, then the result of Theorem 10 still holds, that is, we can obtain the Corollary 13 as follows.

Corollary 13. *Let S be a nonempty compact subset of R^n and satisfy the following assumptions:*

(i) *For every $m = 1, 2, \cdots$, the two function sequences φ_m, $\psi_m : S \longrightarrow S$ and a set-valued mapping sequence $H_m : S \longrightarrow T(S)$ are satisfied by*

$$\begin{aligned} &\sup_{s\in S}\|\varphi_m(s) - \varphi(s)\| \longrightarrow 0\,(m \longrightarrow \infty), \\ &\sup_{s\in S}\|\psi_m(s) - \psi(s)\| \longrightarrow 0\,(m \longrightarrow \infty), \\ &\sup_{s\in S} h(H_m(s), H(s)) \longrightarrow 0\,(m \longrightarrow \infty), \end{aligned} \tag{19}$$

where $\varphi, \psi : S \longrightarrow S$ and $H : S \longrightarrow T(S)$ are continuous. And for all $s \in S$, $H(s)$ is a nonempty convex compact set.

(ii) *For every $m = 1, 2, \cdots$, A_m is a nonempty subset of S and*

$$h(A_m, A) \longrightarrow 0(m \longrightarrow \infty), \tag{20}$$

where A is a nonempty compact set of S

(iii) *For every $m = 1, 2, \cdots$, $s_m \in A_m$ is satisfied with $\psi_m(s_m) \in H_m(s_m)$, we have*

$$\langle \varphi_m(s_m), h - \psi_m(s_m)\rangle + \varepsilon_m \geq 0, \text{for all } h \in H_m(s_m), \tag{21}$$

where $\varepsilon_m \geq 0$ and $\varepsilon_m \longrightarrow 0\,(m \longrightarrow \infty)$

Then,

(1) There exists a convergent subsequence $\{s_{m_k}\}$ of $\{s_m\}$ which converges to some $s^* \in A(m \longrightarrow \infty)$

(2) For all $h \in H(s^*)$, we have

$$\langle \varphi(s^*), h - \psi(s^*)\rangle \geq 0 \tag{22}$$

(3) If the solution of IQVI problems is a singleton set, there is $s_m \longrightarrow s^*$

In Theorem 10, if $A = A_m$ $(m = 1, 2, \cdots)$, then the result of Theorem 10 still holds, that is, we can obtain the Corollary 14 as follows.

Corollary 14. *Let S be a nonempty compact subset of R^n and all the following assumptions be satisfied:*

(i) *For every $m = 1, 2, \cdots$, the two function sequences φ_m, $\psi_m : S \longrightarrow S$ and a set-valued mapping sequence $H_m : S \longrightarrow T(S)$ are satisfied by*

$$\begin{aligned} &\sup_{s\in S}\|\varphi_m(s) - \varphi(s)\| \longrightarrow 0(m \longrightarrow \infty), \\ &\sup_{s\in S}\|\psi_m(s) - \psi(s)\| \longrightarrow 0(m \longrightarrow \infty), \\ &\sup_{s\in S} h(H_m(s), H(s)) \longrightarrow 0(m \longrightarrow \infty), \end{aligned} \tag{23}$$

where $\varphi, \psi : S \longrightarrow S$ and $H : S \longrightarrow T(S)$ are continuous. And for all $s \in S$, $H(s)$ is a nonempty convex compact set.

(ii) *A is a nonempty compact set of S:*

(iii) *For every $m = 1, 2, \cdots$, $s_m \in A_m$ is an ε_m-approximate solution of function sequences for the IQVI problem, which satisfies $\psi_m(s_m) \in H_m(s_m)$, we have*

$$\langle \varphi_m(s_m), h - \psi_m(s_m)\rangle + \varepsilon_m \geq 0, \text{for all } h \in H_m(s_m), \tag{24}$$

where $\varepsilon_m \geq 0$ and $\varepsilon_m \longrightarrow 0\,(m \longrightarrow \infty)$

Then,

(1) There exists a convergent subsequence $\{s_{m_k}\}$ of $\{s_m\}$ which converges to some $s^* \in A(m \longrightarrow \infty)$

(2) For all $h \in H(s^*)$, we have

$$\langle \varphi(s^*), h - \psi(s^*)\rangle \geq 0 \tag{25}$$

(3) If the solution of IQVI problems is singleton set, there is $s_m \longrightarrow s^*$

4. Generic Convergence of Inverse Quasivariational Inequality

Let S be a nonempty compact subset of R^n, $T(S)$ be a set of all nonempty compact subsets in S. The function space U of the IQVI problem is as follows:

$$U = \left\{ u = \{\varphi, \psi, H\}: \begin{array}{l} \varphi : S \longrightarrow S \text{ is a continuous mapping with monotone quasiconcave;} \\ \psi : S \longrightarrow S \text{ is a continuous mapping with monotone quasiconvex;} \\ (\varphi, \psi) \text{ is a strongly monotone couple;} \\ H : S \longrightarrow T(S) \text{ is a continuous set-valued mapping, and for any } s \in S, H(s) \text{ is a nonempty convex compact set;} \\ \text{there is } s \in S \text{ such that } \psi(s) \in H(s), \text{ and for any } h \in H(s), \text{ we have } \varphi(s), h - \psi(s) \geq 0. \end{array} \right\} \quad (26)$$

For any $u_1 = (\varphi_1, \psi_1, H_1)$, $u_2 = (\varphi_2, \psi_2, H_2) \in U$, we define the distance on U

$$\ell(u_1, u_2) = \sup_{s\in S} \|\varphi_1(s) - \varphi_2(s)\| + \sup_{s\in S} \|\psi_1(s) - \psi_2(s)\| + \sup_{s\in S} h(H_1(s), H_2(s)). \quad (27)$$

Theorem 15. *(U, ℓ) is a complete metric space.*

Proof. Clearly, (U, ℓ) is a metric space. Therefore, we only need to prove that (U, ℓ) is complete.

First, we assume that any Cauchy sequence in U is $\{u_m = (\varphi_m, \psi_m, H_m)\}_{m=1}^{\infty}$. This means that for every $\theta > 0$, there exists an integer N_1 such that any $p, m \geq N_1$, we have

$$\ell(u_m, u_p) = \sup_{s\in S} \left\|\varphi_m(s) - \varphi_p(s)\right\| + \sup_{s\in S} \left\|\psi_m(s) - \psi_p(s)\right\| + \sup_{s\in S} h\left(H_m(s), H_p(s)\right) < \theta. \quad (28)$$

Then, there exist $\varphi, \psi : S \longrightarrow S$ and $H : S \longrightarrow T(S)$ such that $\lim_{p\longrightarrow\infty} \varphi_p(s) = \varphi(s)$, $\lim_{p\longrightarrow\infty} \psi_p(s) = \psi(s)$, $\lim_{p\longrightarrow\infty} H_p(s) = H(s)$, we can obtain

$$\begin{aligned} &\sup_{s\in S} \|\varphi(s) - \varphi_m(s)\| \leq \theta, \\ &\sup_{s\in S} \|\psi(s) - \psi_m(s)\| \leq \theta, \\ &\sup_{s\in S} h(H(s), H_m(s)) \leq \theta. \end{aligned} \quad (29)$$

Because $s_m \longrightarrow s$, $\varphi_m \longrightarrow \varphi$, and $\psi_m \longrightarrow \psi$, it is clear that φ and ψ are continuous at s and $H(s)$ is a convex set. Since $H_m(s)$ is the Cauchy sequence in $T(S)$. According to a theorem in [20], $(T(S), h)$ is the complete metric space. Therefore, there exists $H : S \longrightarrow T(S)$ such that $H_m(s) \longrightarrow H(s)$, for any $s \in S$, $H(s)$ is a nonempty compact set, then $H(s)$ is a nonempty convex compact set and H is continuous at s.

Next, we verify that φ (or ψ) is monotone quasiconcave (or monotone quasiconvex) and (φ, ψ) is a strongly monotone couple. Since φ_m, ψ_m are monotone, (φ_m, ψ_m) is a strongly monotone couple, and φ, ψ are continuous at s, that is, $\varphi_m \longrightarrow \varphi$, $\psi_m \longrightarrow \psi$, then φ, ψ are monotone and (φ, ψ) is a strongly monotone couple. Again, because φ_m (or ψ_m) is quasiconcave (or quasiconvex), for every $s_1, s_2 \in S$, $\rho \in (0, 1)$, we have

$$\begin{aligned} &\varphi_m(\rho s_1 + (1 - \rho)s_2) \geq \min\{\varphi_m(s_1), \varphi_m(s_2)\}, \\ &\psi_m(\rho s_1 + (1 - \rho)s_2) \leq \max\{\psi_m(s_1), \psi_m(s_2)\}. \end{aligned} \quad (30)$$

Since φ, ψ are continuous at s, then there exists a real parameter $\theta > 0$ such that

$$\begin{aligned} &\varphi(s) - \theta < \varphi_m(s) < \theta + \varphi(s), \\ &\psi(s) - \theta < \psi_m(s) < \theta + \psi(s). \end{aligned} \quad (31)$$

Therefore, we can obtain

$$\begin{aligned} \varphi(\rho s_1 + (1 - \rho)s_2) + \theta &> \varphi_m(\rho s_1 + (1 - \rho)s_2) \\ &\geq \min\{\varphi_m(s_1), \varphi_m(s_2)\} \\ &> \min\{\varphi(s_1) - \theta, \varphi(s_2) - \theta\}, \\ \psi(\rho s_1 + (1 - \rho)s_2) - \theta &< \psi_m(\rho s_1 + (1 - \rho)s_2) \\ &\leq \max\{\psi_m(s_1), \psi_m(s_2)\} \\ &< \max\{\psi(s_1) + \theta, \psi(s_2) + \theta\}. \end{aligned} \quad (32)$$

By the arbitrariness of θ, then φ is quasiconcave and ψ is quasiconvex at s.

Again, according to $u_m = (\varphi_m, \psi_m, H_m) \in U$, there exists $s_m \in S$ such that $\psi_m(s_m) \in H_m(s_m)$, we have

$$\langle \varphi_m(s_m), h - \psi_m(s_m) \rangle \geq 0, \text{ for any } h \in H_m(s_m). \quad (33)$$

Since S is a compact set, there exists $s \in S$ such that s is the convergence point of $\{s_m\}$. It may be assumed that $s_m \longrightarrow s$ and $H_m(s) \longrightarrow H(s)$, H is continuous at s, then

$$h(H_m(s_m), H(s)) \le h(H_m(s_m), H(s_m)) + h(H(s_m), H(s)) \longrightarrow 0. \tag{34}$$

Because ψ is continuous at s, $\psi_m(s) \longrightarrow \psi(s)$, we define the distance on S is d, then

$$\begin{aligned} d(\psi(s), H(s)) &\le d(\psi(s), \psi_m(s_m)) + d(\psi_m(s_m), H_m(s_m)) \\ &\quad + h(H_m(s_m), H(s)) \longrightarrow 0. \end{aligned} \tag{35}$$

Since $H(s)$ is a nonempty convex compact set, we can obtain the result that $\psi(s) \in H(s)$.

Finally, by the contradiction method, we suppose that there exists $h^0 \in H(s)$ satisfying $\langle \varphi(s), h^0 - \psi(s) \rangle < 0$; thus, there exists a small enough $\beta > 0$ such that

$$\left\langle \varphi(s), h^0 - \psi(s) \right\rangle + \beta < 0. \tag{36}$$

Because $h(H_m(s_m), H(s)) \longrightarrow 0$ and $h^0 \in H(s)$, there exists $h_m \in H_m(s_m)$ such that $h_m \longrightarrow h^0$. Since $\varphi(\cdot), \psi(\cdot)$ are continuous at s and $\varphi_m \longrightarrow \varphi, \psi_m \longrightarrow \psi, s_m \longrightarrow s, h_m \longrightarrow h^0$, there exists $m_1 \ge 0$ such that for any $m \ge m_1$, we can obtain

$$\langle \varphi_m(s_m), h_m - \psi_m(s_m) \rangle < \left\langle \varphi(s), h^0 - \psi(s) \right\rangle + \beta < 0. \tag{37}$$

Then, this conflicts with (33). So $u = (\varphi, \psi, H) \in U$. Therefore, the metric space (U, ℓ) is complete. The proof is completed. □

For all $u = (\varphi, \psi, H) \in U$, we define

$$\Omega(u) = \{s \in S : \psi(s) \in H(s), \text{for all}\, h \in H(s), \langle \varphi(s), h - \psi(s) \rangle \ge 0\}. \tag{38}$$

Then, we can see that $\Omega(u) \ne \varnothing$ by the definition of U. A set-valued mapping $\Omega : U \longrightarrow P_0(S)$ is defined by $u \longrightarrow \Omega(u)$.

Lemma 16. *Ω is a usco mapping on U.*

Proof. Since S be a nonempty compact subset, according to Lemma 6, we only need to prove that Ω is closed, which means that we have to prove that $Graph(\Omega)$ is closed, that is, for each $u_m \in U$, $u_m \longrightarrow u$ and for each $s_m \in \Omega(u_m)$ and $s_m \longrightarrow s$, then $s \in \Omega(u)$.

For all $m = 1, 2, \cdots$, since $s_m \in \Omega(u_m)$, we have

$$\langle \varphi_m(s_m), h - \psi_m(s_m) \rangle \ge 0, \text{for all}\, h \in H_m(s_m). \tag{39}$$

Since $\varphi(\cdot), \psi(\cdot)$ are continuous at s and $\varphi_m \longrightarrow \varphi, \psi_m \longrightarrow \psi, s_m \longrightarrow s$, therefore,

$$\begin{aligned} \|\varphi_m(s_m) - \varphi(s)\| &\le \|\varphi_m(s_m) - \varphi(s_m)\| + \|\varphi(s_m) - \varphi(s)\| \longrightarrow 0, \\ \|\psi_m(s_m) - \psi(s)\| &\le \|\psi_m(s_m) - \psi(s_m)\| + \|\psi(s_m) - \psi(s)\| \longrightarrow 0. \end{aligned} \tag{40}$$

Then, we can obtain $\varphi_m(s_m) \longrightarrow \varphi(s)$, $\psi_m(s_m) \longrightarrow \psi(s)$. Since $h - \psi_m(s_m) \longrightarrow h - \psi(s)$, we have

$$\langle \varphi(s), h - \psi(s) \rangle = \lim_{m \longrightarrow \infty} \langle \varphi_m(s_m), h - \psi_m(s_m) \rangle \ge 0. \tag{41}$$

Thus, $s \in \Omega(u)$. The proof is completed. □

Theorem 17. *There exists a dense residual subset Q of U such that for all $u = (\varphi, \psi, H) \in Q$, $\Omega(u)$ is a singleton set.*

Proof. First, because U is a complete metric space and Ω is a usco mapping on U, by the Fort lemma, there exists a dense residual subset Q of U such that for all $u = (\varphi, \psi, H) \in Q$, the set-valued mapping Ω is l.s.c. at u.

For all $u = (\varphi, \psi, H) \in Q$, we suppose that $\Omega(u)$ is not a singleton set; thus, there exist $s_1, s_2 \in \Omega(u)$, where $s_1 \ne s_2$. Simultaneously, there exist two open neighbourhoods K of s_1 and L of s_2, respectively, such that $K \bigcap L \ne \varnothing$ and $K \bigcap \Omega(u) \ne \varnothing$. Let $s_1 \in K \bigcap \Omega(u)$, then for all $h \in H(s_1)$, we can obtain

$$\langle \varphi(s_1), h - \psi(s_1) \rangle \ge 0. \tag{42}$$

For each $s \in S$, $m = 1, 2, \cdots$, we define

$$\begin{aligned} \varphi_m(s) &= \varphi(s) + \frac{1}{m}(\psi(s) - \psi(s_1)), \\ \psi_m(s) &= \psi(s) + \frac{1}{m}(\psi(s) - \psi(s_1)). \end{aligned} \tag{43}$$

Therefore, it is easy to see that φ_m, ψ_m is continuous.

Next, because φ, ψ are monotone and for all $s, l \in S$, by Definition 4, we can obtain

$$\begin{aligned} \langle \varphi_m(s) - \varphi_m(l), s - l \rangle &= \left\langle \varphi(s) - \varphi(l) + \frac{1}{m}(\psi(s) - \psi(l)), s - l \right\rangle \\ &= \langle \varphi(s) - \varphi(l), s - l \rangle + \frac{1}{m} \langle \psi(s) - \psi(l), s - l \rangle \ge 0. \end{aligned} \tag{44}$$

Similarly,

$$\langle \psi_m(s) - \psi_m(l), s - l \rangle \ge 0, \tag{45}$$

thus, φ_m and ψ_m are monotone.

Since (φ, ψ) is a strongly monotone couple on S, by Definition 5, there exists a constant $\delta > 0$ and for each $s, l \in S$, we have

$$\langle \varphi(s) - \varphi(l), \psi(s) - \psi(l) \rangle \ge \delta \|s - l\|^2. \tag{46}$$

Then,

$$\begin{aligned}\langle \varphi_m(s)-\varphi_m(l), \psi_m(s)-\psi_m(l)\rangle &= \left\langle \varphi(s)-\varphi(l)+\frac{1}{m}(\psi(s)-\psi(l)), \frac{m+1}{m}(\psi(s)-\psi(l))\right\rangle \\ &= \frac{m+1}{m}\langle \varphi(s)-\varphi(l), \psi(s)-\psi(l)\rangle + \frac{m+1}{m^2}\|\psi(s)-\psi(l)\|^2 \\ &\geq \delta\|s-l\|^2 \geq 0.\end{aligned} \tag{47}$$

Therefore, (φ_m, ψ_m) is a strongly monotone couple on S.

Again, because φ is quasiconcave and ψ is quasiconvex, so for all $\rho \in (0,1)$, according to Definition 7 we can obtain

$$\begin{aligned}\varphi_m(\rho s_1+(1-\rho)s_2) &= \varphi(\rho s_1+(1-\rho)s_2) + \frac{1}{m}(\psi(\rho s_1+(1-\rho)s_2)-\psi(s_1)) \geq \varphi(\rho s_1+(1-\rho)s_2) \\ &\geq \min\{\varphi(s_1), \varphi(s_2)\}.\end{aligned} \tag{48}$$

Similarly,

$$\begin{aligned}\psi_m(\rho s_1+(1-\rho)s_2) &= \psi(\rho s_1+(1-\rho)s_2) + \frac{1}{m}(\psi(\rho s_1+(1-\rho)s_2)-\psi(s_1)) \\ &= \frac{m+1}{m}\psi(\rho s_1+(1-\rho)s_2) - \frac{1}{m}\psi(s_1) \\ &\leq \frac{m+1}{m}\max\{\psi(s_1).\psi(s_2)\}.\end{aligned} \tag{49}$$

Then, φ_m is quasiconcave and ψ_m is quasiconvex. Finally, for all $h \in H_m(s_1)$, we have

$$\langle \varphi_m(s_1), h-\psi_m(s_1)\rangle = \langle \varphi(s_1), h-\psi(s_1)\rangle \geq 0. \tag{50}$$

Therefore, $s_1 \in \Omega(u_m)$, $u_m \in U$. Obviously, $\ell(u_m, u) \longrightarrow 0$ $(m \longrightarrow \infty)$.

Note that $s_2 \in L$, then $L \bigcap \Omega(u) \neq \varnothing$. Since the set-valued mapping Ω is l.s.c. at u, there exists a sufficiently large integer m_0 such that $L \bigcap \Omega(u_{m_0}) \neq \varnothing$. Take $s_{m_0} \in L \bigcap \Omega(u_{m_0})$, we have

$$\left\langle \varphi_{m_0}(s_{m_0}), h-\psi_{m_0}(s_{m_0})\right\rangle \geq 0, \text{for all } h \in H_{m_0}(s_{m_0}). \tag{51}$$

Let $h=\psi(s_1)$, then

$$\begin{aligned}\left\langle \varphi_{m_0}(s_{m_0}), \psi(s_1)-\psi_{m_0}(s_{m_0})\right\rangle &= \Big\langle \varphi(s_{m_0}) + \frac{1}{m_0}(\psi(s_{m_0}) \\ &\quad -\psi(s_1)), \psi(s_1)-\psi(s_{m_0}) - \frac{1}{m_0}(\psi(s_{m_0})-\psi(s_1))\Big\rangle \\ &= \left\langle \varphi(s_{m_0}), \frac{m_0+1}{m_0}(\psi(s_1)-\psi(s_{m_0}))\right\rangle \\ &\quad - \frac{m_0+1}{m_0^2}\|\psi(s_1)-\psi(s_{m_0})\|^2 \geq 0,\end{aligned} \tag{52}$$

that is,

$$\langle \varphi(s_{m_0}), \psi(s_1)-\psi(s_{m_0})\rangle \geq \frac{1}{m_0}\|\psi(s_1)-\psi(s_{m_0})\|^2 > 0. \tag{53}$$

Because $s_1 \in \Omega(u)$, for every $h \in H(s_1)$, we have

$$\langle \varphi(s_1), h-\psi(s_1)\rangle \geq 0. \tag{54}$$

Let $h=\psi(s_{m_0})$, then

$$\langle \varphi(s_1), \psi(s_{m_0})-\psi(s_1)\rangle \geq 0, \tag{55}$$

that is,

$$\begin{aligned}\langle \varphi(s_1)-\varphi(s_{m_0}), \psi(s_1)-\psi(s_{m_0})\rangle &= \langle \varphi(s_1), \psi(s_1)-\psi(s_{m_0})\rangle \\ &\quad - \langle \varphi(s_{m_0}), \psi(s_1)-\psi(s_{m_0})\rangle = -[\langle \varphi(s_1), \psi(s_{m_0})-\psi(s_1)\rangle \\ &\quad + \langle \varphi(s_{m_0}), \psi(s_1)-\psi(s_{m_0})\rangle] < 0.\end{aligned} \tag{56}$$

This contradicts with the fact that (φ, ψ) is a strongly monotone couple on S. Therefore, for any $u=(\varphi, \psi, H) \in Q$, $\Omega(u)$ is a singleton set. This completes the proof. □

Theorem 18. *There exists a dense residual subset Q of U such that for all $u=(\varphi, \psi, H) \in Q$, there must be $s_m \longrightarrow s$, $\psi(s) \in H(s)$, and $\langle \varphi(s), h-\psi(s)\rangle \geq 0$ for each $h \in H(s)$.*

Proof. According to Theorem 17, there exists a dense residual subset Q of U such that for any $u=(\varphi, \psi, H) \in Q$, the solution of the IQVI problem is a singleton set. By conclusion (3) of Theorem 10, there must be $s_m \longrightarrow s$. This completes the proof. □

Remark 19. Theorem 18 shows that the solution set of the IQVI problem has generic convergence in the case of perturbation with the objective function on S.

5. Conclusion

In this paper, we mainly obtained two new results for IQVI problems: one is the approximation theorem, and the other is the generic convergence theorem. According to the approximation theorem, we can see that the approximate solution representing bounded rationality can converge to the exact solution representing complete rationality from the perspective of Simon's bounded rationality. Especially, notice Theorem 10, these conditions are general, that is, the objective functions φ_m, ψ_m, and H_m $(m = 1, 2, 3, \cdots)$ are not necessarily continuous and the set of feasible solutions A_m is not necessarily compact. Hence, Theorem 10 provides a unified theoretical framework for the convergence of the approximate solution on IQVI problems. Moreover, the generic convergence of the IQVI problems implies that a certain sequence is found to converge to an exact solution, rather than a subsequence converging to an exact solution. And on the meaning of the Baire category, we obtained that most of IQVI problems have generic convergence under the perturbation with the objective function. The results obtained in this paper are new and different from the literatures [2, 3].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

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