Research Article

Generalized Fractional Integral Inequalities for MT-Non-Convex and pq -Convex Functions

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Fractional integral inequalities have a wide range of applications in pure and applied mathematics. In the present research, we establish generalized fractional integral inequalities for MT-non-convex functions and pq -convex functions. Our results extended many inequalities already existing in the literature.

1. Introduction

The study of convex analysis is not modern part of mathematics, but some ancient mathematicians also used the interesting geometry of convex functions and convex sets. However, the subject of convex analysis was started in the mid of 20th century. Many remarkable facts and generalization of convex analysis have been obtained quite recently.

Convex analysis is one of the appealing subjects for the researchers of geometry and analysis. The interesting geometric, differentiability, and other facilitating properties of convex functions make it distinct from other subjects. Moreover, the convex function and convex set have diverse applications in mathematical physics, technology, economics, and optimization theory.

In the last decades, the connection of convexity got stronger due to rapid development in fractional calculus. Therefore, nowadays, it is appreciable to seek new fractional integral inequalities. Simply, we can say that convexity plays a concrete role in fractional integral inequalities and symmetry theory because of its interesting geometric features.

There are many well-known integral inequalities related to convex functions, like Jensen's inequality [1, 2], Opial-type inequality [3], Simpson inequality [4], Ostrowski inequality [5–7], Hermite–Hadamard inequality [8, 9], Olsen

integral inequality [10], Fejér-type inequality [1, 11], Hardy inequality [12], and so on. One of the remarkable inequalities for convex function is the Hermite–Hadamard-type inequality.

$$\varphi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v \varphi(\eta) d\eta \leq \frac{\varphi(u) + \varphi(v)}{2}, \quad (1)$$

where $\varphi: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $u, v \in I$ with $u < v$. Note that several integral inequalities can be obtained from equation (1). There are several versions of inequality equation (1) in the literature, for example, inequalities of Hermite–Hadamard type for functions whose derivatives' absolute values are quasi-convex are presented in [8]. In [13], the authors established Hermite–Hadamard-type inequalities for p -convex functions via fractional integrals. In [14], the generalized Hermite–Hadamard inequalities are presented. A new version of the Hermite–Hadamard inequality for Riemann–Liouville fractional integrals is presented in [9]. The MT-convex functions via classical and Riemann–Liouville fractional integrals are studied in [15] by Park. He also established Hermite–Hadamard inequality for MT-convex functions. Mohammad in [16] established Hermite–Hadamard-type inequalities on differentiable co-ordinates for the same class of functions as studied by Park

in [15], and Liua et al. [17] developed this inequality for the same class of functions for classical integrals and fractional integrals. For more details, we refer the readers to [18–20].

First of all, we recall few definitions.

Definition 1 (p -convex set) [21]. I is said to be p -convex set, if $[\eta u^p + (1 - \eta)v^p]^{(1/p)} \in I$ for all $u, v \in I$ and $\eta \in [0, 1]$.

Definition 2 (p -convex) [13, 22]. A function $\varphi: I \rightarrow \mathbb{R}$ is said to be p -convex function, if

$$\varphi(\eta u^p + (1 - \eta)v^p)^{(1/p)} \leq \eta \varphi(u) + (1 - \eta) \varphi(v), \quad (2)$$

for all $u, v \in I$ and $\eta \in [0, 1]$.

Definition 3 (MT-convex) [5, 15–17]. A function $\varphi: I \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be MT-convex on I , if

$$\varphi(\eta u + (1 - \eta)v) \leq \frac{\sqrt{\eta}}{2\sqrt{1-\eta}} \varphi(u) + \frac{\sqrt{1-\eta}}{2\sqrt{\eta}} \varphi(v) \quad (3)$$

holds for all $u, v \in I$ and $\eta \in [0, 1]$.

Now we are ready to extend the form of convexities.

Definition 4 (MT-non-convex). A function $\varphi: I \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be MT-non-convex. Let I be p -convex set, if

$$(\eta u^p + (1 - \eta)v^p)^{(1/p)} \leq \frac{\sqrt{\eta}}{2\sqrt{1-\eta}} \varphi(u) + \frac{\sqrt{1-\eta}}{2\sqrt{\eta}} \varphi(v) \quad (4)$$

holds for all $u, v \in I$ and $\eta \in [0, 1]$.

Remark 1. In the above definition, for $p=1$, we get MT-convex function, and for $p=-1$, we get harmonically MT-convex function.

Definition 5 ((p,q) convex). A function $\varphi: I \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be pq -convex. Let I be p -convex set if

$$\varphi(\eta u^p + (1 - \eta)v^p)^{(1/p)} \leq (\eta \varphi(u)^q + (1 - \eta) \varphi(v)^q)^{(1/p)} \quad (5)$$

holds for all $u, v \in I$ and $\eta \in [0, 1]$.

Definition 6 (Riemann–Liouville fractional integral) [9]. Let $\varphi \in L^1[u, v]$ and $\gamma > 0$. The right side and left side Riemann–Liouville fractional integrals are initiated by

$$\begin{aligned} J_{u+}^{\gamma} \varphi(s) &= \frac{1}{\Gamma(\gamma)} \int_u^s (s - \eta)^{\gamma-1} \varphi(\eta) d\eta, \quad s > u, \\ J_{v-}^{\gamma} \varphi(s) &= \frac{1}{\Gamma(\gamma)} \int_s^v (\eta - s)^{\gamma-1} \varphi(\eta) d\eta, \quad s < v, \end{aligned} \quad (6)$$

respectively. For details, we refer the readers to [6, 23].

Now, we define some special functions:

(1) Gamma function:

$$\Gamma(z) = \int_0^{\infty} e^{-\eta} \eta^{z-1} d\eta, \quad z > 0. \quad (7)$$

(2) Beta function:

$$\beta(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = \int_0^1 \eta^{u-1} (1 - \eta)^{v-1} d\eta \quad (u, v > 0) \quad (8)$$

(see [24–27]).

(3) The hypergeometric function [18]:

$$\begin{aligned} {}_2F_1(u, v, w, x) &= \frac{1}{\beta(v, w-v)} \int_0^1 \eta^{v-1} (1 - \eta)^{w-v-1} \\ &\quad \cdot (1 - d\eta)^{-u} d\eta, \quad w > v > 0, |x| < 1. \end{aligned} \quad (9)$$

In [28], Raina introduced a function initiated by

$$F_{\varrho, \lambda}^{\varsigma}(u) = F_{\varrho, \lambda}^{\varsigma(0), \varsigma(1), \dots}(u) = \sum_{k=0}^{\infty} \frac{\varsigma(k)}{\Gamma(\varrho k + \lambda)} u^k, \quad (10)$$

where $\varrho, \lambda \in \mathbb{R}^+$; $|u| < \mathbb{R}$ and the coefficients $\varsigma(k) \in \mathbb{R}^+$. By using equation (10), Raina and Agarwal [28, 29] initiated the following left and right-side fractional integral operators:

$$\begin{aligned} (J_{\varrho, \lambda, u+; \omega}^{\varsigma})(z) &= \int_u^z (z - \eta)^{\lambda-1} F_{\varrho, \lambda}^{\varsigma}[\omega(z - \eta)^{\varrho}] \varphi(\eta) d\eta, \quad (z > u), \\ (J_{\varrho, \lambda, v-; \omega}^{\varsigma})(z) &= \int_v^z (\eta - z)^{\lambda-1} F_{\varrho, \lambda}^{\varsigma}[\omega(\eta - z)^{\varrho}] \varphi(\eta) d\eta, \quad (z < v), \end{aligned} \quad (11)$$

where $\omega \in \mathbb{R}$ and $\lambda, \varrho \in \mathbb{R}^+$.

This paper is organized as follows. In Section 2, we will derive generalized fractional integral inequalities for MT-non-convex function. However, the last section is dedicated to establish generalized fractional integral inequalities for (p,q) convex function.

2. Fractional Integral Inequalities for MT-Non-Convex Function

The following lemma is useful to derive our main results.

Lemma 1 (see [22]). Let $\lambda, \varphi: I \subset \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a differentiable mapping on I° , $u, v \in I$ such that $u < v$. If $\varphi' \in L^1[u, v]$, $p > 0$, then we obtain

$$\begin{aligned}
& \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2F_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p - u^p)^{\varrho}] (v^p - u^p)^{\lambda}} \left[(J_{\varrho, \lambda, u^p+; \omega}^{\varsigma} \varphi \circ \phi)(v^p) + (J_{\varrho, \lambda, v^p-; \omega}^{\varsigma} \varphi \circ \phi)(u^p) \right] \\
&= \frac{(v^p - u^p)}{2pF_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p - u^p)^{\varrho}]} \int_0^1 \frac{(1-\eta)^{\lambda} F_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p - u^p)^{\varrho} (1-\eta)^{\varrho}]}{(1-\eta)v^p)^{(1/p)-1}} \varphi' \left((\eta u^p + (1-\eta)v^p)^{(1/p)} \right) d\eta \\
&+ \int_0^1 \frac{\eta^{\lambda} F_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p - u^p)^{\varrho} \eta^{\varrho}]}{(\eta u^p + (1-\eta)v^p)^{(1/p)-1}} \varphi' \left((\eta u^p + (1-\eta)v^p)^{(1/p)} \right) d\eta,
\end{aligned} \tag{12}$$

where $\phi(c) = c^{(1/p)}$.

Theorem 1. Let $\lambda, \varrho \in \mathbb{R}^+$; $\varphi: I \subseteq \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a differentiable mapping on I° , $u, v \in I^{\circ}$ such that $u < v$. If $|\varphi'|$ is MT-non-convex on $[u, v]$, $p > 0$, then we obtain

$$\begin{aligned}
& \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2F_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p - u^p)^{\varrho}] (v^p - u^p)^{\lambda}} \left[(J_{\varrho, \lambda, u^p+; \omega}^{\sigma} \varphi \circ \phi)(v^p) + (J_{\varrho, \lambda, v^p-; \omega}^{\varsigma} \varphi \circ \phi)(u^p) \right] \\
&\leq \left(F_{\varrho, \lambda+1}^{C_1} [|\omega|(v^p - u^p)^{\varrho}] + F_{\varrho, \lambda+1}^{C_3} [|\omega|(v^p - u^p)^{\varrho}] \right) |\varphi'(u)| \\
&+ \left(F_{\varrho, \lambda+1}^{C_2} [|\omega|(v^p - u^p)^{\varrho}] + F_{\varrho, \lambda+1}^{C_4} [|\omega|(v^p - u^p)^{\varrho}] \right) |\varphi'(v)|,
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
C_1 &= \varsigma(k) v^{1-p} \frac{\varrho k + \lambda - (1/2)}{3(\varrho k + \lambda + 1)} {}_2F_1 \left(1 - \frac{1}{p}, \frac{3}{2}; \varrho k + \lambda + 2, 1 - \frac{u^p}{v^p} \right), \\
C_2 &= \varsigma(k) v^{1-p} \frac{\varrho k + \lambda + (1/2)}{\varrho k + \lambda + 1} {}_2F_1 \left(1 - \frac{1}{p}, \frac{1}{2}; \varrho k + \lambda + 2, 1 - \frac{u^p}{v^p} \right), \\
C_3 &= \varsigma(k) v^{1-p} \frac{(5/4)}{(\varrho k + \lambda + (3/2))(\varrho k + \lambda + 1)} {}_2F_1 \left(1 - \frac{1}{p}, \varrho k + \lambda + \frac{3}{2}; \varrho k + \lambda + 2, 1 - \frac{u^p}{v^p} \right), \\
C_4 &= \varsigma(k) v^{1-p} \frac{1}{4(\varrho k + \lambda + (1/2))(\varrho k + \lambda + 1)} {}_2F_1 \left(1 - \frac{1}{p}, \varrho k + \lambda + \frac{1}{2}; \varrho k + \lambda + 2, 1 - \frac{u^p}{v^p} \right).
\end{aligned} \tag{14}$$

Proof. Employing Lemma 1 and definition of MT-non-convexity of $|\varphi'|$, we obtain

$$\begin{aligned}
& \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2F_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p - u^p)^{\varrho}] (v^p - u^p)^{\lambda}} \left[(J_{\varrho, \lambda, u^p+; \omega}^{\varsigma} \varphi \circ \phi)(v^p) + (J_{\varrho, \lambda, v^p-; \omega}^{\varsigma} \varphi \circ \phi)(u^p) \right] \\
&= \frac{(v^p - u^p)}{2pF_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p - u^p)^{\varrho}]} \left[\int_0^1 (1 - \eta^{\lambda}) F_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p - u^p)^{\varrho} (1 - \eta)^{\varrho}] \right. \\
&\quad \left. \left(\eta u^p + (1 - \eta)v^p \right) \frac{1}{p} - 1 \varphi' \left(\eta u^p + (1 - \eta)v^p \right) \frac{1}{p} \right] d\eta \\
&\quad + \int_0^1 \eta^{\lambda} F_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p - u^p)^{\varrho} \eta^{\varrho}] \times (\eta u^p + (1 - \eta)v^p)^{(1/p)-1} \varphi' \left((\eta u^p + (1 - \eta)v^p)^{(1/p)} \right) d\eta \Big| \\
&\leq \frac{v^p - u^p}{2pF_{\varrho, \lambda+1}^{\varsigma} [|\omega|(v^p - u^p)^{\varrho}]} \sum_{k=0}^{\infty} \frac{\varsigma(k) |\omega|^k (v^p - u^p)^{\varrho k}}{\Gamma(\varrho k + \lambda + 1)} \\
&\quad \times \int_0^1 \left[|(1 - \eta)^{\varrho k + \lambda} + \eta^{\varrho k + \lambda}| \right] (\eta u^p + (1 - \eta)v^p)^{(1/p)-1} + \left[\frac{\sqrt{\eta}}{2\sqrt{1 - \eta}} |\varphi'(u)| \frac{\sqrt{1 - \eta}}{2\sqrt{\eta}} |\varphi'(v)| \right] d\eta \\
&\leq \frac{v^p - u^p}{2pF_{\varrho, \lambda+1}^{\varsigma} [|\omega|(v^p - u^p)^{\varrho}]} \sum_{k=0}^{\infty} \frac{\varsigma(k) |\omega|^k (v^p - u^p)^{\varrho k}}{\Gamma(\varrho k + \lambda + 1)} \\
&\quad \times \left[\int_0^1 (1 - \eta)^{\varrho k + \lambda} (\eta u^p + (1 - \eta)v^p)^{(1/p)-1} \left[\frac{\sqrt{\eta}}{2\sqrt{1 - \eta}} |\varphi'(u)| \frac{\sqrt{1 - \eta}}{2\sqrt{\eta}} |\varphi'(v)| \right] d\eta \right. \\
&\quad \left. + \int_0^1 \eta^{\varrho k + \lambda} (\eta u^p + (1 - \eta)v^p)^{(1/p)-1} \right. \\
&\quad \left. \cdot \left[\frac{\sqrt{\eta}}{2\sqrt{1 - \eta}} |\varphi'(u)| \left(\frac{\sqrt{1 - \eta}}{2\sqrt{\eta}} |\varphi'(v)| \right) d\eta \right] \right] \\
&\left| \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2F_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p - u^p)^p] (v^p - u^p)^{\lambda}} \left[J_{\varrho, \lambda, u^p+; \omega}^{\varsigma} \varphi \circ \phi)(v^p) (J_{\varrho, \lambda, v^p-; \omega}^{\varsigma} \varphi \circ \phi)(u^p) \right] \right| \\
&\leq \frac{v^p - u^p}{2pF_{\varrho, \lambda+1}^{\varsigma} [|\omega|(v^p - u^p)^p]} \sum_{k=0}^{\infty} \frac{\varsigma(k) |\omega|^k (v^p - u^p)^{\varrho k}}{\Gamma(\varrho k + \lambda + 1)} \\
&\quad \times \left[\int_0^1 \frac{1}{2} \eta^{(1/2)} (1 - \eta)^{\varrho k + \lambda - (1/2)} |\varphi'(u)| (\eta u^p + (1 - \eta)v^p)^{(1/p)-1} d\eta \right. \\
&\quad + \int_0^1 \frac{1}{2} \eta^{-(1/2)} (1 - \eta)^{\varrho k + \lambda + (1/2)} |\varphi'(v)| (\eta u^p + (1 - \eta)v^p)^{(1/p)-1} d\eta \\
&\quad + \int_0^1 \frac{1}{2} \eta^{\varrho k + \lambda + (1/2)} (1 - \eta)^{-(1/2)} |\varphi'(u)| (\eta u^p + (1 - \eta)v^p)^{(1/p)-1} d\eta \\
&\quad \left. + \int_0^1 \frac{1}{2} \eta^{\varrho k + \lambda - (1/2)} (1 - \eta)^{(1/2)} |\varphi'(v)| (\eta u^p + (1 - \eta)v^p)^{(1/p)-1} d\eta \right].
\end{aligned} \tag{15}$$

So,

$$\begin{aligned}
& \left| \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2F_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p - u^p)^p]} (v^p - u^p)^{\lambda} \left[J_{\varrho, \lambda, u^p+; \omega}^{\varsigma} \varphi \circ \phi \right] (v^p) \left(J_{\varrho, \lambda, v^p-; \omega}^{\varsigma} \varphi \circ \phi \right) (u^p) \right| \\
& \leq \frac{v^p - u^p}{2pF_{\varrho, \lambda+1}^{\varsigma} [|\omega|(v^p - u^p)^p]} \sum_{k=0}^{\infty} \frac{\varsigma(k) |\omega|^k (v^p - u^p)^{\varrho k}}{\Gamma(\varrho k + \lambda + 1)} \\
& \quad \times \left[v^{1-p} \frac{1}{2} \beta\left(\frac{3}{2}, \varrho k + \lambda + \frac{1}{2}\right) {}_2F_1\left(1 - \frac{1}{p}, \frac{3}{2}; \varrho k + \lambda + 2, 1 - \frac{u^p}{v^p}\right) |\varphi'(u)| \right. \\
& \quad + v^{1-p} \frac{1}{2} \beta\left(\frac{1}{2}, \varrho k + \lambda + \frac{3}{2}\right) {}_2F_1\left(1 - \frac{1}{p}, \frac{1}{2}; \varrho k + \lambda + 2, 1 - \frac{u^p}{v^p}\right) |\varphi'(v)| \\
& \quad + v^{1-p} \frac{1}{2} \beta\left(\varrho k + \lambda + \frac{3}{2}, \frac{1}{2}\right) {}_2F_1\left(1 - \frac{1}{p}, \varrho k + \lambda + \frac{3}{2}; \varrho k + \lambda + 2, 1 - \frac{u^p}{v^p}\right) |\varphi'(u)| \\
& \quad \left. + v^{1-p} \frac{1}{2} \beta\left(\varrho k + \lambda + \frac{1}{2}, \frac{3}{2}\right) {}_2F_1\left(1 - \frac{1}{p}, \varrho k + \lambda + \frac{1}{2}; \varrho k + \lambda + 2, 1 - \frac{u^p}{v^p}\right) |\varphi'(v)| \right].
\end{aligned} \tag{16}$$

From here,

$$\begin{aligned}
& \left| \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2F_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p - u^p)^p]} (v^p - u^p)^{\lambda} \left[J_{\varrho, \lambda, u^p+; \omega}^{\varsigma} \varphi \circ \phi \right] (v^p) + \left(J_{\varrho, \lambda, v^p-; \omega}^{\varsigma} \varphi \circ \phi \right) (u^p) \right| \\
& \leq \frac{v^p - u^p}{2pF_{\varrho, \lambda+1}^{\varsigma} [|\omega|(v^p - u^p)^p]} \sum_{k=0}^{\infty} \frac{\varsigma(k) |\omega|^k (v^p - u^p)^{\varrho k}}{\Gamma(\varrho k + \lambda + 1)} \\
& \quad \times \left[v^{1-p} \frac{\varrho k + \lambda - (1/2)}{3(\varrho k + \lambda + 1)} \left[{}_2F_1\left(1 - \frac{1}{p}, \frac{3}{2}; \varrho k + \lambda + 2, 1 - \frac{u^p}{v^p}\right) \right] |\varphi'(u)| \right. \\
& \quad + v^{1-p} \frac{\varrho k + \lambda + (1/2)}{\varrho k + \lambda + 1} \left[{}_2F_1\left(1 - \frac{1}{p}, \frac{1}{2}; \varrho k + \lambda + 2, 1 - \frac{u^p}{v^p}\right) \right] |\varphi'(v)| \\
& \quad + v^{1-p} \frac{(5/4)}{(\varrho k + \lambda + (3/2))(\varrho k + \lambda + 1)} \left[{}_2F_1\left(1 - \frac{1}{p}, \varrho k + \lambda + \frac{3}{2}; \varrho k + \lambda + 2, 1 - \frac{u^p}{v^p}\right) \right] |\varphi'(u)| \\
& \quad \left. + v^{1-p} \frac{1}{4(\varrho k + \lambda + (1/2))(\varrho k + \lambda + 1)} \left[{}_2F_1\left(1 - \frac{1}{p}, \varrho k + \lambda + \frac{1}{2}; \varrho k + \lambda + 2, 1 - \frac{u^p}{v^p}\right) \right] |\varphi'(v)| \right].
\end{aligned} \tag{17}$$

Simple calculations yield equation (13).

□ *Remark 2.* In Theorem 1, we see the following:

- (1) For $p = 1$, we have the inequality for MT-convex function:

$$\begin{aligned} & \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2F_{\varrho, \lambda+1}^{\varsigma} [|\omega|(v-u)^{\varrho}] (v-u)^{\lambda}} \left[v \left(J_{\varrho, \lambda, u+; \omega}^{\varsigma} \varphi \circ \phi \right) + u \left(J_{\varrho, \lambda, v-; \omega}^{\varsigma} \varphi \circ \phi \right) \right] \\ & \leq \left(F_{\varrho, \lambda+1}^{C_1} [|\omega|(v-u)^{\varrho}] + F_{\varrho, \lambda+1}^{C_3} [|\omega|(v-u)^{\varrho}] \right) |\varphi'(u)| \\ & \quad + \left(F_{\varrho, \lambda+1}^{C_2} [|\omega|(v-u)^{\varrho}] + F_{\varrho, \lambda+1}^{C_4} [|\omega|(v-u)^{\varrho}] \right) |\varphi'(v)|. \end{aligned} \quad (18)$$

- (2) For $p = -1$, we have the inequality of harmonically MT-convex function:

$$\begin{aligned} & \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2F_{\varrho, \lambda+1}^{\varsigma} [|\omega|(u-v/uv)^{\varrho}] (u-v/uv)^{\lambda}} \left[\frac{1}{v} \left(J_{\varrho, \lambda, (1/u)+; \omega}^{\varsigma} \varphi \circ \phi \right) + \frac{1}{u} \left(J_{\varrho, \lambda, (1/v)-; \omega}^{\varsigma} \varphi \circ \phi \right) \right] \\ & \leq \left(F_{\varrho, \lambda+1}^{C_1} \left[|\omega| \left(\frac{u-v}{uv} \right)^{\varrho} \right] + F_{\varrho, \lambda+1}^{C_3} \left[|\omega| \left(\frac{u-v}{uv} \right)^{\varrho} \right] \right) |\varphi'(u)| \\ & \quad + \left(F_{\varrho, \lambda+1}^{C_2} \left[|\omega| \left(\frac{u-v}{uv} \right)^{\varrho} \right] + F_{\varrho, \lambda+1}^{C_4} \left[|\omega| \left(\frac{u-v}{uv} \right)^{\varrho} \right] \right) |\varphi'(v)|. \end{aligned} \quad (19)$$

Theorem 2. Let $\lambda, \varrho \in \mathbb{R}^+$, $\varphi: I \subset \mathbb{R} \rightarrow \mathbb{R}$, be a MT-non-convex function on $u, v, \varrho \in I$ such that $u < v$. If $\varphi \in L[u, v]$, $p > 0$, then we get

$$\begin{aligned} \varphi \left(\frac{1}{2} u^p + \frac{1}{2} v^p \right)^{(1/p)} & \leq \frac{1}{2F_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p-u^p)^{\varrho} (v^p-u^p)^{\lambda}]} \left[J_{\varrho, \lambda, u^p+; \omega}^{\varsigma} \varphi \circ \phi(v^p) + J_{\varrho, \lambda, v^p-; \omega}^{\varsigma} \varphi \circ \phi(u^p) \right] \\ & \leq \frac{1}{4\sqrt{\eta}\sqrt{1-\eta}} \varphi(u) + \frac{1}{4\sqrt{\eta}\sqrt{1-\eta}} \varphi'(v), \end{aligned} \quad (20)$$

where $\phi(c) = c^{(1/p)}$.

and substituting $c = (\xi u^p + (1-\eta)v^p)^{(1/p)}$ and $d = ((1-\eta)u^p + \eta v^p)^{(1/p)}$, equation (21) yields

Proof. Since φ is MT-non-convex on $[u, v]$, for all $c, d \in [u, v]$,

$$\varphi \left(\frac{1}{2} c^p + \frac{1}{2} d^p \right)^{(1/p)} \leq \frac{\varphi(c) + \varphi(d)}{2}, \quad (21)$$

$$\varphi \left(\frac{1}{2} u^p + \frac{1}{2} v^p \right)^{(1/p)} \leq \varphi(\eta u^p + (1-\eta)v^p)^{(1/p)} + \varphi((1-\eta)u^p + \eta v^p)^{(1/p)}. \quad (22)$$

Multiply inequality equation (22) by $\eta^{\lambda-1} F_{\varrho, \lambda}^{\varsigma} [\omega(v^p-u^p)^{\varrho} \eta^{\varrho}]$, and after that, integrating it over $\eta \in [0, 1]$, then we get

$$\begin{aligned}
2F_{\varrho,\lambda+1}^{\varsigma} \left[\omega (v^p - u^p)^{\varrho} \eta^{\varrho} \right] \varphi \left(\frac{1}{2} u^p + \frac{1}{2} v^p \right)^{(1/p)} &\leq \int_0^1 \eta^{\lambda-1} F_{\varrho,\lambda}^{\varsigma} \left[\omega (v^p - u^p)^{\varrho} \eta^{\varrho} \right] \varphi (\eta u^p + (1-\eta) v^p)^{(1/p)} d\eta \\
&+ \int_0^1 \eta^{\lambda-1} F_{\varrho,\lambda}^{\varsigma} \left[\omega (v^p - u^p)^{\varrho} \eta^{\varrho} \right] \varphi ((1-\eta) u^p + \eta v^p)^{(1/p)} d\eta \\
&= \int_0^1 \eta^{\lambda-1} \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k (v^p - u^p)^{\varrho k} \eta^{\varrho k}}{\Gamma(\varrho k + \lambda)} \varphi (\eta u^p + (1-\eta) v^p)^{(1/p)} d\eta \\
&+ \int_0^1 \eta^{\lambda-1} \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k (v^p - u^p)^{\varrho k} \eta^{\varrho k}}{\Gamma(\varrho k + \lambda)} \varphi ((1-\eta) u^p + \eta v^p)^{(1/p)} d\eta \\
&= \int_0^1 \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k}{\Gamma(\varrho k + \lambda)} \eta^{\varrho k + \lambda - 1} \varphi (\eta u^p + (1-\eta) v^p)^{(1/p)} (v^p - u^p)^{\varrho k} d\eta \\
&+ \int_0^1 \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k}{\Gamma(\varrho k + \lambda)} \eta^{\varrho k + \lambda - 1} \varphi ((1-\eta) u^p + \eta v^p)^{(1/p)} (v^p - u^p)^{\varrho k} d\eta \\
&= \int_{u^p}^{v^p} \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k}{\Gamma(\varrho k + \lambda)} \frac{(v^p - c)^{\varrho k} (v^p - c)^{\lambda-1}}{(v^p - u^p)^{\varrho k} (v^p - u^p)^{\lambda-1}} (v^p - u^p)^{\varrho k} \varphi(c^{(1/p)}) \frac{dc}{v^p - u^p} \quad (23) \\
&+ \int_{u^p}^{v^p} \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k}{\Gamma(\varrho k + \lambda)} \frac{(c - u^p)^{\varrho k} (c - u^p)^{\lambda-1}}{(v^p - u^p)^{\varrho k} (v^p - u^p)^{\lambda-1}} (v^p - u^p)^{\varrho k} \varphi(c^{(1/p)}) \frac{dc}{v^p - u^p} \\
&= \frac{1}{(v^p - u^p)^{\lambda}} \int_{u^p}^{v^p} \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k (v^p - c)^{\varrho k}}{\Gamma(\varrho k + \lambda)} (v^p - c)^{\lambda-1} \varphi(c^{(1/p)}) dc \\
&+ \frac{1}{(v^p - u^p)^{\lambda}} \int_{u^p}^{v^p} \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k (c - u^p)^{\varrho k}}{\Gamma(\varrho k + \lambda)} (c - u^p)^{\lambda-1} \varphi(c^{(1/p)}) dc \\
&= \frac{1}{(v^p - u^p)^{\lambda}} \left(J_{\varrho,\lambda,u^p+;\omega}^{\varsigma} \varphi \circ \phi \right) (v^p) \\
&+ \frac{1}{(v^p - u^p)^{\lambda}} \left(J_{\varrho,\lambda,v^p-;\omega}^{\varsigma} \varphi \circ \phi \right) (u^p) \\
2F_{\varrho,\lambda+1}^{\varsigma} \left[\omega (v^p - u^p)^p \right] \varphi \left(\frac{1}{2} u^p + \frac{1}{2} v^p \right)^{(1/p)} &\leq \frac{1}{(v^p - u^p)^{\lambda}} \left[\left(J_{\varrho,\lambda,u^p+;\omega}^{\varsigma} \varphi \circ \phi \right) (v^p) + \left(J_{\varrho,\lambda,v^p-;\omega}^{\varsigma} \varphi \circ \phi \right) (u^p) \right],
\end{aligned}$$

which is left side of inequality equation (23). Now we have to prove right-hand side of inequality equation (22); applying definition of MT-non-convexity of φ ,

$$\varphi (\eta u^p + (1-\eta) v^p)^{(1/p)} \leq \frac{\sqrt{\eta}}{2\sqrt{1-\eta}} \varphi (u) + \frac{\sqrt{1-\eta}}{2\sqrt{\eta}} \varphi (v),$$

(24)

and

$$\varphi ((1-\eta) u^p + \eta v^p)^{(1/p)} \leq \frac{\sqrt{\eta}}{2\sqrt{1-\eta}} \varphi (u) + \frac{\sqrt{1-\eta}}{2\sqrt{\eta}} \varphi (v), \quad (25)$$

and by adding equations (24) and (25), we get

$$\begin{aligned} & \varphi(\eta u^p + (1-\eta)v^p)^{(1/p)} + \varphi((1-\eta)u^p + \eta v^p)^{(1/p)} \\ & \leq \frac{1}{2\sqrt{\eta}\sqrt{1-\eta}} \varphi(u) + \frac{1}{2\sqrt{\eta}\sqrt{1-\eta}} \varphi(v). \end{aligned} \quad (26)$$

Multiply inequality equation (26) by $\eta^{\lambda-1} F_{\varrho,\lambda}^{\varsigma}[\omega(v^p - u^p)^{\varrho} \eta^{\varrho}]$, and after that, integrating it over $\eta \in [0, 1]$, then we get

$$\begin{aligned} & \frac{1}{(v^p - u^p)^{\lambda}} \left[(J_{\varrho,\lambda,u^p+;\omega}^{\varsigma} \varphi \circ \phi)(v^p) + (J_{\varrho,\lambda,v^p-;\omega}^{\varsigma} \varphi \circ \phi)(u^p) \right] \\ & \leq 2F_{\varrho,\lambda+1}^{\sigma} [\omega(v^p - u^p)^{\varrho}] \frac{1}{4\sqrt{\eta}\sqrt{1-\eta}} \varphi(u) + \frac{1}{4\sqrt{\eta}\sqrt{1-\eta}} \varphi(v). \end{aligned} \quad (27)$$

Combining equations (23) and (27) completes equation (20). $\square$

(1) For $p = 1$, we have inequality for MT-convex function:

Remark 3. In Theorem 2, we see the following:

$$\begin{aligned} & \varphi\left(\frac{1}{2}u + \frac{1}{2}v\right), \\ & \leq \frac{1}{2F_{\varrho,\lambda+1}^{\varsigma} [\omega(v-u)^{\varrho} (v-u)^{\lambda}]} \left[(v)J_{\varrho,\lambda,u+;\omega}^{\varsigma} (\varphi \circ \phi) + (u)J_{\varrho,\lambda,v-;\omega}^{\varsigma} (\varphi \circ \phi) \right], \\ & \leq \frac{1}{4\sqrt{\eta}\sqrt{1-\eta}} \varphi(u) + \frac{1}{4\sqrt{\eta}\sqrt{1-\eta}} \varphi(v). \end{aligned} \quad (28)$$

(2) For $p = -1$, we have the inequality of harmonically MT-convex function:

$$\begin{aligned} & \varphi\left(\frac{u+v}{2uv}\right), \\ & \leq \frac{1}{2F_{\varrho,\lambda+1}^{\varsigma} [\omega(u-v/uv)^{\varrho} (u-v/uv)^{\lambda}]} \left[\left(\frac{1}{v}\right)J_{\varrho,\lambda,(1/v)+;\omega}^{\varsigma} (\varphi \circ \phi) + \left(\frac{1}{u}\right)J_{\varrho,\lambda,(1/v)-;\omega}^{\varsigma} (\varphi \circ \phi) \right], \\ & \leq \frac{1}{4\sqrt{\eta}\sqrt{1-\eta}} \varphi(u) + \frac{1}{4\sqrt{\eta}\sqrt{1-\eta}} \varphi(v). \end{aligned} \quad (29)$$

3. Fractional Integral Inequalities for $(p-q)$ Convex Functions

In this section, we will develop fractional integral inequality for $(p-q)$ convex function.

Theorem 3. Let $\lambda, \epsilon, \mathbb{R}^+, \varphi: I, \subseteq \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, be a differentiable mapping on I° $u, v \in I^\circ$ such that $x < y$. If $|\varphi'|$ is $(p-q)$ convex on $[u, v]$, $p > 0$, then we obtain

$$\begin{aligned} & \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2F_{\varrho, \lambda+1}^\varsigma [\omega(v^p - u^p)^\varrho]} (v^p - u^p)^\lambda \left[(J_{\varrho, \lambda, u^p+; \omega}^\varsigma \varphi \circ \phi)(v^p) + (J_{\varrho, \lambda, v^p-; \omega}^\varsigma \varphi \circ \phi)(u^p) \right], \\ & \leq \left(F_{\varrho, \lambda+1}^{C_1} [|\omega|(v^p - u^p)^\varrho] + F_{\varrho, \lambda+1}^{C_3} [|\omega|(v^p - u^p)^\varrho] \right) |\varphi'(u)|, \\ & \quad + \left(F_{\varrho, \lambda+1}^{C_2} [|\omega|(v^p - u^p)^\varrho] + F_{\varrho, \lambda+1}^{C_4} [|\omega|(v^p - u^p)^\varrho] \right) |\varphi'(v)|, \end{aligned} \quad (30)$$

where

$$\begin{aligned} C_1 &= \varsigma(k) v^{1-p} \frac{\varrho k + \lambda}{((1/q) + 1)(\varrho k + \lambda + (1/q) + 1)_2} F_1 \left(1 - \frac{1}{p}, \frac{1}{q} + 1, \varrho k + \lambda + 1, 1 - \frac{u^p}{v^p} \right) \\ C_2 &= \varsigma(k) v^{1-p} \frac{\varrho k + \lambda + (1/q)}{\varrho k + \lambda + (1/q) + 1_2} F_1 \left(1 - \frac{1}{p}, \varrho k + \lambda + \frac{1}{q} + 1, \varrho k + \lambda + \frac{1}{q} + 1, 1 - \frac{u^p}{v^p} \right) \\ C_3 &= \varsigma(k) v^{1-p} \frac{1}{\varrho k + \lambda + (1/q) + 1_2} F_1 \left(1 - \frac{1}{p}, \varrho k + \lambda + \frac{1}{q} + 1, 1, 1 - \frac{u^p}{v^p} \right) \\ C_4 &= \varsigma(k) v^{1-p} \frac{(1/q)}{(\varrho k + \lambda + 1)(\varrho k + \lambda + (1/q) + 1)_2} F_1 \left(1 - \frac{1}{p}, 1, \frac{1}{q} - 1, 1 - \frac{u^p}{v^p} \right). \end{aligned} \quad (31)$$

Proof. By making use of Lemma 1 and $(p-q)$ -convexity of $|\varphi'|$, we obtain

$$\begin{aligned} & \left| \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2(v^p - u^p)^\lambda F_{\varrho, \lambda+1}^\varsigma [\omega(v^p - u^p)^\varrho]} \left[(J_{\varrho, \lambda, u^p+; \omega}^\varsigma \varphi \circ \phi)(v^p) + (J_{\varrho, \lambda, v^p-; \omega}^\varsigma \varphi \circ \phi)(u^p) \right] \right| \\ &= \left| \frac{v^p - u^p}{2p F_{\varrho, \lambda+1}^\varsigma [\omega(v^p - u^p)^\varrho]} \int_0^1 \frac{(1-\eta)^\lambda F_{\varrho, \lambda+1}^\varsigma [\omega(v^p - u^p)^\varrho (1-\eta)^\varrho]}{\times (\eta u^p + (1-\eta)v^p)^{(1/p)-1}} \varphi'(\eta u^p + (1-\eta)v^p)^{(1/p)} d\eta \right| \\ & \quad + \left| \int_0^1 \frac{\eta^\lambda F_{\varrho, \lambda+1}^\varsigma [\omega(v^p - u^p)^\varrho \eta^\varrho]}{\times (\eta u^p + (1-\eta)v^p)^{(1/p)-1}} \varphi'(\eta u^p + (1-\eta)v^p)^{(1/p)} d\eta \right| \\ & \leq \frac{v^p - u^p}{2p F_{\varrho, \lambda+1}^\varsigma [\omega(v^p - u^p)^\varrho]} \sum_{k=0}^\infty \frac{\varsigma(k) |\omega|^k (v^p - u^p)^{\varrho k}}{\Gamma(\varrho k + \lambda + 1)} \\ & \quad \times \int_0^1 [(1-\eta)^{\varrho k + \lambda} + \eta^{\varrho k + \lambda}] (\eta u^p + (1-\eta)v^p)^{(1/p)-1} (\eta |\varphi'(u)^q| + (1-\eta) |\varphi'(v)^q|)^{(1/q)} d\eta. \end{aligned} \quad (32)$$

Moreover, we observe that

$$\begin{aligned}
 &\leq \frac{v^p - u^p}{2pF_{\varrho, \lambda+1}^{\varsigma} [|\omega|(v^p - u^p)^{\varrho}] |} \sum_{k=0}^{\infty} \frac{\varsigma(k) |\omega|^k (v^p - u^p)^{\varrho k}}{\Gamma(\varrho k + \lambda + 1)} \\
 &\quad \times \left[\int_0^1 (1-\eta)^{\varrho k + \lambda} (\eta u^p + (1-\eta)v^p)^{(1/p)-1} \left(\eta |\varphi'(u)^q| + (1-\eta) |\varphi'(v)^q| \right)^{(1/q)} d\eta \right. \\
 &\quad \left. + \int_0^1 \eta^{\varrho k + \lambda} (\eta u^p + (1-\eta)v^p)^{(1/p)-1} \left(\eta |\varphi'(u)^q| + (1-\eta) |\varphi'(v)^q| \right)^{(1/q)} d\eta \right],
 \end{aligned} \tag{33}$$

so that

$$\begin{aligned}
 &\left| \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2(v^p - u^p)^{\lambda} F_{\varrho, \lambda+1}^{\varsigma} [\omega(v^p - u^p)^{\varrho}]} \left[(J_{\varrho, \lambda, u^p+; \omega}^{\varsigma} \varphi \circ \phi)(v^p) + (J_{\varrho, \lambda, v^p-; \omega}^{\varsigma} \varphi \circ \phi)(u^p) \right] \right| \\
 &\leq \frac{v^p - u^p}{2pF_{\varrho, \lambda+1}^{\varsigma} [|\omega|(v^p - u^p)^{\varrho}] |} \sum_{k=0}^{\infty} \frac{\varsigma(k) |\omega|^k (v^p - u^p)^{\varrho k}}{\Gamma(\varrho k + \lambda + 1)} \\
 &\quad \times \left[\int_0^1 \eta^{(1/q)} (1-\eta)^{\varrho k + \lambda} |\varphi'(u)^q|^{(1/q)} (\eta u^p + (1-\eta)v^p)^{(1/p)-1} d\eta \right. \\
 &\quad + \int_0^1 (1-\eta)^{\varrho k + \lambda + (1/q)} |\varphi'(v)^q|^{(1/q)} (\eta u^p + (1-\eta)v^p)^{(1/p)-1} d\eta \\
 &\quad + \int_0^1 \eta^{\varrho k + \lambda + (1/q)} |\varphi'(u)^q|^{(1/q)} (\eta u^p + (1-\eta)v^p)^{(1/p)-1} d\eta \\
 &\quad \left. + \int_0^1 \eta^{\varrho k + \lambda} (1-\eta)^{(1/q)} |\varphi'(v)^q|^{(1/q)} (\eta u^p + (1-\eta)v^p)^{(1/p)-1} d\eta \right].
 \end{aligned} \tag{34}$$

Now

$$\begin{aligned}
& \left| \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2(v^p - u^p)^\lambda F_{\varrho, \lambda+1}^\varsigma [\omega(v^p - u^p)^\varrho]} \left[(J_{\varrho, \lambda, u^p+; \omega}^\varsigma \varphi \circ \phi)(v^p) + (J_{\varrho, \lambda, v^p-; \omega}^\varsigma \varphi \circ \phi)(u^p) \right] \right| \\
& \leq \frac{v^p - u^p}{2p F_{\varrho, \lambda+1}^\varsigma [|\omega|(v^p - u^p)^\varrho]} \sum_{k=0}^{\infty} \frac{\varsigma(k) |\omega|^k (v^p - u^p)^{\varrho k}}{\Gamma(\varrho k + \lambda + 1)} \\
& \quad \times \left[v^{1-p} \beta \left(\frac{1}{q} + 1, \varrho k + \lambda + 1 \right) {}_2F_1 \left(1 - \frac{1}{p}, \frac{1}{q} + 1, \varrho k + \lambda + \frac{1}{q} + 2, 1 - \frac{u^p}{v^p} \right) |\varphi'(u)| \right. \\
& \quad + v^{1-p} \beta \left(1, \varrho k + \lambda + \frac{1}{q} + 1 \right) {}_2F_1 \left(1 - \frac{1}{p}, 1, \varrho k + \lambda + \frac{1}{q} + 2, 1 - \frac{u^p}{v^p} \right) |\varphi'(v)| \\
& \quad + v^{1-p} \beta \left(\varrho k + \lambda + \frac{1}{q} + 1, 1 \right) {}_2F_1 \left(1 - \frac{1}{p}, \varrho k + \lambda + \frac{1}{q} + 1, \varrho k + \lambda + \frac{1}{q} + 2, 1 - \frac{1}{p} \right) |\varphi'(u)| \\
& \quad \left. + v^{1-p} \beta \left(\varrho k + \lambda + 1, \frac{1}{q} + 1 \right) {}_2F_1 \left(1 - \frac{1}{p}, \varrho k + \lambda + 1, \varrho k + \lambda + \frac{1}{q} + 2, 1 - \frac{u^p}{v^p} \right) |\varphi'(v)| \right].
\end{aligned} \tag{35}$$

From here,

$$\begin{aligned}
& \left| \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2(v^p - u^p)^\lambda F_{\varrho, \lambda+1}^\varsigma [\omega(v^p - u^p)^\varrho]} \left[(J_{\varrho, \lambda, u^p+; \omega}^\varsigma \varphi \circ \phi)(v^p) + (J_{\varrho, \lambda, v^p-; \omega}^\varsigma \varphi \circ \phi)(u^p) \right] \right| \\
& \leq \frac{v^p - u^p}{2p F_{\varrho, \lambda+1}^\varsigma [|\omega|(v^p - u^p)^\varrho]} \sum_{k=0}^{\infty} \frac{\varsigma(k) |\omega|^k (v^p - u^p)^{\varrho k}}{\Gamma(\varrho k + \lambda + 1)} \\
& \quad \times \left[v^{1-p} \left(\frac{\varrho k + \lambda}{((1/q) + 1)(\varrho k + \lambda + (1/q) + 1)} \right) {}_2F_1 \left(1 - \frac{1}{p}, \frac{1}{q} + 1, \varrho k + \lambda + \frac{1}{q} + 2, 1 - \frac{u^p}{v^p} \right) |\varphi'(u)| \right. \\
& \quad + v^{1-p} \left(\frac{\varrho k + \lambda + (1/q)}{\varrho k + \lambda + (1/q) + 1} \right) {}_2F_1 \left(1 - \frac{1}{p}, 1, \varrho k + \lambda + \frac{1}{q} + 2, 1 - \frac{u^p}{v^p} \right) |\varphi'(v)| \\
& \quad + v^{1-p} \left(\frac{1}{\varrho k + \lambda + (1/q) + 1} \right) {}_2F_1 \left(1 - \frac{1}{p}, \varrho k + \lambda + \frac{1}{q} + 1, \varrho k + \lambda + \frac{1}{q} + 2, 1 - \frac{u^p}{v^p} \right) |\varphi'(u)| \\
& \quad \left. + v^{1-p} \left(\frac{(1/q)}{(\varrho k + \lambda + 1)(\varrho k + \lambda + (1/q) + 1)} \right) {}_2F_1 \left(1 - \frac{1}{p}, \varrho k + \lambda + 1, \varrho k + \lambda + \frac{1}{q} + 2, 1 - \frac{u^p}{v^p} \right) |\varphi'(v)| \right],
\end{aligned} \tag{36}$$

and

$$\left| \frac{\varphi(u) + \varphi(v)}{2} - \frac{1}{2(v^p - u^p)^\lambda F_{\varrho, \lambda+1}^\varsigma [\omega(v^p - u^p)^\varrho]} \left[(J_{\varrho, \lambda, u^p+; \omega}^\varsigma \varphi \circ \phi)(v^p) + (J_{\varrho, \lambda, v^p-; \omega}^\varsigma \varphi \circ \phi)(u^p) \right] \right|$$

$$\leq \frac{v^p - u^p}{2p F_{\varrho, \lambda+1}^\varsigma [\omega(v^p - u^p)^\varrho]} \sum_{k=0}^{\infty} \frac{\varsigma(k) |\omega|^k (v^p - u^p)^{\varrho k}}{\Gamma(\varrho k + \lambda + 1)} (C_1 |\varphi'(u)| + C_2 |\varphi'(v)| + C_3 |\varphi'(u)| + C_4 |\varphi'(v)|),$$
(37)

which is the required solution. $\square$

(2) Similarly, for $q = 1$ and $p = 1$ in equation (30), we get classical fractional integral of Hermite–Hadamard inequality.

Remark 4

(1) If one puts $q = 1$ in equation (30), one has Theorem 5 in [22].

Theorem 4. Let $\lambda \in \mathbb{R}^+$, $\varphi: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$, be a $(p-q)$ convex function on $u, v, \in I$ with $u < v$; if $f \in L[u, v]$, $p > 0$, then we obtain

$$\varphi\left(\frac{1}{2}u^p + \frac{1}{2}v^p\right)^{(1/p)} \leq \frac{1}{2F_{\varrho, \lambda+1}^\varsigma [\omega(v^p - u^p)^\varrho (v^p - u^p)^\lambda]} \left[J_{\varrho, \lambda, u^p+; \omega}^\varsigma \varphi \circ \phi(v^p) + J_{\varrho, \lambda, v^p-; \omega}^\varsigma \varphi \circ \phi(u^p) \right]$$

$$\leq \frac{(\varphi(u)^q)^{(1/q)} + (\varphi(v)^q)^{(1/q)}}{2},$$
(38)

where $g(c) = c^{(1/p)}$.

and substituting $c = (\eta u^p + (1 - \eta)v^p)^{(1/p)}$ and $d = ((1 - \eta)u^p + \eta v^p)^{(1/p)}$, then equation (39) yields

Proof. Since φ is $(p-q)$ convex on $[u, v]$, for all $c, d \in [u, v]$,

$$\varphi\left(\frac{1}{2}c^p + \frac{1}{2}d^p\right)^{(1/p)} \leq \frac{1}{2}[\varphi(c)^q + \varphi(d)^q]^{(1/q)}, \quad (39)$$

$$2\varphi\left(\frac{1}{2}u^p + \frac{1}{2}v^p\right)^{(1/p)} \leq \varphi\left[\left((\eta u^p + (1 - \eta)v^p)^{(1/p)}\right)^q\right]^{(1/q)} + \varphi\left[\left((1 - \eta)u^p + \eta v^p\right)^{(1/p)}\right]^{(1/q)}. \quad (40)$$

Multiply inequality equation (40) by $\eta^{\lambda-1} F_{\varrho, \lambda}^\varsigma [\omega(v^p - u^p)^\varrho \eta^\varrho]$, and after that, integrating over $\eta \in [0, 1]$, we get

$$\begin{aligned}
& 2F_{\varrho, \lambda+1}^{\varsigma} \left[\omega (v^p - u^p)^{\varrho} \eta^{\varrho} \right] \varphi \left(\frac{1}{2} u^p + \frac{1}{2} v^p \right)^{(1/p)} \\
& \leq \int_0^1 \eta^{\lambda-1} F_{\varrho, \lambda}^{\varsigma} \left[\omega (v^p - u^p)^{\varrho} \eta^{\varrho} \right] \left[\varphi \left((\eta u^p + (1-\eta) v^p)^{(1/p)} \right)^q \right]^{(1/q)} d\eta \\
& \quad + \int_0^1 \eta^{\lambda-1} F_{\varrho, \lambda}^{\varsigma} \left[\omega (v^p - u^p)^{\varrho} \eta^{\varrho} \right] \left[\varphi \left(((1-\eta) u^p + \eta v^p)^{(1/p)} \right)^q \right]^{(1/q)} d\eta \\
& = \int_0^1 \eta^{\lambda-1} \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k (v^p - u^p)^{\varrho k} \eta^{\varrho k}}{\Gamma(\varrho k + \lambda)} \left[\varphi \left((\eta u^p + (1-\eta) v^p)^{(1/p)} \right)^q \right]^{(1/q)} d\eta \\
& \quad + \int_0^1 \eta^{\lambda-1} \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k (v^p - u^p)^{\varrho k} \eta^{\varrho k}}{\Gamma(\varrho k + \lambda)} \left[\varphi \left(((1-\eta) u^p + \eta v^p)^{(1/p)} \right)^q \right]^{(1/q)} d\eta \\
& = \int_0^1 \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k}{\Gamma(\varrho k + \lambda)} \eta^{\varrho k + \lambda - 1} \left[\varphi \left((\eta u^p + (1-\eta) v^p)^{(1/p)} \right)^q \right]^{(1/q)} (v^p - u^p)^{\varrho k} d\eta \\
& \quad + \int_0^1 \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k}{\Gamma(\varrho k + \lambda)} \eta^{\varrho k + \lambda - 1} \left[\varphi \left(((1-\eta) u^p + \eta v^p)^{(1/p)} \right)^q \right]^{(1/q)} (v^p - u^p)^{\varrho k} d\eta \\
& = \int_{u^p}^{v^p} \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k}{\Gamma(\varrho k + \lambda)} \frac{(v^p - c)^{\varrho k} (v^p - c)^{\lambda-1}}{(v^p - u^p)^{\varrho k} (v^p - u^p)^{\lambda-1}} (v^p - u^p)^{\varrho k} \left[\varphi \left(c^{(1/p)} \right)^q \right]^{(1/q)} \frac{dc}{v^p - u^p} \\
& \quad + \int_{u^p}^{v^p} \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k}{\Gamma(\varrho k + \lambda)} \frac{(c - u^p)^{\varrho k} (c - u^p)^{\lambda-1}}{(v^p - u^p)^{\varrho k} (v^p - u^p)^{\lambda-1}} (v^p - u^p)^{\varrho k} \left[\varphi \left(c^{(1/p)} \right)^q \right]^{(1/q)} \frac{dc}{v^p - u^p} \\
& = \frac{1}{(v^p - u^p)^{\lambda}} \int_{u^p}^{v^p} \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k (v^p - c)^{\varrho k}}{\Gamma(\varrho k + \lambda)} (v^p - c)^{\lambda-1} \left[\varphi \left(c^{(1/p)} \right)^q \right]^{(1/q)} dc \\
& \quad + \frac{1}{(v^p - u^p)^{\lambda}} \int_{u^p}^{v^p} \sum_{k=0}^{\infty} \frac{\varsigma(k) \omega^k (c - u^p)^{\varrho k}}{\Gamma(\varrho k + \lambda)} (c - u^p)^{\lambda-1} \left[\varphi \left(c^{(1/p)} \right)^q \right]^{(1/q)} dc \\
& = \frac{1}{(v^p - u^p)^{\lambda}} \left(J_{\varrho, \lambda, u^p+; \omega}^{\varsigma} \varphi \circ \phi \right) (v^p) \\
& \quad + \frac{1}{(v^p - u^p)^{\lambda}} \left(J_{\varrho, \lambda, v^p-; \omega}^{\varsigma} \varphi \circ \phi \right) (u^p).
\end{aligned} \tag{41}$$

So, we have left-hand side of inequality equation (38).

$$2F_{\varrho, \lambda+1}^{\varsigma} \left[\omega (v^p - u^p)^p \varphi \left(\frac{1}{2} u^p + \frac{1}{2} v^p \right)^{(1/p)} \right] \leq \frac{1}{(v^p - u^p)^{\lambda}} \left(J_{\varrho, \lambda, u^p+; \omega}^{\varsigma} \varphi \circ \phi \right) (v^p) + \left(J_{\varrho, \lambda, v^p-; \omega}^{\varsigma} \varphi \circ \phi \right) (u^p). \tag{42}$$

Now we have to prove other side of equation (40) from pq -convexity of φ .

and

$$\begin{aligned}
& \varphi \left(((1-\eta) u^p + \eta v^p)^{(1/p)} \right) \leq [(1-\eta) \varphi(u)^q + \eta \varphi(v)^q]^{(1/q)}, \\
& \varphi \left((\eta u^p + (1-\eta) v^p)^{(1/p)} \right) \leq [\eta \varphi(u)^q + (1-\eta) \varphi(v)^q]^{(1/q)}, \\
& \text{and by adding inequality equations (43) and (44), we get}
\end{aligned} \tag{43}$$

$$\begin{aligned} \varphi(\eta u^p + (1-\eta)v^p)^{(1/p)} + \varphi((1-\eta)u^p + \eta v^p)^{(1/p)} &\leq [\eta\varphi(u)^q + (1-\eta)\varphi(v)^q]^{(1/q)} + [(1-\eta)\varphi(u)^q + \eta\varphi(v)^q]^{(1/q)} \\ \varphi(\eta u^p + (1-\eta)v^p)^{(1/p)} + \varphi((1-\eta)u^p + \eta v^p)^{(1/p)} &\leq [\varphi(u)^q]^{(1/q)} + [\varphi(v)^q]^{(1/q)}. \end{aligned} \quad (45)$$

Multiply inequality equation (45) by $\eta^{\lambda-1} F_{\varrho,\lambda}^{\varsigma}[\omega(v^p - u^p)^{\varrho}\eta^{\varrho}]$, and after that, integrating inequality over $\eta \in [0, 1]$, we get

$$\frac{1}{(v^p - u^p)^{\lambda}} \left[\left(J_{\varrho,\lambda,u^p+;\omega}^{\varsigma} \varphi \circ \varphi \right) (v^p) + \left(J_{\varrho,\lambda,v^p-;\omega}^{\varsigma} \varphi \circ \varphi \right) (u^p) \right] \leq 2 F_{\varrho,\lambda+1}^{\varsigma} [\omega(v^p - u^p)^{\varrho}] \frac{[\varphi(u)^q]^{(1/q)} + [\varphi(v)^q]^{(1/q)}}{2}. \quad (46)$$

Combining equations (42) and (45) completes equation (38). $\square$

Remark 5

- (1) If one puts $q = 1$ in equation (39), one has [22, Theorem 5].
- (2) Similarly, for $q = 1$ and $p = 1$ in equation (39), we get classical fractional integral of H-H inequality.

4. Conclusion

Fractional integral inequalities are derived for MT-non-convex functions and $(p - q)$ convex functions. With the help of several lemmas, the integral inequalities are derived in generalized fractional integral operator. The remarks at the end are also given to verify the extension of results.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this study.

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