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Research Article

A Novel Approach on Decision Support System Based on the Aczel-Alsina Aggregation Operators and Their Applications to Supplier Selection Problems

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Induced aggregation operators are more suitable for aggregating the individual preference relations into a collective fuzzy preference relation. Therefore, in this paper, we introduce the notion of some new types of induced aggregation operators, based on the Aczel-Alsina operations. We construct some induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina ordered weighted averaging/geometric (I-IVITrFAAOWA/G) operator, induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina hybrid averaging/geometric (I-IVITrFAAHA/G) operator. Moreover, some dominant properties of these developed operators are studied in detail. Based on these proposed approaches, a model is a build up for multicriteria decision making (MCDM), and their stepwise algorithm is being presented. Finally, in utilizing the developed approach, an illustrative example is solved with the help of proposed operators. In the end, we utilize an applicable example for supplier selection to prove the proposed methods and compare the result with existing methods, which shows the superiority, competence, and ability of the developed model.

1. Introduction

Multicriteria group decision-making (MCGDM) problems are of consequence in most kinds of fields such as engineering, economics, medical diagnosis, and management. Traditionally, it has been assumed that the information, which accesses the alternatives in terms of criteria, and weight are expressed in real numbers. Multicriteria group decision making (MCGDM) has develop into tremendously precious in preceding few decades in the learning of decision support systems [1–3]. The convolution of DM problems increases time to time in this world of global antagonism as the socio-economic structure becomes more dynamic. Therefore, making sensible and successful decisions in this situation is very complicated for a single decision expert. In this real world, group DM models are often used to combine the opinions of team of experienced experts in order to generate highly accurate and optimal values. Hence, MCGDM have an excellent ability and systematic method for improving and evaluating various competing criteria in all aspect of DM in order to achieve more appropriate and possible DM outcomes. In DM issues, the knowledge based about even a

fact is often unclear, making the decision-making task more complicated and ambiguous. To resolve this restriction, Zadeh [4] in 1965 first established the idea of fuzzy set (FS) by considering the membership degree MD. This concept has been investigated in a various real-world problems, including clustering analysis [5], decision-making problems (DMPs) [6], and medical diagnosis [7]. Atanassov [8] developed the intuitionistic fuzzy set (IFS) as an extension of the FS, and its constraint is that the sum of the MD and the N-MD will be less than or equal to one. IFS has been a hot research topic for researchers who examined its hybrid structure in a variety of ways. The idea of intuitionistic fuzzy weighted averaging (IFWA) aggregation operators was first investigated by Xu [9]. Xu and Yager [10] developed the definition of intuitionistic fuzzy weighted geometric (IFWG) aggregation operators. The graphical techniques for rating score and accuracy functions were created by Ali et al. [11]. He et al. [12] explored the concept of intuitionistic fuzzy neutral averaging operators (IFNAO). He et al. [13] invented the notions of geometric relationship averaging operator and proposed its implementation in DM. Also Atanassov [14] presented preference relation based on IVIF. Further, Wan and Dong [15] work on extension of best-worst method based on IFS. Zaho et al. [16] were the first to apply the generalized intuitionistic fuzzy weighted averaging (GIFWA), generalized intuitionistic fuzzy order weighted averaging (IFOWA), and generalized intuitionistic fuzzy hybrid averaging (GIFHA) operators to DM. However, it was discovered that these aggregation process was performed by using Archimedean t-norms and t-conorm by operators. As an equivalent to algebraic product and sum, Einstein-based t-norm and t-conorm provides the best estimate for product and sum of intuitionistic fuzzy numbers (IFNs). Wang and Liu and Wang and Zhang [17, 18] provided IF Einstein weighted averaging (IFEWA) and IF Einstein weighted geometric (IFEWG) operators by using the concept of Einstein operation. After that, Abbas et al. [19] developed the idea of intuitionistic fuzzy rough Einstein weighted averaging (IFREWA), intuitionistic fuzzy rough Einstein hybrid averaging (IFREHA), and intuitionistic fuzzy rough ordered weighted averaging (IFREOWA) aggregation operators based on the TOPSIS method. Yager and Filev [20, 21] developed an expansion of the OWA operator, called the induced ordered weighted averaging (IOWA) operator. It takes the argument in pairs (OWA pairs), in which one component is used to induce an ordering over the second component and then aggregates. Wei [22] developed two new aggregation operators: the induced intuitionistic fuzzy ordered weighted geometric (I-IFOWG) operator and the induced interval-valued intuitionistic fuzzy ordered weighted geometric (I-IIFOWG) operator. Su et al. [23] also constructed induced intuitionistic fuzzy hybrid averaging (I-IFHA) operator and induced interval-valued intuitionist fuzzy hybrid averaging (I-IIFHA) operator. Shakeel at al [24] developed the induced Pythagorean trapezoidal fuzzy Einstein ordered weighted averaging operator and the induced Pythagorean trapezoidal fuzzy Einstein hybrid averaging (I-PTFEHA) operator. In [25], Klement et al. discussed the triangular norms, and in [26, 27], Aczel-Alsina (AA) presented t-norm as well as t-conorm. After that, some new idea of AA was developed by Senapati et al. in [28]. Senapati et al. introduced the IF Aczel-Alsina weighted averaging (IFAAWA) operator, the IF Aczel-Alsina orders weighted averaging (IFAAOWA) operator, and the IF Aczel-Alsina hybrid averaging (IFAAHA) operator. Senapati et al. [29, 30] developed interval-valued intuitionistic fuzzy aggregation operators, such as the IVIF Aczel-Alsina weighted averaging/geometric operator, the IVIF Aczel–Alsina order weighted averaging/geometric operator, and the IVIF Aczel–Alsina hybrid averaging/geometric operator and built up several features of such operators. Senapati [31] also defined picture fuzzy aggregation operator such as the PF Aczel–Alsina weighted average (PFAAWA) operator, the PF Aczel–Alsina order weighted average (PFAAOWA) operator, and the PF Aczel–Alsina hybrid average (PFAAHA) operator with their properties. Later on, Senapati et al. [32] originated a few new aggregation operators for aggregating hesitant fuzzy information, namely, the HF Aczel-Alsina weighted averaging (HFAAWA) operator, HF Aczel-Alsina ordered weighted averaging (HFAAOWA) operator, HF Aczel-Alsina hybrid averaging (HFAAHA) operator, HF Aczel-Alsina weighted geometric (HFAAWG) operator, HF Aczel-Alsina ordered weighted geometric (HFAAOWG) operator, HF Aczel–Alsina hybrid geometric (HFAAHG) operator, and HF Aczel–Alsina weighted Bonferroni mean (HFAAWBM) operator.

After that, Ban [33, 34] initiated the concept of trapezoidal fuzzy numbers (TrFNs) and interval-valued trapezoidal fuzzy numbers (I-VTrFNs). Later on, Wang and Zhang [35] expanded the definition of intuitionistic trapezoidal fuzzy numbers (ITrFNs) and interval-valued intuitionistic trapezoidal fuzzy numbers (IVITrFNs). Wei [36] developed the intuitionistic trapezoidal fuzzy ordered weighted averaging (ITrFOWA) operator and the intuitionistic trapezoidal fuzzy hybrid aggregation (ITrFHA) operator. Wan and Dong [37] defined the anticipation and expectant score of intuitionistic trapezoidal fuzzy numbers from the geometric angle. As for our information and based from the above review, no implementation of the MCGDM approach to review of induced interval-valued intuitionist trapezoidal fuzzy Aczel-Alsina (I-IVITrFAA) averaging/geometric aggregation operators. We proposed the average/geometric aggregation operators such as I-IVITrF Aczel-Alsina ordered weighted averaging (I-IVITrFAAOWA) operator, I-IVITrF Aczel-Alsina hybrid averaging (I-IVITrFAAHA) operator, I-IVITrF Aczel-Alsina ordered weighted geometric (I-IVITrFAAOWG) operator, and I-IVITrF Aczel-Alsina hybrid geometric (I-IVITrFAAHG) operator for solving MCGDM. To solve a MCGDM process, the weight of the attributes plays an important role in making decisions under the aggregation approaches.

The paper provided the following: in the next section, we briefly discuss some fundamental ideas of IFSs, I-VIFS, TrFNs, IVITrFNs, IFAA, and I-VIFAA aggregation operators. In Section 3, we outline AA operation laws for the I-VITrFNs. In Section 4, we develop I-IVITrFAAOWA operator, I-IVITrFAAHA operator, I-IVITrFAAOWG operator, and I-IVITrFAAOWG operator. In Section 5, we apply the

I-IVITrFAAOWA operator to build up certain methodologies for managing MCGDM issues. Section 6 provides an example of choosing greatest alternative of the proposed method. In Section 7, we compare the developed method with the existing methods. At end of this paper, a few conclusions and upcoming research are mentioned in Section 8.

2. Preliminaries

In this section, we will define basic definition, results, and operational laws which are important for the improvement of this paper.

2.1. IFSs. Atanassov [8] introduced the idea of IFSs which is a generalization of FSs. FSs discussed only M-D where IFSs discussed M-D and NM-D. Moreover, the sum of M-D and NM-D is less than or equal to 1.

Definition 1 (see [8]). Let a set Uz be fixed. An (IFS)l_e^e in Uz is an object having the form

$$A_e^e = \left\{ \left\langle u_z, \mu_{l_e^e}(u_z), \mathfrak{I}_{l_e^e}(u_z) \right\rangle | u_z \in \mathrm{Uz} \right\}, \tag{1}$$

where $\mu_{\iota_e^e} : \mathrm{Uz} \longrightarrow [0,1]$ and $\mathfrak{I}_{\iota_e^e} : \mathrm{Uz} \longrightarrow [0,1]$ represent the M-D and N-MD such that $0 \le \mu_{\iota_e^e}(u_z) + \mathfrak{I}_{\iota_e^e}(u_z) \le 1$ for each (IFS)ι_e^e in Uz,$\pi_{\iota_e^e}(u_z) = 1 - \mu_{\iota_e^e}(u_z) - \mathfrak{I}_{\iota_e^e}(u_z)$, for all $u_z \in \mathrm{Uz}\pi_{\iota_e^e}(u_z)$ is called the degree of indeterminacy of u_z to ι_e^e.

Definition 2. Atanassov [8] let $\iota_e^e = \{\langle u_z, \mu_{\iota_e^e}(u_z), \mathfrak{I}_{\iota_e^e}(u_z)\rangle | u_z \in \mathrm{Uz}\}$ and $G_e^e = \{\langle u_z, \mu_{G_e^e}(u_z), \mathfrak{I}_{G_e^e}(u_z)\rangle | u_z \in \mathrm{Uz}\}$ are two IFSs over the universe Uz, then the operational laws between A_e^e and G_e^e are defined as

(i) $\iota_e^e \subseteq G_e^e$, if $\mu_{\iota_e^e}(u_z) \le \mu_{G_e^e}(u_z)$ and $\mathfrak{I}_{\iota_e^e}(u_z) \ge \mathfrak{I}_{G_e^e}(u_z) \forall u_z \in Uz$

(ii) $\iota_e^e = G_e^e \Leftrightarrow \iota_e^e \subseteq G_e^e, G_e^e \subseteq \iota_e^e$

(iii) $\iota_e^e \cup G_e^e = \{(u_z, \max\{\mu_{\iota_e^e}(u_z), \mu_{G_e^e}(u_z)\}, \min \mathfrak{I}_{\iota_e^e}(u_z), \mathfrak{I}_{G_e^e}(u_z)\})/u_z \in Uz\}$

(iv) $\iota_e^e \cap G_e^e = \{(u_z, \min\{\mu_{\iota_e^e}(u_z), \mu_{G_e^e}(u_z)\}, \max \mathfrak{I}_{\iota_e^e}(u_z), \mathfrak{I}_{G_e^e}(u_z)\})/u_z \in Uz\}$

(v) $(\iota_e^e)^c = \{(u_z, \mathfrak{I}_{\iota_e^e}(u_z), \mu_{\iota_e^e}(u_z)/u_z \in Uz\}$

(vi) $\iota_e^e \oplus G_e^e = \left\{ \begin{array}{c} (u_z, \mu_{\iota_e^e}(u_z) + \mu_{G_e^e}(u_z) - \mu_{\iota_e^e}(u_z).\mu_{G_e^e}(u_z), \\ \mathfrak{I}_{\iota_e^e}(u_z)\mathfrak{I}_{G_e^e}(u_z)/u_z \in Uz \end{array} \right\}$

(vii) $\iota_e^e \otimes G_e^e = \left\{ \begin{array}{c} (u_z, \mu_{\iota_e^e}(u_z).\mu_{G_e^e}(u_z), \\ \mathfrak{I}_{\iota_e^e}(u_z) + \mathfrak{I}_{G_e^e}(u_z) - \mathfrak{I}_{\iota_e^e}(u_z).\mathfrak{I}_{G_e^e}(u_z)/u_z \in Uz \end{array} \right\}$

Definition 3 (see [26]). Let ι_e^e and G_e^e be any two IFSs, then the generalized intersection and the generalized union are defined in the following ways:

$$\iota_e^e \cap_{T,S} G_e^e = \left\{ \left\langle \begin{array}{c} u_z, T\left\{\mu_{\iota_e^e}(u_z), \mu_{G_e^e}(u_z)\right\}, \\ S\left\{\mathfrak{I}_{\iota_e^e}(u_z), \mathfrak{I}_{G_e^e}(u_z)\right\} \end{array} \right\rangle / u_z \in Uz \right\},$$
$$\iota_e^e \cup_{S,T} G_e^e = \left\{ \left\langle \begin{array}{c} u_z, S\left\{\mu_{\iota_e^e}(u_z), \mu_{G_e^e}(u_z)\right\}, \\ T\left\{\mathfrak{I}_{\iota_e^e}(u_z), \mathfrak{I}_{G_e^e}(u_z)\right\} \end{array} \right\rangle / u_z \in Uz \right\}, \tag{2}$$

where T presents a t-norm, and S presents a t-conorm.

2.2. I-VIFSs

Definition 4 (see [38]). Let $\iota_e^e = (\mu_{\iota_e^e}, \mathfrak{I}_{\iota_e^e}) = \langle [\mu_{\iota_e^e}^{lr}, \mu_{\iota_e^e}^{ur}][\mathfrak{I}_{\iota_e^e}^{lr}, \mathfrak{I}_{\iota_e^e}^{ur}] \rangle$ be an I-VIFNs, where $\mu_{\iota_e^e} = [\mu_{\iota_e^e}^{lr}, \mu_{\iota_{e1}^e}^{ur}]$ and $\mathfrak{I}_{\iota_e^e} = [\mathfrak{I}_{\iota_e^e}^{lr}, \mathfrak{I}_{\iota_e^e}^{ur}]$ represent an I-VIFNs, hence $\mu_{\iota_e^e} \subset [0,1]$ and $\mathfrak{I}_{\iota_e^e} \subset [0,1]$, such that $0 \le \mu_{\iota_e^e} + \mathfrak{I}_{\iota_e^e} \le 1$.

Definition 5. Let $\iota_e^e = (\mu_{\iota_e^e}, \mathfrak{I}_{\iota_e^e}) = ([\mu_{\iota_e^e}^{lr}, \mu_{\iota_e^e}^{ur}][\mathfrak{I}_{\iota_e^e}^{lr}, \mathfrak{I}_{\iota_e^e}^{ur}])$, $\iota_{e_1}^e = (\mu_{\iota_{e_1}^e}, \mathfrak{I}_{\iota_{e_1}^e}) = ([\mu_{\iota_{e_1}^e}^{lr}, \mu_{\iota_{e_1}^e}^{ur}][\mathfrak{I}_{\iota_{e_1}^e}^{lr}, \mathfrak{I}_{\iota_{e_1}^e}^{ur}])$ $\iota_{e_2}^e = (\mu_{\iota_{e_2}^e}, \mathfrak{I}_{\iota_{e_2}^e}) = ([\mu_{\iota_{e_2}^e}^{lr}, \mu_{\iota_{e_2}^e}^{ur}][\mathfrak{I}_{\iota_{e_2}^e}^{lr}, \mathfrak{I}_{\iota_{e_2}^e}^{ur}])$ be any two I-IVFNs, numbers, and $\partial \ge 0$. Then,

(1) $\iota_{e_1}^e \oplus \iota_{e_2}^e$
$$= \left(\begin{array}{c} [((\mu_{\iota_{e_1}^e}^{lr}) + (\mu_{\iota_{e_2}^e}^{lr}) - (\mu_{\iota_{e_1}^e}^{lr})(\mu_{\iota_{e_2}^e}^{lr}), (\mathfrak{I}_{\iota_{e_1}^e}^{lr})(\mathfrak{I}_{\iota_{e_2}^e}^{lr})], \\ [((\mu_{\iota_{e_1}^e}^{ur}) + (\mu_{\iota_{e_2}^e}^{ur}) - (\mu_{\iota_{e_1}^e}^{ur})(\mu_{\iota_{e_2}^e}^{ur}), (\mathfrak{I}_{\iota_{e_1}^e}^{ur})(\mathfrak{I}_{\iota_{e_2}^e}^{ur})] \end{array} \right)$$

(2) $\iota_{e_1}^e \otimes \iota_{e_2}^e$
$$= \left(\begin{array}{c} [(\mu_{\iota_{e_1}^e}^{lr})(\mu_{\iota_{e_2}^e}^{lr}), (\mathfrak{I}_{\iota_{e_1}^e}^{lr}) + (\mathfrak{I}_{\iota_{e_2}^e}^{lr}) - (\mathfrak{I}_{\iota_{e_1}^e}^{lr})(\mathfrak{I}_{\iota_{e_2}^e}^{lr})], \\ [(\mu_{\iota_{e_1}^e}^{ur})(\mu_{\iota_{e_2}^e}^{ur}), ((\mathfrak{I}_{\iota_{e_1}^e}^{ur}) + (\mathfrak{I}_{\iota_{e_2}^e}^{ur}) - (\mathfrak{I}_{\iota_{e_1}^e}^{ur})(\mathfrak{I}_{\iota_{e_2}^e}^{ur})] \end{array} \right)$$

(3) $\partial(\iota_e^e) = ([1 - (1 - \mu_{\iota_e^e}^{lr})^{\partial}, (\mathfrak{I}_{\iota_e^e}^{lr})^{\partial}], [1 - (1 - \mu_{\iota_e^e}^{ur})^{\partial}, (\mathfrak{I}_{\iota_e^e}^{ur})^{\partial}])$

(4) $(\iota_e^e)^{\partial} = ([\mu_{\iota_e^e}^{lr\partial}, (1 - (1 - \mathfrak{I}_{\iota_e^e}^{lr})^{\partial}], [\mu_{\iota_e^e}^{ur\partial}, (1 - (1 - \mathfrak{I}_{\iota_e^e}^{ur})^{\partial}])$

Example 6. Let $\iota_e^e = ([0.45, 0.04], [0.35, 0.23])$, $\iota_{e_1}^e = ([0.34, 0.21], [0.05, 0.34])$, $\iota_{e_2}^e = ([0.27, 0.34], [0.21, 0.31])$ be any three I-VIFNs, and $\partial = 0.4$; then, we verify the above results such that,

(1) $\iota_{e_1}^e \oplus \iota_{e_2}^e$
$$= \left(\begin{array}{c} [((\mu_{\iota_{e_1}^e}^{lr}) + (\mu_{\iota_{e_2}^e}^{lr}) - (\mu_{\iota_{e_1}^e}^{lr})(\mu_{\iota_{e_2}^e}^{lr}), (\mathfrak{I}_{\iota_{e_1}^e}^{lr})(\mathfrak{I}_{\iota_{e_2}^e}^{lr})] \\ , [((\mu_{\iota_{e_1}^e}^{ur}) + (\mu_{\iota_{e_2}^e}^{ur}) - (\mu_{\iota_{e_1}^e}^{ur})(\mu_{\iota_{e_2}^e}^{ur}), (\mathfrak{I}_{\iota_{e_1}^e}^{ur})(\mathfrak{I}_{\iota_{e_2}^e}^{ur})] \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\begin{array}{c} ((0.34) + (0.27) - (0.34)(0.27), \\ (0.05)(0.21) \end{array} \right] \\ , \left[\begin{array}{c} (0.21) + (0.34) - (0.21)(0.34), \\ (0.34)(0.31) \end{array} \right] \end{array} \right)$$

$$= \left(\begin{array}{c} [0.61 - .0.091, 0.0105] \\ , [0.55 - 0.0714, 0.1054] \end{array} \right)$$

$$= \left(\begin{array}{c} [0.519, 0.0105] \\ , [0.4786, 0.1054] \end{array} \right)$$

(2) $\iota_{e_1}^e \otimes \iota_{e_2}^e$

$$= \left(\begin{array}{c} [(\mu_{\iota_{e_1}^e}^{lr})(\mu_{\iota_{e_2}^e}^{lr}), (\mathfrak{I}_{\iota_{e_1}^e}^{lr}) + (\mathfrak{I}_{\iota_{e_2}^e}^{lr}) - (\mathfrak{I}_{\iota_{e_1}^e}^{lr})(\mathfrak{I}_{\iota_{e_2}^e}^{lr})], \\ [(\mu_{\iota_{e_1}^e}^{ur})(\mu_{\iota_{e_2}^e}^{ur}), ((\mathfrak{I}_{\iota_{e_1}^e}^{ur}) + (\mathfrak{I}_{\iota_{e_2}^e}^{ur}) - (\mathfrak{I}_{\iota_{e_1}^e}^{ur})(\mathfrak{I}_{\iota_{e_2}^e}^{ur})] \end{array} \right)$$

$$= \left(\begin{array}{c} [(0.34)(0.27), (0.05) + (0.21) - (0.05)(0.21)], \\ [(0.21)(0.34), (0.34) + (0.31) - (0.34)(0.31))] \end{array} \right)$$

$$= \left(\begin{array}{c} [0.0918, (0.26 - 0.0105)], \\ [0.0714, (0.65 - 0.1054] \end{array} \right)$$

$$= ([0.0918, 0.2495], [0.0714, 0.6394])$$

(3) $$\partial \iota_e^e = \left(\begin{array}{c} [1 - (1 - \mu_{\iota_e^e}^{lr})^{\partial}, (\mathfrak{I}_{\iota_e^e}^{lr})^{\partial}], \\ [1 - (1 - \mu_{\iota_e^e}^{ur})^{\partial}, (\mathfrak{I}_{\iota_e^e}^{ur})^{\partial}] \end{array} \right)$$

$$\cdot \left(\begin{array}{c} [1 - (1 - 0.45)^{0.4}, (0.35)^{0.4}], \\ [1 - (1 - 0.04)^{0.4}, (0.23)^{0.4}] \end{array} \right)$$

$$= \left(\begin{array}{c} [1 - 0.7873, 0.6570], \\ [1 - 0.9838, 0.5550] \end{array} \right).$$

$$= ([0.2127, 0.6570], [0.0162, 0.5550])$$

(4) $$(\iota_e^e)^{\partial} = \left(\begin{array}{c} [\mu_{\iota_e^e}^{lr\partial}, (1 - (1 - \mathfrak{I}_{\iota_e^e}^{lr})^{\partial}], \\ [\mu_{\iota_e^e}^{ur\partial}, (1 - (1 - \mathfrak{I}_{\iota_e^e}^{ur})^{\partial}] \end{array} \right)$$

$$= \left(\begin{array}{c} [0.45^{0.4}, (1 - (1 - 0.35)^{0.4}] \\ [0.04^{0.4}, (1 - (1 - 0.23)^{0.4}] \end{array} \right)$$

$$= ([0.7265, 0.1582][0.2759, 0.0992])$$

Definition 7 (see [39]). Let $\iota_e^e = \langle [\mu_{\iota_e^e}^{lr}, \mu_{\iota_e^e}^{ur}][\mathfrak{I}_{\iota_e^e}^{lr}, \mathfrak{I}_{\iota_e^e}^{ur}] \rangle$ be an $I - VIFNs$, a $Scr(\iota_e^e)$ can be defined as follows:

$$Scr(\iota_e^e) = \left(\frac{\mu_{\iota_e^e}^{lr} - \mathfrak{I}_{\iota_e^e}^{lr} + \mu_{\iota_e^e}^{ur} - \mathfrak{I}_{\iota_e^e}^{ur}}{2} \right) Scr(\iota_e^e) \in [-1, 1]. \quad (3)$$

Example 8. Let $\iota_{e_1}^e = ([0.37, 0.31][0.16, 0.23])$, $\iota_{e_2}^e = ([0.20, 0.34][0.40, 0.54])$.

$\iota_{e_3}^e = ([0.46, 0.32][0.32, 0.24])$, $\iota_{e_4}^e = ([0.46, 0.21][0.20, 0.10])$ be $I - VIFNs$. Then, we verify the above results such that,

$$\begin{aligned} \mathrm{Scr}_1\left(\iota_{e_1}^e\right) &= \left(\frac{0.37 - 0.16 + 0.31 - 0.23}{2}\right) = 0.1450, \\ \mathrm{Scr}_2\left(\iota_{e_2}^e\right) &= \left(\frac{0.20 - 0.40 + 0.34 - 0.54}{2}\right) = -0.2100, \\ \mathrm{Scr}_3\left(\iota_{e_3}^e\right) &= \left(\frac{0.46 - 0.32 + 0.32 - 0.24}{2}\right) = 0.1100, \\ \mathrm{Scr}_4\left(\iota_{e_4}^e\right) &= \left(\frac{0.46 - 0.20 + 0.21 - 0.10}{2}\right) = 0.1910. \end{aligned} \quad (4)$$

In Figure 1, we supposed some interval-valued of intuitionistic fuzzy numbers and apply them score function to find out the highest score values for ranking process.

Definition 9 (see [39]). Let $\iota_e^e = \langle [\mu_{\iota_e^e}^{lr}, \mu_{\iota_e^e}^{ur}][\mathfrak{I}_{\iota_e^e}^{lr}, \mathfrak{I}_{\iota_e^e}^{ur}] \rangle$ be an I-VIFNs; an accuracy function $\mathrm{Acr}(\iota_e^e)$ can be defined as follows:

$$\mathrm{Acr}(\iota_e^e) = \left(\frac{\mu_{\iota_e^e}^{lr} + \mathfrak{I}_{\iota_e^e}^{lr} + \mu_{\iota_e^e}^{ur} + \mathfrak{I}_{\iota_e^e}^{ur}}{2} \right) \mathrm{Acr}(\iota_e^e) \in [0, 1]. \quad (5)$$

Example 10. Let

$$\begin{aligned} \iota_{e_1}^e &= ([0.37, 0.31][0.16, 0.23]), \\ \iota_{e_2}^e &= ([0.20, 0.34][0.40, 0.54]), \\ \iota_{e_3}^e &= ([0.46, 0.32][0.32, 0.24]), \\ \iota_{e_4}^e &= ([0.46, 0.21][0.20, 0.10]), \end{aligned} \quad (6)$$

be $I - VIFNs$. Then, we verify the above results such that,

$$\begin{aligned} \mathrm{Acr}_1\left(\iota_{e_1}^e\right) &= \left(\frac{0.37 + 0.16 + 0.31 + 0.23}{2}\right) = 0.5400, \\ \mathrm{Acr}_2\left(\iota_{e_2}^e\right) &= \left(\frac{0.20 + 0.40 + 0.34 + 0.54}{2}\right) = 0.7400, \\ \mathrm{Acr}_3\left(\iota_{e_3}^e\right) &= \left(\frac{0.46 + 0.32 + 0.32 + 0.24}{2}\right) = 0.6700, \\ \mathrm{Acr}_4\left(\iota_{e_4}^e\right) &= \left(\frac{0.46 + 0.20 + 0.21 + 0.10}{2}\right) = 0.4900. \end{aligned} \quad (7)$$

In Figure 2, we supposed some interval-valued of intuitionistic fuzzy numbers and apply their accuracy function to find out the highest accuracy values for ranking process.

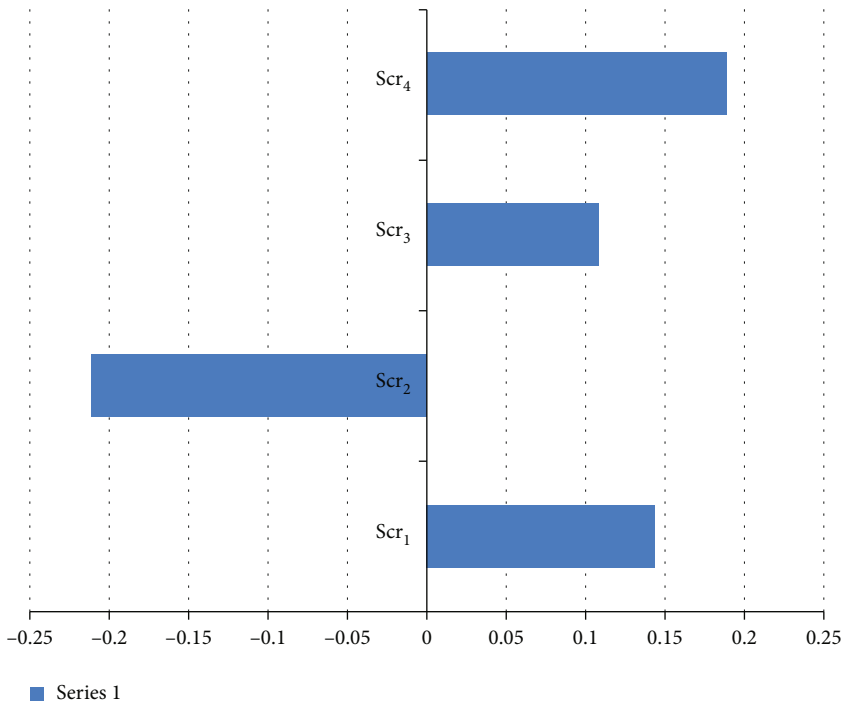

Figure 1: Graphical approach of score function.

2.3. ITFNs and I-VITFNs

Definition 11 (see [40]). Let ι_e^e be ITFNs, its membership function

$$\mu_{\iota_e^e}(l) = \begin{cases} \dfrac{x - a^b}{b^a - a^b}\mu_{\iota_e^e}, a^b \le l \le b^a, \\ \mu_{\iota_e^e}, b^a \le l \le c^d, \\ \dfrac{d^c - x}{d^c - c^d}\mu_{\iota_e^e} c^d \le l \le d^c, \\ 1, \text{otherwise}. \end{cases} \tag{8}$$

Its nonmembership function is

$$\mathfrak{I}_{\iota_e^e}(l) = \begin{cases} \dfrac{d^c - l + \mathfrak{I}_{\iota_e^e}(l - a^b)}{b^a - a^b} a^b \le l \le b^a, \\ \mathfrak{I}_{\iota_e^e}, b^a \le l \le c^d, \\ \dfrac{l - c^d + \mathfrak{I}_{\iota_e^e}(d^c - l)}{d^c - c^d}, c^d \le l \le d^c, \\ 0, \text{otherwise}, \end{cases} \tag{9}$$

where $0 \le \mu_{\iota_e^e} \le 1$; $0 \le \mathfrak{I}_{\iota_e^e} \le 1$; $0 \le (\mu_{\iota_e^e}) + (\mathfrak{I}_{\iota_{e1}^e}) \le 1$; $a^b, b^a, c^d, d^c \in R$. Then, ι_e^e is called ITFNs. For convenience, $A_e^e = \langle [a^b, b^a, c^d, d^c][\mu_{\iota_e^e}, \mathfrak{I}_{\iota_e^e}] \rangle$.

Definition 12 (see [40]). Let

$$\iota_e^e = \left\langle \begin{matrix} [a^b, b^a, \\ c^d, d^c] \\ [\mu_{\iota_e^e}, \mathfrak{I}_{\iota_e^e}] \end{matrix} \right\rangle, \quad \iota_{e_1}^e = \left\langle \begin{matrix} [a_1^b, b_1^a, \\ c_1^d, d_1^c] \\ [\mu_{\iota_{e_1}^e}, \mathfrak{I}_{\iota_{e_1}^e}] \end{matrix} \right\rangle, \tag{10}$$

$$\iota_{e_2}^e = \left\langle \begin{matrix} [a_2^b, b_2^a, \\ c_2^d, d_2^c] \\ [\mu_{\iota_{e_2}^e}, \mathfrak{I}_{\iota_{e_2}^e}] \end{matrix} \right\rangle \tag{11}$$

be the two ITFNs, and $\partial \ge 0$. Then,

(1) $\iota_{e_1}^e \oplus \iota_{e_2}^e = \left(\begin{matrix} [a_1^b + a_2^b, b_1^a + b_2^a, c_1^d + c_2^d, d_1^a + d_2^a]; \\ (\mu_{\iota_{e_1}^e}) + (\mu_{\iota_{e_2}^e}) - (\mu_{\iota_{e_1}^e})(\mu_{\iota_{e_2}^e}), (\mathfrak{I}_{\iota_{e_1}^e})(\mathfrak{I}_{\iota_{e_2}^e}) \end{matrix} \right)$

(2) $\iota_{e_1}^e \otimes \iota_{e_2}^e = \left(\begin{matrix} [a_1^b a_2^b, b_1^a b_2^a, c_1^d c_2^d, d_1^a d_2^a]; (\mu_{\iota_{e_1}^e})(\mu_{\iota_{e_2}^e}), \\ (\mathfrak{I}_{\iota_{e_1}^e}) + (\mathfrak{I}_{\iota_{e_2}^e}) - (\mathfrak{I}_{\iota_{e_1}^e})(\mathfrak{I}_{\iota_{e_2}^e}) \end{matrix} \right)$

(3) $\partial(\iota_e^e) = ([\partial a^b, \partial b^a, \partial c^d, \partial d^a]; 1 - (1 - \mu_{\iota_e^e})^\partial; (\mathfrak{I}_{\iota_e^e})^\partial)$

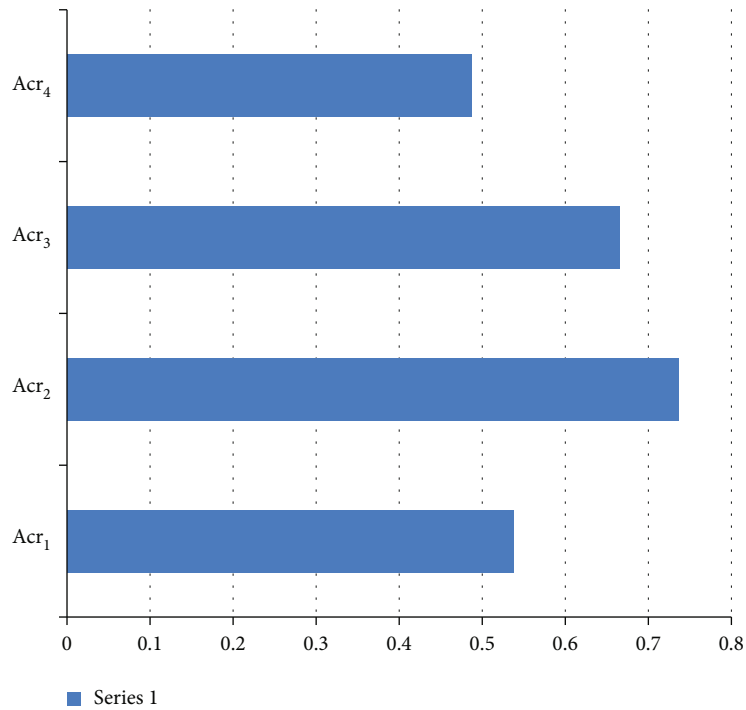

Figure 2: Graphical approach of accuracy function.

(4) $(\iota_e^e)^{\partial} = ([a^{b\partial}, b^{a\partial}, c^{d\partial}, d^{a\partial}]; \mu_{\iota_e^e}, 1-(1-\mathfrak{I}_{\iota_e^e})^{\partial})$

Definition 13 (see [40]). Let

$$\iota_e^e = \left\langle \begin{matrix} [a^b, b^a, c^d, d^c], \\ [\mu_{\iota_e^e}^{lr}, \mu_{\iota_e^e}^{ur}] \\ [\mathfrak{I}_{\iota_e^e}^{lr}, \mathfrak{I}_{\iota_e^e}^{ur}] \end{matrix} \right\rangle, \quad \iota_{e_1}^e = \left\langle \begin{matrix} [a_1^b, b_1^a, c_1^d, d_1^c], \\ [\mu_{\iota_{e_1}^e}^{lr}, \mu_{\iota_{e_1}^e}^{ur}] \\ [\mathfrak{I}_{\iota_{e_1}^e}^{lr}, \mathfrak{I}_{\iota_{e_1}^e}^{ur}] \end{matrix} \right\rangle, \tag{12}$$

$$\iota_{e_2}^e = \left\langle \begin{matrix} [a_2^b, b_2^a, c_2^d, d_2^c], \\ [\mu_{\iota_{e_2}^e}^{lr}, \mu_{\iota_{e_2}^e}^{ur}] \\ [\mathfrak{I}_{\iota_{e_2}^e}^{lr}, \mathfrak{I}_{\iota_{e_2}^e}^{ur}] \end{matrix} \right\rangle \tag{13}$$

be any three I-VITrNs, and $\partial \geq 0$. Then,

(1) $\iota_{e_1}^e \oplus \iota_{e_2}^e = \left(\begin{matrix} [a_1^b + a_2^b, b_1^a + b_2^a, c_1^d + c_2^d, d_1^a + d_2^a], \\ [(\mu_{\iota_{e_1}^e}^{lr}) + (\mu_{\iota_{e_2}^e}^{lr}) - (\mu_{\iota_{e_1}^e}^{lr})(\mu_{\iota_{e_2}^e}^{lr})), (\mathfrak{I}_{\iota_{e_1}^e}^{lr})(\mathfrak{I}_{\iota_{e_2}^e}^{lr})], \\ [(\mu_{\iota_{e_1}^e}^{ur}) + (\mu_{\iota_{e_2}^e}^{ur}) - (\mu_{\iota_{e_1}^e}^{ur})(\mu_{\iota_{e_2}^e}^{ur})), (\mathfrak{I}_{\iota_{e_1}^e}^{ur})(\mathfrak{I}_{\iota_{e_2}^e}^{ur})], \end{matrix} \right)$

(2) $\iota_{e_1}^e \otimes \iota_{e_2}^e = \left(\begin{matrix} [a_1^b a_2^b, b_1^a b_2^a, c_1^d c_2^d, d_1^a d_2^a], \\ [(\mu_{\iota_{e_1}^e}^{lr})(\mu_{\iota_{e_2}^e}^{lr}), \mathfrak{I}_{\iota_{e_1}^e}^{lr} + \mathfrak{I}_{\iota_{e_2}^e}^{lr} - (\mathfrak{I}_{\iota_{e_1}^e}^{lr})(\mathfrak{I}_{\iota_{e_2}^e}^{lr})], \\ [(\mu_{\iota_{e_1}^e}^{ur})(\mu_{\iota_{e_2}^e}^{ur}), \mathfrak{I}_{\iota_{e_1}^e}^{ur} + \mathfrak{I}_{\iota_{e_2}^e}^{ur} - (\mathfrak{I}_{\iota_{e_1}^e}^{ur})(\mathfrak{I}_{\iota_{e_2}^e}^{ur})], \end{matrix} \right)$

(3) $\partial(\iota_e^e) = \left(\begin{matrix} [\partial a^b, \partial b^a, \partial c^d, \partial d^a], [1-(1-\mu_{\iota_e^e}^{lr})^{\partial}, (\mathfrak{I}_{\iota_e^e}^{lr})^{\partial}], \\ [1-(1-\mu_{\iota_e^e}^{ur})^{\partial}, (\mathfrak{I}_{\iota_e^e}^{ur})^{\partial}] \end{matrix} \right)$

(4) $(\iota_e^e)^{\partial} = \left(\begin{matrix} [a^{b\partial}, b^{a\partial}, c^{d\partial}, d^{a\partial}], [(\mu_{\iota_e^e}^{lr})^{\partial}, 1-(1-\mathfrak{I}_{\iota_e^e}^{lr})^{\partial}], \\ [(\mu_{\iota_e^e}^{ur})^{\partial}, 1-(1-\mathfrak{I}_{\iota_e^e}^{ur})^{\partial}] \end{matrix} \right)$

Example 14. Let

$$\iota_e^e = \left\langle \begin{matrix} [0.30, 0.40, \\ 0.20, 0.50], \\ [0.40, 0.30] \end{matrix} \right\rangle, \quad \iota_{e_1}^e = \left\langle \begin{matrix} [0.50, 0.20] \\ [0.20, 0.30, \\ 0.10, 0.40], \\ [0.50, 0.20] \end{matrix} \right\rangle, \quad [0.30, 0.60] \tag{14}$$

$$\iota_{e_2}^{e} = \left\langle \begin{matrix} [0.40, 0.50, \\ 0.60, 0.60], \\ [0.40, 0.30] \end{matrix} \right\rangle \\ [0.50, 0.20] \tag{15}$$

be any three I-VITrNs, and $\partial = 2$. We verify the above operational laws such that,

$$\begin{aligned} \iota_{e_1}^{e} \oplus \iota_{e_2}^{e} &= \begin{pmatrix} \left[a_1^b + a_2^b, b_1^a + b_2^a, c_1^d + c_2^d, d_1^a + d_2^a\right], \\ \left[\left(\mu_{\iota_{e_1}^e}^{lr}\right) + \left(\mu_{\iota_{e_2}^e}^{lr}\right) - \left(\mu_{\iota_{e_1}^e}^{lr}\right)\left(\mu_{\iota_{e_2}^e}^{lr}\right)\right), \left(\mathfrak{I}_{\iota_{e_1}^e}^{lr}\right)\left(\mathfrak{I}_{\iota_{e_2}^e}^{lr}\right)\right] \\ \left[\left(\mu_{\iota_{e_1}^e}^{ur}\right) + \left(\mu_{\iota_{e_2}^e}^{ur}\right) - \left(\mu_{\iota_{e_1}^e}^{ur}\right)\left(\mu_{\iota_{e_2}^e}^{ur}\right)\right), \left(\mathfrak{I}_{\iota_{e_1}^e}^{ur}\right)\left(\mathfrak{I}_{\iota_{e_2}^e}^{ur}\right)\right] \end{pmatrix} \\ &= \begin{pmatrix} [0.20 + 0.40, 0.30 + 0.50, \\ 0.10 + 0.60, 0.40 + 0.60], \\ [(0.50) + (0.40) - (0.50)(0.40), (0.30)0.50)] \\ [(0.20) + (0.30) - (0.20)(0.30), (0.60)0.20)] \end{pmatrix} \\ &= \begin{pmatrix} [0.60, 0.80, 0.70, 0.10], \\ [(0.90) - (0.20), (0.15)] \\ [(0.50) - (0.06), (0.12)] \end{pmatrix} = \begin{pmatrix} [0.60, 0.80, 0.70, 0.10], \\ [(0.70, 0.15][(0.44, 0.12] \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \iota_{e_1}^{e} \otimes \iota_{e_2}^{e} &= \begin{pmatrix} \left[a_1^b a_2^b, b_1^a b_2^a, c_1^d c_2^d, d_1^a d_2^a\right]; \\ \left[\left(\mu_{\iota_{e_1}^e}^{lr}\right)\left(\mu_{\iota_{e_2}^e}^{lr}\right), \mathfrak{I}_{\iota_{e_1}^e}^{lr} + \mathfrak{I}_{\iota_{e_2}^e}^{lr} - \left(\mathfrak{I}_{\iota_{e_1}^e}^{lr}\right)\left(\mathfrak{I}_{\iota_{e_2}^e}^{lr}\right)\right], \\ \left[\left(\mu_{\iota_{e_1}^e}^{ur}\right)\left(\mu_{\iota_{e_2}^e}^{ur}\right), \mathfrak{I}_{\iota_{e_1}^e}^{ur} + \mathfrak{I}_{\iota_{e_2}^e}^{ur} - \left(\mathfrak{I}_{\iota_{e_1}^e}^{ur}\right)\left(\mathfrak{I}_{\iota_{e_2}^e}^{ur}\right)\right] \end{pmatrix} \\ &= \begin{pmatrix} [(0.20)(0.40), (0.30)(0.50), \\ (0.10)(0.60), (0.40)(0.60)] \\ [(0.5)(0.40), 0.30 + 0.50 - (0.30)(0.50)] \\ [(0.2)(0.30), 0.60 + 0.20 - (0.60)(0.20)] \end{pmatrix} \\ &= \begin{pmatrix} [0.08, 0.15, 0.06, 0.24] \\ [0.20, 0.80 - 0.15] \\ [0.06, 0.80 - 0.12] \end{pmatrix} = \begin{pmatrix} [0.08, 0.15, 0.06, 0.24] \\ [0.20, 0.65][0.06, 0.68] \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \partial(\iota_e^e) &= \begin{pmatrix} \left[\partial a^b, \partial b^a, \partial c^d, \partial d^a\right], \left[1 - \left(1 - \mu_{\iota_e^e}^{lr}\right)^{\partial}, \left(\mathfrak{I}_{\iota_e^e}^{lr}\right)^{\partial}\right], \\ \left[1 - \left(1 - \mu_{\iota_e^e}^{ur}\right)^{\partial}, \left(\mathfrak{I}_{\iota_e^e}^{ur}\right)^{\partial}\right] \end{pmatrix} \\ &= \begin{pmatrix} [2(0.30), 2(0.40), 2(0.20), 2(0.50)], \\ \left[1 - (1 - 0.4)^3, (0.5)^3\right] \\ \left[1 - (1 - 0.3)^3, (0.2)^3\right] \end{pmatrix} \\ &= \begin{pmatrix} [0.60, 0.80, 0.40, 0.10], \\ [0.78, 0.13][0.66, 0.01] \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} (\iota_e^e)^{\partial} &= \begin{pmatrix} \left[a^{b\partial}, b^{a\partial}, c^{d\partial}, d^{a\partial}\right], \left[\left(\mu_{\iota_e^e}^{lr}\right)^{\partial}, 1 - \left(1 - \mathfrak{I}_{\iota_e^e}^{lr}\right)^{\partial}\right], \\ \left[\left(\mu_{\iota_e^e}^{ur}\right)^{\partial}, 1 - \left(1 - \mathfrak{I}_{\iota_e^e}^{ur}\right)^{\partial}\right] \end{pmatrix} \\ &= \begin{pmatrix} \left[(0.30)^2, (0.40)^2, (0.20)^{02}, 0.50^2\right], \\ \left[(0.40)^2, 1 - (1 - 0.50)^2\right] \\ \left[(0.3)^2, 1 - (1 - 0.20)^2\right] \end{pmatrix} \\ &= \begin{pmatrix} [0.09, 0.16, 0.04, 0.25], \\ [0.16, 0.75][0.09, 0.36] \end{pmatrix}. \end{aligned} \tag{16}$$

Definition 15 (see [40]). Let

$$\iota_e^e = \left\langle \begin{matrix} \left[a^b, b^a, c^d, d^c\right], \\ \left[\mu_{\iota_e^e}^{lr}, \mu_{\iota_e^e}^{ur}\right] \\ \left[\mathfrak{I}_{\iota_e^e}^{lr}, \mathfrak{I}_{\iota_e^e}^{ur}\right] \end{matrix} \right\rangle \tag{17}$$

be I-VITFNs; a score function $\mathrm{Scr}(\iota_e^e)$ can be defined as follows:

$$\mathrm{Scr}(\iota_e^e) = \begin{pmatrix} \dfrac{\left[a^b + b^a + c^d + d^c\right]}{4}. \\ \dfrac{\mu_{\iota_e^e}^{lr} - \mathfrak{I}_{\iota_e^e}^{lr} + \mu_{\iota_e^e}^{ur} - \mathfrak{I}_{\iota_e^e}^{ur}}{2} \end{pmatrix} \mathrm{Scr}(\iota_e^e) \in [-1, 1]. \tag{18}$$

Definition 16 (see [40]). Let

$$\iota_e^e = \left\langle \begin{matrix} \left[a^b, b^a, c^d, d^c\right], \\ \left[\mu_{\iota_e^e}^{lr}, \mu_{\iota_e^e}^{ur}\right] \\ \left[\mathfrak{I}_{\iota_e^e}^{lr}, \mathfrak{I}_{\iota_e^e}^{ur}\right] \end{matrix} \right\rangle \tag{19}$$

be I-VITFNs; an accuracy function $\mathrm{Acr}(\iota_e^e)$ can be defined as follows:

$$\mathrm{Acr}(\iota_e^e) = \begin{pmatrix} \dfrac{\left[a^b + b^a + c^d + d^c\right]}{4}. \\ \dfrac{\mu_{\iota_e^e}^{lr} + \mathfrak{I}_{\iota_e^e}^{lr} + \mu_{\iota_e^e}^{ur} + \mathfrak{I}_{\iota_e^e}^{ur}}{2} \end{pmatrix} \mathrm{Acr}(\iota_e^e) \in [-1, 1]. \tag{20}$$

2.4. The Aczel-Alsina Operations of IFNs. In this section, we will present A-A operations on IFNs [28]. Let T presents a t-norm and S presents a t-conorm be A-A product T_A and A-A sum $_A$, respectively. A-A product $(\iota_e^e \otimes G_e^e)$ and A-A sum $(\iota_e^e \oplus G_e^e)$ over two IFSs ι_e^e and G_e^e defined in the following

way $\mu^{lr}_{\iota^e_e}, \mu^{ur}_{\iota^e_e}$:

$$\iota^e_e \otimes G^e_e = \left\{ \left\langle \begin{matrix} u_z, T_A\left\{\mu_{\iota^e_e}(u_z), \mu_{G^e_e}(u_z)\right\}, \\ S_A\left\{\mathfrak{I}_{\iota^e_e}(u_z), \mathfrak{I}_{G^e_e}(u_z)\right\} \end{matrix} \right\rangle / u_z \in \mathrm{Uz} \right\}, \tag{21}$$

$$\iota^e_e \oplus G^e_e = \left\{ \left\langle \begin{matrix} u_z, S_A\left\{\mu_{\iota^e_e}(u_z), \mu_{G^e_e}(u_z)\right\}, \\ T_A\left\{\mathfrak{I}_{\iota^e_e}(u_z), \mathfrak{I}_{G^e_e}(u_z)\right\} \end{matrix} \right\rangle / u_z \in \mathrm{Uz} \right\}. \tag{22}$$

Definition 17 (see [28]). Let $\iota^e_e = [\mu_{\iota^e_e}, \mathfrak{I}_{\iota^e_e}]$, $\iota^e_{e_1} = [\mu_{\iota^e_{e_1}}, \mathfrak{I}_{\iota^e_{e_1}}]$ and $\iota^e_{e_2} = [\mu_{\iota^e_{e_2}}, \mathfrak{I}_{\iota^e_{e_2}}]$ be any three IFNs, $\xi \geq 1$, and $\partial > 0$. Then, the A-A t-norm and t-conorm operation of IFNs are defined as

(i) $$\iota^e_{e_1} \oplus \iota^e_{e_2} = \left\{ \begin{matrix} \left(1 - e^{-\left((-\log(1-\mu_{\iota^e_{e_1}}))^{\xi} + (-\log(1-\mu_{\iota^e_{e_2}}))^{\xi}\right)^{1/\xi}}, \\ e^{-\left((-\log \mathfrak{I}_{\iota^e_{e_1}})^{\xi} + (-\log \mathfrak{I}_{\iota^e_{e_2}})^{\xi}\right)^{1/\xi}} \right) \end{matrix} \right\}$$

(ii) $$\iota^e_{e_1} \otimes \iota^e_{e_2} = \left\{ \begin{matrix} \left(e^{-\left((-\log \mu_{\iota^e_{e_1}})^{\xi} + (-\log \mu_{\iota^e_{e_2}})^{\xi}\right)^{1/\xi}}, \\ 1 - e^{-\left((-\log(1-\mathfrak{I}_{\iota^e_{e_1}}))^{\xi} + (-\log(1-\mathfrak{I}_{\iota^e_{e_2}}))^{\xi}\right)^{1/\xi}} \right) \end{matrix} \right\}$$

(iii) $$\partial(\iota^e_e) = \left\{\left(1 - e^{-\left(\partial(-\log(1-\mu_{\iota^e_e}))^{\xi}\right)^{1/\xi}}, e^{-\left(\partial(-\log \mathfrak{I}_{\iota^e_e})^{\xi}\right)^{1/\xi}}\right)\right\}$$

(iv) $$(\iota^e_e)^{\partial} = \left\{\left(e^{-\left(\partial(-\log \mu_{\iota^e_e})^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\partial(-\log(1-\mathfrak{I}_{\iota^e_e}))^{\xi}\right)^{1/\xi}}\right)\right\}$$

3. The Aczel-Alsina Operations of I-VITrFNs

In this section, we will present the operational laws of I-VITrFNs based on A-A aggregation operators.

Definition 18. Let

$$\iota^e_e = \left\langle \begin{matrix} \left[a^b, b^a, \right. \\ \left. c^d, d^c\right], \\ \left[\mu^{lr}_{\iota^e_e}, \mu^{ur}_{\iota^e_e}\right] \\ \left[\mathfrak{I}^{lr}_{\iota^e_e}, \mathfrak{I}^{ur}_{\iota^e_e}\right] \end{matrix} \right\rangle, \tag{23}$$

$$\iota^e_{e_1} = \left\langle \begin{matrix} \left[a^b_1, b^a_1, \right. \\ \left. c^d_1, d^c_1\right], \\ \left[\mu^{lr}_{\iota^e_{e_1}}, \mu^{ur}_{\iota^e_{e_1}}\right] \\ \left[\mathfrak{I}^{lr}_{\iota^e_{e_1}}, \mathfrak{I}^{ur}_{\iota^e_{e_1}}\right] \end{matrix} \right\rangle,$$

$$\iota^e_{e_2} = \left\langle \begin{matrix} \left[a^b_2, b^a_2, \right. \\ \left. c^d_2, d^c_2\right], \\ \left[\mu^{lr}_{\iota^e_{e_2}}, \mu^{ur}_{\iota^e_{e_2}}\right] \\ \left[\mathfrak{I}^{lr}_{\iota^e_{e_2}}, \mathfrak{I}^{ur}_{\iota^e_{e_2}}\right] \end{matrix} \right\rangle \tag{24}$$

be any three I-VITrFNs, $\partial > 0$. Then, the Aczel-Alsina operation laws on I-VITrFNs are defined as

(1) $$\iota^e_{e_1} \oplus \iota^e_{e_2} = \left\langle \begin{matrix} [a^b_1 + a^b_2, b^a_1 + b^a_2, c^d_1 + c^d_2, d^c_1 + d^c_2], \\ \left[1 - e^{-\left((-\log(1-\mu^{lr}_{\iota^e_{e_1}}))^{\xi} + (-\log(1-\mu^{lr}_{\iota^e_{e_2}}))^{\xi}\right)^{1/\xi}}, \right. \\ \left. e^{-\left((-\log \mathfrak{I}^{lr}_{\iota^e_{e_1}})^{\xi} + (-\log \mathfrak{I}^{lr}_{\iota^e_{e_2}})^{\xi}\right)^{1/\xi}}\right] \\ \left[1 - e^{-\left((-\log(1-\mu^{ur}_{\iota^e_{e_1}}))^{\xi} + (-\log(1-\mu^{ur}_{\iota^e_{e_2}}))^{\xi}\right)^{1/\xi}}, \right. \\ \left. e^{-\left((-\log \mathfrak{I}^{ur}_{\iota^e_{e_1}})^{\xi} + (-\log \mathfrak{I}^{ur}_{\iota^e_{e_2}})^{\xi}\right)^{1/\xi}}\right] \end{matrix} \right\rangle$$

(2) $$\iota^e_{e_1} \otimes \iota^e_{e_2} = \left\langle \begin{matrix} [a^b_1 a^b_2, b^a_1 b^a_2, c^d_1 c^d_2, d^c_1 d^c_2], \\ \left[e^{-\left((-\log(\mu^{lr}_{\iota^e_{e_1}}))^{\xi} + (-\log(\mu^{lr}_{\iota^e_{e_2}}))^{\xi}\right)^{1/\xi}}, \right. \\ \left. 1 - e^{-\left((-\log(1-\mathfrak{I}^{lr}_{\iota^e_{e_1}}))^{\xi} + (-\log(1-\mathfrak{I}^{lr}_{\iota^e_{e_2}}))^{\xi}\right)^{1/\xi}}\right] \\ \left[e^{-\left((-\log(\mu^{ur}_{\iota^e_{e_1}}))^{\xi} + (-\log(\mu^{ur}_{\iota^e_{e_2}}))^{\xi}\right)^{1/\xi}}, \right. \\ \left. 1 - e^{-\left((-\log(1-\mathfrak{I}^{ur}_{\iota^e_{e_1}}))^{\xi} + (-\log(1-\mathfrak{I}^{ur}_{\iota^e_{e_2}}))^{\xi}\right)^{1/\xi}}\right] \end{matrix} \right\rangle$$

(3) $$\partial(\iota^e_e) = \left\langle \begin{matrix} [\partial(a^b), \partial(b^a), \partial(c^d), \partial(d^c)], \\ \left[1 - e^{-\left(\partial(-\log(1-\mu^{lr}_{\iota^e_e}))^{\xi}\right)^{1/\xi}}, e^{-\left(\partial(-\log(\mathfrak{I}^{lr}_{\iota^e_e}))^{\xi}\right)^{1/\xi}}\right] \\ \left[1 - e^{-\left(\partial(-\log(1-\mu^{ur}_{\iota^e_e}))^{\xi}\right)^{1/\xi}}, e^{-\left(\partial(-\log(\mathfrak{I}^{ur}_{\iota^e_e}))^{\xi}\right)^{1/\xi}}\right] \end{matrix} \right\rangle$$

(4) $$(\iota^e_e)^{\partial} = \left\langle \begin{matrix} [(a^b)^{\partial}, (b^a)^{\partial}, (c^d)^{\partial}, (c^d)^{\partial}], \\ \left[e^{-\left(\partial(-\log(\mu^{lr}_{\iota^e_e}))^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\partial(-\log(1-\mathfrak{I}^{lr}_{\iota^e_e}))^{\xi}\right)^{1/\xi}}\right] \\ \left[e^{-\left(\partial(-\log(\mu^{ur}_{\iota^e_e}))^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\partial(-\log(1-\mathfrak{I}^{ur}_{\iota^e_e}))^{\xi}\right)^{1/\xi}}\right] \end{matrix} \right\rangle$$

Example 19. Let

$$\iota^e_e = \left\langle \begin{matrix} [0.2, 0.4, \\ 0.5, 0.6], \\ [0.6, 0.3] \\ [0.5, 0.4] \end{matrix} \right\rangle,$$

$$\iota^e_{e_1} = \left\langle \begin{matrix} [0.3, 0.5, \\ 0.6, 0.7], \\ [0.2, 0.7] \\ [0.1, 0.8] \end{matrix} \right\rangle, \tag{25}$$

$$\iota_{e_2}^{e} = \left\langle \begin{matrix} [0.4, 0.3, \\ 0.2, 0.1], \\ [0.8, 0.3] \end{matrix} \right\rangle \\ [0.6, 0.3] \tag{26}$$

be any three I-VITrFNs; then, using Aczel-Alsina operation on I-VITrFNs as defined in above operation laws for $\xi = 2$ and $\partial = 3$, we get

$$(1)\ \iota_{e_1}^{e} \oplus \iota_{e_2}^{e} = \left(\begin{matrix} [a_1^b + a_2^b, b_1^a + b_2^a, c_1^d + c_2^d, d_1^c + d_2^c], \\ \left\langle \begin{matrix} [1 - e^{-((-\log(1-\mu_{\iota_{e_1}^{lr}}))^{\xi} + (-\log(1-\mu_{\iota_{e_2}^{lr}}))^{\xi})^{1/\xi}}, \\ e^{-((-\log(\mathfrak{I}_{\iota_{e_1}^{lr}}))^{\xi} + (-\log(\mathfrak{I}_{\iota_{e_2}^{lr}}))^{\xi})^{1/\xi}}] \\ [1 - e^{-((-\log(1-\mu_{\iota_{e_1}^{ur}}))^{\xi} + (-\log(1-\mu_{\iota_{e_2}^{ur}}))^{\xi})^{1/\xi}}, \\ e^{-((-\log(\mathfrak{I}_{\iota_{e_1}^{ur}}))^{\xi} + (-\log(\mathfrak{I}_{\iota_{e_2}^{ur}}))^{\xi})^{1/\xi}}] \end{matrix} \right\rangle \\ [0.3 + 0.4, 0.5 + 0.3, 0.6 + 0.2, 0.7 + 0.1] \\ \left\langle \begin{matrix} [1 - e^{-((-\log(1-0.2))^2 + (-\log(1-0.8))^2)^{1/2}}, \\ e^{-((-\log(0.1))^2 + (-\log(0.6))^2)^{1/2}}] \\ [1 - e^{-((-\log(1-0.7))^2 + (-\log(1-0.3))^2)^{1/2}}, \\ e^{-((-\log(0.8))^2 + (-\log(0.3))^2)^{1/2}}] \end{matrix} \right\rangle \\ = \left\langle \begin{matrix} [0.7, 0.8, 0.8, 0.9], \\ [0.8031, 0.0946] \\ [0.7151, 0.2939] \end{matrix} \right\rangle \end{matrix} \right)$$

$$(2)\ \iota_{e_1}^{e} \otimes \iota_{e_2}^{e} = \left(\begin{matrix} [a_1^b a_2^b, b_1^a b_2^a, c_1^d c_2^d, d_1^c d_2^c], \\ \left\langle \begin{matrix} [e^{-((-\log(\mu_{\iota_{e_1}^{lr}}))^{\xi} + (-\log(\mu_{\iota_{e_2}^{lr}}))^{\xi})^{1/\xi}}, \\ 1 - e^{-((-\log(1-\mathfrak{I}_{\iota_{e_1}^{lr}}))^{\xi} + (-\log(1-\mathfrak{I}_{\iota_{e_2}^{lr}}))^{\xi})^{1/\xi}}] \\ [e^{-((-\log(\mu_{\iota_{e_1}^{ur}}))^{\xi} + (-\log(\mu_{\iota_{e_2}^{ur}}))^{\xi})^{1/\xi}}, \\ 1 - e^{-((-\log(1-\mathfrak{I}_{\iota_{e_1}^{ur}}))^{\xi} + (-\log(1-\mathfrak{I}_{\iota_{e_2}^{ur}}))^{\xi})^{1/\xi}}] \end{matrix} \right\rangle \\ [(0.3)(0.4), (0.5)(0.3), (0.6)(0.2), (0.7)(0.1)] \\ = \begin{matrix} [e^{-((-\log(0.2))^2 + (-\log(0.8))^2)^{1/2}}, \\ 1 - e^{-((-\log(1-0.1))^2 + (-\log(1-0.6))^2)^{1/2}}] \\ [e^{-((-\log(0.7))^2 + (-\log(0.3))^2)^{1/2}}, \\ 1 - e^{-((-\log(1-0.8))^2 + (-\log(1-0.3))^2)^{1/2}}] \end{matrix} \\ = \left\langle \begin{matrix} [0.12, 0.15, 0.12, 0.07], \\ [0.1969, 0.6024][0.2849, 0.8077 \end{matrix} \right\rangle \end{matrix} \right)$$

$$(3)\ \partial(\iota_e^e) = \left(\left\langle \begin{matrix} [\partial(a^b), \partial(b^a), \partial(c^d), \partial(d^c)], \\ [1 - e^{-(\partial(-\log(1-\mu_{\iota_e^{lr}}))^{\xi})^{1/\xi}}, e^{-(\partial(-\log(\mathfrak{I}_{\iota_e^{lr}}))^{\xi})^{1/\xi}}] \\ [1 - e^{-(\partial(-\log(1-\mu_{\iota_e^{ur}}))^{\xi})^{1/\xi}}, e^{-(\partial(-\log(\mathfrak{I}_{\iota_e^{ur}}))^{\xi})^{1/\xi}}] \end{matrix} \right\rangle \right) \\ = \left(\left\langle \begin{matrix} [0.3(0.2), 0.3(0.4), 0.3(0.5), 0.3(0.6)], \\ [1 - e^{-(3(-\log(1-0.6))2)^{1/2}}, e^{-(3(-\log(0.5))^2)^{1/2}}] \\ [1 - e^{-(3(-\log(1-0.3))^2)^{1/2}}, e^{-(3(-\log(0.4))^2)^{1/2}}] \end{matrix} \right\rangle \right) \\ = \left\langle \begin{matrix} [0.06, 0.12, 0.15, 0.18], \\ [0.7955, 0.3040][0.4609, 0.2045 \end{matrix} \right\rangle$$

$$(4)\ (\iota_e^e)^{\partial} = \left\langle \begin{matrix} [(a^b)^{\partial}, (b^a)^{\partial}, (c^d)^{\partial}, (c^d)^{\partial}], \\ [e^{-(\partial(-\log(\mu_{\iota_e^{lr}}))^{\xi})^{1/\xi}}, 1 - e^{-(\partial(-\log(1-\mathfrak{I}_{\iota_e^{lr}}))^{\xi})^{1/\xi}}] \\ [e^{-(\partial(-\log(\mu_{\iota_e^{ur}}))^{\xi})^{1/\xi}}, 1 - e^{-(\partial(-\log(1-\mathfrak{I}_{\iota_e^{ur}}))^{\xi})^{1/\xi}}] \end{matrix} \right\rangle \\ = \left\langle \begin{matrix} [(0.2)^{0.3}, (0.4)^{0.3}, (0.5)^{0.3}, (0.6)^{0.3}] \\ [e^{-(3(-\log(0.6))^2)^{1/2}}, 1 - e^{-(3(-\log(1-0.5))^2)^{1/2}}], \\ [e^{-(3(-\log(0.3))^2)^{1/2}}, 1 - e^{-(3(-\log(1-0.4))^2)^{1/2}}] \end{matrix} \right\rangle \\ = \left\langle \begin{matrix} [0.62, 0.76, 0.81, 0.86], \\ [0.4128, 0.6990][0.1243, 0.5872 \end{matrix} \right\rangle$$

Theorem 20. *Let*

$$\iota_e^e = \left\langle \begin{matrix} [a^b, b^a, \\ c^d, d^c]; \\ [\mu_{\iota_e^e}^{lr}, \mu_{\iota_e^e}^{ur}] \end{matrix} \right\rangle, \\ [\mathfrak{I}_{\iota_e^e}^{lr}, \mathfrak{I}_{\iota_e^e}^{ur}] \\ \iota_{e_1}^e = \left\langle \begin{matrix} [a_1^b, b_1^a, \\ c_1^d, d_1^c] \\ [\mu_{\iota_{e_1}^e}^{lr}, \mu_{\iota_{e_1}^e}^{ur}] \end{matrix} \right\rangle, \\ [\mathfrak{I}_{\iota_{e_1}^e}^{lr}, \mathfrak{I}_{\iota_{e_1}^e}^{ur}] \tag{27}$$

$$\iota_{e_2}^e = \left\langle \begin{matrix} [a_2^b, b_2^a, \\ c_2^d, d_2^c] \\ [\mu_{\iota_{e_2}^e}^{lr}, \mu_{\iota_{e_2}^e}^{ur}] \end{matrix} \right\rangle \\ [\mathfrak{I}_{\iota_{e_2}^e}^{lr}, \mathfrak{I}_{\iota_{e_2}^e}^{ur}] \tag{28}$$

be any three I-VITrFNs, $\partial > 0$. Then, we define the following results.

(1) $\iota_{e_1}^e \oplus \iota_{e_2}^e = \iota_{e_2}^e \oplus \iota_{e_1}^e$

(2) $\iota_{e_1}^e \otimes \iota_{e_2}^e = \iota_{e_2}^e \otimes \iota_{e_1}^e$

(3) $\partial(\iota_{e_1}^e \oplus \iota_{e_2}^e) = \partial(\iota_{e_1}^e) \oplus \partial(\iota_{e_2}^e),\ \partial > 0$

(4) $(\partial_1 \oplus \partial_2)\iota_e^e = \partial_1(\iota_e^e) \oplus \partial_2(\iota_e^e), \partial_1, \partial_2 > 0$

(5) $(\iota_{e_1}^e \otimes \iota_{e_2}^e)^{\partial} = (\iota_{e_1}^e))^{\partial} \otimes (\iota_{e_2}^e)^{\partial},\ \partial > 0$

Proof. For the three $\tilde{\alpha}, \tilde{\alpha}_1$, and $\tilde{\alpha}_2$, and $\partial, \partial_1, \partial_2 > 0$, as in Definition 18 such that

(1) $\iota_{e_1}^e \oplus \iota_{e_2}^e$

$$\left[a_1^b + a_2^b, b_1^a + b_2^a, c_1^d + c_2^d, d_1^c + d_2^c\right]$$

$$= \left\langle \begin{array}{c} \left[1 - e^{-\left((-\log(1-\mu_{\iota_{e_1}^e}^{lr}))^{\xi} + (-\log(1-\mu_{\iota_{e_2}^e}^{lr}))^{\xi}\right)^{1/\xi}}, e^{-\left((-\log(\mathfrak{T}_{\iota_{e_1}^e}^{lr}))^{\xi} + (-\log(\mathfrak{T}_{\iota_{e_2}^e}^{lr}))^{\xi}\right)^{1/\xi}}\right] \\ \left[1 - e^{-\left((-\log(1-\mu_{\iota_{e_1}^e}^{ur}))^{\xi} + (-\log(1-\mu_{\iota_{e_2}^e}^{ur}))^{\xi}\right)^{1/\xi}}, e^{-\left((-\log(\mathfrak{T}_{\iota_{e_1}^e}^{ur}))^{\xi} + (-\log(\mathfrak{T}_{\iota_{e_2}^e}^{ur}))^{\xi}\right)^{1/\xi}}\right] \end{array} \right\rangle$$

$$\left[a_2^b + a_1^b, b_2^a + b_1^a, c_2^d + c_1^d, d_2^c + d_1^c\right]$$

$$= \left\langle \begin{array}{c} \left[1 - e^{-\left((-\log(1-\mu_{\iota_{e_2}^e}^{lr}))^{\xi} + (-\log(1-\mu_{\iota_{e_1}^e}^{lr}))^{\xi}\right)^{1/\xi}}, e^{-\left((-\log(\mathfrak{T}_{\iota_{e_2}^e}^{lr}))^{\xi} + (-\log(\mathfrak{T}_{\iota_{e_1}^e}^{lr}))^{\xi}\right)^{1/\xi}}\right] \\ \left[1 - e^{-\left((-\log(1-\mu_{\iota_{e_2}^e}^{ur}))^{\xi} + (-\log(1-\mu_{\iota_{e_1}^e}^{ur}))^{\xi}\right)^{1/\xi}}, e^{-\left((-\log(\mathfrak{T}_{\iota_{e_2}^e}^{ur}))^{\xi} + (-\log(\mathfrak{T}_{\iota_{e_1}^e}^{ur}))^{\xi}\right)^{1/\xi}}\right] \end{array} \right\rangle$$

$$= \iota_{e_2}^e \oplus \iota_{e_1}^e$$

(2) $\iota_{e_1}^e \otimes \iota_{e_2}^e$

$$\left[a_1^b a_2^b, b_1^a b_2^a, c_1^d c_2^d, d_1^c d_2^c\right]$$

$$= \left\langle \begin{array}{c} \left[e^{-\left((-\log(\mu_{\iota_{e_1}^e}^{lr}))^{\xi} + (-\log(\mu_{A_{e2}^e}^{lr}))^{\xi}\right)^{1/\xi}}, 1 - e^{-\left((-\log(1-\mathfrak{T}_{\iota_{e_1}^e}^{lr}))^{\xi} + (-\log(1-\mathfrak{T}_{A_{e2}^e}^{lr}))^{\xi}\right)^{1/\xi}}\right] \\ \left[e^{-\left((-\log(\mu_{\iota_{e_1}^e}^{ur}))^{\xi} + (-\log(\mu_{A_{e2}^e}^{ur}))^{\xi}\right)^{1/\xi}}, 1 - e^{-\left((-\log(1-\mathfrak{T}_{\iota_{e_1}^e}^{ur}))^{\xi} + (-\log(1-\mathfrak{T}_{A_{e2}^e}^{ur}))^{\xi}\right)^{1/\xi}}\right] \end{array} \right\rangle$$

$$\left[a_2^b a_1^b, b_2^a b_1^a, c_2^d c_1^d, d_2^c d_1^c\right]$$

$$= \left\langle \begin{array}{c} \left[e^{-\left((-\log(\mu_{\iota_{e_2}^e}^{lr}))^{\xi} + (-\log(\mu_{\iota_{e_1}^e}^{lr}))^{\xi}\right)^{1/\xi}}, 1 - e^{-\left((-\log(1-\mathfrak{T}_{\iota_{e_2}^e}^{lr}))^{\xi} + (-\log(1-\mathfrak{T}_{\iota_{e_1}^e}^{lr}))^{\xi}\right)^{1/\xi}}\right] \\ \left[e^{-\left((-\log(\mu_{\iota_{e_2}^e}^{ur}))^{\xi} + (-\log(\mu_{\iota_{e_1}^e}^{ur}))^{\xi}\right)^{1/\xi}}, 1 - e^{-\left((-\log(1-\mathfrak{T}_{\iota_{e_2}^e}^{ur}))^{\xi} + (-\log(1-\mathfrak{T}_{\iota_{e_1}^e}^{ur}))^{\xi}\right)^{1/\xi}}\right] \end{array} \right\rangle$$

$$= \iota_{e_2}^e \otimes \iota_{e_1}^e$$

(3) $\partial(\iota_{e_1}^e \oplus \iota_{e_2}^e)$

$$\left[a_1^b + a_2^b, b_1^a + b_2^a, c_1^d + c_2^d, d_1^c + d_2^c\right]$$

$$= \partial\left\langle \begin{array}{c} \left[1 - e^{-\left((-\log(1-\mu_{\iota_{e_1}^e}^{lr}))^{\xi} + (-\log(1-\mu_{\iota_{e_2}^e}^{lr}))^{\xi}\right)^{1/\xi}}, e^{-\left((-\log(\mathfrak{T}_{\iota_{e_1}^e}^{lr}))^{\xi} + (-\log(\mathfrak{T}_{\iota_{e_2}^e}^{lr}))^{\xi}\right)^{1/\xi}}\right] \\ \left[1 - e^{-\left((-\log(1-\mu_{\iota_{e_1}^e}^{ur}))^{\xi} + (-\log(1-\mu_{\iota_{e_2}^e}^{ur}))^{\xi}\right)^{1/\xi}}, e^{-\left((-\log(\mathfrak{T}_{\iota_{e_1}^e}^{ur}))^{\xi} + (-\log(\mathfrak{T}_{\iota_{e_2}^e}^{ur}))^{\xi}\right)^{1/\xi}}\right] \end{array} \right\rangle$$

$$\partial\left[a_1^b + a_2^b, b_1^a + b_2^a, c_1^d + c_2^d, d_1^c + d_2^c\right]$$

$$= \left\langle \begin{array}{c} \left[1 - e^{-\left(\partial\left((-\log(1-\mu_{\iota_{e_1}^e}^{lr}))^{\xi} + (-\log(1-\mu_{\iota_{e_2}^e}^{lr}))^{\xi}\right)^{1/\xi}\right)}, e^{-\left(\partial\left((-\log(\mathfrak{T}_{\iota_{e_1}^e}^{lr}))^{\xi} + (-\log(\mathfrak{T}_{\iota_{e_2}^e}^{lr}))^{\xi}\right)^{1/\xi}\right)}\right] \\ \left[1 - e^{-\left(\partial\left((-\log(1-\mu_{\iota_{e_1}^e}^{ur}))^{\xi} + (-\log(1-\mu_{\iota_{e_2}^e}^{ur}))^{\xi}\right)^{1/\xi}\right)}, e^{-\left(\partial\left((-\log(\mathfrak{T}_{\iota_{e_1}^e}^{ur}))^{\xi} + (-\log(\mathfrak{T}_{\iota_{e_2}^e}^{ur}))^{\xi}\right)^{1/\xi}\right)}\right] \end{array} \right\rangle$$

$$\left[\partial(a_1^b, b_1^a, c_1^d, d_1^c) + \partial(a_2^b, b_2^a, c_2^d, d_2^c)\right]$$

$$= \left\langle \begin{array}{c} \left[1 - e^{-\left(\partial\left((-\log(1-\mu_{\iota_{e_1}^e}^{lr}))^{\xi} + (-\log(1-\mathfrak{T}_{\iota_{e_1}^e}^{lr}))^{\xi}\right)^{1/\xi}\right)}, e^{-\left(\partial\left((-\log(\mu_{\iota_{e_1}^e}^{lr}))^{\xi} + (-\log(\mathfrak{T}_{\iota_{e_2}^e}^{lr}))^{\xi}\right)^{1/\xi}\right)}\right] \\ \oplus \\ \left[1 - e^{-\left(\partial\left((-\log(1-\mu_{\iota_{e_1}^e}^{ur}))^{\xi} + (-\log(1-\mu_{\iota_{e_1}^e}^{ur}))^{\xi}\right)^{1/\xi}\right)}, e^{-\left(\partial\left((-\log(\mathfrak{T}_{\iota_{e_1}^e}^{ur}))^{\xi} + (-\log(\mathfrak{T}_{\iota_{e_2}^e}^{ur}))^{\xi}\right)^{1/\xi}\right)}\right] \end{array} \right\rangle$$

$$= \partial\iota_{e_1}^e \oplus \partial\iota_{e_2}^e$$

(4) $\partial_1\iota_e^e \oplus \partial_2\iota_e^e$

$$\partial_1\left[a^b, b^a, c^d, d^c\right] + \partial_2\left[a^b, b^a, c^d, d^c\right]$$

$$= \left\langle \begin{array}{c} \left[1 - e^{-\left(\partial_1\left((-\log(1-\mu_{\iota_e^e}^{lr}))^{\xi} + (-\log(1-\mu_{\iota_e^e}^{ur}))^{\xi}\right)^{1/\xi}\right)}, e^{-\left(\partial_1\left((-\log(\mathfrak{T}_{\iota_e^e}^{lr}))^{\xi} + (-\log(\mathfrak{T}_{\iota_e^e}^{ur}))^{\xi}\right)^{1/\xi}\right)}\right] \oplus \\ \left[1 - e^{-\left(\partial_2\left((-\log(1-\mu_{\iota_e^e}^{lr}))^{\xi} + (-\log(1-\mu_{\iota_e^e}^{ur}))^{\xi}\right)^{1/\xi}\right)}, e^{-\left(\partial_2\left((-\log(\mathfrak{T}_{\iota_e^e}^{lr}))^{\xi} + (-\log(\mathfrak{T}_{\iota_e^e}^{ur}))^{\xi}\right)^{1/\xi}\right)}\right] \end{array} \right\rangle$$

$$(\partial_1 + \partial_2)\left[a^b, b^a, c^d, d^c\right]$$

$$= \left\langle \left[1 - e^{-\left(\partial_1 + \partial_2\left((-\log(1-\mu_{\iota_e^e}^{lr}))^{\xi} + (-\log(1-\mu_{\iota_e^e}^{ur}))^{\xi}\right)^{1/\xi}\right)}, e^{-\left(\partial_1 + \partial_2\left((-\log(\mathfrak{T}_{\iota_e^e}^{lr}))^{\xi} + (-\log(\mathfrak{T}_{\iota_e^e}^{lr}))^{\xi}\right)^{1/\xi}\right)}\right] \right\rangle$$

$$= (\partial_1 \oplus \partial_2)\iota_e^e$$

(5) $(\iota_{e_1}^e \otimes \iota_{e_2}^e)^{\partial}$

$$\left[a_1^b a_2^b, b_1^a b_2^a, c_1^d c_2^d, d_1^c d_2^c\right]^{\partial},$$

$$= \left\langle \begin{array}{c} \left[e^{-\left((-\log(\mu_{\iota_{e_1}^e}^{lr}))^{\xi} + (-\log(\mu_{\iota_{e_2}^e}^{lr}))^{\xi}\right)^{1/\xi}}, 1 - e^{-\left((-\log(1-\mathfrak{T}_{\iota_{e_1}^e}^{lr}))^{\xi} + (-\log(1-\mathfrak{T}_{\iota_{e_2}^e}^{lr}))^{\xi}\right)^{1/\xi}}\right] \\ \left[e^{-\left(\partial(-\log(\mu_{\iota_{e_1}^e}^{ur}))^{\xi} + (-\log(\mu_{\iota_{e_2}^e}^{ur}))^{\xi}\right)^{1/\xi}}, 1 - e^{-\left((-\log(1-\mathfrak{T}_{\iota_{e_1}^e}^{ur}))^{\xi} + (-\log(1-\mathfrak{T}_{\iota_{e_2}^e}^{ur}))^{\xi}\right)^{1/\xi}}\right] \end{array} \right\rangle$$

$$\left[(a_1^b a_2^b)^{\partial}, (b_1^a b_2^a)^{\partial}, (c_1^d c_2^d)^{\partial}, (d_1^c d_2^c)^{\partial}\right],$$

$$= \left\langle \begin{array}{c} \left[e^{-\left(\partial\left((-\log(\mu_{\iota_{e_1}^e}^{lr}))^{\xi}\right)^{1/\xi}\right)}, 1 - e^{-\left(\partial\left((-\log(1-\mathfrak{T}_{\iota_{e_1}^e}^{lr}))^{\xi}\right)^{1/\xi}\right)}, e^{\left(\partial(-\log(\mu_{\iota_{e_2}^e}^{lr}))^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\partial(-\log(1-\mathfrak{T}_{\iota_{e_2}^e}^{lr}))^{\xi}\right)^{1/\xi}}\right] \otimes \\ \left[e^{-\left(\partial(-\log(\mu_{\iota_{e_1}^e}^{ur}))^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\partial(-\log(1-\mathfrak{T}_{\iota_{e_1}^e}^{ur}))^{\xi}\right)^{1/\xi}}, e^{\left(\partial(-\log(\mu_{\iota_{e_2}^e}^{ur}))^{\xi}\right)^{1/\xi}}, -e^{-\left(\left(\partial(-\log(1-\mathfrak{T}_{\iota_{e_2}^e}^{ur}))^{\xi}\right)^{1/\xi}\right)}\right] \end{array} \right\rangle$$

$$= (\iota_{e_1}^e)^{\partial} \otimes (\iota_{e_2}^e)^{\partial}$$

□

4. Induced Interval-Valued Intuitionistic Trapezoidal Fuzzy Aczel-Alsina Ordered Weighted Averaging (I-IVITrFAAOWA) Operator

Definition 21. Let $\langle u_{e_j}^e, \iota_{e_j}^e \rangle = \langle u_{e_j}^e, [a_j^b, b_j^a, c_j^d, d_j^c][\mu_{\iota_{ej}^e}^{lr}, \mu_{\iota_{ej}^e}^{ur}][\mathfrak{I}_{\iota_{ej}^e}^{lr}, \mathfrak{I}_{\iota_{ej}^e}^{ur}]\rangle (j = 1, 2, \cdots, n)$ be a collection of I-VITrFNs. Then, induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina ordered weighted averaging $(I - IVIFTrAAO\ WA)$ operator of dimension n is mapping $I - IVIFTrAAO\ WA, \Omega^n \longrightarrow \Omega$ such that

$$\text{I-IVIFTrAAOWA}\left\langle \left(u_{e_1}^e, \iota_{e_1}^e\right), \left(u_{e_2}^e, \iota_{e_2}^e\right), \cdots, \left(u_{e_n}^e, \iota_{e_n}^e\right) \right\rangle = \oplus_{j=1}^n \left(\varepsilon_j^\iota\right) \left(\iota_{e_{\sigma(n)}}^e\right) = \varepsilon_1^\iota \iota_{e_{\sigma(1)}}^e \oplus \varepsilon_2^\iota \iota_{e_{\sigma(2)}}^e \oplus, \cdots, \oplus \varepsilon_n^\iota \iota_{e_{\sigma(n)}}^e. \tag{29}$$

$\varepsilon^\iota = (\varepsilon_1^\iota, \varepsilon_2^\iota, \cdots, \varepsilon_n^\iota)^T$ be the weighting vectors of ι_{ej}^e $(j = 1, 2, \cdots, n)$ with $\varepsilon_j^\iota \in [0, 1]$ and $\sum_{j=1}^n \varepsilon_j^\iota = 1$.

Also, $\iota_{e_{\sigma(j)}}^e$ is the $\iota_{e_j}^e$ value of the I-VIFTrNs, and pairs $\langle u_{e_j}^e, \iota_{e_j}^e \rangle$ having the jıh largest $u_{e_j}^e$ and $u_{e_j}^e$ in $\langle u_{e_j}^e, \iota_{e_j}^e \rangle$ is referred to as the order inducing variable and $\iota_{e_j}^e$ as the interval-valued intuitionistict trapezoidal fuzzy argument variable.

Theorem 22. *Let $\langle u_{e_j}^e, \iota_{e_j}^e \rangle (j = 1, 2, \cdots, n)$ be a collection of I-VITrFNs. Then, induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina ordered weighted averaging (I-IVIFTrAAOWA) operator of dimension n is mapping $I - I\ VITrFAAOWA: \Omega^n \longrightarrow \Omega$ with the corresponding vector $\varepsilon^\iota = (\varepsilon_1^\iota, \varepsilon_2^\iota, \cdots, \varepsilon_n^\iota)^T$ is the weight vector such that $\varepsilon_j^\iota \in [0, 1]$ and $\sum_{j=1}^n \varepsilon_j^\iota = 1$, such that,*

$$\text{I-IVIFTrAAOWA}\left\langle \left(u_{e_1}^e, \iota_{e_1}^e\right), \left(u_{e_2}^e, \iota_{e_2}^e\right), \cdots, \left(u_{e_n}^e, \iota_{e_n}^e\right) \right\rangle = \oplus_{j=1}^n \left(\varepsilon_j^\iota\right) \left(\iota_{e_{\sigma(n)}}^e\right) \left\langle \begin{array}{c} \left[\sum_{j=1}^n \varepsilon_j^\iota a_{(\sigma)_j}^b, \sum_{j=1}^n \varepsilon_j^\iota b_{(\sigma)j}^a, \sum_{j=1}^n \varepsilon_j^\iota c_{(\sigma)_j}^d, \sum_{j=1}^n \varepsilon_j^\iota d_{(\sigma)_j}^c \right] \\ \left[1 - e^{-\left(\sum_{j=1}^n \varepsilon_j^\iota \left(-\log\left(1-\mu_{\iota_{e(\sigma)j}^e}^{lr}\right)\right)^\xi\right)^{1/\xi}}, e^{-\left(\sum_{j=1}^n \varepsilon_j^\iota \left(-\log \mathfrak{I}_{\iota_{e(\sigma)j}^e}^{lr}\right)^\xi\right)^{1/\xi}}\right] \\ \left[1 - e^{-\left(\sum_{j=1}^n \varepsilon_j^\iota \left(-\log\left(1-\mu_{\iota_{e(\sigma)j}^e}^{ur}\right)\right)^\xi\right)^{1/\xi}}, e^{-\left(\sum_{j=1}^n \varepsilon_j^\iota \left(-\log \mathfrak{I}_{\iota_{e(\sigma)j}^e}^{ur}\right)^\xi\right)^{1/\xi}}\right] \end{array} \right\rangle. \tag{30}$$

Also, $\iota_{e_{\sigma(j)}}^e$ is the $\iota_{e_j}^e$ value of the I-VITrFNs, and pairs $\langle u_{e_j}^e, \iota_{e_j}^e \rangle$ having the jıh largest $u_{e_j}^e$ and $u_{e_j}^e$ in $\langle u_{e_j}^e, \iota_{e_j}^e \rangle$ is referred to as the order inducing variable and $\iota_{e_j}^e$ as the interval-valued intuitionistic trapezoidal fuzzy argument variable.

Proof. By mathematical induction, we can prove that, Equation (30) holds for all positive integer n. First, we show that Equation (22) holds for $n = 2$, since

$$\varepsilon_1^\iota \iota_{e_{\sigma(1)}}^e = \left\langle \begin{array}{c} \left[\varepsilon_1^\iota a_{(\sigma)_1}^b, \varepsilon_1^\iota b_{(\sigma)_1}^a, \varepsilon_1^\iota c_{(\sigma)_1}^d, \varepsilon_1^\iota d_{(\sigma)_1}^c\right]; \\ \left[1 - e^{-\left(\varepsilon_1^\iota \left(-\log\left(1-\mu_{\iota_{e(\sigma)1}^e}^{lr}\right)\right)^\xi\right)^{1/\xi}}, e^{-\left(\varepsilon_1^\iota \left(-\log \mathfrak{I}_{\iota_{e(\sigma)1}^e}^{lr}\right)^\xi\right)^{1/\xi}}\right] \\ \left[1 - e^{-\left(\varepsilon_1^\iota \left(-\log\left(1-\mu_{\iota_{e(\sigma)1}^e}^{ur}\right)\right)^\xi\right)^{1/\xi}}, e^{-\left(\varepsilon_1^\iota \left(-\log \mathfrak{I}_{\iota_{e(\sigma)1}^e}^{ur}\right)^\xi\right)^{1/\xi}}\right] \end{array} \right\rangle,$$

$$\varepsilon_2^\iota \iota_{e_{\sigma(2)}}^e = \left\langle \begin{array}{c} \left[\varepsilon_2^\iota a_{(\sigma)_2}^b, \varepsilon_2^\iota b_{(\sigma)_2}^a, \varepsilon_2^\iota c_{(\sigma)_2}^d, \varepsilon_2^\iota d_{(\sigma)_2}^c\right]; \\ \left[1 - e^{-\left(\varepsilon_2^\iota \left(-\log\left(1-\mu_{\iota_{e(\sigma)2}^e}^{lr}\right)\right)^\xi\right)^{1/\xi}}, e^{-\left(\varepsilon_2^\iota \left(-\log \mathfrak{I}_{\iota_{e(\sigma)2}^e}^{lr}\right)^\xi\right)^{1/\xi}}\right] \\ \left[1 - e^{-\left(\varepsilon_2^\iota \left(-\log\left(1-\mu_{\iota_{e(\sigma)2}^e}^{ur}\right)\right)^\xi\right)^{1/\xi}}, e^{-\left(\varepsilon_2^\iota \left(-\log \mathfrak{I}_{\iota_{e(\sigma)2}^e}^{ur}\right)^\xi\right)^{1/\xi}}\right] \end{array} \right\rangle. \tag{31}$$

Based on Definition 18, we get

$$\text{I-IVITrFAAOWA}(\iota_{e1}^e, \iota_{e2}^e)$$

$$= \varepsilon_1^\iota \iota_{e(\sigma)1}^e \oplus \varepsilon_2^\iota \iota_{e(\sigma)2}^e$$

$$\left[\varepsilon_1^t a^b_{(\sigma)_1}, \varepsilon_1^t b^a_{(\sigma)_1}, \varepsilon_1^t c^d_{(\sigma)_1}, \varepsilon_1^t d^c_{(\sigma)_1}\right]$$

$$= \left\langle \begin{bmatrix} 1-e^{-\left(\varepsilon_1^t\left(-\log\left(1-\mu^{lr}_{t^e_{e(\sigma)1}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\varepsilon_1^t\left(-\log \mathfrak{T}^{lr}_{t^e_{e(\sigma)1}}\right)^{\xi}\right)^{1/\xi}} \\ 1-e^{-\left(\varepsilon_1^t\left(-\log\left(1-\mu^{ur}_{t^e_{e(\sigma)1}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\varepsilon_1^t\left(-\log \mathfrak{T}^{ur}_{t^e_{e(\sigma)1}}\right)^{\xi}\right)^{1/\xi}} \end{bmatrix} \right\rangle \oplus$$

$$\left[\varepsilon_2^t a^b_{(\sigma)_2}, \varepsilon_2^t b^a_{(\sigma)_2}, \varepsilon_2^t c^d_{(\sigma)_2}, \varepsilon_2^t d^c_{(\sigma)_2}\right]$$

$$\cdot \left\langle \begin{bmatrix} 1-e^{-\left(\varepsilon_2^t\left(-\log\left(1-\mu^{lr}_{t^e_{e(\sigma)2}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\varepsilon_2^t\left(-\log \mathfrak{T}^{lr}_{t^e_{e(\sigma)2}}\right)^{\xi}\right)^{1/\xi}} \\ 1-e^{-\left(\varepsilon_2^t\left(-\log\left(1-\mu^{ur}_{t^e_{e(\sigma)2}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\varepsilon_2^t\left(-\log \mathfrak{T}^{ur}_{t^e_{e(\sigma)2}}\right)^{\xi}\right)^{1/\xi}} \end{bmatrix} \right\rangle$$

$$\begin{bmatrix} \varepsilon_1^t a^b_{(\sigma)_1} + \varepsilon_2^t a^b_{(\sigma)_2}, \varepsilon_1^t b^a_{(\sigma)_1} + \varepsilon_2^t b^a_{(\sigma)_2}, \\ \varepsilon_1^t c^d_{(\sigma)_1} + \varepsilon_2^t c^d_{(\sigma)_2}, \varepsilon_1^t d^c_{(\sigma)_1} + \varepsilon_2^t d^c_{(\sigma)_2} \end{bmatrix}$$

$$= \left\langle \begin{matrix} \begin{bmatrix} 1-e^{-\left(\varepsilon_1^t\left(-\log\left(1-\mu^{lr}_{t^e_{e(\sigma)1}}\right)\right)^{\xi}+\varepsilon_2^t\left(-\log\left(1-\mu^{lr}_{t^e_{e(\sigma)2}}\right)\right)^{\xi}\right)^{1/\xi}}, \\ e^{-\left(\varepsilon_1^t\left(-\log \mathfrak{T}^{lr}_{t^e_{e(\sigma)1}}\right)^{\xi}+\varepsilon_2^t\left(-\log \mathfrak{T}^{lr}_{t^e_{e(\sigma)2}}\right)^{\xi}\right)^{1/\xi}} \end{bmatrix} \\ \begin{bmatrix} 1-e^{-\left(\varepsilon_1^t\left(-\log\left(1-\mu^{ur}_{t^e_{e(\sigma)1}}\right)\right)^{\xi}+\varepsilon_2^t\left(-\log\left(1-\mu^{ur}_{t^e_{e(\sigma)2}}\right)\right)^{\xi}\right)^{1/\xi}}, \\ e^{-\left(\varepsilon_1^t\left(-\log \mathfrak{T}^{ur}_{t^e_{e(\sigma)1}}\right)^{\xi}+\varepsilon_2^t\left(-\log \mathfrak{T}^{ur}_{t^e_{e(\sigma)2}}\right)^{\xi}\right)^{1/\xi}} \end{bmatrix} \end{matrix} \right\rangle$$

$$\begin{bmatrix} \sum_{j=1}^{2} \varepsilon_j^t a^b_{(\sigma)_j}, \sum_{j=1}^{2} \varepsilon_j^t b^a_{(\sigma)j}, \\ \sum_{j=1}^{2} \varepsilon_j^t c^d_{(\sigma)_j}, \sum_{j=1}^{2} \varepsilon_j^t d^c_{(\sigma)_j} \end{bmatrix}$$

$$= \left\langle \begin{matrix} \begin{bmatrix} 1-e^{-\left(\sum_{j=1}^{2} \varepsilon_j^t\left(-\log\left(1-\mu^{lr}_{t^e_{e(\sigma)j}}\right)\right)^{\xi}\right)^{1/\xi}}, \\ e^{-\left(\sum_{j=1}^{2} \varepsilon_j^t\left(-\log \mathfrak{T}^{lr}_{t^e_{e(\sigma)j}}\right)^{\xi}\right)^{1/\xi}} \end{bmatrix} \\ \begin{bmatrix} 1-e^{-\left(\sum_{j=1}^{2} \varepsilon_j^t\left(-\log\left(1-\mu^{ur}_{t^e_{e(\sigma)j}}\right)\right)^{\xi}\right)^{1/\xi}}, \\ e^{-\left(\sum_{j=1}^{2} \varepsilon_j^t\left(-\log \mathfrak{T}^{ur}_{t^e_{e(\sigma)j}}\right)^{\xi}\right)^{1/\xi}} \end{bmatrix} \end{matrix} \right\rangle \tag{32}$$

Hence, (22) is true for $n = 2$.

Let, we want to show that Equation (30) is true for $n = k$, then we have

$$\text{I-IVITrFAAOWA}\left\langle \left(u^e_{e_1}, t^e_{e_1}\right), \left(u^e_{e_2}, t^e_{e_2}\right), \cdots, \left(u^e_{e_k}, t^e_{e_k}\right) \right\rangle$$

$$\left[\sum_{j=1}^{k} \varepsilon_j^t a^b_{(\sigma)_j}, \sum_{j=1}^{k} \varepsilon_j^t b^a_{(\sigma)j}, \sum_{j=1}^{k} \varepsilon_j^t c^d_{(\sigma)_j}, \sum_{j=1}^{k} \varepsilon_j^t d^c_{(\sigma)_j}\right]$$

$$= \left\langle \begin{bmatrix} 1-e^{-\left(\sum_{j=1}^{k} \varepsilon_j^t\left(-\log\left(1-\mu^{lr}_{t^e_{e(\sigma)j}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\sum_{j=1}^{k} \varepsilon_j^t\left(-\log \mathfrak{T}^{lr}_{t^e_{e(\sigma)j}}\right)^{\xi}\right)^{1/\xi}} \\ 1-e^{-\left(\sum_{j=1}^{k} \varepsilon_j^t\left(-\log\left(1-\mu^{ur}_{t^e_{e(\sigma)j}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\sum_{j=1}^{k} \varepsilon_j^t\left(-\log \mathfrak{T}^{ur}_{t^e_{e(\sigma)j}}\right)^{\xi}\right)^{1/\xi}} \end{bmatrix} \right\rangle \tag{33}$$

Now, for $n = k + 1$, we get

$$\text{I-IVITrFAAOWA}\left\langle \left(u^e_{e_1}, t^e_{e_1}\right), \left(u^e_{e_2}, t^e_{e_2}\right), \cdots, \left(u^e_{e_k}, t^e_{e_{k+1}}\right) \right\rangle$$

$$\begin{bmatrix} \sum_{j=1}^{k+1} \varepsilon_j^t a^b_{(\sigma)_j}, \sum_{j=1}^{k+1} \varepsilon_j^t b^a_{(\sigma)j}, \\ \sum_{j=1}^{k+1} \varepsilon_j^t c^d_{(\sigma)_j}, \sum_{j=1}^{k+1} \varepsilon_j^t d^c_{(\sigma)_j} \end{bmatrix}$$

$$= \left\langle \begin{bmatrix} 1-e^{-\left(\sum_{j=1}^{k} \varepsilon_j^t\left(-\log\left(1-\mu^{lr}_{t^e_{e(\sigma)j}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\sum_{j=1}^{k} \varepsilon_j^t\left(-\log \mathfrak{T}^{lr}_{t^e_{e(\sigma)j}}\right)^{\xi}\right)^{1/\xi}} \\ 1-e^{-\left(\sum_{j=1}^{k} \varepsilon_j^t\left(-\log\left(1-\mu^{ur}_{t^e_{e(\sigma)j}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\sum_{j=1}^{k} \varepsilon_j^t\left(-\log \mathfrak{T}^{ur}_{t^e_{e(\sigma)j}}\right)^{\xi}\right)^{1/\xi}} \end{bmatrix} \right\rangle \oplus$$

$$= \left\langle \begin{bmatrix} 1-e^{-\left(\varepsilon_{k+1}^t\left(-\log\left(1-\mu^{lr}_{t^e_{ek+1}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\varepsilon_{k+1}^t\left(-\log \mathfrak{T}^{lr}_{t^e_{e(k+1)}}\right)^{\xi}\right)^{1/\xi}} \\ 1-e^{-\left(\varepsilon_{k+1}^t\left(-\log\left(1-\mu^{ur}_{t^e_{e(k+1)}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\varepsilon_{k+1}^t\left(-\log \mathfrak{T}^{ur}_{t^e_{e(k+1)}}\right)^{\xi}\right)^{1/\xi}} \end{bmatrix} \right\rangle$$

$$\begin{bmatrix} \sum_{j=1}^{k+1} \varepsilon_j^t a^b_{(\sigma)_j}, \sum_{j=1}^{k+1} \varepsilon_j^t b^a_{(\sigma)j}, \\ \sum_{j=1}^{k+1} \varepsilon_j^t c^d_{(\sigma)_j}, \sum_{j=1}^{k+1} \varepsilon_j^t d^c_{(\sigma)_j} \end{bmatrix}$$

$$= \left\langle \begin{matrix} \begin{bmatrix} 1-e^{-\left(\sum_{j=1}^{k+1} \varepsilon_j^t\left(-\log\left(1-\mu^{lr}_{t^e_{e(\sigma)j}}\right)\right)^{\xi}\right)^{1/\xi}}, \\ e^{-\left(\sum_{j=1}^{k+1} \varepsilon_j^t\left(-\log \mathfrak{T}^{lr}_{t^e_{e(\sigma)j}}\right)^{\xi}\right)^{1/\xi}} \end{bmatrix} \\ \begin{bmatrix} 1-e^{-\left(\sum_{j=1}^{k+1} \varepsilon_j^t\left(-\log\left(1-\mu^{ur}_{t^e_{e(\sigma)j}}\right)\right)^{\xi}\right)^{1/\xi}}, \\ e^{-\left(\sum_{j=1}^{k+1} \varepsilon_j^t\left(-\log \mathfrak{T}^{ur}_{t^e_{e(\sigma)j}}\right)^{\xi}\right)^{1/\xi}} \end{bmatrix} \end{matrix} \right\rangle \tag{34}$$

Thus, (30) is legitimate for $n = k + 1$. □

As consequences, we might come to conclusion that Equation (30) holds for all n.

Theorem 23 (Commutativity). *Let*

$$\begin{aligned} &I\text{-}IVITrFAAOWA\left\langle \left(u_{e_1}^e, \iota_{e_1}^e\right), \left(u_{e_2}^e, \iota_{e_2}^e\right), \cdots, \left(u_{e_n}^e, \iota_{e_n}^e\right) \right\rangle \\ &\quad = I\text{-}IVITrFAAOWA_{\varepsilon^l}\left\langle \left(u_{e_1}^{*e}, \iota_{e_1}^{*e}\right), \left(u_{e_2}^{*e}, \iota_{e_2}^{*e}\right), \cdots, \left(u_{e_n}^{*e}, \iota_{e_n}^{*e}\right) \right\rangle, \end{aligned} \tag{35}$$

where

$$\left\langle \begin{matrix} \left(u_{e_1}^{*e}, \iota_{e_1}^{*e}\right), \\ \left(u_{e_2}^{*e}, \iota_{e_2}^{*e}\right), \cdots, \left(u_{e_n}^{*e}, \iota_{en}^{*e}\right) \end{matrix} \right\rangle \tag{36}$$

is any permutation of

$$\left\langle \begin{matrix} \left(u_{e_1}^{e}, \iota_{e_1}^{e}\right), \\ \left(u_{e_2}^{e}, \iota_{e_2}^{e}\right), \cdots, \left(u_{e_n}^{e}, \iota_{en}^{e}\right) \end{matrix} \right\rangle. \tag{37}$$

Proof. As we know that

$$\begin{aligned} &\text{I-IVITrFAAOWA}\left\langle \left(u_{e_1}^e, \iota_{e_1}^e\right), \left(u_{e_2}^e, \iota_{e_2}^e\right), \cdots, \left(u_{e_n}^e, \iota_{e_n}^e\right) \right\rangle \\ &\quad = \varepsilon_1^l \iota_{e_{\sigma(1)}}^e \oplus \varepsilon_2^l \iota_{e_{\sigma(2)}}^e \oplus, \cdots, \oplus \varepsilon_n^l \iota_{e_{\sigma(n)}}^e, \\ &\text{I-IVITrFAAOWA}_{\varepsilon^l}\left\langle \left(u_{e_1}^{*e}, \iota_{e_1}^{*e}\right), \left(u_{e_2}^{*e}, \iota_{e_2}^{*e}\right), \cdots, \left(u_{e_n}^{*e}, \iota_{e_n}^{*e}\right) \right\rangle \\ &\quad = \varepsilon_1^l \iota_{e_{\sigma(1)}}^{*e} \oplus \varepsilon_2^l \iota_{e_{\sigma(2)}}^{*e} \oplus, \cdots, \oplus \varepsilon_n^l \iota_{e_{\sigma(n)}}^{*e}, \end{aligned} \tag{38}$$

since

$$\left\langle \begin{matrix} \left(u_{e_1}^{*e}, \iota_{e_1}^{*e}\right), \\ \left(u_{e_2}^{*e}, \iota_{e_2}^{*e}\right), \cdots, \left(u_{e_n}^{*e}, \iota_{e_n}^{*e}\right) \end{matrix} \right\rangle \tag{39}$$

is any permutation of

$$\left\langle \begin{matrix} \left(u_{e_1}^{e}, \iota_{e_1}^{e}\right), \\ \left(u_{e_2}^{e}, \iota_{e_2}^{e}\right), \cdots, \left(u_{e_n}^{e}, \iota_{e_n}^{e}\right) \end{matrix} \right\rangle. \tag{40}$$

□

Theorem 24 (Idempotency). *If all*

$$\iota_{ej}^e = \left\langle \begin{matrix} u_{e_j}^e, \left[a_j^b, b_j^a, c_j^d, d_j^c\right] \\ \left[\mu_{\iota_{ej}^e}^{lr}, \mu_{\iota_{ej}^e}^{ur}\right] \left[\mathfrak{I}_{\iota_{ej}^e}^{lr}, \mathfrak{I}_{\iota_{ej}^e}^{ur}\right] \end{matrix} \right\rangle \tag{41}$$

$$(j = 1, 2, \cdots, n)$$

are equal, $\iota_{e_{(j)}}^e = \iota_e^e$ *for all* $nj(j = 1, 2, \cdots, n)$, *then*

$$I\text{-}IVITrFAAOWA\left\langle \left(u_{e_1}^e, \iota_{e_1}^e\right), \left(u_{e_2}^e, \iota_{e_2}^e\right), \cdots, \left(u_{e_n}^e, \iota_{e_n}^e\right) \right\rangle = \iota_e^e. \tag{42}$$

Proof. Since $\iota_{ej}^e = \iota_e^e$ for all j, we have

$$\begin{aligned} &\text{I} - \text{IVITrFAAOWA}\left\langle \left(u_{e_1}^e, \iota_{e_1}^e\right), \left(u_{e_2}^e, \iota_{e_2}^e\right), \cdots, \left(u_{e_n}^e, \iota_{e_n}^e\right) \right\rangle \\ &\quad = \oplus_{j=1}^n \left(\varepsilon_j^l \iota_{e\sigma_{(j)}}^e\right) = \varepsilon_1^l \iota_{e_{\sigma(1)}}^e \oplus \varepsilon_2^l \iota_{e_{\sigma(2)}}^e \oplus, \cdots, \oplus \varepsilon_n^l \iota_{e_{\sigma(n)}}^e \\ &\quad = \oplus_{j=1}^n \left(\varepsilon_j^l \iota_e^e\right) = \varepsilon_1^l \iota_e^e \oplus \varepsilon_2^l \iota_e^e \oplus, \cdots, \oplus \varepsilon_n^l \iota_e^e \\ &\qquad \left[\varepsilon_j^l a_{(\sigma)_j}^b, \varepsilon_j^l b_{(\sigma)\,j}^a, \varepsilon_j^l c_{(\sigma)_j}^d, \varepsilon_j^l d_{(\sigma)_j}^c\right], \\ &\quad = \left\langle \begin{matrix} \left[1 - e^{-\left(\sum_{j=1}^n \varepsilon_j^l \left(-\log\left(1-\mu_{\iota_{e_{(j)}}^e}^{lr}\right)\right)^\xi\right)^{1/\xi}}, e^{-\left(\sum_{j=1}^n \varepsilon_j^l \left(-\log \mathfrak{I}_{\iota_{e_{(j)}}^e}^{lr}\right)^\xi\right)^{1/\xi}}\right] \\ \left[1 - e^{-\left(\sum_{j=1}^n \varepsilon_j^l \left(-\log\left(1-\mu_{\iota_{e_{(j)}}^e}^{ur}\right)\right)^\xi\right)^{1/\xi}}, e^{-\left(\sum_{j=1}^n \varepsilon_j^l \left(-\log \mathfrak{I}_{\iota_{e_{(j)}}^e}^{ur}\right)^\xi\right)^{1/\xi}}\right] \end{matrix} \right\rangle \\ &\qquad \left[a_j^b, b_j^a, c_j^d, d_j^c\right]; \\ &\quad = \left\langle \begin{matrix} \left[1 - e^{-\left(\left(-\log\left(1-\mu_{\iota_e^e}^{lr}\right)\right)^\xi\right)^{1/\xi}}, e^{-\left(\left(-\log \mathfrak{I}_{\iota_e^e}^{lr}\right)^\xi\right)^{1/\xi}}\right] \\ \left[1 - e^{-\left(\left(-\log\left(1-\mu_{\iota_e^e}^{ur}\right)\right)^\xi\right)^{1/\xi}}, e^{-\left(\left(-\log \mathfrak{I}_{\iota_e^e}^{ur}\right)^\xi\right)^{1/\xi}}\right] \end{matrix} \right\rangle \\ &\qquad \left[a_j^b, b_j^a, c_j^d, d_j^c\right]; \\ &\quad = \left\langle \begin{matrix} \left[1 - e^{\log\left(1-\mu_{\iota_e^e}^{lr}\right)}, e^{\log\left(\mathfrak{I}_{\iota_e^e}^{lr}\right)}\right] \\ \left[1 - e^{\log\left(1-\mu_{\iota_e^e}^{ur}\right)}, e^{\log\left(\mathfrak{I}_{\iota_e^e}^{ur}\right)}\right] \end{matrix} \right\rangle \\ &\quad = \left(\left[a_j^b, b_j^a, c_j^d, d_j^c\right]; \left[\mu_{\iota_e^e}^{lr}, \mu_{\iota_e^e}^{ur}\right] \left[\mathfrak{I}_{\iota_e^e}^{lr}, \mathfrak{I}_{\iota_e^e}^{ur}\right]\right) = \iota_e^e. \end{aligned} \tag{43}$$

Thus,

$$\text{I-IVITrFAAOWA}_{\varepsilon^l}\left\langle \left(u_{e_1}^e, \iota_{e_1}^e\right), \left(u_{e_2}^e, \iota_{e_2}^e\right), \cdots, \left(u_{e_n}^e, \iota_{e_n}^e\right) \right\rangle = \iota_e^e. \tag{44}$$

□

Theorem 25 (Monotonicity). *If* $\iota_{ej}^e \le \iota_{ej}^{e*}$ *for all* $j(j = 1, 2, \cdots, n)$, *then*

$$\begin{aligned} &I - IVITrFAAOWA_{\varepsilon^l}\left\langle \left(u_{e_1}^e, \iota_{e_1}^e\right), \left(u_{e_2}^e, \iota_{e_2}^e\right), \cdots, \left(u_{e_n}^e, \iota_{e_n}^e\right) \right\rangle \\ &\quad \le I - IVITrFAAOWA_{\varepsilon^l}\left\langle \left(u_{e_1}^e, \iota_{e_1}^{*e}\right), \left(u_{e_2}^e, \iota_{e_2}^{*e}\right), \cdots, \left(u_{e_n}^e, \iota_{e_n}^{*e}\right) \right\rangle. \end{aligned} \tag{45}$$

Proof. Let

$$\begin{aligned}&\text{I-IVITrFAAOWA}\left\langle\left(u_{e_1}^e, \iota_{e_1}^e\right), \left(u_{e_2}^e, \iota_{e_2}^e\right), \cdots, \left(u_{en}^e, \iota_{en}^e\right)\right\rangle\\&\quad = \varepsilon_1^I \iota_{e_{\sigma(1)}}^e \oplus \varepsilon_2^I \iota_{e_{\sigma(2)}}^e \oplus, \cdots, \oplus \varepsilon_n^I \iota_{e_{\sigma(n)}}^e\end{aligned}$$

$$\begin{aligned}&\text{I-IVITrFAAOWA}_{\varepsilon^I}\left\langle\left(u_{e_1}^e, \iota_{e_1}^{*e}\right), \left(u_{e_2}^e, \iota_{e_2}^{*e}\right), \cdots, \left(u_{e_n}^e, \iota_{e_n}^{*e}\right)\right\rangle\\&\quad = \varepsilon_1^I \iota_{e_{\sigma(1)}}^{*e} \oplus \varepsilon_2^I \iota_{e_{\sigma(2)}}^{*e} \oplus, \cdots, \oplus \varepsilon_n^I \iota_{e_{\sigma(n)}}^{*e}\end{aligned} \tag{46}$$

Since $\iota_{ej}^e \le \iota_{ej}^{e*}$ for all j, it follows that $(\iota_{e_{\sigma(j)}}^e) \le (\iota_{e_{\sigma(j)}}^{*e})$ $(j = 1, 2, \cdots, n)$, then

$$\begin{aligned}&\text{I-IVITrFAAOWA}_{\varepsilon^I}\left\langle\left(u_{e_1}^e, \iota_{e_1}^e\right), \left(u_{e_2}^e, \iota_{e_2}^e\right), \cdots, \left(u_{e_n}^e, \iota_{e_n}^e\right)\right\rangle\\&\quad \le \text{I-IVITrFAAOWA}_{\varepsilon^I}\left\langle\left(u_{e_1}^e, \iota_{e_1}^{*e}\right), \left(u_{e_2}^e, \iota_{e_2}^{*e}\right), \cdots, \left(u_{en}^e, \iota_{e_n}^{*e}\right)\right\rangle.\end{aligned} \tag{47}$$

□

Example 26. Consider the following collection of I-IVITrFNs in pairs form such that

$$\begin{aligned}\left\langle u_{e_1}^e, \iota_{e_1}^e\right\rangle &= \langle 0.1, [0.7, 0.4, 0.5, 0.3]\,; [0.50, 0.20][0.40, 0.20]\rangle,\\ \left\langle u_{e_2}^e, \iota_{e_2}^e\right\rangle &= \langle 0.8, [0.1, 0.5, 0.2, 0.6]\,; [0.20, 0.60][0.10, 0.20]\rangle,\\ \left\langle u_{e_3}^e, \iota_{e_3}^e\right\rangle &= \langle 0.3, [0.2, 0.4, 0.2, 0.3]\,; [0.40, 0.30][0.30, 0.20]\rangle,\\ \left\langle u_{e_4}^e, \iota_{e_4}^e\right\rangle &= \langle 0.5, [0.6, 0.1, 0.5, 0.7]\,; [0.60, 0.20][0.30, 0.10]\rangle.\end{aligned} \tag{48}$$

Performing the ordering of the I-IVITrF pairs with respect to the first component, we have

$$\begin{aligned}\left\langle u_{e_2}^e, \iota_{e_2}^e\right\rangle &= \langle 0.8, [0.1, 0.5, 0.2, 0.6]\,; [0.20, 0.60][0.10, 0.20]\rangle,\\ \left\langle u_{e_4}^e, \iota_{e_4}^e\right\rangle &= \langle 0.5, [0.6, 0.1, 0.5, 0.7]\,; [0.60, 0.20][0.30, 0.10]\rangle,\\ \left\langle u_{e_3}^e, \iota_{e_3}^e\right\rangle &= \langle 0.3, [0.2, 0.4, 0.2, 0.3]\,; [0.40, 0.30][0.30, 0.20]\rangle,\\ \left\langle u_{e_1}^e, \iota_{e_1}^e\right\rangle &= \langle 0.1, [0.7, 0.4, 0.5, 0.3]\,; [0.50, 0.20][0.40, 0.20]\rangle.\end{aligned} \tag{49}$$

This ordering includes the ordered I-VITrF arguments.

$$\begin{aligned}\left\langle u_{e_{\sigma(1)}}^e, \iota_{e_{\sigma(1)}}^e\right\rangle = \left\langle u_{e_2}^e, \iota_{e_2}^e\right\rangle &= \left\langle \begin{matrix}0.8, [0.1, 0.5, 0.2, 0.6]\,;\\ [0.20, 0.60][0.10, 0.20]\end{matrix}\right\rangle,\\ \left\langle u_{e_{\sigma(2)}}^e, \iota_{e_{\sigma(2)}}^e\right\rangle = \left\langle u_{e_4}^e, \iota_{e_4}^e\right\rangle &= \left\langle \begin{matrix}0.5, [0.6, 0.1, 0.5, 0.7]\,;\\ [0.60, 0.20][0.30, 0.10]\end{matrix}\right\rangle,\\ \left\langle u_{e_{\sigma(3)}}^e, \iota_{e_{\sigma(3)}}^e\right\rangle = \left\langle u_{e_3}^e, \iota_{e_3}^e\right\rangle &= \left\langle \begin{matrix}0.3, [0.2, 0.4, 0.2, 0.3]\,;\\ [0.40, 0.30][0.30, 0.20]\end{matrix}\right\rangle,\\ \left\langle u_{e_{\sigma(4)}}^e, \iota_{e_{\sigma(4)}}^e\right\rangle = \left\langle u_{e_1}^e, \iota_{e_1}^e\right\rangle &= \left\langle \begin{matrix}0.1, [0.7, 0.4, 0.5, 0.3]\,;\\ [0.50, 0.20][0.40, 0.20]\end{matrix}\right\rangle.\end{aligned} \tag{50}$$

If the associated weighting vector is $\varepsilon^I = (0.10, 0.20, 0.30, 0.40)$ and $N = 2$ then, we get an aggregate value by Equation (9).

$$\begin{aligned}&\text{I-IVITrFAAOWA}\left\langle\left(u_{e_1}^e, \iota_{e_1}^e\right), \left(u_{e_2}^e, \iota_{e_2}^e\right), \left(u_{e_3}^e, \iota_{e_3}^e\right)\left(u_{e_4}^e, \iota_{e_4}^e\right)\right\rangle\\&\quad \left[{}_{j=1}^{4}\varepsilon_j^I a_{(\sigma)_j}^b, {}_{j=1}^{4}\varepsilon_j^I b_{(\sigma)\,j}^a, {}_{j=1}^{4}\varepsilon_j^I c_{(\sigma)_j}^d, {}_{j=1}^{4}\varepsilon_j^I d_{(\sigma)_j}^c\right]\\&\quad = \left\langle \begin{matrix}\left[1 - e^{-\left({}_{j=1}^{4}\varepsilon_j^I\left(-\log\left(1-\mu_{\iota_{e(\sigma)j}^e}^{Ir}\right)\right)\right)^{\xi}}, e^{-\left({}_{j=1}^{4}\varepsilon_j^I\left(-\log \mathfrak{I}_{\iota_{e(\sigma)j}^e}^{Ir}\right)\right)^{\xi}}\right]^{1/\xi}\\ \left[1 - e^{-\left({}_{j=1}^{4}\varepsilon_j^I\left(-\log\left(1-\mu_{\iota_{e(\sigma)j}^e}^{ur}\right)\right)\right)^{\xi}}, e^{-\left({}_{j=1}^{4}\varepsilon_j^I\left(-\log \mathfrak{I}_{\iota_{e(\sigma)j}^e}^{ur}\right)\right)^{\xi}}\right]^{1/\xi}\end{matrix}\right\rangle\\&\quad = \left\langle \begin{matrix}\left[\begin{matrix}0.1(0.1) + 0.2(0.6) + 0.3(0.3) + 0.4(0.7),\\ 0.1(0.5) + 0.2(0.1) + 0.3(0.4) + 0.4(0.4),\\ 0.1(0.2) + 0.2(0.5) + 0.3(0.3) + 0.4(0.5),\\ 0.1(0.6) + 0.2(0.7) + 0.3(0.3) + 0.4(0.3),\end{matrix}\right]\\ 1 - e^{\begin{matrix}-\left(0.1(-\log(1-0.2))^2\right)^{1/2} + \left(0.2(-\log(1-0.6))^2\right)^{1/2}\\ +\left(0.3(-\log(1-0.4))^2\right)^{1/2} + \left(0.4(-\log(1-0.5))^2\right)^{1/2}\end{matrix}}, e^{\begin{matrix}-\left(0.1(-\log 0.1)^2\right)^{1/2} + 0.2(-\log 0.3)^2\right)^{1/2}\\ +0.3(-\log 0.3)^2\right)^{1/2} + 0.4(-\log 0.4)^2\right)^{1/2}\end{matrix}}\end{matrix}\right\rangle\\&\quad = \left\langle \begin{matrix}[0.47, 0.35, 0.38, 0.41]\,;\\ \left[1 - e^{\begin{matrix}-\left(0.1(-\log(1-0.6))^2\right)^{1/2} + \left(0.2(-\log(1-0.2))^2\right)^{1/2}\\ +\left(0.3(-\log(1-0.3))^2\right)^{1/2} + \left(0.4(-\log(1-0.2))^2\right)^{1/2}\end{matrix}}, e^{\begin{matrix}-\left(0.1(-\log 0.2)^2\right)^{1/2} + 0.2(-\log 0.1)^2\right)^{1/2}\\ +0.3(-\log 0.2)^2\right)^{1/2} + 0.4(-\log 0.2)^2\right)^{1/2}\end{matrix}}\right]\end{matrix}\right\rangle\\&\quad = \left\langle \begin{matrix}[0.47, 0.35, 0.38, 0.41]\,;\\ [0.6984, 0.0816]\,[0.5162, 0.0321]\end{matrix}\right\rangle.\end{aligned} \tag{51}$$

4.1. Induced Interval-Valued Intuitionistic Trapezoidal Fuzzy Aczel Alsina Hybrid Averaging (I-IVITrFAAHA) Operator

Definition 27. Let $\langle u^e_{e_j}, \iota^e_{e_j}\rangle = \langle u^e_{e_j}, [a^b_j, b^a_j, c^d_j, d^c_j]; [\mu^{lr}_{\iota^e_{ej}}, \mu^{ur}_{\iota^e_{ej}}][\mathfrak{I}^{lr}_{\iota^e_{ej}}, \mathfrak{I}^{ur}_{\iota^e_{ej}}]\rangle$ $(j = 1, 2, \cdots, n)$ be a collection of I-VITrFNs; then, induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina hybrid averaging (I-IVITrFAAHA) operator of dimension n is mapping I-IVITrFAAHA, $\Omega^n \longrightarrow \Omega$ such that

$$\begin{aligned}&\text{I-IVITrFAAHA}\left\langle \left(u^e_{e_1}, \iota^e_{e_1}\right), \left(u^e_{e_2}, \iota^e_{e_2}\right), \cdots, \left(u^e_{e_n}, \iota^e_{e_n}\right)\right\rangle \\ &\quad = \oplus^n_{j=1}\left(\varepsilon^l_j\right)\left(\dot{\iota}^e_{e_{\sigma(n)}}\right) = \varepsilon^l_1 \dot{\iota}^e_{e_{\sigma(1)}} \oplus \varepsilon^l_2 \dot{\iota}^e_{e_{\sigma(2)}} \oplus, \cdots, \oplus \varepsilon^l_n \dot{\iota}^e_{e_{\sigma(n)}},\end{aligned} \tag{52}$$

where $\dot{\iota}^e_{e\sigma(j)}$ is the weighted interval-valued intuitionistic trapezoidal fuzzy values, $\dot{\iota}^e_{e\sigma(j)}$, and $\dot{\iota}^e_{e\sigma(j)} = n\varepsilon^l_j \iota^e_{ej}(j = 1, 2, \cdots, n)$ of the interval-valued intuitionistic trapezoidal fuzzy ordered weighted averaging pair $\langle u^e_{e_j}, \iota^e_{e_j}\rangle$ having the $j\imath h$ largest $u^e_{e_j}$ and $u^e_{e_j}$ in $\langle u^e_{e_j}, \iota^e_{e_j}\rangle$ is referred to as the order inducing variable and $\iota^e_{e_j}$ as the interval-valued intuitionistic trapezoidal fuzzy argument variable. $\varepsilon^l = (\varepsilon^l_1, \varepsilon^l_2, \cdots, \varepsilon^l_n)^T$ is the weighting vector of $\iota^e_{ej}(j = 1, 2, \cdots, n)$ with $\varepsilon^l_j \in [0, 1]$ and $\sum^n_{j=1} \varepsilon^l_j = 1$.

Theorem 28. *Let $\langle u^e_{e_j}, \iota^e_{e_j}\rangle$ $(j = 1, 2, \cdots, n)$ be a collection of I-VITrFNs. then induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina hybrid averaging (I-IVITrFAAHA) operator is defined as*

$$\begin{aligned}&I\text{-}IVITrFAAHA\left\langle \left(u^e_{e_1}, \iota^e_{e_1}\right), \left(u^e_{e_2}, \iota^e_{e_2}\right), \cdots, \left(u^e_{e_n}, \iota^e_{e_n}\right)\right\rangle \\ &\quad = \oplus^n_{j=1}\left(\varepsilon^l_j\right)\left(\dot{\iota}^e_{e_{\sigma(\mathfrak{I})}}\right) = \varepsilon^l_1 \dot{\iota}^e_{e_{\sigma(1)}} \oplus \varepsilon^l_2 \dot{\iota}^e_{e_{\sigma(2)}} \oplus, \cdots, \oplus \varepsilon^l_n \dot{\iota}^e_{e_{\sigma(n)}} \\ &\qquad \left[\sum^n_{j=1}\varepsilon^l_j a^{\cdot b}_{(\sigma)_j}, \sum^n_{j=1}\varepsilon^l_j b^{\cdot a}_{(\sigma)j}, \sum^n_{j=1}\varepsilon^l_j c^{\cdot d}_{(\sigma)_j}, \sum^n_{j=1}\varepsilon^l_j d^{\cdot c}_{(\sigma)_j}\right] \\ &\cdot\left\langle \begin{bmatrix} 1 - e^{-\left(\sum^n_{j=1}\varepsilon^l_j\left(-\log\left(1-\mu^{lr}_{\dot{\iota}^e_{e(\sigma)j}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\sum^n_{j=1}\varepsilon^l_j\left(-\log \mathfrak{I}^{lr}_{\dot{\iota}^e_{e(\sigma)j}}\right)^{\xi}\right)^{1/\xi}} \\ 1 - e^{-\left(\sum^n_{j=1}\varepsilon^l_j\left(-\log\left(1-\mu^{ur}_{\dot{\iota}^e_{e(\sigma)j}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\sum^n_{j=1}\varepsilon^l_j\left(-\log \mathfrak{I}^{ur}_{\dot{\iota}^e_{e(\sigma)j}}\right)^{\xi}\right)^{1/\xi}} \end{bmatrix}\right\rangle.\end{aligned} \tag{53}$$

Theorem 29. *The I-VITrFAAWA operator is the special case of the I-IVITrFAAHA operator.*

Proof. Let $\varepsilon^l = ((1/n), (1/n), \cdots, (1/n))$, then

$$\begin{aligned}&\text{I-IVITrFAAHA}_{\varepsilon^l,\varepsilon^l}\left\langle \left(u^e_{e_1}, \iota^e_{e_1}\right), \left(u^e_{e_2}, \iota^e_{e_2}\right), \cdots, \left(u^e_{e_n}, \iota^e_{e_n}\right)\right\rangle \\ &\quad = \left(\varepsilon^l_1 \dot{\iota}^e_{e\sigma(1)}\right) \oplus \left(\varepsilon^l_2 \dot{\iota}^e_{e\sigma(2)}\right), \cdots, \left(\varepsilon^l_n \dot{\iota}^e_{e\sigma(n)}\right) \\ &\quad = \frac{1}{n}\left(\left(\dot{\iota}^e_{e\sigma(1)}\right) \oplus \left(\dot{\iota}^e_{e\sigma(2)}\right), \cdots, \left(\dot{\iota}^e_{e\sigma(n)}\right)\right) = (\varepsilon^l_1 \iota^e_{e1} \oplus \varepsilon^l_2 \iota^e_{e2}, \cdots, \varepsilon^l_n \iota^e_{en}) \\ &\quad = \text{I-VITrFAAWA}_{\varepsilon^l}(\iota^e_{e1}, \iota^e_{e2}, \cdots, \iota^e_{en}).\end{aligned} \tag{54}$$

□

Theorem 30. *The I-IVITrFAAOWA operator is the special case of the I-IVITrFAAHA operator.*

Proof. Let $\varepsilon^l = (1/n, 1/n, \cdots, 1/n)$, then $\dot{\iota}^e_{ej} = \iota^e_{ej}(j = 1, 2, \cdots, n)$

$$\begin{aligned}&\text{I-IVITrFAAHA}_{\varepsilon^l,\varepsilon^l}\left\langle \left(u^e_{e_1}, \iota^e_{e_1}\right), \left(u^e_{e_2}, \iota^e_{e_2}\right), \cdots, \left(u^e_{e_n}, \iota^e_{e_n}\right)\right\rangle \\ &\quad = \left(\varepsilon^l_1 \dot{\iota}^e_{e\sigma(1)}\right) \oplus \left(\varepsilon^l_2 \dot{\iota}^e_{e\sigma(2)}\right), \cdots, \left(\varepsilon^l_n \dot{\iota}^e_{e\sigma(n)}\right) \\ &\quad = \left(\varepsilon^l_1 \iota^e_{e\sigma(1)}\right) \oplus \left(\varepsilon^l_2 \iota^e_{e\sigma(2)}\right), \cdots, \left(\varepsilon^l_n \iota^e_{e\sigma(n)}\right) \\ &\quad = \text{I-VITrFAAOWA}_{\varepsilon^l}\left\langle \left(u^e_{e_1}, \iota^e_{e_1}\right), \left(u^e_{e_2}, \iota^e_{e_2}\right), \cdots, (u^e_{en}, \iota^e_{en})\right\rangle.\end{aligned} \tag{55}$$

□

4.2. Induced Interval-Valued Intuitionistic Trapezoidal Fuzzy Aczel-Alsina Ordered Weighted Geometric (I-IVITrFAAWG) Operator

Definition 31. Let $\langle u^e_{e_j}, \iota^e_{e_j}\rangle = \langle u^e_{e_j}, [a^b_j, b^a_j, c^d_j, d^c_j]; [\mu^{lr}_{\iota^e_{ej}}, \mu^{lr}_{\iota^e_{ej}}][\mathfrak{I}^{lr}_{\iota^e_{ej}}, \mathfrak{I}^{ur}_{\iota^e_{ej}}]\rangle$ be a collection of I-VITrFNs. Then, induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina weighted geometric (I-IVITrFAAWG) operator of dimension n is mapping I-IVITrFAAWG, $\Omega^n \longrightarrow \Omega$ such that

$$\begin{aligned}&\text{I-IVIFTrAAOWG}\left\langle \left(u^e_{e_1}, \iota^e_{e_1}\right), \left(u^e_{e_2}, \iota^e_{e_2}\right), \cdots, \left(u^e_{e_n}, \iota^e_{e_n}\right)\right\rangle \\ &\quad = \otimes^n_{j=1}\left(\iota^e_{e_{\sigma(n)}}\right)^{\varepsilon^l_j} = \left(\iota^e_{e_{\sigma(1)}}\right)^{\varepsilon^l_1} \otimes \left(\iota^e_{e_{\sigma(2)}}\right)^{\varepsilon^l_2} \otimes, \cdots, \otimes \left(\iota^e_{e_{\sigma(n)}}\right)^{\varepsilon^l_n}.\end{aligned} \tag{56}$$

$\varepsilon^l = (\varepsilon^l_1, \varepsilon^l_2, \cdots, \varepsilon^l_n)^T$ be the weighting vectors of $\iota^e_{ej}(j = 1, 2, \cdots, n)$ with $\varepsilon^l_j \in [0, 1]$ and $\sum^n_{j=1}\varepsilon^l_j = 1$..

Also $\iota^e_{e_{\sigma(j)}}$ is the $\iota^e_{e_j}$ value of the I-VIFTrNs, and pairs $\langle u^e_{e_j}, \iota^e_{e_j}\rangle$ having the $j\imath h$ largest $u^e_{e_j}$ and $u^e_{e_j}$ in $\langle u^e_{e_j}, \iota^e_{e_j}\rangle$ is referred to as the order inducing variable and $\iota^e_{e_j}$ as the interval-valued intuitionistict trapezoidal fuzzy argument variable.

Theorem 32. *Let $\langle u^e_{e_j}, \iota^e_{e_j}\rangle$ $(j = 1, 2, \cdots, n)$ be a collection of I-VITrFNs. Then, induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina ordered weighted averaging (I-*

IVIFTrAAOWG) operator of dimension n is mapping I − I VITrFAAOWG, $\Omega^n \longrightarrow \Omega$ with the corresponding vector $\varepsilon^{\iota} = (\varepsilon_1^{\iota}, \varepsilon_2^{\iota}, \cdots, \varepsilon_n^{\iota})^T$ is the weight vector such that $\varepsilon_j^{\iota} \in [0, 1]$ and $\sum_{j=1}^{n} \varepsilon_j^{\iota} = 1$, such that,

$$\begin{aligned}
&\text{I-IVIFTrAAOWG}\left\langle \left(u_{e_1}^e, \iota_{e_1}^e\right), \left(u_{e_2}^e, \iota_{e_2}^e\right), \cdots, \left(u_{e_n}^e, \iota_{e_n}^e\right) \right\rangle \\
&\qquad \left[\Pi_{j=1}^{n}\left(a_{(\sigma)_j}^b\right)^{\varepsilon_j^{\iota}}, \Pi_{j=1}^{n}\left(b_{(\sigma)j}^a\right)^{\varepsilon_j^{\iota}}, \Pi_{j=1}^{n}\left(c_{(\sigma)_j}^d\right)^{\varepsilon_j^{\iota}}, \Pi_{j=1}^{n}\left(d_{(\sigma)_j}^c\right)^{\varepsilon_j^{\iota}}\right] \\
&= \otimes_{j=1}^{n}\left(\iota_{e_{\sigma(n)}}^e\right)^{\varepsilon_j^{\iota}} \left\langle \begin{array}{c} \left[e^{-\left(\Pi_{j=1}^{n}\left(-\log\left(\mu_{\iota_{e(\sigma)j}^e}^{lr}\right)^{\varepsilon_j^{\iota}}\right)^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\Pi_{j=1}^{n}\varepsilon_j^{\iota}\left(-\log\left(1-\mathfrak{T}_{\iota_{e(\sigma)j}^e}^{lr}\right)^{\varepsilon_j^{\iota}}\right)^{\xi}\right)^{1/\xi}} \right] \\ \left[e^{-\left(\Pi_{j=1}^{n}\left(-\log\left(\mu_{\iota_{e(\sigma)j}^e}^{ur}\right)^{\varepsilon_j^{\iota}}\right)^{\xi}\right)}, 1 - e^{-\left(\Pi_{j=1}^{n}\varepsilon_j^{\iota}\left(-\log\left(1-\mathfrak{T}_{\iota_{e(\sigma)j}^e}^{ur}\right)^{\varepsilon_j^{\iota}}\right)^{\xi}\right)^{1/\xi}}, \right] \end{array} \right\rangle.
\end{aligned} \tag{57}$$

Also $\iota_{e_{\sigma(j)}}^e$ is the $\iota_{e_j}^e$ value of the I-VITrFNs, and pairs $\langle u_{e_j}^e, \iota_{e_j}^e \rangle$ having the jih largest $u_{e_j}^e$ and $u_{e_j}^e$ in $\langle u_{e_j}^e, \iota_{e_j}^e \rangle$ is referred to as the order inducing variable and $\iota_{e_j}^e$ as the interval-valued intuitionistic trapezoidal fuzzy argument variable.

Proof. By mathematical induction, we can prove that Equation (56) holds for all positive integer n. First, we show that Equation (56) holds for $n = 2$, since

$$\begin{aligned}
&\left(\iota_{e_{\sigma(1)}}^e\right)^{\varepsilon_1^{\iota}} = \left\langle \begin{array}{c} \left[\left(a_{(\sigma)_1}^b\right)^{\varepsilon_1^{\iota}}, \left(b_{(\sigma)_1}^a\right)^{\varepsilon_1^{\iota}}, \left(c_{(\sigma)_1}^d\right)^{\varepsilon_1^{\iota}}, \left(d_{(\sigma)_1}^c\right)^{\varepsilon_1^{\iota}}\right]; \\ \left[e^{-\left(\left(-\log\left(\mu_{\iota_{e(\sigma)1}^e}^{lr}\right)^{\varepsilon_1^{\iota}}\right)^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\left(-\log\left(1-\mathfrak{T}_{\iota_{e(\sigma)1}^e}^{lr}\right)^{\varepsilon_1^{\iota}}\right)^{\xi}\right)^{1/\xi}} \right] \\ \left[e^{-\left(\left(-\log\left(\mu_{\iota_{e(\sigma)1}^e}^{ur}\right)^{\varepsilon_1^{\iota}}\right)^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\left(-\log\left(1-\mathfrak{T}_{\iota_{e(\sigma)1}^e}^{ur}\right)^{\varepsilon_1^{\iota}}\right)^{\xi}\right)^{1/\xi}} \right] \end{array} \right\rangle, \\
&\left(\iota_{e_{\sigma(2)}}^e\right)^{\varepsilon_2^{\iota}} = \left\langle \begin{array}{c} \left[\left(a_{(\sigma)_2}^b\right)^{\varepsilon_2^{\iota}}, \left(b_{(\sigma)_2}^a\right)^{\varepsilon_2^{\iota}}, \left(c_{(\sigma)_2}^d\right)^{\varepsilon_2^{\iota}}, \left(d_{(\sigma)_2}^c\right)^{\varepsilon_2^{\iota}}\right]; \\ \left[e^{-\left(\left(-\log\left(\mu_{\iota_{e(\sigma)2}^e}^{lr}\right)^{\varepsilon_2^{\iota}}\right)^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\left(-\log\left(1-\mathfrak{T}_{\iota_{e(\sigma)2}^e}^{lr}\right)^{\varepsilon_2^{\iota}}\right)^{\xi}\right)^{1/\xi}} \right] \\ \left[e^{-\left(\left(-\log\left(\mu_{\iota_{e(\sigma)2}^e}^{ur}\right)^{\varepsilon_2^{\iota}}\right)^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\left(-\log\left(1-\mu_{\iota_{e(\sigma)2}^e}^{ur}\right)^{\varepsilon_2^{\iota}}\right)^{\xi}\right)^{1/\xi}} \right] \end{array} \right\rangle.
\end{aligned} \tag{58}$$

Based on Definition 18, we get

$$\begin{aligned}
&\text{I-IVITrFAAOWG}\left(\iota_{e_1}^e, \iota_{e_2}^e\right) = \left(\iota_{e(\sigma)1}^e\right)^{\varepsilon_1^{\iota}} \otimes \left(\iota_{e(\sigma)2}^e\right)^{\varepsilon_2^{\iota}} \\
&\qquad \left[\left(a_{(\sigma)_1}^b\right)^{\varepsilon_1^{\iota}}, \left(b_{(\sigma)_1}^a\right)^{\varepsilon_1^{\iota}}, \left(c_{(\sigma)_1}^d\right)^{\varepsilon_1^{\iota}}, \left(d_{(\sigma)_1}^c\right)^{\varepsilon_1^{\iota}}\right]; \\
&= \left\langle \begin{array}{c} \left[e^{-\left(\left(-\log\left(\mu_{\iota_{e(\sigma)1}^e}^{lr}\right)^{\varepsilon_1^{\iota}}\right)^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\left(-\log\left(1-\mathfrak{T}_{\iota_{e(\sigma)1}^e}^{lr}\right)^{\varepsilon_1^{\iota}}\right)^{\xi}\right)^{1/\xi}} \right] \\ \left[e^{-\left(\left(-\log\left(\mu_{\iota_{e(\sigma)1}^e}^{ur}\right)^{\varepsilon_1^{\iota}}\right)^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\left(-\log\left(1-\mathfrak{T}_{\iota_{e(\sigma)1}^e}^{ur}\right)^{\varepsilon_1^{\iota}}\right)^{\xi}\right)^{1/\xi}} \right] \end{array} \right\rangle \otimes \\
&\qquad \left[\left(a_{(\sigma)_2}^b\right)^{\varepsilon_2^{\iota}}, \left(b_{(\sigma)_2}^a\right)^{\varepsilon_2^{\iota}}, \left(c_{(\sigma)_2}^d\right)^{\varepsilon_2^{\iota}}, \left(d_{(\sigma)_2}^c\right)^{\varepsilon_2^{\iota}}\right]; \\
&\cdot \left\langle \begin{array}{c} \left[e^{-\left(\left(-\log\left(\mu_{\iota_{e(\sigma)2}^e}^{lr}\right)^{\varepsilon_2^{\iota}}\right)^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\left(-\log\left(1-\mathfrak{T}_{\iota_{e(\sigma)2}^e}^{lr}\right)^{\varepsilon_2^{\iota}}\right)^{\xi}\right)^{1/\xi}} \right] \\ \left[e^{-\left(\left(-\log\left(\mu_{\iota_{e(\sigma)2}^e}^{ur}\right)^{\varepsilon_2^{\iota}}\right)^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\left(-\log\left(1-\mu_{\iota_{e(\sigma)2}^e}^{ur}\right)^{\varepsilon_2^{\iota}}\right)^{\xi}\right)^{1/\xi}} \right] \end{array} \right\rangle
\end{aligned}$$

$$\left[\begin{array}{l}\left(a^b_{(\sigma)_1}\right)^{\varepsilon^i_1}\left(a^b_{(\sigma)_2}\right)^{\varepsilon^i_2},\left(b^a_{(\sigma)_1}\right)^{\varepsilon^i_1}\left(b^a_{(\sigma)_2}\right)^{\varepsilon^i_2},\\ \left(c^d_{(\sigma)_1}\right)^{\varepsilon^i_1}\left(c^d_{(\sigma)_2}\right)^{\varepsilon^i_2},\left(d^c_{(\sigma)_1}\right)^{\varepsilon^i_1}\left(d^c_{(\sigma)_2}\right)^{\varepsilon^i_2}\end{array}\right]$$

$$=\left\langle\left[\begin{array}{l}e^{-\left(\left(-\log\left(\mu^{lr}_{t^e_{e(\sigma)1}}\right)^{\varepsilon^i_1}\right)^{\xi}\left(-\log\left(\mu^{lr}_{t^e_{e(\sigma)2}}\right)^{\varepsilon^i_2}\right)^{\xi}\right)^{1/\xi}},\\ 1-e^{-\left(\left(-\log\left(\left(1-\mathfrak{T}^{lr}_{t^e_{e(\sigma)1}}\right)^{\varepsilon^i_1}\right)^{\xi}\left(-\log\left(\left(1-\mathfrak{T}^{lr}_{t^e_{e(\sigma)2}}\right)^{\varepsilon^i_2}\right)\right)^{\xi}\right)^{1/\xi}}\end{array}\right]\right\rangle.$$

$$\left[\begin{array}{l}e^{-\left(\left(-\log\left(\mu^{ur}_{t^e_{e(\sigma)1}}\right)^{\varepsilon^i_1}\right)^{\xi}\left(-\log\left(\mu^{ur}_{t^e_{e(\sigma)2}}\right)^{\varepsilon^i_2}\right)^{\xi}\right)^{1/\xi}},\\ 1-e^{-\left(\left(-\log\left(1-\mathfrak{T}^{ur}_{t^e_{e(\sigma)1}}\right)^{\varepsilon^i_1}\right)^{\xi}\left(-\log\left(1-\mathfrak{T}^{ur}_{t^e_{e(\sigma)2}}\right)^{\varepsilon^i_2}\right)^{\xi}\right)^{1/\xi}}\end{array}\right]$$

$$\left[\Pi^2_{j=1}\left(a^b_{(\sigma)_j}\right)^{\varepsilon^i_j},\Pi^2_{j=1}\left(b^a_{(\sigma)j}\right)^{\varepsilon^i_j},\Pi^2_{j=1}\left(c^d_{(\sigma)_j}\right)^{\varepsilon^i_j},\Pi^2_{j=1}\left(d^c_{(\sigma)_j}\right)^{\varepsilon^i_j}\right]$$

$$=\left\langle\left[\begin{array}{l}e^{-\left(\Pi^2_{j=1}\left(-\log\left(\mu^{lr}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}},1-e^{-\left(\Pi^2_{j=1}\varepsilon^i_j\left(-\log\left(1-\mathfrak{T}^{lr}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}}\\ e^{-\left(\Pi^2_{j=1}\left(-\log\left(\mu^{ur}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}},1-e^{-\left(\Pi^2_{j=1}\varepsilon^i_j\left(-\log\left(1-\mathfrak{T}^{ur}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}},\end{array}\right]\right\rangle \tag{59}$$

Hence, (17) is true for $n=2$.

Let we want to show that Equation (17) is true for $n=k$, then we have

$$\text{I-IVITrFAAOWG}\left\langle\left(u^e_{e_1},t^e_{e_1}\right),\left(u^e_{e_2},t^e_{e_2}\right),\cdots,\left(u^e_{e_k},t^e_{e_k}\right)\right\rangle$$
$$\left[\Pi^k_{j=1}\left(a^b_{(\sigma)_j}\right)^{\varepsilon^i_j},\Pi^k_{j=1}\left(b^a_{(\sigma)j}\right)^{\varepsilon^i_j},\Pi^k_{j=1}\left(c^d_{(\sigma)_j}\right)^{\varepsilon^i_j},\Pi^k_{j=1}\left(d^c_{(\sigma)_j}\right)^{\varepsilon^i_j}\right]$$
$$=\left\langle\left[\begin{array}{l}e^{-\left(\Pi^k_{j=1}\left(-\log\left(\mu^{lr}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}},1-e^{-\left(\Pi^k_{j=1}\varepsilon^i_j\left(-\log\left(1-\mathfrak{T}^{lr}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}}\\ e^{-\left(\Pi^k_{j=1}\left(-\log\left(\mu^{ur}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}},1-e^{-\left(\Pi^k_{j=1}\varepsilon^i_j\left(-\log\left(1-\mathfrak{T}^{ur}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}}\end{array}\right]\right\rangle. \tag{60}$$

Now, for $n=k+1$, we get

$$\text{I-IVITrFAAOWG}\left\langle\left(u^e_{e_1},t^e_{e_1}\right),\left(u^e_{e_2},t^e_{e_2}\right),\cdots,\left(u^e_{e_k},t^e_{e_{k+1}}\right)\right\rangle$$
$$\left[\Pi^{k+1}_{j=1}\left(a^b_{(\sigma)_j}\right)^{\varepsilon^i_j},\Pi^{k+1}_{j=1}\left(b^a_{(\sigma)j}\right)^{\varepsilon^i_j},\Pi^{k+1}_{j=1}\left(c^d_{(\sigma)_j}\right)^{\varepsilon^i_j},\Pi^{k+1}_{j=1}\left(d^c_{(\sigma)_j}\right)^{\varepsilon^i_j}\right]$$
$$=\left\langle\left[\begin{array}{l}e^{-\left(\Pi^k_{j=1}\left(-\log\left(\mu^{lr}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}},1-e^{-\left(\Pi^k_{j=1}\varepsilon^i_j\left(-\log\left(1-\mathfrak{T}^{lr}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}}\\ e^{-\left(\Pi^k_{j=1}\left(-\log\left(\mu^{ur}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}},1-e^{-\left(\Pi^k_{j=1}\varepsilon^i_j\left(-\log\left(1-\mathfrak{T}^{ur}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}},\end{array}\right]\right\rangle\otimes$$

$$\left\langle\left[\begin{array}{l}e^{-\left(\left(-\log\left(\mu^{lr}_{t^e_{e(k+1)}}\right)^{\varepsilon^i_{k+1}}\right)^{\xi}\right)^{1/\xi}},1-e^{-\left(\left(-\log\left(1-\mathfrak{T}^{lr}_{t^e_{ek+1}}\right)^{\varepsilon^i_{k+1}}\right)^{\xi}\right)^{1/\xi}}\\ e^{-\left(\left(-\log\left(\mu^{ur}_{t^e_{e(k+1)}}\right)^{\varepsilon^i_{k+1}}\right)^{\xi}\right)^{1/\xi}},1-e^{-\left(\left(-\log\left(1-\mathfrak{T}^{ur}_{t^e_{e(k+1)}}\right)^{\varepsilon^i_{k+1}}\right)^{\xi}\right)^{1/\xi}}\end{array}\right]\right\rangle$$

$$\left[\Pi^{k+1}_{j=1}\left(a^b_{(\sigma)_j}\right)^{\varepsilon^i_j},\Pi^{k+1}_{j=1}\left(b^a_{(\sigma)j}\right)^{\varepsilon^i_j},\Pi^{k+1}_{j=1}\left(d^c_{(\sigma)_j}\right)^{\varepsilon^i_j},\Pi^{k+1}_{j=1}\left(d^c_{(\sigma)_j}\right)^{\varepsilon^i_j}\right]$$
$$=\left\langle\left[\begin{array}{l}e^{-\left(\Pi^{k+1}_{j=1}\left(-\log\left(\mu^{lr}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}},1-e^{-\left(\Pi^{k+1}_{j=1}\varepsilon^i_j\left(-\log\left(1-\mathfrak{T}^{lr}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}}\\ e^{-\left(\Pi^{k+1}_{j=1}\left(-\log\left(\mu^{ur}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}},1-e^{-\left(\Pi^{k+1}_{j=1}\varepsilon^i_j\left(-\log\left(1-\mathfrak{T}^{ur}_{t^e_{e(\sigma)j}}\right)^{\varepsilon^i_j}\right)^{\xi}\right)^{1/\xi}}\end{array}\right]\right\rangle \tag{61}$$

□

Theorem 33 (Commutativity). *Let*

$$I\text{-}IVITrFAAOWG\left\langle\left(u^e_{e_1},t^e_{e_1}\right),\left(u^e_{e_2},t^e_{e_2}\right),\cdots,\left(u^e_{e_n},t^e_{e_n}\right)\right\rangle$$
$$=I\text{-}IVITrFAAOWG_{\varepsilon^i}\left\langle\left(u^{*e}_{e_1},t^{*e}_{e_1}\right),\left(u^{*e}_{e_2},t^{*e}_{e_2}\right),\cdots,\left(u^{*e}_{e_n},t^{*e}_{e_n}\right)\right\rangle, \tag{62}$$

where

$$\left\langle\begin{array}{c}\left(u^{*e}_{e_1},t^{*e}_{e_1}\right),\\ \left(u^{*e}_{e_2},t^{*e}_{e_2}\right),\cdots,\left(u^{*e}_{e_n},t^{*e}_{en}\right)\end{array}\right\rangle \tag{63}$$

is any permutation of

$$\left\langle \begin{matrix} \left(u_{e_1}^{e}, \imath_{e_1}^{e}\right), \\ \left(u_{e_2}^{e}, \imath_{e_2}^{e}\right), \cdots, \left(u_{e_n}^{e}, \imath_{en}^{e}\right) \end{matrix} \right\rangle. \tag{64}$$

Proof. As we know that

$$\begin{aligned} &\text{I-IVITrFAAOWG}\left\langle \left(u_{e_1}^{e}, \imath_{e_1}^{e}\right), \left(u_{e_2}^{e}, \imath_{e_2}^{e}\right), \cdots, \left(u_{e_n}^{e}, \imath_{e_n}^{e}\right) \right\rangle \\ &\quad = \left(\imath_{e_{\sigma(1)}}^{e}\right)^{\varepsilon_1^{\imath}} \otimes \left(\imath_{e_{\sigma(2)}}^{e}\right)^{\varepsilon_2^{\imath}} \otimes, \cdots, \otimes \left(\imath_{e_{\sigma(n)}}^{e}\right)^{\varepsilon_n^{\imath}}, \\ &\text{I-IVITrFAAOWG}_{\varepsilon^{\imath}}\left\langle \left(u_{e_1}^{*e}, \imath_{e_1}^{*e}\right), \left(u_{e_2}^{*e}, \imath_{e_2}^{*e}\right), \cdots, \left(u_{e_n}^{*e}, \imath_{e_n}^{*e}\right) \right\rangle \\ &\quad = \left(\imath_{e_{\sigma(1)}}^{*e}\right)^{\varepsilon_1^{\imath}} \otimes \left(\imath_{e_{\sigma(2)}}^{*e}\right)^{\varepsilon_2^{\imath}} \otimes, \cdots, \otimes \left(\imath_{e_{\sigma(n)}}^{*e}\right)^{\varepsilon_n^{\imath}}, \end{aligned} \tag{65}$$

since

$$\left\langle \begin{matrix} \left(u_{e_1}^{*e}, \imath_{e_1}^{*e}\right), \\ \left(u_{e_2}^{*e}, \imath_{e_2}^{*e}\right), \cdots, \left(u_{e_n}^{*e}, \imath_{e_n}^{*e}\right) \end{matrix} \right\rangle \tag{66}$$

is any permutation of

$$\left\langle \begin{matrix} \left(u_{e_1}^{e}, \imath_{e_1}^{e}\right), \\ \left(u_{e_2}^{e}, \imath_{e_2}^{e}\right), \cdots, \left(u_{e_n}^{e}, \imath_{e_n}^{e}\right) \end{matrix} \right\rangle. \tag{67}$$

□

Theorem 34 (Idempotency). *If all* $\imath_{ej}^{e} = (u_{e_j}^{e}, [a_j^b, b_j^a, c_j^d, d_j^c]; [\mu_{\imath_{ej}^{e}}^{lr}, \mu_{\imath_{ej}^{e}}^{ur}][\mathfrak{I}_{\imath_{ej}^{e}}^{lr}, \mathfrak{I}_{\imath_{ej}^{e}}^{ur}])$ *are equal,* $\imath_{e_{(j)}}^{e} = \imath_{e}^{e}$ *for all* $j(j = 1, 2, \cdots, n)$, *then*

$$\textit{I-IVITrFAAOWG}\left\langle \left(u_{e_1}^{e}, \imath_{e_1}^{e}\right), \left(u_{e_2}^{e}, \imath_{e_2}^{e}\right), \cdots, \left(u_{e_n}^{e}, \imath_{e_n}^{e}\right) \right\rangle = \imath_{e}^{e}. \tag{68}$$

Proof. Since $\imath_{ej}^{e} = \imath_{e}^{e}$ for all j we have

$$\begin{aligned} &\text{I}-\text{IVITrFAAOWG}\left\langle \left(u_{e_1}^{e}, \imath_{e_1}^{e}\right), \left(u_{e_2}^{e}, \imath_{e_2}^{e}\right), \cdots, \left(u_{e_n}^{e}, \imath_{e_n}^{e}\right) \right\rangle \\ &\quad = \otimes_{j=1}^{n} \left(\imath_{e\sigma(j)}^{e}\right)^{\varepsilon_j^{\imath}} = \left(\imath_{e_{\sigma(1)}}^{e}\right)^{\varepsilon_1^{\imath}} \otimes \left(\imath_{e_{\sigma(2)}}^{e}\right)^{\varepsilon_2^{\imath}} \otimes, \cdots, \otimes \left(\imath_{e_{\sigma(n)}}^{e}\right)^{\varepsilon_n^{\imath}} \\ &\quad = \otimes_{j=1}^{n} \left(\imath_{e}^{e}\right)^{\varepsilon_j^{\imath}} = \left(\imath_{e}^{e}\right)^{\varepsilon_1^{\imath}} \otimes \left(\imath_{e}^{e}\right)^{\varepsilon_2^{\imath}} \otimes, \cdots, \otimes \left(\imath_{e}^{e}\right)^{\varepsilon_n^{\imath}} \end{aligned}$$

$$\begin{aligned} &\left[\left(a_{(\sigma)_j}^{b}\right)^{\varepsilon_j^{\imath}}, \left(b_{(\sigma)j}^{a}\right)^{\varepsilon_j^{\imath}}, \left(c_{(\sigma)_j}^{d}\right)^{\varepsilon_j^{\imath}}, \left(d_{(\sigma)_j}^{c}\right)^{\varepsilon_j^{\imath}}\right] \\ &= \left\langle \begin{matrix} \left[e^{-\left(\Pi_{j=1}^{k}\left(-\log\left(\mu_{\imath_{e(\sigma)j}^{e}}^{lr}\right)^{\varepsilon_j^{\imath}}\right)^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\Pi_{j=1}^{k}\varepsilon_j^{\imath}\left(-\log\left(1-\mathfrak{I}_{\imath_{e(\sigma)j}^{e}}^{lr}\right)^{\varepsilon_j^{\imath}}\right)^{\xi}\right)^{1/\xi}}\right] \\ \left[e^{-\left(\Pi_{j=1}^{k}\left(-\log\left(\mu_{\imath_{e(\sigma)j}^{e}}^{ur}\right)^{\varepsilon_j^{\imath}}\right)^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\Pi_{j=1}^{k}\varepsilon_j^{\imath}\left(-\log\left(1-\mathfrak{I}_{\imath_{e(\sigma)j}^{e}}^{ur}\right)^{\varepsilon_j^{\imath}}\right)^{\xi}\right)^{1/\xi}}\right] \end{matrix} \right\rangle \\ &\quad \left[a_j^b, b_j^a, c_j^d, d_j^c\right]; \\ &= \left\langle \begin{matrix} \left[e^{-\left(\left(-\log \mu_{\imath_e^e}^{lr}\right)^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\left(-\log\left(1-\mathfrak{I}_{\imath_e^e}^{lr}\right)\right)^{\xi}\right)^{1/\xi}}\right] \\ \left[e^{-\left(\left(-\log \mu_{\imath_e^e}^{ur}\right)^{\xi}\right)^{1/\xi}}, 1 - e^{-\left(\left(-\log\left(1-\mathfrak{I}_{\imath_e^e}^{ur}\right)\right)^{\xi}\right)^{1/\xi}}\right] \end{matrix} \right\rangle \\ &\quad \left[a_j^b, b_j^a, c_j^d, d_j^c\right]; \\ &= \left\langle \begin{matrix} \left[e^{\log\left(\mu_{\imath_e^e}^{lr}\right)}, 1 - e^{\log\left(1-\mathfrak{I}_{\imath_e^e}^{lr}\right)}\right] \\ \left[e^{\log\left(\mu_{\imath_e^e}^{ur}\right)}, 1 - e^{\log\left(1-\mathfrak{I}_{\imath_e^e}^{ur}\right)}\right] \end{matrix} \right\rangle \\ &= \left(\left[a_j^b, b_j^a, c_j^d, d_j^c\right]; \left[\mu_{\imath_e^e}^{lr}, \mu_{\imath_e^e}^{ur}\right]\left[\mathfrak{I}_{\imath_e^e}^{lr}, \mathfrak{I}_{\imath_e^e}^{ur}\right]\right) = \imath_{e}^{e} \end{aligned} \tag{69}$$

Thus,

$$\text{I-IVITrFAAOWG}_{\varepsilon^{\imath}}\left\langle \left(u_{e_1}^{e}, \imath_{e_1}^{e}\right), \left(u_{e_2}^{e}, \imath_{e_2}^{e}\right), \cdots, \left(u_{e_n}^{e}, \imath_{e_n}^{e}\right) \right\rangle = \imath_{e}^{e}. \tag{70}$$

□

Theorem 35 (Monotonicity). *If* $\imath_{ej}^{e} \le \imath_{ej}^{e*}$ *for all* $j(j = 1, 2, \cdots, n)$, *then*

$$\begin{aligned} &\textit{I-IVITrFAAOWG}_{\varepsilon^{\imath}}\left\langle \left(u_{e_1}^{e}, \imath_{e_1}^{e}\right), \left(u_{e_2}^{e}, \imath_{e_2}^{e}\right), \cdots, \left(u_{e_n}^{e}, \imath_{e_n}^{e}\right) \right\rangle \\ &\quad \le \textit{I-IVITrFAAOWG}_{\varepsilon^{\imath}}\left\langle \left(u_{e_1}^{e}, \imath_{e_1}^{*e}\right), \left(u_{e_2}^{e}, \imath_{e_2}^{*e}\right), \cdots, \left(u_{e_n}^{e}, \imath_{e_n}^{*e}\right) \right\rangle, \end{aligned} \tag{71}$$

Proof. Let

$$\begin{aligned} &\text{I-IVITrFAAOWG}\left\langle \left(u_{e_1}^{e}, \imath_{e_1}^{e}\right), \left(u_{e_2}^{e}, \imath_{e_2}^{e}\right), \cdots, \left(u_{en}^{e}, \imath_{en}^{e}\right) \right\rangle \\ &\quad = \left(\imath_{e_{\sigma(1)}}^{e}\right)^{\varepsilon_1^{l}} \otimes \left(\imath_{e_{\sigma(2)}}^{e}\right)^{\varepsilon_2^{l}} \otimes, \cdots, \otimes \left(\imath_{e_{\sigma(n)}}^{e}\right)^{\varepsilon_n^{l}}, \\ &\text{I-IVITrFAAOWG}_{\varepsilon^{\imath}}\left\langle \left(u_{e_1}^{e}, \imath_{e_1}^{*e}\right), \left(u_{e_2}^{e}, \imath_{e_2}^{*e}\right), \cdots, \left(u_{e_n}^{e}, \imath_{e_n}^{*e}\right) \right\rangle \\ &\quad = \left(\imath_{e_{\sigma(1)}}^{*e}\right)^{\varepsilon_1^{l}} \otimes \left(\imath_{e_{\sigma(2)}}^{*e}\right)^{\varepsilon_2^{l}} \otimes, \cdots, \otimes \left(\imath_{e_{\sigma(n)}}^{*e}\right)^{\varepsilon_n^{l}}. \end{aligned} \tag{72}$$

Since $\imath_{ej}^{e} \le \imath_{ej}^{e*}$ for all j, it follows that $(\imath_{e_{\sigma(j)}}^{e}) \le (\imath_{e_{\sigma(j)}}^{*e})$ $(j = 1, 2, \cdots, n)$, then

Table 1: Decision matrix of expert -1 (H^1).

1	β_1	β_2
α_1	(0.5,[0.5,0.2,0.1,0.4][0.3,0.6][0.4,0.3])	(0.3,[0.3,0.2,0.1,0.5][0.4,0.2][0.4,0.3])
α_2	(0.5,[0.6,0.3,0.2,0.5][0.3,0.5][0.3,0.4])	(0.6,[0.5,0.2,0.1,0.8][0.3,0.6][0.4,0.3])
α_3	(0.2,[0.1,0.6,0.4,0.3][0.4,0.2][0.3,0.2])	(0.4,[0.2,0.7,0.1,0.4][0.3,0.3][0.3,0.4])
α_4	(0.5,[0.2,0.3,0.1,0.4][0.4,0.3][0.5,0.5])	(0.6,[0.5,0.6,0.1,0.5][0.4,0.2][0.4,0.1])
	β_3	β_4
α_1	(0.4,[0.1,0.3,0.6,0.8][0.3,0.2][0.4,0.1])	(0.2,[0.2,0.7,0.1,0.4][0.4,0.2][0.3,0.5])
α_2	(0.3,[0.2,0.7,0.1,0.4][0.5,0.7][0.3,0.1])	(0.1,[0.1,0.3,0.4,0.5][0.3,0.5][0.4,0.6])
α_3	(0.3,[0.3,0.2,0.8,0.5][0.3,0.3][0.4,0.6])	(0.7,[0.5,0.8,0.1,0.3][0.3,0.4][0.6,0.3])
α_4	(0.7,[0.5,0.3,0.1,0.4][0.6,0.4][0.4,0.3])	(0.8,[0.6,0.3,0.4,0.2][0.3,0.1][0.5,0.2])

Table 2: Decision matrix of expert -2 (H^2).

2	β_1	β_2
α_1	(0.1,[0.4,0.2,0.1,0.3][0.6,0.4][0.4,0.3])	(0.2,[0.2,0.3,0.5,0.4][0.4,0.1][0.3,0.2])
α_2	(0.5,[0.2,0.6,0.3,0.4][0.3,0.5][0.4,0.6])	(0.7,[0.1,0.6,0.1,0.8][0.3,0.6][0.5,0.3])
α_3	(0.1,[0.6,0.3,0.1,0.5][0.4,0.1][0.5,0.6])	(0.5,[0.6,0.3,0.7,0.4][0.3,0.3][0.3,0.4])
α_4	(0.2,[0.1,0.7,0.3,0.4][0.4,0.2][0.6,0.1])	(0.1,[0.3,0.5,0.1,0.8][0.4,0.3][0.4,0.2])
	β_3	β_4
α_1	(0.4,[0.3,0.4,0.1,0.2][0.4,0.3][0.4,0.2])	(0.6,[0.6,0.3,0.5,0.4][0.3,0.6][0.5,0.2])
α_2	(0.2,[0.2,0.3,0.6,0.4][0.3,0.2][0.5,0.6])	(0.3,[0.2,0.7,0.1,0.8][0.6,0.3][0.3,0.1])
α_3	(0.3,[0.5,0.4,0.1,0.7][0.8,0.1][0.7,0.2])	(0.8,[0.5,0.3,0.5,0.4][0.3,0.4][0.4,0.2])
α_4	(0.4,[0.7,0.6,0.3,0.4][0.3,0.1][0.6,0.5])	(0.3,[0.2,0.6,0.1,0.8][0.4,0.2][0.3,0.5])

Table 3: Decision matrix of expert -3 (H^3).

3	β_1	β_2
α_1	(0.6,[0.6,0.3,0.1,0.4][0.4,0.1][0.3,0.2])	(0.4,[0.4,0.3,0.1,0.4][0.6,0.4][0.4,0.3])
α_2	(0.2,[0.2,0.5,0.2,0.1][0.3,0.6][0.5,0.3])	(0.3,[0.2,0.8,0.1,0.3][0.3,0.5][0.4,0.6])
α_3	(0.4,[0.8,0.3,0.1,0.4][0.8,0.1][0.7,0.2])	(0.3,[0.9,0.3,0.7,0.4][0.6,0.1][0.5,0.4])
α_4	(0.5,[0.1,0.2,0.8,0.7][0.6,0.3][0.6,0.1])	(0.4,[0.1,0.5,0.1,0.2][0.7,0.3][0.4,0.2])
	β_3	β_4
α_1	(0.9,[0.3,0.2,0.1,0.5][0.3,0.2][0.5,0.3])	(0.7,[0.4,0.3,0.2,0.4][0.3,0.2][0.5,0.6])
α_2	(0.5,[0.1,0.3,0.6,0.3][0.5,0.3][0.3,0.1])	(0.1,[0.5,0.7,0.1,0.6][0.4,0.3][0.4,0.2])
α_3	(0.7,[0.7,0.3,0.3,0.4][0.7,0.3][0.4,0.2])	(0.6,[0.2,0.8,0.3,0.5][0.4,0.3][0.3,0.4])
α_4	(0.8,[0.2,0.3,0.1,0.4][0.3,0.1][0.6,0.5])	(0.6,[0.6,0.3,0.5,0.4][0.4,0.2][0.3,0.5])

$$\text{I-IVITrFAAOWG}_{\varepsilon^{\iota}}\left\langle\left(u_{e_1}^{e},\iota_{e_1}^{e}\right),\left(u_{e_2}^{e},\iota_{e_2}^{e}\right),\cdots,\left(u_{e_n}^{e},\iota_{e_n}^{e}\right)\right\rangle \leq \text{I-IVITrFAAOWG}_{\varepsilon^{\iota}}\left\langle\left(u_{e_1}^{e},\iota_{e_1}^{*e}\right),\left(u_{e_2}^{e},\iota_{e_2}^{*e}\right),\cdots,\left(u_{en}^{e},\iota_{e_n}^{*e}\right)\right\rangle. \tag{73}$$

□

4.3. Induced Interval-Valued Intuitionistic Trapezoidal Fuzzy Aczel-Alsina Hybrid Geometric (I-IVITrFAAHG) Operator

Definition 36. Let $\langle u_{e_j}^{e},\iota_{e_j}^{e}\rangle=\langle u_{e_j}^{e},[a_j^b,b_j^a,c_j^d,d_j^c];[\mu_{\iota_{ej}^{e}}^{lr},\mu_{\iota_{ej}^{e}}^{ur}][\mathfrak{J}_{\iota_{ej}^{e}}^{lr},\mathfrak{J}_{\iota_{ej}^{e}}^{ur}]\rangle$ $(j=1,2,\cdots,n)$ be a collection of I-VITrFNs, then induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina hybrid geometric (I-IVITrFAAHG) operator of dimension n is mapping I-IVITrFAAHAG, $\Omega^n\longrightarrow\Omega$ such that

$$\text{I-IVITrFAAHG}\left\langle\left(u_{e_1}^{e},\iota_{e_1}^{e}\right),\left(u_{e_2}^{e},\iota_{e_2}^{e}\right),\cdots,\left(u_{e_n}^{e},\iota_{e_n}^{e}\right)\right\rangle = \otimes_{j=1}^{n}\left(\iota_{e_{\sigma(n)}}^{\cdot e}\right)^{\varepsilon_j^{\iota}}=\left(\iota_{e_{\sigma(1)}}^{\cdot e}\right)^{\varepsilon_1^{\iota}}\otimes\left(\iota_{e_{\sigma(2)}}^{\cdot e}\right)^{\varepsilon_2^{\iota}}\otimes,\cdots,\otimes\left(\iota_{e_{\sigma(n)}}^{\cdot e}\right)^{\varepsilon_n^{\iota}}, \tag{74}$$

where $\iota_{e\sigma(j)}^{\cdot e}$ is the weighted interval-valued intuitionistic trapezoidal fuzzy values $\iota_{e\sigma(j)}^{\cdot e}$, and $\iota_{e\sigma(j)}^{\cdot e}=(\iota_{ej}^{e})^{n\varepsilon_j^{\iota}}$.

$(j=1,2,\cdots,n)$ of the interval-valued intuitionistic trapezoidal fuzzy ordered weighted averaging pair $\langle u_{e_j}^{e},\iota_{e_j}^{e}\rangle$ having the *jth* largest $u_{e_j}^{e}$ and $u_{e_j}^{e}$ in $\langle u_{e_j}^{e},\iota_{e_j}^{e}\rangle$ is referred to as the order inducing variable and $\iota_{e_j}^{e}$ as the interval-valued intuitionistic trapezoidal fuzzy argument variable. and $\varepsilon^{\iota}=(\varepsilon_1^{\iota},\varepsilon_2^{\iota},\cdots,\varepsilon_n^{\iota})^T$ is the weighting vector of $\iota_{ej}^{e}(j=1,2,\cdots,n)$ with $\varepsilon_j^{\iota}\in[0,1]$ and $\sum_{j=1}^{n}\varepsilon_j^{\iota}=1$.

Theorem 37. *Let $\langle u_{e_j}^{e},\iota_{e_j}^{e}\rangle$ $(j=1,2,\cdots,n)$, be a collection of I-VITrFNs. then induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina hybrid geometric (I-IVITrFAAHG) operator is defined as*

$$\begin{aligned}&\textit{I-IVITrFAAHG}\left\langle\left(u_{e_1}^{e},\iota_{e_1}^{e}\right),\left(u_{e_2}^{e},\iota_{e_2}^{e}\right),\cdots,\left(u_{e_n}^{e},\iota_{e_n}^{e}\right)\right\rangle\\&=\otimes_{j=1}^{n}\left(\iota_{e_{\sigma(n)}}^{\cdot e}\right)^{\varepsilon_j^{\iota}}=\left(\iota_{e_{\sigma(1)}}^{\cdot e}\right)^{\varepsilon_1^{\iota}}\otimes\left(\iota_{e_{\sigma(2)}}^{\cdot e}\right)^{\varepsilon_2^{\iota}}\otimes,\cdots,\otimes\left(\iota_{e_{\sigma(n)}}^{\cdot e}\right)^{\varepsilon_n^{\iota}}\\&\quad\left[\Pi_{j=1}^{n}\left(a_{(\sigma)_j}^{b}\right)^{\varepsilon_j^{\iota}},\Pi_{j=1}^{n}\left(b_{(\sigma)j}^{a}\right)^{\varepsilon_j^{\iota}},\Pi_{j=1}^{n}\left(c_{(\sigma)_j}^{d}\right)^{\varepsilon_j^{\iota}},\Pi_{j=1}^{n}\left(d_{(\sigma)_j}^{c}\right)^{\varepsilon_j^{\iota}}\right]\\&\times\left\langle\begin{bmatrix}-\left(\Pi_{j=1}^{n}\left(-\log\left(\mu_{\iota_{e(\sigma)j}^{e}}^{lr}\right)^{\varepsilon_j^{\iota}}\right)^{\xi}\right)^{1/\xi}\\e\end{bmatrix},\begin{bmatrix}-\left(\Pi_{j=1}^{n}\varepsilon_j^{\iota}\left(-\log\left(1-\mathfrak{J}_{\iota_{e(\sigma)j}^{e}}^{lr}\right)^{\varepsilon_j^{\iota}}\right)^{\xi}\right)^{1/\xi}\\1-e\end{bmatrix}\right\rangle.\\&\quad\begin{bmatrix}-\left(\Pi_{j=1}^{n}\left(-\log\left(\mu_{\iota_{e(\sigma)j}^{e}}^{ur}\right)^{\varepsilon_j^{\iota}}\right)^{\xi}\right)^{1/\xi}\\e\end{bmatrix},\begin{bmatrix}-\left(\Pi_{j=1}^{n}\varepsilon_j^{\iota}\left(-\log\left(1-\mathfrak{J}_{\iota_{e(\sigma)j}^{e}}^{ur}\right)^{\varepsilon_j^{\iota}}\right)^{\xi}\right)^{1/\xi}\\1-e\end{bmatrix}\end{aligned} \tag{75}$$

5. MCGDM Techniques on I-IVITrFAAWA Operator

In terms of I-VITrF informations, we suggest an approach for solving MCGDM problems. A decision matrix can be used to express MCGDM, where the attributes (criteria) are expressed by columns, and alternatives are expressed by row. Suppose we have a set which consists m number of alternatives represented by $\alpha=(\alpha_1,\alpha_1,\cdots,\alpha_m)$ and a set consists n number of attributes (criteria) represented as $\beta=(\beta_1,\beta_1,\cdots\beta_n)$. Let $M=\{m_1,m_2,\cdots,m_p\}$ be the set of decision makers and $\lambda=(\lambda_1,\lambda_1,\cdots,\lambda_p)^T$ be the weighting vector of DMs $\lambda_k\in[0,1]$ and $\sum_{k=1}^{p}\lambda_k=1$. DMs provided their assessment report for each alternative $\alpha_i(i=1,2,,..,m)$ against their attributes (criteria) $\beta_j(j=1,2,\cdots,n)$. Let ε_j^{ι} represent the weight vector for attribute such that $\sum_{j=1}^{n}\varepsilon_j^{\iota}=1$ and $\varepsilon_j^{\iota}\in[0,1]$ .The I-IVITrF decision matrix is assumed to be denoted by $\iota_e^e=[\langle u_{e_j}^{e},\iota_{e_{ij}}^{e}\rangle]_{m\times n}=\langle u_{e_j}^{e},[a_j^b,b_j^a,c_j^d,d_j^c];[\mu_{\iota_{ej}^{e}}^{lr},\mu_{\iota_{ej}^{e}}^{ur}][\mathfrak{J}_{\iota_{ej}^{e}}^{lr},\mathfrak{J}_{\iota_{ej}^{e}}^{ur}]\rangle$ where $\mu_{ij}^{lr},\mu_{ij}^{ur}\in$ M-D and $\mathfrak{J}_{ij}^{lr}$, $\mathfrak{J}_{ij}^{ur}\in$ N-MD consider by decision makers such that,

$$\begin{bmatrix}\beta_1 & \beta_2 & \cdots, & \beta_n\\ a_1\left\langle\begin{matrix}u_{e_{11}}^{e},\left[a_{11}^b,b_{11}^a,c_{11}^d,d_{11}^c\right]\\\left[\mu_{11}^{lr},\mu_{11}^{ur}\right]\left[\mathfrak{J}_{11}^{lr},\mathfrak{J}_{11}^{ur}\right]\end{matrix}\right\rangle, & \left\langle\begin{matrix}u_{e_{12}}^{e},\left[a_{12}^b,b_{12}^a,c_{12}^d,d_{12}^c\right]\\\left[\mu_{12}^{lr},\mu_{12}^{ur}\right]\left[\mathfrak{J}_{12}^{lr},\mathfrak{J}_{12}^{ur}\right]\end{matrix}\right\rangle, & \cdots, & \left\langle\begin{matrix}u_{e_{1n}}^{e},\left[a_{1n}^b,b_{1n}^a,c_{1n}^d,d_{1n}^c\right]\\\left[\mu_{1n}^{lr},\mu_{1n}^{ur}\right]\left[\mathfrak{J}_{1n}^{lr},\mathfrak{J}_{1n}^{ur}\right]\end{matrix}\right\rangle\\ a_2\left\langle\begin{matrix}u_{e_{21}}^{e},\left[a_{21}^b,b_{21}^a,c_{21}^d,d_{21}^c\right]\\\left[\mu_{21}^{lr},\mu_{21}^{ur}\right]\left[\mathfrak{J}_{21}^{lr},\mathfrak{J}_{21}^{ur}\right]\end{matrix}\right\rangle, & \left\langle\begin{matrix}u_{e_{22}}^{e},\left[a_{22}^b,b_{22}^a,c_{22}^d,d_{22}^c\right]\\\left[\mu_{22}^{lr},\mu_{22}^{ur}\right]\left[\mathfrak{J}_{22}^{lr},\mathfrak{J}_{22}^{ur}\right]\end{matrix}\right\rangle, & \cdots, & \left\langle\begin{matrix}u_{e_{2n}}^{e},\left[a_{2n}^b,b_{2n}^a,c_{2n}^d,d_{2n}^c\right]\\\left[\mu_{2n}^{lr},\mu_{2n}^{ur}\right]\left[\mathfrak{J}_{2n}^{lr},\mathfrak{J}_{2n}^{ur}\right]\end{matrix}\right\rangle\\ \vdots & \vdots & \vdots & \vdots\\ a_m\left\langle\begin{matrix}u_{e_{m1}}^{e},\left[a_{m_1}^b,b_{m_1}^a,c_{m_1}^d,d_{m_1}^c\right]\\\left[\mu_{m_1}^{lr},\mu_{m_1}^{ur}\right]\left[\mathfrak{J}_{m_1}^{lr},\mathfrak{J}_{m_1}^{ur}\right]\end{matrix}\right\rangle, & \left\langle\begin{matrix}u_{e_{m1}}^{e},\left[a_{m_2}^b,b_{m_2}^a,c_{m_2}^d,d_{m_2}^c\right]\\\left[\mu_{m_2}^{lr},\mu_{m_2}^{ur}\right]\left[\mathfrak{J}_{m_2}^{lr},\mathfrak{J}_{m_2}^{ur}\right]\end{matrix}\right\rangle, & \cdots, & \left\langle\begin{matrix}u_{e_{mn}}^{e},\left[a_{mn}^b,b_{mn}^a,c_{mn}^d,d_{mn}^c\right]\\\left[\mu_{mn}^{lr},\mu_{mn}^{ur}\right]\left[\mathfrak{J}_{mn}^{lr},\mathfrak{J}_{mn}^{ur}\right]\end{matrix}\right\rangle\end{bmatrix}. \tag{76}$$

Table 4: Decision matrix of expert -4 (H^4).

4	β_1	β_2
α_1	(0.4,[0.5,0.3,0.2,0.4][0.4,0.3][0.6,0.3])	(0.5,[0.2,0.3,0.5,0.4][0.4,0.1][0.5,0.2])
α_2	(0.4,[0.2,0.8,0.1,0.3][0.6,0.3][0.5,0.3])	(0.1,[0.3,0.2,0.7,0.8][0.4,0.2][0.5,0.3])
α_3	(0.5,[0.8,0.3,0.5,0.6][0.5,0.3][0.6,0.2])	(0.7,[0.7,0.3,0.1,0.4][0.4,0.3][0.3,0.4])
α_4	(0.7,[0.2,0.8,0.1,0.4][0.3,0.1][0.6,0.5])	(0.4,[0.4,0.3,0.3,0.5][0.3,0.2][0.7,0.1])
	β_3	β_4
α_1	(0.6,[0.5,0.3,0.1,0.5][0.4,0.3][0.4,0.2])	(0.7,[0.8,0.5,0.1,0.5][0.4,0.6][0.6,0.4])
α_2	(0.8,[0.4,0.2,0.1,0.3][0.3,0.6][0.7,0.3])	(0.7,[0.2,0.7,0.3,0.8][0.7,0.3][0.4,0.6])
α_3	(0.9,[0.8,0.3,0.6,0.7][0.3,0.4][0.8,0.2])	(0.2,[0.3,0.3,0.2,0.4][0.6,0.1][0.7,0.3])
α_4	(0.6,[0.2,0.9,0.1,0.3][0.4,0.1][0.3,0.5])	(0.1,[0.8,0.3,0.5,0.4][0.6,0.3][0.5,0.2])

Table 5: Decision matrix of expert -5 (H^5).

5	β_1	β_2
α_1	(0.5,[0.5,0.2,0.1,0.4][0.3,0.6][0.4,0.3])	(0.4,[0.1,0.3,0.6,0.8][0.3,0.2][0.4,0.1])
α_2	(0.6,[0.5,0.2,0.1,0.8][0.3,0.6][0.4,0.3])	(0.5,[0.6,0.3,0.2,0.5][0.3,0.5][0.3,0.4])
α_3	(0.7,[0.5,0.8,0.1,0.3][0.3,0.4][0.6,0.3])	(0.4,[0.2,0.7,0.1,0.4][0.3,0.3][0.3,0.4])
α_4	(0.8,[0.6,0.3,0.4,0.2][0.3,0.1][0.5,0.2])	(0.7,[0.5,0.3,0.1,0.4][0.6,0.4][0.4,0.3])
	β_3	β_4
α_1	(0.3,[0.3,0.2,0.1,0.5][0.4,0.2][0.4,0.3])	(0.2,[0.2,0.7,0.1,0.4][0.4,0.2][0.3,0.5])
α_2	(0.3,[0.2,0.7,0.1,0.4][0.5,0.7][0.3,0.1])	(0.1,[0.1,0.3,0.4,0.5][0.3,0.5][0.4,0.6])
α_3	(0.3,[0.3,0.2,0.8,0.5][0.3,0.3][0.4,0.6])	(0.2,[0.1,0.6,0.4,0.3][0.4,0.2][0.3,0.2])
α_4	(0.6,[0.5,0.6,0.1,0.5][0.4,0.2][0.4,0.1])	(0.5,[0.2,0.3,0.1,0.4][0.4,0.3][0.5,0.5])

The following steps are developed for solving MCGDM problem with I-IVITrFAAOWA operators.

Step 1. In this step, we construct the I-IVITrF matrix $H^k = (z_{ij})^k_{m\times n}$ in which the decision makers give their opinion related to each alternative with respect to each criteria.

Step 2. In this step, we perform the ordering of the decision matrix with respect to the first component.

Step 3. We utilize the decision information given in the I-IVITrF matrix $H^{(k)} = (z^{(k)}_{ij})_{m\times n}$. Assume that $\xi = 2$, and apply the I-IVITrFAAOWA operator to derive the individual overall preference values $z^{(k)}_i$ of the alternative α_i, and ε^t_j be the weight vector attribute $\sum_{j=1}^{n}\varepsilon^t_j = 1$, $\varepsilon^t_j \in [0, 1]$ such that,

$$\text{I-IVITrFAAOWA}\left\langle \left(u^e_{e_1}, t^e_{e_1}\right), \cdots, \left(u^e_{e_n}, t^e_{e_n}\right)\right\rangle = \oplus_{j=1}^{n}\left(\varepsilon^t_j\right)\left(t^e_{e_{\sigma(n)}}\right)$$

$$\left[\sum_{j=1}^{n}\varepsilon^t_j a^b_{(\sigma)_j}, \sum_{j=1}^{n}\varepsilon^t_j b^a_{(\sigma)j}, \sum_{j=1}^{n}\varepsilon^t_j c^d_{(\sigma)_j}, \sum_{j=1}^{n}\varepsilon^t_j d^c_{(\sigma)_j}\right] = \left\langle \begin{bmatrix} 1-e^{-\left(\sum_{j=1}^{n}\varepsilon^t_j\left(-\log\left(1-\mu^{lr}_{t^e_{e(\sigma)j}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\sum_{j=1}^{n}\varepsilon^t_j\left(-\log \mathfrak{T}^{lr}_{t^e_{e(\sigma)j}}\right)^{\xi}\right)^{1/\xi}} \\ 1-e^{-\left(\sum_{j=1}^{n}\varepsilon^t_j\left(-\log\left(1-\mu^{ur}_{t^e_{e(\sigma)j}}\right)\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\sum_{j=1}^{n}\varepsilon^t_j\left(-\log \mathfrak{T}^{ur}_{t^e_{e(\sigma)j}}\right)^{\xi}\right)^{1/\xi}} \end{bmatrix}\right\rangle \tag{77}$$

Step 4. We utilize the I-VITrFAAOWA operator to aggregate all the individual overall preference values into the collective overall preference values, where λ is weighting vector of decision makers such that $\lambda_k \in [0, 1]$ and $\sum_{k=1}^{p}\lambda_k = 1$..

Step 5. In this step, we calculate the scores function to aggregate the value of each alternative α_i.

Step 6. In ascending order, the alternative with the greatest value is our best choice.

Table 6: Decision matrix of expert -6 (H^6).

6	β_1	β_2
α_1	(0.6,[0.6,0.3,0.5,0.4][0.3,0.6][0.5,0.2])	(0.4,[0.3,0.4,0.1,0.2][0.4,0.3][0.4,0.2])
α_2	(0.7,[0.1,0.6,0.1,0.8][0.3,0.6][0.5,0.3])	(0.5,[0.2,0.6,0.3,0.4][0.3,0.5][0.4,0.6])
α_3	(0.8,[0.5,0.3,0.5,0.4][0.3,0.4][0.4,0.2])	(0.5,[0.6,0.3,0.7,0.4][0.3,0.3][0.3,0.4])
α_4	(0.4,[0.7,0.6,0.3,0.4][0.3,0.1][0.6,0.5])	(0.3,[0.2,0.6,0.1,0.8][0.4,0.2][0.3,0.5])
	β_3	β_4
α_1	(0.2,[0.2,0.3,0.5,0.4][0.4,0.1][0.3,0.2])	(0.1,[0.4,0.2,0.1,0.3][0.6,0.4][0.4,0.3])
α_2	(0.3,[0.2,0.7,0.1,0.8][0.6,0.3][0.3,0.1])	(0.2,[0.2,0.3,0.6,0.4][0.3,0.2][0.5,0.6])
α_3	(0.3,[0.5,0.4,0.1,0.7][0.8,0.1][0.7,0.2])	(0.1,[0.6,0.3,0.1,0.5][0.4,0.1][0.5,0.6])
α_4	(0.2,[0.1,0.7,0.3,0.4][0.4,0.2][0.6,0.1])	(0.1,[0.3,0.5,0.1,0.8][0.4,0.3][0.4,0.2])

Table 7: Decision matrix of expert -7 (H^7).

7	β_1	β_2
α_1	(0.9,[0.3,0.2,0.1,0.5][0.3,0.2][0.5,0.3])	(0.7,[0.4,0.3,0.2,0.4][0.3,0.2][0.5,0.6])
α_2	(0.5,[0.1,0.3,0.6,0.3][0.5,0.3][0.3,0.1])	(0.3,[0.2,0.8,0.1,0.3][0.3,0.5][0.4,0.6])
α_3	(0.7,[0.7,0.3,0.3,0.4][0.7,0.3][0.4,0.2])	(0.6,[0.2,0.8,0.3,0.5][0.4,0.3][0.3,0.4])
α_4	(0.8,[0.2,0.3,0.1,0.4][0.3,0.1][0.6,0.5])	(0.6,[0.6,0.3,0.5,0.4][0.4,0.2][0.3,0.5])
	β_3	β_4
α_1	(0.6,[0.6,0.3,0.1,0.4][0.4,0.1][0.3,0.2])	(0.4,[0.4,0.3,0.1,0.4][0.6,0.4][0.4,0.3])
α_2	(0.2,[0.2,0.5,0.2,0.1][0.3,0.6][0.5,0.3])	(0.1,[0.5,0.7,0.1,0.6][0.4,0.3][0.4,0.2])
α_3	(0.4,[0.8,0.3,0.1,0.4][0.8,0.1][0.7,0.2])	(0.3,[0.9,0.3,0.7,0.4][0.6,0.1][0.5,0.4])
α_4	(0.5,[0.1,0.2,0.8,0.7][0.6,0.3][0.6,0.1])	(0.4,[0.1,0.5,0.1,0.2][0.7,0.3][0.4,0.2])

6. Numerical Example

This example is taken from [38], with the help of this example we will explain our proposed method. A company wants to select a supplier and now there are four suppliers as candidates $\alpha_i = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $(i = 1, 2, 3, 4)$. We evaluate each supplier from four aspects $\beta_j = (\beta_1, \beta_2, \beta_3, \beta_4)(j = 1, 2, 3, 4)$ which are the following:

β_1 is the production cost, β_2 is the production quality, β_3 is the supplier's service performance, and β_4 is the risk factor.

The weight vector of attributes is $\varepsilon^t = (0.10, 0.20, 0.30, 0.40)^T$. There are four experts, and the weight vector of the experts is $\lambda = (0.40, 0.30, 0.20, 0.10)^T$. Then, the decision matrix $(H^{(k)} = (h_{ij}^{(k)})_{4\times4})(k = 1, 2, 3, 4)$ are shown in Tables 1–4, and our goal is to rank four suppliers and select the best one.

Step 1. The decision makers give their decision in the following tables.

Perform the ordering of the decision matrix with respect to the first component in the following tables.

From Tables 5–8, we introduced induced aggregation operators perform the ordering of the decision matrix with respect to the first component.

Step 3. We utilize the decision information given in the I-VITrF matrix $H^{(k)} = (h_{ij}^{(k)})_{m\times n}(k = 1, 2, 3, 4)$ and the I-IVITrFAAOWA operator to derive the individual overall preference values $H_i^{(k)}$ of the alternative α_i.

In Table 9, we apply IVITrFAAOWA operators to find out overall preference values.

Step 4. We utilize the IVITrFAAOWA operator to aggregate all the individual overall preference values into the collective overall preference values. Consider that $\lambda = (0.40, 0.30, 0.20, 0.10)^T$.

Table 10 shows that the collective values by applying the IVITrFAAOWA operators for the ranking process.

Step 5. Calculate score values of alternatives α_i.

$$s(h_1) = 0.4400, s(h_2) = 0.3800,$$

$$s(h_3) = 0.4100, s(h_4) = 0.1900. \quad (78)$$

Step 6. Ranking the score values

$$\alpha_1 \geq \alpha_3 \geq \alpha_2 \geq \alpha_1. \quad (79)$$

Table 8: Decision matrix of expert -8 (H^8).

8	β_1	β_2
α_1	(0.7,[0.8,0.5,0.1,0.5][0.4,0.6][0.6,0.4])	(0.6,[0.5,0.3,0.1,0.5][0.4,0.3][0.4,0.2])
α_2	(0.8,[0.4,0.2,0.1,0.3][0.3,0.6][0.7,0.3])	(0.7,[0.2,0.7,0.3,0.8][0.7,0.3][0.4,0.6])
α_3	(0.9,[0.8,0.3,0.6,0.7][0.3,0.4][0.8,0.2])	(0.7,[0.7,0.3,0.1,0.4][0.4,0.3][0.3,0.4])
α_4	(0.7,[0.2,0.8,0.1,0.4][0.3,0.1][0.6,0.5])	(0.6,[0.2,0.9,0.1,0.3][0.4,0.1][0.3,0.5])
	β_3	β_4
α_1	(0.5,[0.2,0.3,0.5,0.4][0.4,0.1][0.5,0.2])	(0.4,[0.5,0.3,0.2,0.4][0.4,0.3][0.6,0.3])
α_2	(0.4,[0.2,0.8,0.1,0.3][0.6,0.3][0.5,0.3])	(0.1,[0.3,0.2,0.7,0.8][0.4,0.2][0.5,0.3])
α_3	(0.5,[0.8,0.3,0.5,0.6][0.5,0.3][0.6,0.2])	(0.2,[0.3,0.3,0.2,0.4][0.6,0.1][0.7,0.3])
α_4	(0.4,[0.4,0.3,0.3,0.5][0.3,0.2][0.7,0.1])	(0.1,[0.8,0.3,0.5,0.4][0.6,0.3][0.5,0.2])

Table 9: Overall preference values.

(a)

$h1^{(1)}$	[0.22,0.43,0.25,0.54]	[0.4223,0.0918]	[0.6447,0.0328]
$h1^{(2)}$	[0.26,0.40,0.23,0.53]	[0.9992,0.0335]	[0.8626,0.0014]
$h1^{(3)}$	[0.46,0.45,0.34,0.38]	[0.7993,0.0037]	[0.9575,0.0001]
$h1^{(4)}$	[0.57,0.25,0.22,0.34]	[0.9029,0.0002]	[0.9886,0.0002]

(b)

$h2^{(1)}$	[0.44,0.33,0.34,0.35]	[0.5160,0.1101]	[0.6784,0.0220]
$h2^{(2)}$	[0.18,0.55,0.25,0.64]	[0.7417,0.0183]	[0.9064,0.0013]
$h2^{(3)}$	[0.53,0.33,0.38,0.50]	[0.8946,0.0009]	[0.9794,0.0003]
$h2^{(4)}$	[0.36,0.59,0.18,0.60]	[0.9420,0.0002]	[0.9953,0.0001]

(c)

$h3^{(1)}$	[0.39,0.27,0.14,0.43]	[0.4918,0.0562]	[0.7090,0.0570]
$h3^{(2)}$	[0.38,0.58,0.26,0.40]	[0.7228,0.0112]	[0.9064,0.0016]
$h3^{(3)}$	[0.55,0.50,0.36,0.44]	[0.9275,0.0051]	[0.9794,0.0001]
$h3^{(4)}$	[0.33,0.34,0.36,0.39]	[0.9718,0.0000]	[0.0053,0.0003]

(d)

$h4^{(1)}$	[0.56,0.38,0.19,0.47]	[0.4759,0.0968]	[0.8105,0.0342]
$h4^{(2)}$	[0.21,0.33,0.15,0.64]	[0.7963,0.0128]	[0.9653,0.0025]
$h4^{(3)}$	[0.58,0.30,0.23,0.24]	[0.9069,0.0010]	[0.9963,0.0001]
$h4^{(4)}$	[0.48,0.53,0.38,0.35]	[0.9527,0.0002]	[0.9993,0.0002]

Table 10: Collective values.

h1	[0.3500,0.3600,0.2500,0.8300]	[0.9709,0.0000]	[0.9915,0.0000]
h2	[0.2600,0.4700,0.2400,0.5500]	[0.9999,0.0000]	[0.9905,0.0054]
h3	[0.5100,0.4100,0.3600,0.4100]	[0.0001,0.0020]	[0.9908,0.0065]
h4	[0.4500,0.4000,0.2400,0.4300]	[0.9999,0.1002]	[0.9986,0.0069]

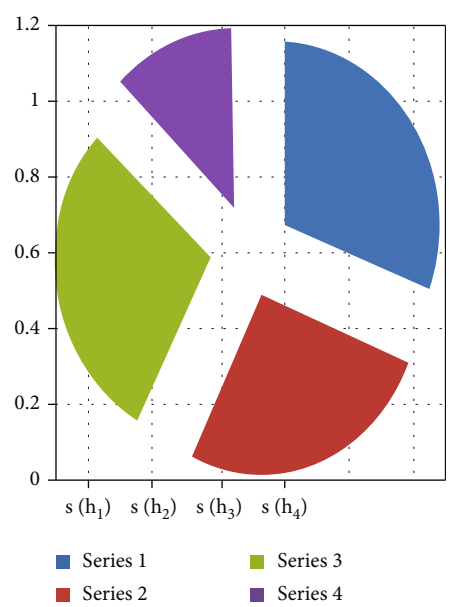

Figure 3: Graphical approach of proposed operator.

In Figure 3, we represent the graphical approach of proposed aggregation operator and using the score function to find out the best alternative.

7. Comparative Analysis

To expand on the advantages of the developed methods, we compare them with the existing methods by solving the same example with same weight $\lambda = (0.10, 0.20, 0.30, 0.40)^T$, such as the interval-valued intuitionistic fuzzy weighted average [39] operator, the intuitionistic fuzzy weighted average [39] operator, the intuitionistic fuzzy Einstein weighted average [41] operator, the intuitionistic fuzzy weighted geometric [10] operator, and the intuitionistic trapezoidal fuzzy weighted geometric [41] operator.

But in our paper, we take the data in the form of I-IVITrF information and used basic concept of the Aczel-Alsina aggregation operators like, I-IVITrFAAOWA operator, I-IVITrFAAHA operator, I-IVITrFAAOWG operator, and I-IVITrFAAHWG operator to aggregate the MCGDM issues. Furthermore, we have used these aggregation

Table 11: The comparison analysis with the different methods.

	Approach	Methods	Ranking result	Best option
(i)	I-VIFWA [39]	Existing method	$\alpha_3 \geq \alpha_2 \geq \alpha_1 \geq \alpha_4$	0.0017
(ii)	IFWA [39]	Existing method	$\alpha_3 \geq \alpha_2 \geq \alpha_1 \geq \alpha_4$	−0.0131
(iii)	IFEWA [41]	Existing method	$\alpha_2 \geq \alpha_1 \geq \alpha_4 \geq \alpha_3$	0.0171
(iv)	IFWG [10]	Existing method	$\alpha_1 \geq \alpha_4 \geq \alpha_2 \geq \alpha_3$	0.0300
(v)	ITFWG [41]	Existing method	$\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$	0.0100
(vi)	I-IVITrFAAOWA	Proposed method	$\alpha_1 \geq \alpha_3 \geq \alpha_2 \geq \alpha_4$	0.4400

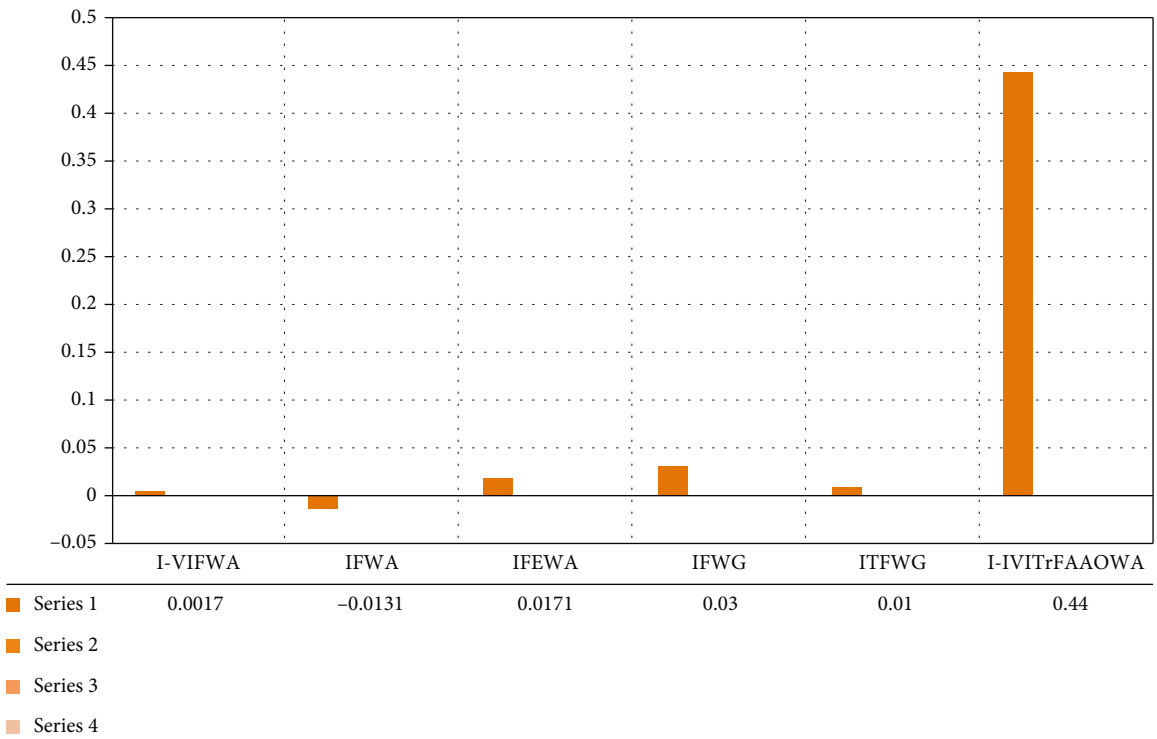

Figure 4: Graphical presentation of comparison analysis.

operators for solving the MCGDM in form of I-IVITrFNs environment. If we compare the score values of proposed aggregation operators with existing aggregation operators. We can see that our proposed work is more accurate and more applicable than existing methods due to highest score value of the proposed method. Applying the above mentioned methods, we obtain the comparison results shown in Table 11.

From Table 11, we observe that the optimal ranking results are different of the different methods, even though the score functions are different in different methods. Therefore, the novel methods proposed by us are reasonable and valid. Moreover, compared the proposed I-IVITrFAAOWA method with the existing methods, there is difference between the score functions obtained from the existing methods such that the highest score value of the proposed method is 0.4400 which is shown in Table 11. Therefore, decision-making processing based on the novel methods can be performed in an evaluation system, which implies that the propose methods have greater advantages than the existing methods.

In Figure 4, we compared all the results of proposed method with existing methods with the help of graphical approach.

8. Conclusions

The objective of this paper is to present some induced operators based on I-VITrFAA aggregation operators and apply them to the MCGDM problems. The MCGDM has an enormous prospective and restraint development used for improving and assessing different contradictory criteria in all aspects of DM in order to achieve more suitable and pragmatic DM result. In DM issues, the effectual erudition concerning a fastidious fact makes the decision-making task more complicated and dynamic. Induced interval-valued intuitionistic fuzzy sets are general mathematical method that can easily handle ambiguous and imprecise knowledge.

In this article we developed operation laws based on the Aczel-Alsina (A-A) and introduced four new types of aggregations operators such as I-IVITrFAAOWA operator, I-IVITrFAAHA operator, I-IVITrFAAOWG operator and I-IVITrFAAHG operator. Finally, examples are given to show the technique's prospective application and improvement. Furthermore, the established approach can be expanded for future research by incorporating other existing fuzzy sets and applying them to various MCGDM problems.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

Authors will pay all the outstanding dues after the acceptance of this manuscript. Yong-Long Wang, Ebenezer Bonyah, Mashael Khayyat, Zubair Ahmad, Muhammad Shakeel, and Waris Khan contributed equally to this work.

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