Research Article

A Novel Approach on Decision Support System Based on the Aczel-Alsina Aggregation Operators and Their Applications to Supplier Selection Problems

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Induced aggregation operators are more suitable for aggregating the individual preference relations into a collective fuzzy preference relation. Therefore, in this paper, we introduce the notion of some new types of induced aggregation operators, based on the Aczel-Alsina operations. We construct some induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina ordered weighted averaging/geometric (I-IVITrFAAOWA/G) operator, induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina hybrid averaging/geometric (I-IVITrFAAHA/G) operator. Moreover, some dominant properties of these developed operators are studied in detail. Based on these proposed approaches, a model is a build up for multicriteria decision making (MCDM), and their stepwise algorithm is being presented. Finally, in utilizing the developed approach, an illustrative example is solved with the help of proposed operators. In the end, we utilize an applicable example for supplier selection to prove the proposed methods and compare the result with existing methods, which shows the superiority, competence, and ability of the developed model.

1. Introduction

Multicriteria group decision-making (MCGDM) problems are of consequence in most kinds of fields such as engineering, economics, medical diagnosis, and management. Traditionally, it has been assumed that the information, which accesses the alternatives in terms of criteria, and weight are expressed in real numbers. Multicriteria group decision making (MCGDM) have develop into tremendously precious in preceding few decades in the learning of decision support systems [1–3]. The convolution of DM problems increases time to time in this world of global antagonism as the socio-economic structure becomes more dynamic. Therefore, making sensible and successful decisions in this situation is very complicated for a single decision expert. In this real world, group DM models are often used to combine the opinions of team of experienced experts in order to generate highly accurate and optimal values. Hence, MCGDM have an excellent ability and systematic method for improving and evaluating various competing criteria in all aspect of DM in order to achieve more appropriate and possible DM outcomes. In DM issues, the knowledge based about even a
fact is often unclear, making the decision-making task more complicated and ambiguous. To resolve this restriction, Zadeh [4] in 1965 first established the idea of fuzzy set (FS) by considering the membership degree MD. This concept has been investigated in various real-world problems, including clustering analysis [5], decision-making problems (DMPs) [6], and medical diagnosis [7]. Atanassov [8] developed the intuitionistic fuzzy set (IFS) as an extension of the FS, and its constraint is that the sum of the MD and the N-MD will be less than or equal to one. IFS has been a hot research topic for researchers who examined its hybrid structure in a variety of ways. The idea of intuitionistic fuzzy weighted averaging (IFWA) aggregation operators was first investigated by Xu [9]. Xu and Yager [10] developed the definition of intuitionistic fuzzy weighted geometric (IFWG) aggregation operators. The graphical techniques for rating score and accuracy functions were created by Ali et al. [11]. He et al. [12] explored the concept of intuitionistic fuzzy neutral averaging operators (IFNAO). He et al. [13] invented the notions of geometric relationship averaging operator and proposed its implementation in DM. Also Atanassov [14] presented preference relation based on IVIF. Further, Wan and Dong [15] work on extension of best-worst method based on IFS. Zaho et al. [16] were the first to apply the generalized intuitionistic fuzzy weighted averaging (GIFWA), generalized intuitionistic fuzzy order weighted averaging (IFOWA), and generalized intuitionistic fuzzy hybrid averaging (GIFHWA) operators to DM. However, it was discovered that these aggregation process was performed by using Archimedean t-norms and t-conorm by operators. As an equivalent to algebraic product and sum, Einstein-based t-norm and t-conorm provides the best estimate for product and sum of intuitionistic fuzzy numbers (IFNs). Wang and Liu and Wang and Zhang [17, 18] provided IF Einstein weighted averaging (IFEWA) and IF Einstein weighted geometric (IFEWG) operators by using the concept of Einstein operation. After that, Abbas et al. [19] developed the idea of intuitionistic fuzzy rough Einstein weighted averaging (IFREWA), intuitionistic fuzzy rough Einstein hybrid averaging (IFREHA), and intuitionistic fuzzy rough ordered weighted averaging (IFREOWA) aggregation operators based on the TOPSIS method. Yager and Filev [20, 21] developed an expansion of the OWA operator, called the induced ordered weighted averaging (IOWA) operator. It takes the argument in pairs (OWA pairs), in which one component is used to induce an ordering over the second component and then aggregates. Wei [22] developed two new aggregation operators: the induced intuitionistic fuzzy ordered weighted geometric (I-IFOWG) operator and the induced interval-valued intuitionistic fuzzy ordered weighted geometric (I-IFOWG) operator. Su et al. [23] also constructed induced intuitionistic fuzzy hybrid averaging (I-IFHA) operator and induced interval-valued intuitionistic fuzzy hybrid averaging (I-IIIFAHA) operator. Sha-keel et al [24] developed the induced Pythagorean trapezoidal fuzzy Einstein ordered weighted averaging operator and the induced Pythagorean trapezoidal fuzzy Einstein hybrid averaging (I-PTFEHA) operator. In [25], Klement et al. discussed the triangular norms, and in [26, 27], Aczel-Alsina (AA) presented t-norm as well as t-conorm. After that, some new idea of AA was developed by Senapati et al. in [28]. Senapati et al. introduced the IF Aczel-Alsina weighted averaging (IFAAWA) operator, the IF Aczel-Alsina orders weighted averaging (IFAAOWA) operator, and the IF Aczel-Alsina hybrid averaging (IFAAHA) operator. Senapati et al. [29, 30] developed interval-valued intuitionistic fuzzy aggregation operators, such as the IVIF Aczel-Alsina weighted averaged/geometric operator, the IVIF Aczel–Alsina order weighted averaged/geometric operator, and the IVIF Aczel–Alsina hybrid averaged/geometric operator and built up several features of such operators. Senapati [31] also defined picture fuzzy aggregation operator such as the PF Aczel–Alsina weighted average (PFAAWA) operator, the PF Aczel–Alsina order weighted average (PFAAOWA) operator, and the PF Aczel–Alsina hybrid average (PFAAHA) operator with their properties. Later on, Senapati et al. [32] originated a few new aggregation operators for aggregating hesitant fuzzy information, namely, the HF Aczel-Alsina weighted averaging (HFAAWA) operator, HF Aczel-Alsina order weighted averaging (HFAAOWA) operator, HF Aczel-Alsina hybrid averaging (HFAAHA) operator, HF Aczel-Alsina weighted geometric (HFAAWG) operator, HF Aczel-Alsina order weighted geometric (HFAAOWG) operator, HF Aczel–Alsina hybrid geometric (HFAAHG) operator, and HF Aczel–Alsina weighted Bonferroni mean (HFAAWBM) operator.

After that, Ban [33, 34] initiated the concept of trapezoidal fuzzy numbers (TrFNs) and interval-valued trapezoidal fuzzy numbers (I-IVTrFNs). Later on, Wang and Zhang [35] expanded the definition of intuitionistic trapezoidal fuzzy numbers (I-IVTrFNs) and interval-valued intuitionistic trapezoidal fuzzy numbers (IVITrFNs). Wei [36] developed the intuitionistic trapezoidal fuzzy ordered weighted averaging (ITrFOWA) operator and the intuitionistic trapezoidal fuzzy hybrid aggregation (ITrFHA) operator. Wan and Dong [37] defined the anticipation and expectant score of intuitionistic trapezoidal fuzzy numbers from the geometric angle. As for our information and based from the above review, no implementation of the MCGDM approach to review of induced interval-valued intuitionist trapezoidal fuzzy Aczel-Alsina (I-IVITrFAA) averaging/geometric aggregation operators. We proposed the average/geometric aggregation operators such as I-IVITrF Aczel-Alsina ordered weighted averaged (I-IVITrFAAOWA) operator, I-IVITrF Aczel-Alsina hybrid averaged (I-IVITrFAAHA) operator, I-IVITrF Aczel-Alsina ordered weighted geometric (I-IVITrFAAOWG) operator, and I-IVITrF Aczel-Alsina hybrid geometric (I-IVITrFAAHG) operator for solving MCGDM. To solve a MCGDM process, the weight of the attributes plays an important role in making decisions under the aggregation approaches.

The paper provided the following: in the next section, we briefly discuss some fundamental ideas of IFSs, I-VIFS, TrFNs, IVITrFNs, IF, and I-VIFAA aggregation operators. In Section 3, we outline AA operation laws for the I-VITrFNs. In Section 4, we develop I-IVITrFAAOWA operator, I-IVITrFAAHA operator, I-IVITrFAAOWG operator, and I-IVITrFAAOWG operator. In Section 5, we apply the
I-IVTrFAAOWA operator to build up certain methodologies for managing MCGDM issues. Section 6 provides an example of choosing the best alternative of the proposed method. In Section 7, we compare the developed method with the existing methods. At end of this paper, a few conclusions and upcoming research are mentioned in Section 8.

2. Preliminaries

In this section, we will define basic definition, results, and operational laws which are important for the improvement of this paper.

2.1. FSs. Atanassov [8] introduced the idea of FSs which is a generalization of FSs. FSs discussed only M-D where FSs discussed M-D and NM-D. Moreover, the sum of M-D and NM-D is less than or equal to 1.

Definition 1 (see [8]). Let a set $U$ be fixed. An (IFS)$t^e$ in $U$ is an object having the form

$$A^e = \left\{ \langle u, \mu^e(u), \Theta^e(u) \rangle | u \in U \right\},$$

where $\mu^e : U \rightarrow [0, 1]$ and $\Theta^e : U \rightarrow [0, 1]$ represent the M-D and N-MD such that $0 \leq \mu^e(u) + \Theta^e(u) \leq 1$ for each (IFS)$t^e$ in $U$, $\mu^e(u) = 1 - \Theta^e(u)$, for all $u \in U$. $\Theta^e(u)$ is called the degree of indeterminacy of $u$ to $t^e$.

Definition 2. Atanassov [8] let $t^e = \{ (u, \mu^e(u), \Theta^e(u)) | u \in U \}$ and $G^e = \{ (u, \mu^e(u), \Theta^e(u)) | u \in U \}$ be any two IFSs over the universe $U$, then the operational laws between $A^e$ and $G^e$ are defined as:

(i) $t^e \subseteq G^e$, if $\mu^e(u) \leq \mu^e(u)$ and $\Theta^e(u) \geq \Theta^e(u)$,

(ii) $t^e \cup G^e = \{ (u, \mu^e(u), \Theta^e(u)) | u \in U \}$

(iii) $t^e \cap G^e = \{ (u, \min \{ \mu^e(u), \mu^e(u) \}, \max \{ \Theta^e(u), \Theta^e(u) \}) | u \in U \}$

(iv) $(t^e)^* = \{ (u, \Theta^e(u), \mu^e(u)) | u \in U \}$

(v) $(t^e)^{\ast} = \{ (u, \Theta^e(u), \mu^e(u)) | u \in U \}$

(vi) $t^e \ast G^e = \{ (u, \mu^e(u) \ast \Theta^e(u), \mu^e \ast \Theta^e(u)) | u \in U \}$

(vii) $t^e \ast G^e = \{ (u, \mu^e(u) \ast \Theta^e(u), \mu^e \ast \Theta^e(u)) | u \in U \}$

Definition 3 (see [26]). Let $t^e$ and $G^e$ be any two IFSs, then the generalized intersection and the generalized union are defined in the following ways:

$$t^e \cap_{T,S} G^e = \left\{ \langle u, T\{\mu^e(u), \mu^e(u)\}, S\{\Theta^e(u), \Theta^e(u)\} \rangle | u \in U \right\},$$

$$t^e \cup_{T,S} G^e = \left\{ \langle u, T\{\mu^e(u), \mu^e(u)\}, S\{\Theta^e(u), \Theta^e(u)\} \rangle | u \in U \right\}$$

where $T$ presents a t-norm, and $S$ presents a t-conorm.

2.2. I-VIFNs

Definition 4 (see [38]). Let $t^e = (\mu^e, \Theta^e, \Gamma^e)$ be an I-VIFNs, where $\mu^e = [\mu^e_{[0,1]}, \mu^e_{[0,1]}]$ and $\Theta^e = [\Theta^e_{[0,1]}, \Theta^e_{[0,1]}]$ represent an I-VIFNs, hence $\mu^e \in [0, 1]$ and $\Theta^e \in [0, 1]$, such that $0 \leq \mu^e + \Theta^e \leq 1$.

Definition 5. Let $t^e = (\mu^e, \Theta^e, \Gamma^e) = ([\mu^e_{[0,1]}, \mu^e_{[0,1]}], [\Theta^e_{[0,1]}, \Theta^e_{[0,1]}])$, $t^e = (\mu^e, \Theta^e, \Gamma^e) = ([\mu^e_{[0,1]}, \mu^e_{[0,1]}], [\Theta^e_{[0,1]}, \Theta^e_{[0,1]}])$, $t^e = (\mu^e, \Theta^e, \Gamma^e) = ([\mu^e_{[0,1]}, \mu^e_{[0,1]}], [\Theta^e_{[0,1]}, \Theta^e_{[0,1]}])$ be any two I-VIFNs, numbers, and $\partial \geq 0$. Then,

(1) $t^e \ast t^e = \left\{ \left[ \left( (\mu^e_{[0,1]} + (\mu^e_{[0,1]} - (\mu^e_{[0,1]} - (\mu^e_{[0,1]} - (\mu^e_{[0,1]} \ast \Theta^e_{[0,1]} \ast \Theta^e_{[0,1]} \ast \Theta^e_{[0,1]})) \right) \right) \right] \right\}$

(2) $t^e = (\mu^e, \Theta^e, \Gamma^e) = \left\{ \left( (\mu^e_{[0,1]} + (\mu^e_{[0,1]} \ast \Theta^e_{[0,1]} \ast \Theta^e_{[0,1]} \ast \Theta^e_{[0,1]})) \right) \right\}$

(3) $\partial(t^e) = (1 - (1 - \mu^e_{[0,1]} \ast \Theta^e_{[0,1]} \ast \Theta^e_{[0,1]} \ast \Theta^e_{[0,1]}))

(4) $(t^e)^d = ([\mu^e_{[0,1]} \ast (1 - (1 - \mu^e_{[0,1]} \ast \Theta^e_{[0,1]} \ast \Theta^e_{[0,1]} \ast \Theta^e_{[0,1]})))]

Example 6. Let $t^e = ([0.45, 0.04], [0.35, 0.23]), t^e = ([0.34, 0.21], [0.05, 0.34]), t^e = ([0.27, 0.34], [0.21, 0.31])$ be any three I-VIFNs, and $\partial = 0.4$; then, we verify the above results such that,

(1) $t^e \ast t^e = \left\{ \left( (\mu^e_{[0,1]} + (\mu^e_{[0,1]} - (\mu^e_{[0,1]} \ast \Theta^e_{[0,1]} \ast \Theta^e_{[0,1]} \ast \Theta^e_{[0,1]})) \right) \right\}$
that,
\[
\text{Scr}_1(\hat{c}_1) = \left(\frac{0.37 - 0.16 + 0.31 - 0.23}{2}\right) = 0.1450,
\]
\[
\text{Scr}_2(\hat{c}_2) = \left(\frac{0.20 - 0.40 + 0.34 - 0.54}{2}\right) = -0.2100,
\]
\[
\text{Scr}_3(\hat{c}_3) = \left(\frac{0.46 - 0.32 + 0.32 - 0.24}{2}\right) = 0.1100,
\]
\[
\text{Scr}_4(\hat{c}_4) = \left(\frac{0.46 - 0.20 + 0.21 - 0.10}{2}\right) = 0.1910.
\]

In Figure 1, we supposed some interval-valued of intuitionistic fuzzy numbers and apply them score function to find out the highest score values for ranking process.

Definition 9 (see [39]). Let \( \hat{c}_i = [\mu_{c_i}^L, \mu_{c_i}^U, \xi_{c_i}^L, \xi_{c_i}^R] \) be an I-VIFNs; an accuracy function \( \text{Acr}(\hat{c}_i) \) can be defined as follows:
\[
\text{Acr}(\hat{c}_i) = \left(\frac{\mu_{c_i}^L + \xi_{c_i}^L + \xi_{c_i}^R}{2}\right) \text{Acr}(\hat{c}_i) \in [0, 1].
\]

Example 10. Let
\[
\hat{c}_1 = [0.37, 0.31][0.16, 0.23],
\]
\[
\hat{c}_2 = [0.20, 0.34][0.40, 0.54],
\]
\[
\hat{c}_3 = [0.46, 0.32][0.32, 0.24],
\]
\[
\hat{c}_4 = [0.46, 0.21][0.20, 0.10],
\]
be \( I - \text{VIFNs} \). Then, we verify the above results such that,
\[
\text{Acr}_1(\hat{c}_1) = \left(\frac{0.37 + 0.16 + 0.31 + 0.23}{2}\right) = 0.5400,
\]
\[
\text{Acr}_2(\hat{c}_2) = \left(\frac{0.20 + 0.40 + 0.34 + 0.54}{2}\right) = 0.7400,
\]
\[
\text{Acr}_3(\hat{c}_3) = \left(\frac{0.46 + 0.32 + 0.32 + 0.24}{2}\right) = 0.6700,
\]
\[
\text{Acr}_4(\hat{c}_4) = \left(\frac{0.46 + 0.20 + 0.21 + 0.10}{2}\right) = 0.4900.
\]

In Figure 2, we supposed some interval-valued of intuitionistic fuzzy numbers and apply their accuracy function to find out the highest accuracy values for ranking process.
2.3. ITFNs and I-VITFNs

Definition 11 (see [40]). Let $\mathbf{i}$ be ITFNs, its membership function

$$
\mu_{\mathbf{i}}(l) = \begin{cases} 
\frac{x - a^b}{b^a - a^b} \mu_{\mathbf{i}}, & a^b \leq l \leq b^a, \\
\mu_{\mathbf{i}}, & b^a \leq l \leq c^d, \\
\frac{d^e - x}{d^e - c^d} \mu_{\mathbf{i}} c^d \leq l \leq d^e, \\
1, & \text{otherwise}
\end{cases}
$$

Its nonmembership function is

$$
I_{\mathbf{i}}(l) = \begin{cases} 
\frac{d^e - l + \mathcal{F}_c (l - d^e)}{b^a - a^b} \mu_{\mathbf{i}}, & a^b \leq l \leq b^a, \\
\mathcal{F}_c, & b^a \leq l \leq c^d, \\
1 - c^d + \mathcal{F}_c (d^e - l) \mu_{\mathbf{i}} c^d \leq l \leq d^e, \\
0, & \text{otherwise,
\end{cases}
$$

where $0 \leq \mu_{\mathbf{i}} \leq 1; 0 \leq \mathcal{F}_c \leq 1; 0 \leq (\mu_{\mathbf{i}}) + (\mathcal{F}_c) \leq 1; a^b, b^a, c^d, d^e \in \mathbb{R}$. Then, $\mathbf{i}$ is called ITFNs. For convenience, $A_{\mathbf{i}} = \langle [a^b, b^a, c^d, d^e] [\mu_{\mathbf{i}}, \mathcal{F}_c] \rangle$.

Definition 12 (see [40]). Let $\mathbf{i}_1, \mathbf{i}_2$ be ITFNs, its membership function

$$
\mu_{\mathbf{i}_1} = \begin{bmatrix} a^b, b^a, c^d, d^e \end{bmatrix}, \quad \mu_{\mathbf{i}_2} = \begin{bmatrix} a_1^b, b_1^a, c_1^d, d_1^e \end{bmatrix}
$$

$
\mu_{\mathbf{i}_1} + \mathcal{F}_c (\mu_{\mathbf{i}_2} - \mu_{\mathbf{i}_1}) \leq 1$

(1) $\mathbf{i}_1 \oplus \mathbf{i}_2 = \left[ a_1^b + a_2^b, b_1^a + b_2^a, c_1^d + c_2^d, d_1^e + d_2^e \right] ; (\mu_{\mathbf{i}_1} + (\mu_{\mathbf{i}_2} - \mu_{\mathbf{i}_1}) (\mathcal{F}_c) (\mathcal{F}_c))$

(2) $\mathbf{i}_1 \otimes \mathbf{i}_2 = \left[ a_1^b b_1^a + a_2^b b_2^a, b_1^a + b_2^a, c_1^d + c_2^d, d_1^e + d_2^e \right] ; (\mu_{\mathbf{i}_1}) (\mu_{\mathbf{i}_2})$

(3) $\partial (\mathbf{i}) = \left[ [a^b, b^a, c^d, d^e] ; 1 - (1 - \mu_{\mathbf{i}})^{\mathcal{F}_c} \right] (\mathcal{F}_c)^{\mathcal{F}_c}$
be any three I-VITrNs, and $\partial \geq 0$. Then, for $\mathcal{F}_{\mathcal{V}}$, $\partial$, and $\mathcal{F}_{\mathcal{V}}$, we have:

\[
\mathcal{F}_{\mathcal{V}} = \left( \left[ a^1 b^1 c^1 d^1 \right], \left[ a^2 b^2 c^2 d^2 \right], \left[ a^3 b^3 c^3 d^3 \right], \left[ a^4 b^4 c^4 d^4 \right] \right)
\]

\[
\partial \mathcal{F}_{\mathcal{V}} = \left( \left[ a^1 b^1 c^1 d^1 \right], \left[ a^2 b^2 c^2 d^2 \right], \left[ a^3 b^3 c^3 d^3 \right], \left[ a^4 b^4 c^4 d^4 \right] \right)
\]

\[
\mathcal{F}_{\mathcal{V}}^\partial = \left( \left[ a^1 b^1 c^1 d^1 \right], \left[ a^2 b^2 c^2 d^2 \right], \left[ a^3 b^3 c^3 d^3 \right], \left[ a^4 b^4 c^4 d^4 \right] \right)
\]

where $a, b, c,$ and $d$ are elements of accuracy function.}

\[
(1) \quad r_i^{\mathcal{V}} = \left( \left[ a^i b^i c^i d^i \right], \left[ a^i b^i c^i d^i \right], \left[ a^i b^i c^i d^i \right], \left[ a^i b^i c^i d^i \right] \right)
\]

\[
(2) \quad r_i^\partial = \left( \left[ a^i b^i c^i d^i \right], \left[ a^i b^i c^i d^i \right], \left[ a^i b^i c^i d^i \right], \left[ a^i b^i c^i d^i \right] \right)
\]

\[
(3) \quad \partial = \left( \left[ a^i b^i c^i d^i \right], \left[ a^i b^i c^i d^i \right], \left[ a^i b^i c^i d^i \right], \left[ a^i b^i c^i d^i \right] \right)
\]

\[
(4) \quad \mathcal{F}_{\mathcal{V}}^\partial = \left( \left[ a^i b^i c^i d^i \right], \left[ a^i b^i c^i d^i \right], \left[ a^i b^i c^i d^i \right], \left[ a^i b^i c^i d^i \right] \right)
\]

**Example 14.**

Let $\mathcal{F}_{\mathcal{V}} = [0.30, 0.40]$, $\partial = [0.20, 0.50]$, and $\mathcal{F}_{\mathcal{V}}^\partial = [0.30, 0.60]$. Then, $r_i^\partial = \mathcal{F}_{\mathcal{V}}^\partial$.

\[
\mathcal{F}_{\mathcal{V}} = \left[ a^1 b^1 c^1 d^1 \right], \left[ a^2 b^2 c^2 d^2 \right], \left[ a^3 b^3 c^3 d^3 \right], \left[ a^4 b^4 c^4 d^4 \right]
\]

\[
\partial = \left[ a^1 b^1 c^1 d^1 \right], \left[ a^2 b^2 c^2 d^2 \right], \left[ a^3 b^3 c^3 d^3 \right], \left[ a^4 b^4 c^4 d^4 \right]
\]

\[
\mathcal{F}_{\mathcal{V}}^\partial = \left[ a^1 b^1 c^1 d^1 \right], \left[ a^2 b^2 c^2 d^2 \right], \left[ a^3 b^3 c^3 d^3 \right], \left[ a^4 b^4 c^4 d^4 \right]
\]
be any three I-VITTrNs, and δ = 2. We verify the above operational laws such that,

\[
\begin{align*}
t_e^{\oplus} & = \left\langle \begin{array}{c}
d_t^0 + d_t^1, b_t^0 + b_t^1, c_t^0 + c_t^1, d_t^0 + d_t^1, \\
0.40, 0.50, \\
0.60, 0.60, \\
0.40, 0.30, \\
0.50, 0.20
\end{array} \right\rangle \\
\end{align*}
\]

(15)

\[
\begin{align*}
t_e \cdot t_e & = \left\langle \begin{array}{c}
m_t^0 + m_t^1, b_t^0 + b_t^1, c_t^0 + c_t^1, d_t^0 + d_t^1, \\
0.40, 0.50, \\
0.60, 0.60, \\
0.40, 0.30, \\
0.50, 0.20
\end{array} \right\rangle \\
\end{align*}
\]

be I-VITFNs; a score function Scr(\textit{t}_e) can be defined as follows:

\[
\begin{align*}
\text{Scr}(t_e) & = \left( \frac{a^b + b^c + d^d + d^d}{4} \right) \\
& \frac{\mu_{t_e}^l - \mathcal{T}_e^l + \mu_{t_e}^r - \mathcal{T}_e^r}{2} \\
\end{align*}
\]

Definition 15 (see [40]). Let

\[
\begin{align*}
\text{Scr}(t_e) & = \left( \frac{a^b + b^c + d^d + d^d}{4} \right) \\
& \frac{\mu_{t_e}^l - \mathcal{T}_e^l + \mu_{t_e}^r - \mathcal{T}_e^r}{2} \\
\end{align*}
\]

be I-VITFNs; an accuracy function Acr(\textit{t}_e) can be defined as follows:

\[
\begin{align*}
\text{Acr}(t_e) & = \left( \frac{a^b + b^c + d^d + d^d}{4} \right) \\
& \frac{\mu_{t_e}^l + \mathcal{T}_e^l + \mu_{t_e}^r + \mathcal{T}_e^r}{2} \\
\end{align*}
\]

Definition 16 (see [40]). Let

\[
\begin{align*}
\text{Acr}(t_e) & = \left( \frac{a^b + b^c + d^d + d^d}{4} \right) \\
& \frac{\mu_{t_e}^l + \mathcal{T}_e^l + \mu_{t_e}^r + \mathcal{T}_e^r}{2} \\
\end{align*}
\]

be I-VITFNs; an accuracy function Acr(\textit{t}_e) can be defined as follows:

\[
\begin{align*}
\text{Acr}(t_e) & = \left( \frac{a^b + b^c + d^d + d^d}{4} \right) \\
& \frac{\mu_{t_e}^l + \mathcal{T}_e^l + \mu_{t_e}^r + \mathcal{T}_e^r}{2} \\
\end{align*}
\]

2.4. The Aczel-Alsina Operations of IFNs. In this section, we will present A-A operations on IFNs [28]. Let T presents a t-norm and S presents a t-conorm be A-A product T_A and A-A sum A_A, respectively. A-A product (\textit{t}_e \oplus \textit{G}_e) and A-A sum (\textit{t}_e \ominus \textit{G}_e) over two IFSs \textit{t}_e and \textit{G}_e defined in the following

\[
\begin{align*}
\partial(t_e) & = \left\langle \begin{array}{c}
\partial^{\oplus} + \partial^c + \partial^d + \partial^d, \\
0.40, 0.50, \\
0.60, 0.60, \\
0.40, 0.30, \\
0.50, 0.20
\end{array} \right\rangle \\
\end{align*}
\]
way \( \mu^\mu_\ast, \mu^\mu_\circ \):

\[
\tau'_c \otimes G'_c = \left\{ \begin{array}{l}
\left( u_c, T_A\left\{ \mu_c(u_c), \mu_{G'_c}(u_c) \right\} \right) \bigg| u_c \in U_c
\end{array} \right\}.
\]

(21)

\[
\tau'_c \otimes G'_c = \left\{ \begin{array}{l}
\left( u_c, S_A\left\{ \mu_c(u_c), \mu_{G'_c}(u_c) \right\} \right) \bigg| u_c \in U_c
\end{array} \right\}.
\]

(22)

**Definition 17** (see [28]). Let \( \tau'_c = [\mu'_c, \mathfrak{F}_c], \tau'_c = [\mu'_c, \mathfrak{F}_c] \) and \( \tau'_c = [\mu'_c, \mathfrak{F}_c] \) be any three IFNs, \( \xi \geq 1 \), and \( \partial > 0 \). Then, the A-A t-norm and t-conorm operation of IFNs are defined as

(i) \( \tau'_c \oplus \tau'_c = \begin{cases}
1 - e^{-((\log (1-\mu'_{c1}))^{\partial} + (\log (1-\mu'_{c2}))^{\partial})^{\xi}}
\end{cases}
\]

(ii) \( \tau'_c \otimes \tau'_c = \begin{cases}
1 - e^{-((\log \mu'_{c1} + (\log \mu'_{c2}))^{\xi})}
\end{cases}
\]

(iii) \( \partial(\tau'_c) = \begin{cases}
(1 - e^{-((\log (1-\mu'_{c1})))^{\partial} + (\log (1-\mu'_{c2})))^{\partial})^{\xi}
\end{cases}
\]

(iv) \( (\tau'_c)^{\partial} = \begin{cases}
(e^{-((\log \mu'_{c1})^{\xi}, 1 - e^{-((\log (1-\mu'_{c2}))^{\xi})})}
\end{cases}
\]

3. The Aczel-Alsina Operations of I-VITrFNs

In this section, we will present the operational laws of I-VITrFNs based on A-A aggregation operators.

**Definition 18.** Let

\[
\tau'_c = \begin{bmatrix}
\mu'^{a'}_{\ast}, \\ \mu'^{b'}_{\ast} \\
\mathfrak{F}_c, \\ \mathfrak{F}_c
\end{bmatrix},
\]

\[
\tau'_c = \begin{bmatrix}
\mu'^{a'_{\ast}}, \\ \mu'^{b'_{\ast}} \\
\mathfrak{F}_c, \\ \mathfrak{F}_c
\end{bmatrix},
\]

be any three I-VITrFNs, \( \partial > 0 \). Then, the Aczel-Alsina operation laws on I-VITrFNs are defined as

(1) \( \tau'_c \oplus \tau'_c = \begin{bmatrix}
[a'^{a'_{\ast}}, b'^{a'_{\ast}}] + [a'^{b'_{\ast}}, b'^{b'_{\ast}}], \\ e^{-((\log (1-\mu'_{c1})))^{\partial} + (\log (1-\mu'_{c2})))^{\partial}}
\end{bmatrix}
\]

(2) \( \tau'_c \otimes \tau'_c = \begin{bmatrix}
[a'^{a'_{\ast}}, b'^{a'_{\ast}}] + [a'^{b'_{\ast}}, b'^{b'_{\ast}}], \\ e^{-((\log (1-\mu'_{c1})))^{\partial} + (\log (1-\mu'_{c2})))^{\partial}}
\end{bmatrix}
\]

(3) \( \partial(\tau'_c) = \begin{bmatrix}
\partial(a'_{\ast}), \partial(b'_{\ast}), \partial(e'_{\ast}), \partial(d'_{\ast})
\end{bmatrix}
\]

(4) \( (\tau'_c)^{\partial} = \begin{bmatrix}
[e^{-((\log \mu'_{c1}))^{\xi}, 1 - e^{-((\log (1-\mu'_{c2}))^{\xi})})]
\end{bmatrix}
\]

**Example 19.** Let

\[
\tau'_c = \begin{bmatrix}
[0.2, 0.4], \\ [0.5, 0.6], \\ [0.6, 0.3], \\ [0.5, 0.4]
\end{bmatrix},
\]

\[
\tau'_c = \begin{bmatrix}
[0.3, 0.5], \\ [0.2, 0.7], \\ [0.1, 0.8]
\end{bmatrix},
\]

(23)
be any three I-VITrFNs; then, using Aczel-Alsina operation on I-VITrFNs as defined in above operation laws for ξ = 2 and \( \bar{d} = 3 \), we get

\[
\left[ 0.4, 0.3, 0.6, 0.3 \right]
\]

(26)

\[
\begin{align*}
\mathcal{F}^\xi_2 &= \left\langle \begin{array}{c}
0.2, 0.1, 0.8, 0.3
\end{array} \right\rangle, \\
\mathcal{F}^\bar{d}_2 &= \left\langle \begin{array}{c}
0.4, 0.3, 0.7, 0.3
\end{array} \right\rangle
\end{align*}
\]

(4) \( (\mathcal{F}^\xi)\bar{d} = \left\langle \begin{array}{c}
(0.2, 0.3, 0.5, 0.6, 0.3)
\end{array} \right\rangle
\]

\[
\begin{align*}
&= \left\langle \begin{array}{c}
0.62, 0.76, 0.81, 0.86
\end{array} \right\rangle, \\
&= \left\langle \begin{array}{c}
0.4128, 0.6990, 0.1243, 0.5872
\end{array} \right\rangle.
\end{align*}
\]

Theorem 20. Let

\[
\mathcal{F}^\xi_2 = \left\langle \begin{array}{c}
\mathcal{F}_2
\end{array} \right\rangle, \\
\mathcal{F}^\bar{d}_2 = \left\langle \begin{array}{c}
\mathcal{F}_2
\end{array} \right\rangle
\]

(27)

(28)

be any three I-VITrFNs, \( \bar{d} > 0 \). Then, we define the following results.

(1) \( \mathcal{F}^\xi_2 \oplus \mathcal{F}^\bar{d}_2 = \mathcal{F}_2 \oplus \mathcal{F}_2 \)

(2) \( \mathcal{F}^\xi_2 \odot \mathcal{F}^\bar{d}_2 = \mathcal{F}_2 \odot \mathcal{F}_2 \)

(3) \( \partial(\mathcal{F}^\xi_2) = \left\langle \begin{array}{c}
0.5(0.3), 0.5(0.3), 0.6(0.2), 0.7(0.7)
\end{array} \right\rangle
\]

(4) \( (\partial \mathcal{F}_2) \partial \mathcal{F}_2 = \partial \mathcal{F}_2 \mathcal{F}_2, \partial > 0
\]

(5) \( (\mathcal{F}^\xi_2 \mathcal{F}^\bar{d}_2)^{\mathcal{F}_2} = (\mathcal{F}_2)^{\mathcal{F}_2}, \partial > 0
\]

Proof. For the three \( \bar{a}_1, \bar{a}_2, \) and \( \partial, \bar{d}_1, \bar{d}_2 > 0, \) as in Definition 18 such that
(1) $r_{c_1} \oplus r_{c_2}$

\[
\begin{align*}
[r_{c_1}^a + a_{c_1}^b, b_{c_1}^a + b_{c_1}^b, c_{c_1}^a + c_{c_1}^b, d_{c_1}^a + d_{c_1}^b] & \in \left(1 - e^{-(\log (1-\rho_{c_1}^a))^4 + (\log (1-\rho_{c_1}^b))^4}\right) \\
\end{align*}
\]

\[
\begin{align*}
[r_{c_2}^a + a_{c_2}^b, b_{c_2}^a + b_{c_2}^b, c_{c_2}^a + c_{c_2}^b, d_{c_2}^a + d_{c_2}^b] & \in \left(1 - e^{-(\log (1-\rho_{c_2}^a))^4 + (\log (1-\rho_{c_2}^b))^4}\right) \\
\end{align*}
\]

\[
\begin{align*}
\partial \left[r_{c_1}^a + a_{c_1}^b, b_{c_1}^a + b_{c_1}^b, c_{c_1}^a + c_{c_1}^b, d_{c_1}^a + d_{c_1}^b\right] & = \left(1 - e^{-(\log (1-\rho_{c_1}^a))^4 + (\log (1-\rho_{c_1}^b))^4}\right) \\
\end{align*}
\]

(2) $r_{c_1} \otimes r_{c_2}$

\[
\begin{align*}
[a_{c_1}^a b_{c_1}^a, b_{c_1}^a b_{c_1}^b, c_{c_1}^a c_{c_1}^b, d_{c_1}^a d_{c_1}^b] & \in \left(1 - e^{-(\log (1-\rho_{c_1}^a))^4 + (\log (1-\rho_{c_1}^b))^4}\right) \\
\end{align*}
\]

\[
\begin{align*}
[a_{c_1}^a b_{c_1}^a, b_{c_2}^a b_{c_1}^b, c_{c_2}^a c_{c_1}^b, d_{c_2}^a d_{c_1}^b] & \in \left(1 - e^{-(\log (1-\rho_{c_1}^a))^4 + (\log (1-\rho_{c_1}^b))^4}\right) \\
\end{align*}
\]

\[
\begin{align*}
[a_{c_1}^a b_{c_1}^a, b_{c_1}^a b_{c_1}^b, c_{c_1}^a c_{c_1}^b, d_{c_2}^a d_{c_1}^b] & \in \left(1 - e^{-(\log (1-\rho_{c_1}^a))^4 + (\log (1-\rho_{c_1}^b))^4}\right) \\
\end{align*}
\]

\[
\begin{align*}
[a_{c_1}^a b_{c_1}^a, b_{c_1}^a b_{c_1}^b, c_{c_1}^a c_{c_1}^b, d_{c_1}^a d_{c_1}^b] & \in \left(1 - e^{-(\log (1-\rho_{c_1}^a))^4 + (\log (1-\rho_{c_1}^b))^4}\right) \\
\end{align*}
\]

\[
\begin{align*}
[a_{c_1}^a b_{c_1}^a, b_{c_1}^a b_{c_1}^b, c_{c_1}^a c_{c_1}^b, d_{c_1}^a d_{c_1}^b] & \in \left(1 - e^{-(\log (1-\rho_{c_1}^a))^4 + (\log (1-\rho_{c_1}^b))^4}\right) \\
\end{align*}
\]

(3) $\partial (r_{c_1} \oplus r_{c_2})$

\[
\begin{align*}
\partial [a_{c_1}^a + a_{c_1}^b, b_{c_1}^a + b_{c_1}^b, c_{c_1}^a + c_{c_1}^b, d_{c_1}^a + d_{c_1}^b] & \in \left(1 - e^{-(\log (1-\rho_{c_1}^a))^4 + (\log (1-\rho_{c_1}^b))^4}\right) \\
\end{align*}
\]

(4) $\partial_1 r_{c_1} \oplus \partial_2 r_{c_2}$

\[
\begin{align*}
\partial_1 [a_{c_1}^a, b_{c_1}^a, c_{c_1}^a, d_{c_1}^a] & \oplus \partial_2 [a_{c_1}^b, b_{c_1}^b, c_{c_1}^b, d_{c_1}^b] \\
\end{align*}
\]

\[
\begin{align*}
[\partial (a_{c_1}^a b_{c_1}^a), \partial (b_{c_1}^a b_{c_1}^b), \partial (c_{c_1}^a c_{c_1}^b), \partial (d_{c_1}^a d_{c_1}^b)] & \in \left(1 - e^{-(\log (1-\rho_{c_1}^a))^4 + (\log (1-\rho_{c_1}^b))^4}\right) \\
\end{align*}
\]

(5) $(r_{c_1} \oplus r_{c_2})^T$

\[
\begin{align*}
[a_{c_1}^a b_{c_1}^a, b_{c_1}^a b_{c_1}^b, c_{c_1}^a c_{c_1}^b, d_{c_1}^a d_{c_1}^b] & \in \left(1 - e^{-(\log (1-\rho_{c_1}^a))^4 + (\log (1-\rho_{c_1}^b))^4}\right) \\
\end{align*}
\]

\[
\begin{align*}
[\partial (a_{c_1}^a b_{c_1}^a), \partial (b_{c_1}^a b_{c_1}^b), \partial (c_{c_1}^a c_{c_1}^b), \partial (d_{c_1}^a d_{c_1}^b)] & \in \left(1 - e^{-(\log (1-\rho_{c_1}^a))^4 + (\log (1-\rho_{c_1}^b))^4}\right) \\
\end{align*}
\]
4. Induced Interval-Valued Intuitionistic Trapezoidal Fuzzy Acelz-Alsina Ordered Weighted Averaging (I–IVITrFAAOWA) Operator

Definition 21. Let \((u_c^j, u_e^j) = (u_c^j, [a^j_c, b^j_c, c^j_c, d^j_c], \mu_{u_c}^j, \mu_{u_e}^j) \mid \Sigma_{u_c}^j, \Sigma_{u_e}^j) \) \((j = 1, 2, \ldots, n)\) be a collection of I-VITrFNs. Then, induced interval-valued intuitionistic trapezoidal fuzzy Acelz-Alsina ordered weighted averaging (I–IVITrFAAOWA) operator of dimension \(n\) is mapping \(I–IVITrFAAOWA : \Omega \rightarrow \Omega\) such that

\[
I–IVITrFAAOWA\left((u_c^1, \epsilon_1^1), (u_c^2, \epsilon_2^1), \ldots, (u_c^n, \epsilon_n^1)\right) = \oplus_{j=1}^{n} \left(\epsilon_1^j \mathbin{\hat{\otimes}} \epsilon_2^j \mathbin{\hat{\otimes}} \ldots \mathbin{\hat{\otimes}} \epsilon_n^j\right) \label{eq:29}
\]

Also, \(\epsilon_1^j\) is the \(\epsilon_1^j\) value of the I-VITrFNs, and pairs \((u_c^j, \epsilon_1^j)\) having the \(j\)th largest \(u_c^j\) and \(u_e^j\) in \((u_c^j, \epsilon_1^j)\) is referred to as the order inducing variable and \(\epsilon_2^j\) as the interval-valued intuitionistic trapezoidal fuzzy argument variable.

Proof. By mathematical induction, we can prove that, Equation (30) holds for all positive integer \(n\). First, we show that Equation (22) holds for \(n = 2\), since

\[
e' = (\epsilon_1', \epsilon_2', \ldots, \epsilon_n')^T \text{ be the weighting vectors of } \epsilon_1^j\quad (j = 1, 2, \ldots, n) \text{ with } \epsilon_j' \in [0, 1] \text{ and } \sum_{j=1}^{n} \epsilon_j' = 1.
\]

Also, \(r^j, i_{r^j}^j\) is the \(r^j\) value of the I-VITrFNs, and pairs \((u_c^j, r^j)\) having the \(j\)th largest \(u_c^j\) and \(u_e^j\) in \((u_c^j, r^j)\) is referred to as the order inducing variable and \(r^j\) as the interval-valued intuitionistic trapezoidal fuzzy argument variable.

Theorem 22. Let \((u_c^j, r^j) (j = 1, 2, \ldots, n)\) be a collection of I-VITrFNs. Then, induced interval-valued intuitionistic trapezoidal fuzzy Acelz-Alsina ordered weighted averaging (I–IVITrFAAOWA) operator of dimension \(n\) is mapping \(I–IVITrFAAOWA : \Omega \rightarrow \Omega\) with the corresponding vector \(e' = (\epsilon_1', \epsilon_2', \ldots, \epsilon_n')^T\) is the weight vector such that \(\epsilon_j' \in [0, 1]\) and \(\sum_{j=1}^{n} \epsilon_j' = 1\), that is,

\[
\left[\sum_{j=1}^{n} \epsilon_j' \sigma_{1} \log \left(1 - \mu_{u_c}^j\right) - \sum_{j=1}^{n} \epsilon_j' \sigma_{1} \log \left(1 - \mu_{u_e}^j\right)\right] \\
\left[\sum_{j=1}^{n} \epsilon_j' \sigma_{2} \log \left(1 - \mu_{u_c}^j\right) - \sum_{j=1}^{n} \epsilon_j' \sigma_{2} \log \left(1 - \mu_{u_e}^j\right)\right] \\
\left[\sum_{j=1}^{n} \epsilon_j' \sigma_{\lambda} \log \left(1 - \mu_{u_c}^j\right) - \sum_{j=1}^{n} \epsilon_j' \sigma_{\lambda} \log \left(1 - \mu_{u_e}^j\right)\right] \\
\left[\sum_{j=1}^{n} \epsilon_j' \sigma_{\mu} \log \left(1 - \mu_{u_c}^j\right) - \sum_{j=1}^{n} \epsilon_j' \sigma_{\mu} \log \left(1 - \mu_{u_e}^j\right)\right]
\right]
\]

Based on Definition 18, we get

\[
I–IVITrFAAOWA(\epsilon_1^j, \epsilon_2^j) = \epsilon_1^j \epsilon_1^j \mathbin{\hat{\otimes}} \epsilon_1^j \mathbin{\hat{\otimes}} \ldots \mathbin{\hat{\otimes}} \epsilon_1^j
\]

(31)
Hence, (22) is true for $n = 2$.

Let, we want to show that Equation (30) is true for $n = k$, then we have

\[
\begin{align*}
1 - \text{IVTFAOWA} & \left( \left( u_1', \xi_1 \right), \left( u_2', \xi_2 \right), \ldots, \left( u_k', \xi_k \right) \right) \\
&= \left\langle \sum_{j=1}^{k} i_j^e \left( \log \left( 1 - \mu_j^e \right) \right)^t, \sum_{j=1}^{k} i_j^e \left( \log \left( 1 - \mu_j^e \right) \right)^t \right\rangle \\
&= \left\langle \sum_{j=1}^{k} i_j^e \left( \log \left( 1 - \mu_j^e \right) \right)^t, \sum_{j=1}^{k} i_j^e \left( \log \left( 1 - \mu_j^e \right) \right)^t \right\rangle \\
&= \left\langle \sum_{j=1}^{k} i_j^e \left( \log \left( 1 - \mu_j^e \right) \right)^t, \sum_{j=1}^{k} i_j^e \left( \log \left( 1 - \mu_j^e \right) \right)^t \right\rangle
\end{align*}
\]

Now, for $n = k + 1$, we get

\[
\begin{align*}
1 - \text{IVTFAOWA} & \left( \left( u_1', \xi_1 \right), \left( u_2', \xi_2 \right), \ldots, \left( u_k', \xi_k \right) \right) \\
&= \left\langle \sum_{j=1}^{k+1} i_j^e \left( \log \left( 1 - \mu_j^e \right) \right)^t, \sum_{j=1}^{k+1} i_j^e \left( \log \left( 1 - \mu_j^e \right) \right)^t \right\rangle \\
&= \left\langle \sum_{j=1}^{k+1} i_j^e \left( \log \left( 1 - \mu_j^e \right) \right)^t, \sum_{j=1}^{k+1} i_j^e \left( \log \left( 1 - \mu_j^e \right) \right)^t \right\rangle \\
&= \left\langle \sum_{j=1}^{k+1} i_j^e \left( \log \left( 1 - \mu_j^e \right) \right)^t, \sum_{j=1}^{k+1} i_j^e \left( \log \left( 1 - \mu_j^e \right) \right)^t \right\rangle
\end{align*}
\]

Hence, (22) is true for $n = 2$. Thus, (30) is legitimate for $n = k + 1$. 

\[\square\]
As consequences, we might come to conclusion that Equation (30) holds for all \( n \).

**Theorem 23** (Commutativity). Let

\[
\text{I-IVITrFAAOWA} \left( \left( u_{1}^{e}, r_{1}^{e} \right), \left( u_{2}^{e}, r_{2}^{e} \right), \ldots, \left( u_{n}^{e}, r_{n}^{e} \right) \right)
\]

\[
= \text{I-IVITrFAAOWA}_{\pi} \left( \left( u_{1}^{e}, r_{1}^{e} \right), \left( u_{2}^{e}, r_{2}^{e} \right), \ldots, \left( u_{n}^{e}, r_{n}^{e} \right) \right),
\]

where

\[
\begin{bmatrix}
\left( u_{1}^{e}, r_{1}^{e} \right), \\
\left( u_{2}^{e}, r_{2}^{e} \right), \\
\vdots \\
\left( u_{n}^{e}, r_{n}^{e} \right)
\end{bmatrix}
\]

is any permutation of

\[
\begin{bmatrix}
\left( u_{1}^{e}, r_{1}^{e} \right), \\
\left( u_{2}^{e}, r_{2}^{e} \right), \\
\vdots \\
\left( u_{n}^{e}, r_{n}^{e} \right)
\end{bmatrix}
\]

**Proof.** As we know that

\[
\text{I-IVITrFAAOWA} \left( \left( u_{1}^{e}, r_{1}^{e} \right), \left( u_{2}^{e}, r_{2}^{e} \right), \ldots, \left( u_{n}^{e}, r_{n}^{e} \right) \right)
\]

\[
= \epsilon_{1}^{e} \epsilon_{1}^{e} \oplus \epsilon_{2}^{e} \epsilon_{2}^{e} \oplus \ldots \oplus \epsilon_{n}^{e} \epsilon_{n}^{e},
\]

\[
\text{I-IVITrFAAOWA}_{\pi} \left( \left( u_{1}^{e}, r_{1}^{e} \right), \left( u_{2}^{e}, r_{2}^{e} \right), \ldots, \left( u_{n}^{e}, r_{n}^{e} \right) \right)
\]

\[
= \epsilon_{1}^{e} \epsilon_{1}^{e} \oplus \epsilon_{2}^{e} \epsilon_{2}^{e} \oplus \ldots \oplus \epsilon_{n}^{e} \epsilon_{n}^{e},
\]

since

\[
\begin{bmatrix}
\left( u_{1}^{e}, r_{1}^{e} \right), \\
\left( u_{2}^{e}, r_{2}^{e} \right), \\
\vdots \\
\left( u_{n}^{e}, r_{n}^{e} \right)
\end{bmatrix}
\]

is any permutation of

\[
\begin{bmatrix}
\left( u_{1}^{e}, r_{1}^{e} \right), \\
\left( u_{2}^{e}, r_{2}^{e} \right), \\
\vdots \\
\left( u_{n}^{e}, r_{n}^{e} \right)
\end{bmatrix}
\]

are equal, \( \epsilon_{ij}^{e} = \epsilon_{ij}^{e} \) for all \( nj(j = 1, 2, \ldots, n) \), then

\[
\text{I-IVITrFAAOWA} \left( \left( u_{1}^{e}, r_{1}^{e} \right), \left( u_{2}^{e}, r_{2}^{e} \right), \ldots, \left( u_{n}^{e}, r_{n}^{e} \right) \right) = \epsilon_{i}^{e}.
\]

(42)

**Theorem 25** (Monotonicity). If \( \epsilon_{ij}^{e} \leq \epsilon_{ij}^{e} \) for all \( j = 1, 2, \ldots, n \), then

\[
1 - \text{I-IVITrFAAOWA} \left( \left( u_{1}^{e}, r_{1}^{e} \right), \left( u_{2}^{e}, r_{2}^{e} \right), \ldots, \left( u_{n}^{e}, r_{n}^{e} \right) \right)
\]

\[
\leq 1 - \text{I-IVITrFAAOWA}_{\pi} \left( \left( u_{1}^{e}, r_{1}^{e} \right), \left( u_{2}^{e}, r_{2}^{e} \right), \ldots, \left( u_{n}^{e}, r_{n}^{e} \right) \right).
\]

(45)
Proof. Let

$$I-\text{IVITrFAAOWA}(u_1^*, e_1^*), (u_2^*, e_2^*), \ldots, (u_n^*, e_n^*)$$

$$= e_1^* e_{(1)}^* + e_2^* e_{(2)}^* + \cdots + e_n^* e_{(n)}^*$$

$$I-\text{IVITrFAAOWA}(u_1^* e_1^*, (u_2^* e_2^*), \ldots, (u_n^* e_n^*))$$

$$= e_1^* e_{(1)}^* + e_2^* e_{(2)}^* + \cdots + e_n^* e_{(n)}^*$$

(46)

Since $r_j^* \leq e_j^*$ for all $j$, it follows that $(r_e e_{(j)}^*) \leq (r_e e_{(j)}^*)$ ($j = 1, 2, \ldots, n$), then

$$I-\text{IVITrFAAOWA}(u_1^* e_1^*, (u_2^* e_2^*), \ldots, (u_n^* e_n^*))$$

$$\leq I-\text{IVITrFAAOWA}(u_1^* e_1^*, (u_2^* e_2^*), \ldots, (u_n^* e_n^*))$$

(47)

Example 26. Consider the following collection of I-IVITrFNs in pairs form such that

$$\langle u_1^*, e_1^* \rangle = \langle 0.1, 0.7, 0.4, 0.5, 0.3 \rangle; [0.50, 0.20], [0.40, 0.20],$$

$$\langle u_2^*, e_2^* \rangle = \langle 0.8, 0.1, 0.5, 0.2, 0.6 \rangle; [0.20, 0.60], [0.10, 0.20],$$

$$\langle u_3^*, e_3^* \rangle = \langle 0.3, 0.2, 0.4, 0.2, 0.3 \rangle; [0.40, 0.30], [0.30, 0.20],$$

$$\langle u_4^*, e_4^* \rangle = \langle 0.5, 0.6, 0.1, 0.5, 0.7 \rangle; [0.60, 0.20], [0.30, 0.10].$$

(48)

Performing the ordering of the I-IVITrF pairs with respect to the first component, we have

$$\langle u_1^*, e_1^* \rangle = \langle 0.8, 0.1, 0.5, 0.2, 0.6 \rangle; [0.20, 0.60], [0.10, 0.20],$$

$$\langle u_2^*, e_2^* \rangle = \langle 0.5, 0.6, 0.1, 0.5, 0.7 \rangle; [0.60, 0.20], [0.30, 0.10],$$

$$\langle u_3^*, e_3^* \rangle = \langle 0.3, 0.2, 0.4, 0.2, 0.3 \rangle; [0.40, 0.30], [0.30, 0.20],$$

$$\langle u_4^*, e_4^* \rangle = \langle 0.1, 0.7, 0.4, 0.5, 0.3 \rangle; [0.50, 0.20], [0.40, 0.20].$$

(49)

This ordering includes the ordered I-VITrF arguments.

$$\langle u_{e_{(1)}}^*, e_{e_{(1)}}^* \rangle = \langle u_{e_{(1)}}^*, e_{e_{(1)}}^* \rangle = \langle 0.8, [0.1, 0.5, 0.2, 0.6] \rangle; [0.20, 0.60], [0.10, 0.20],$$

$$\langle u_{e_{(2)}}^*, e_{e_{(2)}}^* \rangle = \langle u_{e_{(2)}}^*, e_{e_{(2)}}^* \rangle = \langle 0.5, [0.6, 0.1, 0.5, 0.7] \rangle; [0.60, 0.20], [0.30, 0.10],$$

$$\langle u_{e_{(3)}}^*, e_{e_{(3)}}^* \rangle = \langle u_{e_{(3)}}^*, e_{e_{(3)}}^* \rangle = \langle 0.3, [0.2, 0.4, 0.2, 0.3] \rangle; [0.40, 0.30], [0.30, 0.20],$$

$$\langle u_{e_{(4)}}^*, e_{e_{(4)}}^* \rangle = \langle u_{e_{(4)}}^*, e_{e_{(4)}}^* \rangle = \langle 0.1, [0.7, 0.4, 0.5, 0.3] \rangle; [0.50, 0.20], [0.40, 0.20].$$

(50)

If the associated weighting vector is $\epsilon^* = (0.10, 0.20, 0.30, 0.40)$ and $N = 2$ then, we get an aggregate value by Equation (9).

$$I-\text{IVITrFAAOWA}(u_1^* e_1^*, (u_2^* e_2^*), (u_3^* e_3^*), (u_4^* e_4^*))$$

$$\begin{pmatrix}
0.1(1.0) + 0.2(0.6) + 0.3(0.3) + 0.4(0.7), \\
0.1(0.5) + 0.2(0.1) + 0.3(0.4) + 0.4(0.4), \\
0.1(0.2) + 0.2(0.5) + 0.3(0.3) + 0.4(0.5), \\
0.1(0.6) + 0.2(0.7) + 0.3(0.3) + 0.4(0.3), \\
-0.1(-log(1-0.2))^2 + 0.2(-log(1-0.6))^2, \\
1 - e^+ 0.3(-log(1-0.4))^2 + 0.4(-log(1-0.5))^2, \\
0.1(-log(0.1))^2 + 0.2(-log(0.3))^2, \\
0.3(-log(0-0.3))^2 + 0.4(-log(0.4))^2, \\
0.47, 0.35, 0.38, 0.41, \\
[0.47, 0.35, 0.38, 0.41] \\
[0.46, 0.35, 0.38, 0.41], \\
[0.698, 0.0816, 0.5162, 0.0321].
\end{pmatrix}$$

(51)
4.1. Induced Interval-Valued Intuitionistic Trapezoidal Fuzzy Aczel Alsina Hybrid Averaging (I-IVITrFAHA) Operator

Definition 27. Let \( \langle u_{ij}, v_{ij} \rangle = (u_{ij}, [a_{ij}, b_{ij}, c_{ij}, d_{ij}]; [\mu_{ij}, \nu_{ij}] | \mathfrak{S}_{ij}^{\alpha}, \mathfrak{S}_{ij}^{\beta}) \) \((j = 1, 2, \cdots, n)\) be a collection of I-VITrFNs; then, induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina hybrid averaging (I-IVITrFAHA) operator of dimension \( n \) is mapping I-IVITrFAHA, \( \Omega^n \rightarrow \Omega \) such that

\[
\text{I-IVITrFAHA} \left( \sum_{i=1}^{n} \left( e_{ij}, \epsilon_{ij} \right) \right) = \sum_{j=1}^{n} \left( e_{ij}, \epsilon_{ij} \right) = e_{11} \epsilon_{11} + e_{22} \epsilon_{22} + \cdots + e_{nn} \epsilon_{nn},
\]

where \( e_{ij}, \epsilon_{ij} \) is the weighted interval-valued intuitionistic trapezoidal fuzzy values, \( e_{ij} \epsilon_{ij} \) and \( e_{ij} \epsilon_{ij} = n e_{ij} j (j = 1, 2, \cdots, n) \) of the interval-valued intuitionistic trapezoidal fuzzy ordered weighted averaging pair \( (u_{ij}, v_{ij}) \) having the \( j \)-th largest \( u_{ij} \) and \( v_{ij} \) in \( (u_{ij}, v_{ij}) \) is referred to as the order inducing variable and \( e_{ij} \epsilon_{ij} \) as the interval-valued intuitionistic trapezoidal fuzzy argument variable. \( e' = (e_1, e_2, \cdots, e_n)^T \) is the weighting vector of \( e_{ij} j = 1, 2, \cdots, n \) with \( e_j \in [0, 1] \) and \( \sum_{j=1}^{n} e_j = 1 \).

Theorem 28. Let \( \langle u_{ij}, v_{ij} \rangle \) \((j = 1, 2, \cdots, n)\) be a collection of I-VITrFNs. then induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina hybrid averaging (I-IVITrFAHA) operator is defined as

\[
\text{I-IVITrFAHA} \left( \sum_{i=1}^{n} \left( e_{ij}, \epsilon_{ij} \right) \right) = \sum_{j=1}^{n} \left( e_{ij}, \epsilon_{ij} \right) = e_{11} \epsilon_{11} + e_{22} \epsilon_{22} + \cdots + e_{nn} \epsilon_{nn},
\]

where \( e_{ij}, \epsilon_{ij} \) is the weighted interval-valued intuitionistic trapezoidal fuzzy values, \( e_{ij} \epsilon_{ij} \) and \( e_{ij} \epsilon_{ij} = n e_{ij} j (j = 1, 2, \cdots, n) \) of the interval-valued intuitionistic trapezoidal fuzzy ordered weighted averaging pair \( (u_{ij}, v_{ij}) \) having the \( j \)-th largest \( u_{ij} \) and \( v_{ij} \) in \( (u_{ij}, v_{ij}) \) is referred to as the order inducing variable and \( e_{ij} \epsilon_{ij} \) as the interval-valued intuitionistic trapezoidal fuzzy argument variable. \( e' = (e_1, e_2, \cdots, e_n)^T \) is the weighting vector of \( e_{ij} j = 1, 2, \cdots, n \) with \( e_j \in [0, 1] \) and \( \sum_{j=1}^{n} e_j = 1 \).

Theorem 29. The I-VITrFAHA operator is the special case of the I-IVITrFAHA operator.

Proof. Let \( \epsilon' = ((1/n), (1/n), \cdots, (1/n)) \), then

\[
\text{I-IVITrFAHA}_{\epsilon'} \left( \sum_{i=1}^{n} \left( u_{ij}, v_{ij} \right) \right) = \left( \epsilon_{11} \epsilon_{11} \right) + \left( \epsilon_{22} \epsilon_{22} \right) + \cdots + \left( \epsilon_{nn} \epsilon_{nn} \right).
\]

Theorem 30. The I-IVITrFAOWA operator is the special case of the I-IVITrFAHA operator.

Proof. Let \( \epsilon' = (1/n, 1/n, \cdots, 1/n) \), then \( e_{ij} = e_{ij} (j = 1, 2, \cdots, n) \)

\[
\text{I-IVITrFAOWA}_{\epsilon'} \left( \sum_{i=1}^{n} \left( u_{ij}, v_{ij} \right) \right) = \left( \epsilon_{11} \epsilon_{11} \right) + \left( \epsilon_{22} \epsilon_{22} \right) + \cdots + \left( \epsilon_{nn} \epsilon_{nn} \right).
\]

4.2. Induced Interval-Valued Intuitionistic Trapezoidal Fuzzy Aczel-Alsina Ordered Weighted Geometric (I-IVITrFAWG) Operator

Definition 31. Let \( \langle u_{ij}, v_{ij} \rangle = (u_{ij}, [a_{ij}, b_{ij}, c_{ij}, d_{ij}]; [\mu_{ij}, \nu_{ij}] | \mathfrak{S}_{ij}^{\alpha}, \mathfrak{S}_{ij}^{\beta}) \) \((j = 1, 2, \cdots, n)\) be a collection of I-VITrFNs. Then, induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina weighted geometric (I-IVITrFAWG) operator of dimension \( n \) is mapping I-IVITrFAWG, \( \Omega^n \rightarrow \Omega \) such that

\[
\text{I-IVITrFAWG} \left( \sum_{i=1}^{n} \left( e_{ij}, \epsilon_{ij} \right) \right) = \sum_{j=1}^{n} \left( e_{ij}, \epsilon_{ij} \right) = e_{11} \epsilon_{11} + e_{22} \epsilon_{22} + \cdots + e_{nn} \epsilon_{nn},
\]

where \( e_{ij}, \epsilon_{ij} \) is the weighted interval-valued intuitionistic trapezoidal fuzzy values, \( e_{ij} \epsilon_{ij} \) and \( e_{ij} \epsilon_{ij} = n e_{ij} j (j = 1, 2, \cdots, n) \) of the interval-valued intuitionistic trapezoidal fuzzy ordered weighted geometric pair \( (u_{ij}, v_{ij}) \) having the \( j \)-th largest \( u_{ij} \) and \( v_{ij} \) in \( (u_{ij}, v_{ij}) \) is referred to as the order inducing variable and \( e_{ij} \epsilon_{ij} \) as the interval-valued intuitionistic trapezoidal fuzzy argument variable.

Theorem 32. Let \( \langle u_{ij}, v_{ij} \rangle \) \((j = 1, 2, \cdots, n)\) be a collection of I-VITrFNs. Then, induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina ordered weighted averaging (I-
By mathematical induction, we can prove that Equation (56) holds for all positive integer \( n \). First, we show that Equation (56) holds for \( n = 2 \), since

\[
\left[ (a_{ii})^6, (b_{ii})^6, (c_{ii})^6, (d_{ii})^6 \right] \cdot \left[ e^{\left( -\log \left( \mu_{G(j)}^i \right) \right)^6}, 1 - e^{\left( -\log \left( \mu_{G(j)}^i \right) \right)^6} \right]^6 = \left[ \left( a_{ij}^6 \right), \left( b_{ij}^6 \right), \left( c_{ij}^6 \right), \left( d_{ij}^6 \right) \right].
\]

Also \( \varepsilon_j^{i(j)} \) is the \( \varepsilon_j^i \) value of the I-VITrFNs, and pairs \( (u_{ij}^i, \varepsilon_j^i) \) having the \( j \)th largest \( u_{ij}^i \) and \( \varepsilon_j^i \) in \( (u_{ij}^i, \varepsilon_j^i) \) are referred to as the order inducing variable and \( \varepsilon_j^i \) as the interval-valued intuitionistic trapezoidal fuzzy argument variable.

Proof. By mathematical induction, we can prove that Equation (56) holds for all positive integer \( n \). First, we show that Equation (56) holds for \( n = 2 \), since

\[
\left[ (a_{ij})^6, (b_{ij})^6, (c_{ij})^6, (d_{ij})^6 \right] \cdot \left[ e^{\left( -\log \left( \mu_{G(j)}^i \right) \right)^6}, 1 - e^{\left( -\log \left( \mu_{G(j)}^i \right) \right)^6} \right]^6 = \left[ \left( a_{ij}^6 \right), \left( b_{ij}^6 \right), \left( c_{ij}^6 \right), \left( d_{ij}^6 \right) \right].
\]

Based on Definition 18, we get

\[
\text{I-VITrFAAOWG} \left( \varepsilon_j^i, \varepsilon_j^s \right) = \left( \varepsilon_j^{i(11)}, \varepsilon_j^{(12)} \right) \odot \left( \varepsilon_j^{i(1)}, \varepsilon_j^{(12)} \right)
\]

\[
\left[ (a_{ij}^6)^6, (b_{ij}^6)^6, (c_{ij}^6)^6, (d_{ij}^6)^6 \right] \cdot \left[ e^{\left( -\log \left( \mu_{G(j)}^i \right) \right)^6}, 1 - e^{\left( -\log \left( \mu_{G(j)}^i \right) \right)^6} \right]^6 = \left[ \left( a_{ij}^6 \right), \left( b_{ij}^6 \right), \left( c_{ij}^6 \right), \left( d_{ij}^6 \right) \right].
\]
then we have
\[ n \leq (d^e_{v_{(r)}})^{k^e_{(r)}}(d^e_{b_{v_{(r)}}})(b^e_{v_{(r)}})^{k^e_{(r)}}. \]

Hence, (17) is true for \( n = 2 \).

Let us want to show that Equation (17) is true for \( n = k \), then we have

\[
\left[ I_{1-VIT{\text{fAA}}OG} \left( u^e_{v_{(r)}}, \bar{u}^e_{v_{(r)}}, e_{v_{(r)}}, \cdots, \bar{u}^e_{v_{(r)}}, e_{v_{(r)}} \right) \right] 
\]

\[
= \left[ I_{1-VIT{\text{fAA}}OG} \left( u^e_{v_{(r)}}, \bar{u}^e_{v_{(r)}}, e_{v_{(r)}}, \cdots, \bar{u}^e_{v_{(r)}}, e_{v_{(r)}} \right) \right].
\]

Now, for \( n = k + 1 \), we get

\[
I_{1-VIT{\text{fAA}}OG} \left( \bar{u}^e_{v_{(r)}}, \bar{u}^e_{v_{(r)}}, e_{v_{(r)}}, \cdots, \bar{u}^e_{v_{(r)}}, e_{v_{(r)}} \right)
\]

\[
= I_{1-VIT{\text{fAA}}OG} \left( \bar{u}^e_{v_{(r)}}, \bar{u}^e_{v_{(r)}}, e_{v_{(r)}}, \cdots, \bar{u}^e_{v_{(r)}}, e_{v_{(r)}} \right).
\]

\[
Theorem 33 \text{ (Commutativity). Let}
\]

\[
I_{1-VIT{\text{fAA}}OG} \left( u^e_{v_{(r)}}, \bar{u}^e_{v_{(r)}}, e_{v_{(r)}}, \cdots, \bar{u}^e_{v_{(r)}}, e_{v_{(r)}} \right)
\]

\[
= I_{1-VIT{\text{fAA}}OG} \left( \bar{u}^e_{v_{(r)}}, \bar{u}^e_{v_{(r)}}, e_{v_{(r)}}, \cdots, \bar{u}^e_{v_{(r)}}, e_{v_{(r)}} \right),
\]

where

\[
\left( u^e_{v_{(r)}}, 1 \right), \left( u^e_{v_{(r)}}, 0 \right), \cdots, \left( u^e_{v_{(r)}}, 1 \right)
\]
is any permutation of
\[
\left\langle \begin{array}{c}
(u_{c_1}^e, e_1^c), \\
u_{c_2}^e, e_2^c, \\
\vdots \\
u_{c_n}^e, e_n^c
\end{array} \rightangle.
\]  
(64)

Proof. As we know that

\[
\text{I-IVTrFAAOWG}\left( (u_{c_1}^e, e_1^c), (u_{c_2}^e, e_2^c), \ldots, (u_{c_n}^e, e_n^c) \right)
\]
\[
= \left( \xi_{c_1} \right)_{e_1} \otimes \left( \xi_{c_2} \right)_{e_2} \otimes \cdots \otimes \left( \xi_{c_n} \right)_{e_n},
\]

\[
\text{I-IVTrFAAOWG}_e\left( (u_{c_1}^e, e_1^c), (u_{c_2}^e, e_2^c), \ldots, (u_{c_n}^e, e_n^c) \right)
\]
\[
= \left( \xi_{c_1} \right)_{e_1} \otimes \left( \xi_{c_2} \right)_{e_2} \otimes \cdots \otimes \left( \xi_{c_n} \right)_{e_n},
\]

since

\[
\left\langle \begin{array}{c}
(u_{c_1}^e, e_1^c), \\
u_{c_2}^e, e_2^c, \\
\vdots \\
u_{c_n}^e, e_n^c
\end{array} \rightangle
\]  
(66)

is any permutation of

\[
\left\langle \begin{array}{c}
(u_{c_1}^e, e_1^c), \\
u_{c_2}^e, e_2^c, \\
\vdots \\
u_{c_n}^e, e_n^c
\end{array} \rightangle.
\]  
(67)

Theorem 35 (Monotonicity). If \( \xi_{c_j} \leq \xi_{e_j} \) for all \( j = 1, 2, \ldots, n \), then

\[
\text{I-IVTrFAAOWG}_e\left( (u_{c_1}^e, e_1^c), (u_{c_2}^e, e_2^c), \ldots, (u_{c_n}^e, e_n^c) \right)
\]
\[
\leq \text{I-IVTrFAAOWG}_e\left( (u_{c_1}^e, e_1^c), (u_{c_2}^e, e_2^c), \ldots, (u_{c_n}^e, e_n^c) \right),
\]

(71)

Proof. Let

\[
\text{I-IVTrFAAOWG}_e\left( (u_{c_1}^e, e_1^c), (u_{c_2}^e, e_2^c), \ldots, (u_{c_n}^e, e_n^c) \right)
\]
\[
= \left( \xi_{c_1} \right)_{e_1} \otimes \left( \xi_{c_2} \right)_{e_2} \otimes \cdots \otimes \left( \xi_{c_n} \right)_{e_n},
\]

I-IVTrFAAOWG\(e\left( (u_{c_1}^e, e_1^c), (u_{c_2}^e, e_2^c), (u_{c_3}^e, e_3^c), \ldots, (u_{c_n}^e, e_n^c) \right) \)
\[
= \left( \xi_{c_1} \right)_{e_1} \otimes \left( \xi_{c_2} \right)_{e_2} \otimes \cdots \otimes \left( \xi_{c_n} \right)_{e_n}.
\]

(72)

Since \( \xi_{c_j} \leq \xi_{e_j} \) for all \( j = 1, 2, \ldots, n \), then

(70)
## Table 1: Decision matrix of expert -1 ($H^1$).

<table>
<thead>
<tr>
<th></th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>(0.5, [0.5, 0.2, 0.1, 0.4], [0.3, 0.6], [0.4, 0.3])</td>
<td>(0.3, [0.3, 0.2, 0.1, 0.5], [0.4, 0.2], [0.4, 0.3])</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>(0.5, [0.6, 0.3, 0.2, 0.5], [0.3, 0.5], [0.3, 0.4])</td>
<td>(0.6, [0.5, 0.2, 0.1, 0.8], [0.3, 0.6], [0.4, 0.3])</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>(0.2, [0.1, 0.6, 0.4, 0.3], [0.4, 0.2], [0.3, 0.2])</td>
<td>(0.4, [0.2, 0.7, 0.1, 0.4], [0.3, 0.3], [0.3, 0.4])</td>
</tr>
<tr>
<td>$\alpha_4$</td>
<td>(0.5, [0.2, 0.3, 0.1, 0.4], [0.4, 0.3], [0.5, 0.5])</td>
<td>(0.6, [0.5, 0.6, 0.1, 0.5], [0.4, 0.2], [0.4, 0.1])</td>
</tr>
</tbody>
</table>

## Table 2: Decision matrix of expert -2 ($H^2$).

<table>
<thead>
<tr>
<th></th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>(0.4, [0.1, 0.3, 0.6, 0.8], [0.3, 0.2], [0.4, 0.1])</td>
<td>(0.2, [0.2, 0.7, 0.1, 0.4], [0.4, 0.2], [0.3, 0.5])</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>(0.3, [0.2, 0.7, 0.1, 0.4], [0.5, 0.7], [0.3, 0.1])</td>
<td>(0.1, [0.1, 0.3, 0.5, 0.4], [0.3, 0.5], [0.4, 0.6])</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>(0.3, [0.5, 0.2, 0.8, 0.5], [0.3, 0.3], [0.4, 0.6])</td>
<td>(0.7, [0.5, 0.8, 0.1, 0.3], [0.3, 0.4], [0.6, 0.3])</td>
</tr>
<tr>
<td>$\alpha_4$</td>
<td>(0.7, [0.5, 0.3, 0.1, 0.4], [0.6, 0.4], [0.4, 0.3])</td>
<td>(0.8, [0.6, 0.3, 0.4, 0.2], [0.3, 0.1], [0.5, 0.2])</td>
</tr>
</tbody>
</table>

## Table 3: Decision matrix of expert -3 ($H^3$).

<table>
<thead>
<tr>
<th></th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>(0.6, [0.6, 0.3, 0.1, 0.4], [0.4, 0.1], [0.3, 0.2])</td>
<td>(0.4, [0.4, 0.3, 0.1, 0.4], [0.6, 0.4], [0.4, 0.3])</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>(0.2, [0.2, 0.5, 0.2, 0.1], [0.3, 0.6], [0.5, 0.3])</td>
<td>(0.3, [0.2, 0.8, 0.1, 0.3], [0.3, 0.5], [0.4, 0.6])</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>(0.4, [0.8, 0.3, 0.1, 0.4], [0.8, 0.1], [0.7, 0.2])</td>
<td>(0.3, [0.9, 0.3, 0.7, 0.4], [0.6, 0.1], [0.5, 0.4])</td>
</tr>
<tr>
<td>$\alpha_4$</td>
<td>(0.5, [0.1, 0.2, 0.8, 0.7], [0.6, 0.3], [0.6, 0.1])</td>
<td>(0.4, [0.1, 0.5, 0.1, 0.2], [0.7, 0.3], [0.4, 0.2])</td>
</tr>
</tbody>
</table>


I-IVITrFAAOWG, \( \left( u_{i}, r_{i}^{e} \right), \left( u_{i}, r_{i}^{e} \right), \ldots, \left( u_{n}, r_{n}^{e} \right) \) 

\[ \leq I-IVITrFAAOWG, \left( u_{i}, r_{i}^{e} \right), \left( u_{i}, r_{i}^{e} \right), \ldots, \left( u_{n}, r_{n}^{e} \right) \]  

Theorem 37. Let \( \left( u_{i}, r_{i}^{e} \right) = \left( u_{i}, [a_{i}, b_{i}, c_{i}, d_{i}] ; [\mu_{i}^{u}, \mu_{i}^{l}] \right) \) and \( \left( u_{i}, r_{i}^{e} \right) = \left( u_{i}, [a_{i}, b_{i}, c_{i}, d_{i}] ; [\mu_{i}^{u}, \mu_{i}^{l}] \right) \) be a collection of I-IVITrFNs, then induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina hybrid geometric (I-IVITrFAAHG) operator of dimension \( n \) is mapping I-IVITrFAAHG, \( \Omega^{n} \rightarrow \Omega \) such that

\[ I-IVITrFAAHG, \left( u_{i}, r_{i}^{e} \right), \left( u_{i}, r_{i}^{e} \right), \ldots, \left( u_{n}, r_{n}^{e} \right) \]  

\[ = \mathcal{O}(n) \left( u_{i}^{e}, r_{i}^{e} \right) = u_{i}^{e} \otimes u_{i}^{e} \otimes \cdots \otimes u_{i}^{e} \]  

where \( r_{i}^{e} \) is the weighted interval-valued intuitionistic trapezoidal fuzzy values \( r_{i}^{e} \) and \( r_{j}^{e} = (r_{j}^{e})^{\left( n \right)} \).

(\( j = 1, 2, \ldots, n \)) of the interval-valued intuitionistic trapezoidal fuzzy ordered weighted averaging pair \( \left( u_{i}^{e}, r_{i}^{e} \right) \) having the \( j \)th largest \( u_{i}^{e} \) and \( u_{i}^{e} \) in \( \left( u_{i}^{e}, r_{i}^{e} \right) \) is referred to as the order inducing variable and \( r_{j}^{e} \) as the interval-valued intuitionistic trapezoidal fuzzy argument variable. and \( e^{e} = (e_{1}, e_{2}, \ldots, e_{n})^{T} \) is the weighting vector of \( r_{j}^{e} \) \( (j = 1, 2, \ldots, n) \) with \( e_{j}^{e} \in [0, 1] \) and \( \sum_{j=1}^{n} e_{j}^{e} = 1 \).

Theorem 37. Let \( \left( u_{i}^{e}, r_{i}^{e} \right) \) \( (j = 1, 2, \ldots, n) \), be a collection of I-IVITrFNs, then induced interval-valued intuitionistic trapezoidal fuzzy Aczel-Alsina hybrid geometric (I-IVITrFAAHG) operator is defined as

\[ I-IVITrFAAHG, \left( u_{i}^{e}, r_{i}^{e} \right), \left( u_{i}^{e}, r_{i}^{e} \right), \ldots, \left( u_{n}^{e}, r_{n}^{e} \right) \]  

\[ = \mathcal{O}(n) \left( u_{i}^{e}, r_{i}^{e} \right) = u_{i}^{e} \otimes u_{i}^{e} \otimes \cdots \otimes u_{i}^{e} \]  

\[ \left[ n_{e}^{n}, n_{e}^{n}, \ldots, n_{e}^{n} \right] \]  

5. MCGDM Techniques on I-IVITrFAAWA Operator

In terms of I-IVITrF informations, we suggest an approach for solving MCGDM problems. A decision matrix can be used to express MCGDM, where the attributes (criteria) are expressed by columns, and alternatives are expressed by row. Suppose we have a set which consists \( m \) number of alternatives represented by \( \alpha = (a_{1}, a_{2}, \ldots, a_{m}) \) and a set consists \( n \) number of attributes (criteria) represented as \( \beta = (\beta_{1}, \beta_{2}, \ldots, \beta_{n}) \). Let \( M = \{m_{1}, m_{2}, \ldots, m_{p}\} \) be the set of decision makers and \( \lambda = (\lambda_{1}, \lambda_{2}, \ldots, \lambda_{p})^{T} \) be the weighing vector of DMs \( \lambda_{k} \in [0, 1] \) and \( \sum_{k=1}^{p} \lambda_{k} = 1 \). DMs provided their assessment report for each alternative \( a_{i}(i = 1, 2, \ldots, m) \) against their attributes (criteria) \( \beta_{j}(j = 1, 2, \ldots, n) \). Let \( e_{j}^{e} \) represent the weight vector for attribute such that \( \sum_{j=1}^{n} e_{j}^{e} = 1 \) and \( e_{j}^{e} \in [0, 1] \). The I-IVITrF decision matrix is assumed to be denoted by \( \lambda^{e} = [\left( u_{i}^{e}, r_{i}^{e} \right)]_{\lambda_{m}, \lambda_{n}} = [u_{i}^{e}, [a_{i}^{e}, b_{i}^{e}, c_{i}^{e}, d_{i}^{e}] ; \mu_{i}^{u}, \mu_{i}^{l}] \) and \( \lambda_{m}, \lambda_{n} \in M-D \) and \( \lambda^{e} \) \( \lambda_{m}, \lambda_{n} \in N-MD \) consider by decision makers such that,
The following steps are developed for solving MCGDM problem with I-IVITrFAAOWA operators.

**Step 1.** In this step, we construct the I-IVITrF matrix $H^k = (z_{ij}^k)_{m \times n}$ in which the decision makers give their opinion related to each alternative with respect to each criteria.

**Step 2.** In this step, we perform the ordering of the decision matrix with respect to the first component.

**Step 3.** We utilize the decision information given in the I-IVITrF matrix $H^k = (z_{ij}^k)_{m \times n}$. Assume that $\xi = 2$, and apply the I-IVITrFAAOWA operator to derive the individual overall preference values $z_{ij}^k$ of the alternative $\alpha_i$, and $\epsilon_j$ be the weight attribute $\sum_{j=1}^{n} \epsilon_j = 1$, $\epsilon_j \in [0, 1]$ such that,

$$
I-IVITrFAAOWA(\left\{u_{\alpha_i}^{e_j}, r_{\alpha_i}^{e_j}\right\}, \cdots, \left\{u_{\alpha_i}^{e_j}, r_{\alpha_i}^{e_j}\right\})
= \oplus_{j=1}^{n} (\epsilon_j) \left(\left[r_{\alpha_i}^{e_j}\right]_{<0}\right)
$$

**Step 4.** We utilize the I-IVITrFAAOWA operator to aggregate all the individual overall preference values into the collective overall preference values, where $\lambda$ is weighting vector of decision makers such that $\lambda_k \in [0, 1]$ and $\sum_{k=1}^{K} \lambda_k = 1$.

**Step 5.** In this step, we calculate the scores function to aggregate the value of each alternative $\alpha_i$.

**Step 6.** In ascending order, the alternative with the greatest value is our best choice.
Step 1. The decision makers give their decision in the following tables.

Perform the ordering of the decision matrix with respect to the first component in the following tables.

From Tables 5–8, we introduced induced aggregation operators perform the ordering of the decision matrix with respect to the first component.

Step 2. We utilize the decision information given in the IVITrF matrix $H^{(k)} = (h_{ij}^{(k)})_{m \times n}(k = 1, 2, 3, 4)$ and the IVITrFAAOWA operator to derive the individual overall preference values $H_i^{(k)}$ of the alternative $\alpha_i$.

In Table 9, we apply IVITrFAAOWA operators to find out overall preference values.

Step 3. We utilize the decision information given in the IVITrF matrix $H^{(k)} = (h_{ij}^{(k)})_{m \times n}(k = 1, 2, 3, 4)$ and the IVITrFAAOWA operator to derive the individual overall preference values $H_i^{(k)}$ of the alternative $\alpha_i$.

Step 4. We utilize the IVITrFAAOWA operator to aggregate all the individual overall preference values into the collective overall preference values. Consider that $\lambda = (0.40, 0.30, 0.20, 0.10)^T$.

Table 10 shows that the collective values by applying the IVITrFAAOWA operators for the ranking process.

Step 5. Calculate score values of alternatives $\alpha_i$.

$$s(h_1) = 0.4400, s(h_2) = 0.3800,$$
$$s(h_3) = 0.4100, s(h_4) = 0.1900. \quad (78)$$

Step 6. Ranking the score values

$$\alpha_1 \geq \alpha_3 \geq \alpha_2 \geq \alpha_4. \quad (79)$$
In Figure 3, we represent the graphical approach of proposed aggregation operator and using the score function to find out the best alternative.

7. Comparative Analysis

To expand on the advantages of the developed methods, we compare them with the existing methods by solving the same example with same weight \( \lambda = (0.10, 0.20, 0.30, 0.40)^T \), such as the interval-valued intuitionistic fuzzy weighted average \([39]\) operator, the intuitionistic fuzzy weighted average \([39]\) operator, the intuitionistic fuzzy Einstein weighted average \([41]\) operator, the intuitionistic fuzzy weighted geometric \([10]\) operator, and the intuitionistic trapezoidal fuzzy weighted geometric \([41]\) operator.

But in our paper, we take the data in the form of I-IVITrF information and used basic concept of the Aczel-Alsina aggregation operators like, I-IVITrFAAOWA operator, I-IVITrFAAHA operator, I-IVITrFAAOWG operator, and I-IVITrFAAHWG operator to aggregate the MCGDM issues. Furthermore, we have used these aggregation operators...
operators for solving the MCGDM in form of I-IVITrFNs environment. If we compare the score values of proposed aggregation operators with existing aggregation operators. We can see that our proposed work is more accurate and more applicable than existing methods due to highest score value of the proposed method. Applying the above mentioned methods, we obtain the comparison results shown in Table 11.

From Table 11, we observe that the optimal ranking results are different of the different methods, even though the score functions are different in different methods. Therefore, the novel methods proposed by us are reasonable and valid. Moreover, compared the proposed I-IVITrFAAOWA method with the existing methods, there is difference between the score functions obtained from the existing methods such that the highest score value of the proposed method is 0.4400 which is shown in Table 11. Therefore, decision-making processing based on the novel methods can be performed in an evaluation system, which implies that the propose methods have greater advantages than the existing methods.

In Figure 4, we compared all the results of proposed method with existing methods with the help of graphical approach.

8. Conclusions

The objective of this paper is to present some induced operators based on I-IVITrFAA aggregation operators and apply them to the MCGDM problems. The MCGDM has an enormous prospective and restraint development used for improving and assessing different contradictory criteria in all aspects of DM in order to achieve more suitable and pragmatic DM result. In DM issues, the effectual erudition concerning a fastidious fact makes the decision-making task more complicated and dynamic. Induced interval-valued intuitionistic fuzzy sets are general mathematical method that can easily handle ambiguous and imprecise knowledge.

<table>
<thead>
<tr>
<th>Approach</th>
<th>Methods</th>
<th>Ranking result</th>
<th>Best option</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) I-VIFWA [39]</td>
<td>Existing method</td>
<td>$\alpha_3 \geq \alpha_2 \geq \alpha_1 \geq \alpha_4$</td>
<td>0.0017</td>
</tr>
<tr>
<td>(ii) IFWA [39]</td>
<td>Existing method</td>
<td>$\alpha_3 \geq \alpha_2 \geq \alpha_1 \geq \alpha_4$</td>
<td>-0.0131</td>
</tr>
<tr>
<td>(iii) IFEWA [41]</td>
<td>Existing method</td>
<td>$\alpha_2 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$</td>
<td>0.0171</td>
</tr>
<tr>
<td>(iv) IFWG [10]</td>
<td>Existing method</td>
<td>$\alpha_1 \geq \alpha_4 \geq \alpha_2 \geq \alpha_3$</td>
<td>0.0300</td>
</tr>
<tr>
<td>(v) ITFWG [41]</td>
<td>Existing method</td>
<td>$\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$</td>
<td>0.0100</td>
</tr>
<tr>
<td>(vi) I-IVITrFAAOWA</td>
<td>Proposed method</td>
<td>$\alpha_1 \geq \alpha_3 \geq \alpha_2 \geq \alpha_4$</td>
<td>0.4400</td>
</tr>
</tbody>
</table>

Figure 4: Graphical presentation of comparison analysis.
In this article we developed operation laws based on the Aczel-Alsina (A-A) and introduced four new types of aggregations operators such as I-IVITrFAAOWA operator, I-IVITrFAAHG operator and I-IVITrFAAHGA operator. Finally, examples are given to show the technique’s prospective application and improvement. Furthermore, the established approach can be expanded for future research by incorporating other existing fuzzy sets and applying them to various MCGDM problems.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

Authors’ Contributions

Authors will pay all the outstanding dues after the acceptance of this manuscript. Yong-Long Wang, Ebenezer Bonyah, Mashael Khayyat, Zubair Ahmad, Muhammad Shakheel, and Waris Khan contributed equally to this work.

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