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Research Article

Fixed Point Results for Single and Multivalued Maps on Partial Extended b-Metric Spaces

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This article is based on the concept of partial extended b-metric spaces, which is inspired by the notions of new extended b-metric spaces and partial metric spaces. Fixed point results for single and multivalued mappings on such spaces are also presented. Few examples are also provided to elaborate the concepts.

1. Introduction and Preliminaries

Many problems in science and engineering defined by nonlinear equations can be solved by reducing them to an equivalent fixed point problem. Banach contraction principal provided a platform for researchers to prove the existence of fixed point of mappings on different abstract spaces. Fixed point theorems are developed for single-valued or set-valued mappings of abstract metric spaces. One can see a lot of literature on metric fixed point theory [1–9]. Recently, Shukla and Panicker [10] investigated some fixed point results on generalized enriched nonexpansive mappings defined on Banach spaces. These results are generalization of many existing results.

A partial metric space was introduced by Matthews [11, 12]. It is the generalization of a metric space, in which $d(x, x) = 0$ may be nonzero and $d(x, x) \leq d(x, y)$ for every $x, y \in X$. Matthews proved a fixed point theorem for contractive mappings in that space and also discussed its topological properties. After this development, many researchers worked in this direction (see [4, 9, 13–15]). Aydi et al. [16] initiated the idea of a partial Hausdorff metric space, and they proved fixed point theorems in partial metric spaces for multivalued mappings. A beautiful piece of work in this aspect can be seen in [17]. Authors used partial metric spaces by incorporating a strictly increasing F-mapping to introduce multivalued F-contractions. This paper extends some results in partial metric spaces. Shukla [18] presented the notion of a partial b-metric space and proved some related fixed point results. A lot of research can be seen in literature on b-metric spaces (see [1, 2, 7, 8, 19–23]), which is a stimulation towards the concept of extended b-metric spaces introduced by Kamran et al. [6] who proved the Banach contraction theorem in the setting of extended b-metric spaces. Afterwards, many authors have focused on the subject and generalized different results of metric spaces in extended b-metric spaces (see [3, 24–28]). Very recently, Aydi et al. [5] have introduced the concept of new extended b-metric spaces by replacing the triangle inequality of extended b-metric spaces with a modified functional inequality. In this setting, they proved few fixed point theorems for nonlinear contractive mappings. Motivated by the above-mentioned research, we initiate the idea of partial extended b-metric spaces and we extend the result of Kamran et al. [6] in setting of the partial extended b-metric spaces. We have also presented few examples to elaborate the idea.

To reach the goal of proving some fixed point results in the new notion, first we give some definitions.

Definition 1 [21]. Consider a nonempty set X with a real number $b \geq 1$. The function $d : X \times X \longrightarrow [0,\infty)$ is called a b-metric if it satisfies the following properties for each $x, y, z \in X$:

($b1$) $d(x,y) = 0 \Leftrightarrow x = y$

($b2$) $d(x,y) = d(y,x)$

($b3$) $d(x,y) \leq b[d(x,z) + d(z,y)]$

Here, the pair (X, d) is called a b-metric space.

In the following, we give the definition of a new extended b-metric space (in the sense of Aydi et al. [5]).

Definition 2. Let X be a nonempty set and given $\theta : X \times X \times X \longrightarrow [1,\infty)$. The function $d_\theta : X \times X \longrightarrow [0,\infty)$ is called a new extended b-metric if it satisfies the following conditions for all $x, y, z \in X$:

(1) $d_\theta(x,y) = 0 \Leftrightarrow x = y$

(2) $d_\theta(x,y) = d_\theta(y,x)$

(3) $d_\theta(x,z) \leq \theta(x,y,z)[d_\theta(x,y) + d_\theta(y,z)]$

The pair (X, d_θ) is called a new extended b-metric space.

Definition 3 [11]. Let M be a nonempty set. The function $p : M \times M \longrightarrow \mathbb{R}^+$ is said to be a partial metric on M if p satisfies the conditions listed below:

(p_1) $w_1 = w_2 \Leftrightarrow p(w_1, w_1) = p(w_2, w_2) = p(w_1, w_2)$

(p_2) $p(w_1, w_1) \leq p(w_1, w_2)$

(p_3) $p(w_1, w_2) = p(w_2, w_1)$

(p_4) $p(w_1, w_2) \leq p(w_1, w_3) + p(w_2, w_3) - p(w_3, w_3)$

The pair (M, p) is called a partial metric space.

Shukla [18] has presented a partial b-metric space as follows.

Definition 4 [18]. Let M be a nonempty set and $b \geq 1$ be a given real number. The function $p_b : M \times M \longrightarrow [0,\infty)$ is called a partial b-metric with coefficient b if for all $w_1, w_2, w_3 \in M$, the following conditions are satisfied:

(PB1): $p_b(w_1, w_1) = p_b(w_2, w_2) = p_b(w_1, w_2)$ if and only if $w_1 = w_2$

(PB2): $p_b(w_1, w_1) \leq p_b(w_1, w_2)$

(PB3): $p_b(w_1, w_2) = p_b(w_2, w_1)$

(PB4): $p_b(w_1, w_2) \leq b[p_b(w_1, w_3) + p_b(w_3, w_2)] - p_b(w_3, w_3)$

The (M, p_b) is called a partial b-metric space.

2. Partial Extended b-Metric Space

In this section, we first elaborate the idea of a partial extended b-metric space (in the sense of Aydi et al. [5]), and then with the help of an example, we explain our new definition.

Definition 5. Let M be nonempty set and $\theta : M \times M \times M \longrightarrow [1,+\infty)$ be a function. The function $p_\theta : M \times M \longrightarrow [0,+\infty)$ is called a partial extended b-metric if for all $w_1, w_2, w_3 \in M$, it satisfies the following:

(PE1): $p_\theta(w_1, w_1) = p_\theta(w_2, w_2) = p_\theta(w_1, w_2) \Leftrightarrow w_1 = w_2$

(PE2): $p_\theta(w_1, w_1) \leq p_\theta(w_1, w_2)$

(PE3): $p_\theta(w_1, w_2) = p_\theta(w_2, w_1)$

(PE4): $p_\theta(w_1, w_3) \leq \theta(w_1, w_2, w_3)[p_\theta(w_1, w_2) + p_\theta(w_2, w_3)] - p_\theta(w_2, w_2)$

The partial extended b-metric space p_θ on M generates a T_0 topology τ_{p_θ} with a base of the family of open p_θ-balls $\{\mathscr{B}(m, \varepsilon) : m \in M, \varepsilon > 0\}$, where $\mathscr{B}(m, \varepsilon) = \{m_1 \in M : p_\theta(m, m_1) < p_\theta(m, m) + \varepsilon\}$, for all $m \in M$ and $\varepsilon > 0$. A sequence $\{h_n\}$ in a partial new extended b-metric space p_θ converges with respect to τ_{p_θ} to a point $h \in M$ if and only if

$$\lim_{n \longrightarrow +\infty} p_\theta(h, h_n) = p_\theta(h, h). \tag{1}$$

Remark 6.

(1) If $\theta(w_1, w_2, w_3) = b$, then the above definition coincides with a partial b-metric space

(2) If $\theta(w_1, w_2, w_3) = 1$, then the partial new extended b-metric becomes a partial metric

The following is an example of a partial extended b-metric space, which shows that the notion of a partial extended b-metric space is a generalization of a partial b-metric space.

Example 7. Consider $M = \mathbb{N}$ and define $p_\theta : M \times M \longrightarrow \mathbb{R}$ by

$$p_\theta(w_1, w_2) = \begin{cases} \max\{w_1, w_2\} \text{ if } w_1 = w_2, \\ \dfrac{1}{w_1} + \max\{w_1, w_2\} \text{ if } w_1 \text{ is even and } w_2 \text{ is odd}, \\ \dfrac{1}{w_2} + \max\{w_1, w_2\} \text{ if } w_2 \text{ is even and } w_1 \text{ is odd}, \\ 1 + \max\{w_1, w_2\} \text{ otherwise.} \end{cases} \tag{2}$$

Then, p_θ is a partial extended b-metric on $\mathbb{N}$, where $\theta : M \times M \times M \longrightarrow [1,\infty)$ is defined by

$$\theta(w_1,w_2,w_3)=\begin{cases}1, \text{if } w_1=w_3 \text{ and } w_2 \text{ is odd or even,}\\ 5pt\ \dfrac{w_1w_3}{w_1+w_3}, \text{if } w_1\neq w_3, w_1 \text{ and } w_3 \text{ are even and } w_2 \text{ odd,}\\ 5pt\ \dfrac{w_2}{2}, \text{if } w_1\neq w_3, w_1 \text{ and } w_3 \text{ are odd and } w_2 \text{ is even,}\\ 5pt\ \dfrac{3}{2}, \text{if } w_1\neq w_3, w_1, w_2, \text{ and } w_3 \text{ all are even or all are odd,}\\ 5pt\ \dfrac{w_1+w_2(1+w_1)}{w_1(1+w_2)}, \text{if } w_1, w_2 \text{ even and } w_3 \text{ odd,}\\ 5pt\ \dfrac{w_3+w_2(w_3+1)}{w_3(w_2+1)}, \text{if } w_1 \text{ odd, } w_3 \text{ and } w_2 \text{ are even,}\\ 5pt\ \dfrac{2+w_3}{1+w_3}, \text{if } w_1\ w_2 \text{ are odd and } w_3 \text{ is even,}\\ 5pt\ \dfrac{w_1+1}{w_1}, \text{if } w_1 \text{ is even, } w_2 \text{ and } w_3 \text{ are odd.}\end{cases} \tag{3}$$

Example 8. Consider the set $M=\{1,2,3\}$, and define the function $\theta : M\times M\times M \longrightarrow [1,\infty)$ by

$$\theta(w_1,w_2,w_3)=2+w_1+w_2+w_2. \tag{4}$$

Define $p_\theta : M\times M \longrightarrow \mathbb{R}^+$ in the following manner:

$$\begin{gathered}p_\theta(1,1)=1=p_\theta(2,2)=2=p_\theta(3,3)=3,\\ p_\theta(1,2)=p_\theta(2,1)=82, p_\theta(1,3)=p_\theta(3,1)=1000, p_\theta(2,3)=p_\theta(3,2)=600.\end{gathered} \tag{5}$$

Then, p_θ is a partial extended b-metric on M. In fact, the first three axioms of Definition 5 hold, so there is only need to check the triangular inequality. To do this, proceed as follows:

$$\begin{aligned}p_\theta(1,2)=82&<\theta(1,2,3)[p_\theta(1,3)+p_\theta(3,2)-p_\theta(3,3)]\\ &=8(1000+600)-3=8\times 1600-3=12797.\end{aligned} \tag{6}$$

Similarly, the remaining pairs can be verified as above; hence, for all $w_1, w_2, w_3 \in M$,

$$p_\theta(w_1,w_3)\le\theta(w_1,w_2,w_3)[p_\theta(w_1,w_2)+p_\theta(w_2,w_3)]-p_\theta(w_2,w_2) \tag{7}$$

holds. Hence, (M,p_θ) is a partial extended b-metric space.

Remark 9. Note that in the above examples, the function θ depends on all the three variables w_1, w_2, and w_3. So from this fact, it is clear that a partial extended b-metric needs not to be a partial extended b-metric.

In partial extended b-metric spaces, the notion of Cauchyness and convergence are defined as follows.

Definition 10. Let (M,p_θ) be a partial extended b-metric space.

(1) A sequence $\{w_n\}$ is said to be a Cauchy sequence if $\lim_{n,m\longrightarrow\infty} p_\theta(w_m,w_n)$ exists and is finite

(2) A sequence $\{w_n\}$ converges to $w\in M$ if and only if

$$\lim_{m\longrightarrow+\infty} p_\theta(w,w_m)=p_\theta(w,w). \tag{8}$$

It is worth to mention that if p_θ is a partial extended b-metric on M, the $d_\theta : M\times M\longrightarrow \mathbb{R}^+$ defined by

$$d_\theta(m,n)=2p_\theta(m,n)-p_\theta(m,m)-p_\theta(n,n) \tag{9}$$

is an extended b-metric on M.

Definition 11. A partial extended b-metric space (M,p_θ) is complete if for each Cauchy sequence $\{w_m\}$ in M, there is $w\in M$ such that

$$\lim_{m\longrightarrow+\infty} p_\theta(w,w_m)=p_\theta(w,w)=\lim_{n,m\longrightarrow\infty} p_\theta(w_m,w_n). \tag{10}$$

Theorem 12. *Let (M,p_θ) be a complete partial extended b-metric space and p_θ be a continuous functional. Consider a self map $T : M\longrightarrow M$ such that*

$$p_\theta(Tw,Tv)\le kp_\theta(w,v)\forall w,v\in M, \tag{11}$$

where $k\in[0,1)$. Also assume that there exists $w_0\in M$ such that

$$\lim_{n\longrightarrow\infty}\theta(w_n,w_{n+1},w_m)<\frac{1}{k}, \text{for all } m>n, \tag{12}$$

where $w_n=T^n w_0$. Then, T has a unique fixed point.

Proof. For an arbitrary $w_0\in M$, take the iterative sequence $\{w_n\}$ defined by $w_n=Tw_{n-1}=T^n w_0 \forall n\in\mathbb{N}$, which satisfies (12). Inequality (11) implies

$$p_\theta(Tw_1,Tw_2)\le kp_\theta(w_1,w_2)\le kp_\theta(Tw_0,Tw_1)\le k^2p_\theta(w_0,w_1). \tag{13}$$

Thus, by applying (11) successively, we obtain

$$p_\theta(w_n,w_{n+1})\le k^n p_\theta(w_0,w_1)\forall n\in\mathbb{N}. \tag{14}$$

□

By use of triangular inequality, we have (for $m > n$)

$$\begin{aligned}
p_\theta(w_n, w_m) &\le \theta(w_n, w_{n+1}, w_m)[p_\theta(w_n, w_{n+1}) + p_\theta(w_{n+1}, w_m)] \\
&\quad - p_\theta(w_{n+1}, w_{n+1}) = \theta(w_n, w_{n+1}, w_m) \\
&\quad \cdot [p_\theta(w_n, w_{n+1})] + \theta(w_n, w_{n+1}, w_m) \\
&\quad \cdot [p_\theta(w_{n+1}, w_m)] - p_\theta(w_{n+1}, w_{n+1}) \\
&\le \theta(w_n, w_{n+1}, w_m)[p_\theta(w_n, w_{n+1})] + \theta(w_n, w_{n+1}, w_m) \\
&\quad \cdot [p_\theta(w_{n+1}, w_m)] \le \theta(w_n, w_{n+1}, w_m)[p_\theta(w_n, w_{n+1})] \\
&\quad + \theta(w_n, w_{n+1}, w_m)[\theta(w_{n+1}, w_{n+2}, w_m) \\
&\quad \cdot [p_\theta(w_{n+1}, w_{n+2}) + p_\theta(w_{n+2}, w_m)] - p_\theta(w_{n+2}, w_{n+2})] \\
&\le \theta(w_n, w_{n+1}, w_m)[p_\theta(w_n, w_{n+1})] + \theta(w_n, w_{n+1}, w_m) \\
&\quad \cdot \theta(w_{n+1}, w_{n+2}, w_m)[p_\theta(w_{n+1}, w_{n+2}) + p_\theta(w_{n+2}, w_m)].
\end{aligned} \tag{15}$$

Continuing this process, we get

$$\begin{aligned}
p_\theta(w_n, w_m) &\le \theta(w_n, w_{n+1}, w_m)[p_\theta(w_n, w_{n+1})] \\
&\quad + \theta(w_n, w_{n+1}, w_m)\theta(w_{n+1}, w_{n+2}, w_m) \\
&\quad \cdot [p_\theta(w_{n+1}, w_{n+2})] + \theta(w_n, w_{n+1}, w_m) \\
&\quad \cdot \theta(w_{n+1}, w_{n+2}, w_m)\theta(w_{n+2}, w_{n+3}, w_m) \\
&\quad \cdot [p_\theta(w_{n+2}, w_{n+3})] + \cdots + \theta(w_n, w_{n+1}, w_m) \\
&\quad \cdot \theta(w_{n+1}, w_{n+2}, w_m) \cdots \theta(w_{m-2}, w_{m-1}, w_m) \\
&\quad \cdot [p_\theta(w_{m-1}, w_m)].
\end{aligned} \tag{16}$$

By using (14), one writes

$$\begin{aligned}
p_\theta(w_n, w_m) &\le \theta(w_n, w_{n+1}, w_m)k^n p_\theta(w_0, w_1) + \theta(w_n, w_{n+1}, w_m) \\
&\quad \cdot \theta(w_{n+1}, w_{n+2}, w_m)k^{n+1} p_\theta(w_0, w_1) + \cdots + \theta \\
&\quad \cdot (w_{n+1}, w_{n+2}, w_m)\theta(w_{n+2}, w_{n+3}, w_m) \cdots \theta \\
&\quad \cdot (w_{m-2}, w_{m-1}, w_m)k^{m-1} p_\theta(w_0, w_1).
\end{aligned} \tag{17}$$

Therefore,

$$\begin{aligned}
p_\theta(w_n, w_m) &\le p_\theta(w_0, w_1)[\theta(w_n, w_{n+1}, w_m)k^n + \theta(w_n, w_{n+1}, w_m) \\
&\quad \cdot \theta(w_{n+1}, w_{n+2}, w_m)k^{n+1} + \cdots + \theta(w_n, w_{n+1}, w_m) \\
&\quad \cdot \theta(w_{n+1}, w_{n+2}, w_m)\theta(w_{n+2}, w_{n+3}, w_m) \cdots \theta \\
&\quad \cdot (w_{m-2}, w_{m-1}, w_m)k^{m-1}] \le p_\theta(w_0, w_1) \\
&\quad \cdot [\theta(w_1, w_2, w_m)\theta(w_2, w_3, w_m) \cdots \theta(w_{n-1}, w_n, w_m) \\
&\quad \cdot \theta(w_n, w_{n+1}, w_m)k^n \cdots + \theta(w_1, w_2, w_m) \\
&\quad \cdot \theta(w_2, w_3, w_m) \cdots \theta(w_{n-1}, w_n, w_m)\theta(w_n, w_{n+1}, w_m) \\
&\quad \cdot \theta(w_{n+1}, w_{n+2}, w_m)k^{n+1} \cdots + \theta(w_1, w_2, w_m) \\
&\quad \cdot \theta(w_2, w_3, w_m) \cdots \theta(w_{n-1}, w_n, w_m)\theta(w_n, w_{n+1}, w_m) \\
&\quad \cdot \theta(w_{n+1}, w_{n+2}, w_m) \cdots \theta(w_{n+2}, w_{n+3}, w_m) \cdots \\
&\quad \cdot \theta(w_{m-2}, w_{m-1}, w_m)k^{m-1}].
\end{aligned} \tag{18}$$

Set

$$S_n = p_\theta(w_0, w_1) \sum_{j=1}^{n} k^j \prod_{i=1}^{j} \theta(w_i, w_{i+1}, w_m). \tag{19}$$

For $m > n$, we conclude that

$$p_\theta(w_n, w_m) \le p_\theta(w_0, w_1)[S_{m-1} - S_{n-1}]. \tag{20}$$

Consider the series

$$p_\theta(w_0, w_1) \sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \theta(w_i, w_{i+1}, w_m). \tag{21}$$

Take

$$v_n = k^n \prod_{i=1}^{n} \theta(w_i, w_{i+1}, w_m). \tag{22}$$

Thus, we have

$$\lim_{n \longrightarrow \infty} \frac{v_{n+1}}{v_n} = k\theta(w_{n+1}, w_{n+2}, w_m). \tag{23}$$

By using the condition

$$k\theta(w_{n+1}, w_{n+2}, w_m) < 1 \tag{24}$$

by ratio test, the series is convergent. So we conclude that $\lim_{m,n \longrightarrow \infty}[S_{m-1} - S_{n-1}] = 0$. Hence, by (20), $\{w_n\}$ is a Cauchy sequence in M. Since M is complete, we have $w_n \longrightarrow \eta \in M$ such that $\lim_{n \longrightarrow \infty} p_\theta(w_n, \eta) = p_\theta(\eta, \eta) = \lim_{m,n \longrightarrow \infty} p_\theta(w_n, w_m) = 0$. Now, we prove that η is a fixed point of T. Consider

$$p_\theta(T(w_n), T(\eta)) \le k p_\theta(w_n, \eta) \Rightarrow p_\theta(w_{n+1}, T(\eta)) \le k p_\theta(w_n, \eta). \tag{25}$$

Taking limit as $n \longrightarrow \infty$, we get $p_\theta(T\eta, \eta) = 0$ due to the fact that p_θ is continuous. We also conclude $p_\theta(\eta, \eta) = 0$ and $p_\theta(T\eta, T\eta) = 0$. Hence, η is a fixed point of T. To verify its uniqueness, assume that there exists another fixed point γ of T, that is, $p_\theta(\gamma, \gamma) = p_\theta(T\gamma, T\gamma) = p_\theta(T\gamma, \gamma)$. By (11), we have

$$p_\theta(\eta, \gamma) = p_\theta(T\eta, T\gamma) \le k p_\theta(\eta, \gamma). \tag{26}$$

Since $k < 1$, $p_\theta(\eta, \gamma) = 0$. Also, we have $p_\theta(\gamma, \gamma) = 0$ and $p_\theta(\eta, \eta) = 0$. Thus, $\eta = \gamma$; that is, the fixed point of T is unique.

Definition 13. Let $T : M \longrightarrow M$ be a self map and $w_0 \in M$. The set $O(w_0) = \{w_0, Tw_0, T^2 w_0 \cdots\}$ is called the orbit of w_0. The function $H : M \longrightarrow \mathbb{R}$ is said to be T-orbitally lower semicontinuous at $w \in M$ if $\{w_n\} \subset O(w_0)$ and $w_n \longrightarrow w$, which implies

$$H(w) \leq \liminf_{n\longrightarrow\infty} H(w_n). \tag{27}$$

Theorem 14. *Let (M, p_θ) be a complete partial extended b-metric space and p_θ represent a continuous functional. Consider a self map $T : M \longrightarrow M$ and $w_0 \in M$ such that*

$$p_\theta(Tw, T^2 w) \leq k p_\theta(w, Tw), \text{for each } w \in O(w_0), \tag{28}$$

where $k \in [0, 1)$. Also for such $w_0 \in M$,

$$\lim_{n\longrightarrow\infty} \theta(w_n, w_{n+1}, w_m) < \frac{1}{k}, \text{for } m > n, \tag{29}$$

where $w_n = T^n w_0$, $n = 1, 2, 3 \cdots$. Then, $T^n w_0 \longrightarrow \eta \in M$. Further, η is a fixed point of T if $H(w) = p_\theta(w, Tw)$ is T-orbitally lower semi continuous at η.

Proof. For a given $w_0 \in M$, define an orbit $O(w_0)$ with $w_1 = Tw_0$, $w_2 = Tw_1 = T(Tw_0) = T^2(w_0), \cdots, w_n = T^n w_0, \cdots$. By the use of successive iteration of the inequality (28), we get

$$\begin{aligned} p_\theta(T^n w_0, T^{n+1} w_0) = p_\theta(w_n, w_{n+1}) &\leq k p_\theta(w_{n-1}, w_n) \\ &\leq k^2 p_\theta(w_{n-2}, w_{n-1}) \cdots \leq k^n p_\theta(w_0, w_1). \end{aligned} \tag{30}$$

Proceeding as in Theorem 12, we can prove $\{w_n\}$ is a Cauchy sequence. As M is complete, then $w_n = T^n w_0 \longrightarrow \eta \in M$. Since H is T-orbitally lower semi continuous at $\eta \in M$,

$$\begin{aligned} p_\theta(\eta, T\eta) &\leq \liminf_{n\longrightarrow\infty} p_\theta(T^n w_0, T^{n+1} w_0), \\ &\leq \liminf_{n\longrightarrow\infty} k^n p_\theta(w_0, w_1) = 0. \end{aligned} \tag{31}$$

Thus, $p_\theta(\eta, T\eta) = 0$. Hence, η is a fixed point of T. □

Example 15. Take $M = [0,\infty)$. Define $p_\theta : M \times M \longrightarrow \mathbb{R}^+$ by

$$p_\theta(w_1, w_2) = \max\{w_1, w_2\} \tag{32}$$

and $\theta : M \times M \times M \longrightarrow [1,\infty)$ by

$$\theta(w_1, w_2, w_3) = w_1 + w_2 + w_3 + 1. \tag{33}$$

Then, p_θ represents a complete partial extended b-metric on M.

Let $T : M \longrightarrow M$ be defined by

$$Tw = \frac{w}{2}. \tag{34}$$

We have

$$p_\theta(Tw, T^2 w) = p_\theta\left(\frac{w}{2}, \frac{w}{4}\right) = \frac{w}{2} = k p_\theta(w, Tw). \tag{35}$$

For each $w \in M$ and $T^n w = w/2^n$, we get

$$\lim_{m,n\longrightarrow\infty} \theta(T^n w, T^{n+1} w, T^m w) = \lim_{m,n\longrightarrow\infty} \left(\frac{w}{2^m} + \frac{w}{2^n} + \frac{w}{2^{n+1}} + 1\right) < 2. \tag{36}$$

Since all the conditions of Theorem 14 are satisfied, the mapping T has a fixed point.

3. Partial Hausdorff Distance via the Extended b-Metric Space

Shukla and Panicker [10] introduced the notion of a partial Hausdorff metric. Baig and Pathak [29] has proved well-known Nadler's fixed point theorem for multivalued mappings on weak partial metric spaces. This is further used by Kanwal et al. [30] for establishing a fixed point result on weak partial b-metric spaces. In this section, we establish a fixed point result for set-valued mappings on partial new extended b-metric spaces. We first give some requisite definitions. Let (M, p_θ) be a partial extended b-metric space. Denote by $CB^{p_\theta}(M)$ the collection of all nonempty bounded and closed subsets of M with respect to the partial extended b-metric p_θ. For $P, Q \in (M, p_\theta)$ and $w \in M$, define the following:

$$\begin{aligned} p_\theta(w, Q) &= \inf\{p_\theta(w, q) | q \in Q\}, \\ \Omega_{p_\theta}(P, Q) &= \sup\{p_\theta(p, Q) | p \in P\}, \\ \Omega_{p_\theta}(Q, P) &= \sup\{p_\theta(q, P) | q \in Q\}. \end{aligned} \tag{37}$$

Remark 16. Let (M, p_θ) be a partial extended b-metric space and B be any nonempty set in M, then

$$w \in \bar{Q} \text{ if and only if } p_\theta(w, Q) = p_\theta(w, w), \tag{38}$$

where $\bar{Q}$ is the closure of Q. Further, Q is closed in (M, p_θ) if and only if $Q = \bar{Q}$.

Definition 17. Let (M, p_θ) be a partial extended b-metric space. For $P, Q \in CB^{p_\theta}(M)$, define

$$H_{p_\theta}(P, Q) = \max\left\{\Omega_{p_\theta}(Q, P), \Omega_{p_\theta}(P, Q)\right\}. \tag{39}$$

The mapping $H_{p_\theta} : CB^{p_\theta}(M) \times CB^{p_\theta}(M) \longrightarrow [0,\infty)$ is called a partial Hausdorff distance function induced by p_θ.

Corollary 18. *If (M, p_θ) is partial extended b-metric space and $P, Q \in CB^{p_\theta}(M)$, then*

$$H_{p_\theta}(P, Q) = 0 \Rightarrow P = Q. \tag{40}$$

Lemma 19. *Let (M, p_θ) be a partial extended b-metric space, $P, Q \in CB^{p_\theta}(M)$, and $h > 1$. Then, for any $w_1 \in P$, there exists $w_2 \in Q$ such that*

$$p_\theta(w_1, w_2) \leq h H_{p_\theta}(P, Q). \tag{41}$$

Theorem 20. *Let (M, p_θ) be a complete partial extended b-metric space so that p_θ is a continuous functional. If $T : M \longrightarrow CB^{p_\theta}(M)$ is a multivalued mapping such that for all $w_1, w_2 \in M$, we have*

$$H_{p_\theta}(Tw_1, Tw_2) \le kp_\theta(w_1, w_2), \tag{42}$$

where $k \in (0, 1)$. Also, there exists $w_0 \in M$ such that for every $\{w_n\}$ with $w_{n+1} \in T(w_n)$, we have $\lim_{n\longrightarrow\infty} \theta(w_n, w_{n+1}, w_{m'}) < 1/\sqrt{k}, \forall m' > n$. Then, T has a fixed point.

Proof. Given $w_0 \in M$ and $w_1 \in Tw_0$, recall that $h = (1/\sqrt{k}) > 1$, so by use of Lemma 19, we have $w_2 \in Tw_1$ with

$$p_\theta(w_1, w_2) \le \frac{1}{\sqrt{k}} H_{p_\theta}(Tw_0, Tw_1). \tag{43}$$

By using (42), we get

$$\begin{aligned} &H_{p_\theta}(Tw_0, Tw_1) \le kp_\theta(w_0, w_1), \\ &\Rightarrow p_\theta(w_1, w_2) \le \sqrt{k} p_\theta(w_0, w_1). \end{aligned} \tag{44}$$

For $w_2 \in Tw_1$, by Lemma 19, we have $w_3 \in Tw_2$ with

$$p_\theta(w_2, w_3) \le \frac{1}{\sqrt{k}} H_{p_\theta}(Tw_1, Tw_2). \tag{45}$$

Again by (42), we get

$$\begin{aligned} &H_{p_\theta}(Tw_1, Tw_2) \le kp_\theta(w_1, w_2), \\ &\Rightarrow p_\theta(w_2, w_3) \le \sqrt{k} p_\theta(w_1, w_2). \end{aligned} \tag{46}$$

Thus,

$$p_\theta(w_2, w_3) \le \left(\sqrt{k}\right)^2 p_\theta(w_1, w_2). \tag{47}$$

Continuing this process, we have a sequence $\{w_n\}$ in M such that $w_n \in Tw_{n-1}$ and

$$p_\theta(w_n, w_{n+1}) \le \sqrt{k} p_\theta(w_{n-1}, w_n) \forall n \in \mathbb{N}. \tag{48}$$

Also,

$$p_\theta(w_n, w_{n+1}) \le \left(\sqrt{k}\right)^n p_\theta(w_0, w_1) \forall n \in \mathbb{N}. \tag{49}$$

By using (PE4) of the partial extended b-metric space, we have (for $m > n$)

$$\begin{aligned} p_\theta(w_n, w_m) &\le \theta(w_n, w_{n+1}, w_m)[p_\theta(w_n, w_{n+1}) + p_\theta(w_{n+1}, w_m)] \\ &\quad - p_\theta(w_{n+1}, w_{n+1}) = \theta(w_n, w_{n+1}, w_m)[p_\theta(w_n, w_{n+1})] \\ &\quad + \theta(w_n, w_{n+1}, w_m)[p_\theta(w_{n+1}, w_m)] - p_\theta(w_{n+1}, w_{n+1}) \\ &\le \theta(w_n, w_{n+1}, w_m)[p_\theta(w_n, w_{n+1})] + \theta(w_n, w_{n+1}, w_m) \\ &\quad \cdot [p_\theta(w_{n+1}, w_m)] \le \theta(w_n, w_{n+1}, w_m)[p_\theta(w_n, w_{n+1})] \\ &\quad + \theta(w_n, w_{n+1}, w_m)[\theta(w_{n+1}, w_{n+2}, w_m)[p_\theta(w_{n+1}, w_{n+2}) \\ &\quad + p_\theta(w_{n+2}, w_m)] - p_\theta(w_{n+2}, w_{n+2})] \le \theta(w_n, w_{n+1}, w_m) \\ &\quad \cdot [p_\theta(w_n, w_{n+1})] + \theta(w_n, w_{n+1}, w_m)\theta(w_{n+1}, w_{n+2}, w_m) \\ &\quad \cdot [p_\theta(w_{n+1}, w_{n+2}) + p_\theta(w_{n+2}, w_m)]. \end{aligned} \tag{50}$$

Continuing this process, we get

$$\begin{aligned} p_\theta(w_n, w_m) &\le \theta(w_n, w_{n+1}, w_m)[p_\theta(w_n, w_{n+1})] + \theta(w_n, w_{n+1}, w_m) \\ &\quad \cdot \theta(w_{n+1}, w_{n+2}, w_m)[p_\theta(w_{n+1}, w_{n+2})] + \theta(w_n, w_{n+1}, w_m) \\ &\quad \cdot \theta(w_{n+1}, w_{n+2}, w_m)\theta(w_{n+2}, w_{n+3}, w_m)[p_\theta(w_{n+2}, w_{n+3})] \\ &\quad + \cdots + \theta(w_n, w_{n+1}, w_m)\theta(w_{n+1}, w_{n+2}, w_m) \cdots \theta \\ &\quad \cdot (w_{m-2}, w_{m-1}, w_m)[p_\theta(w_{m-1}, w_m)]. \end{aligned} \tag{51}$$

By using (49), one writes

$$\begin{aligned} p_\theta(w_n, w_m) &\le \theta(w_n, w_{n+1}, w_m)\left(\sqrt{k}\right)^n p_\theta(w_0, w_1) \\ &\quad + \theta(w_n, w_{n+1}, w_m)\theta(w_{n+1}, w_{n+2}, w_m) \\ &\quad \cdot \left(\sqrt{k}\right)^{n+1} p_\theta(w_0, w_1) + \cdots + \theta(w_n, w_{n+1}, w_m) \\ &\quad \cdot \theta(w_{n+1}, w_{n+2}, w_m) \cdots \theta(w_{m-2}, w_{m-1}, w_m) \\ &\quad \cdot \left(\sqrt{k}\right)^{n+m-1} p_\theta(w_0, w_1). \end{aligned} \tag{52}$$

This further gives

$$\begin{aligned} p_\theta(w_n, w_m) &\le p_\theta(w_0, w_1)\Big[\theta(w_1, w_2, w_m)\theta(w_2, w_3, w_m) \cdots \theta \\ &\quad \cdot (w_n, w_{n+1}, w_m)\left(\sqrt{k}\right)^n + \theta(w_1, w_2, w_m) \\ &\quad \cdot \theta(w_2, w_3, w_m) \cdots ;\theta(w_n, w_{n+1}, w_m) \\ &\quad \cdot \theta(w_{n+1}, w_{n+2}, w_m)\left(\sqrt{k}\right)^{n+1} + \cdots + \theta(w_1, w_2, w_m) \\ &\quad \cdot \theta(w_2, w_3, w_m) \cdots \theta(w_n, w_{n+1}, w_m) \cdots \theta \\ &\quad \cdot (w_{m-2}, w_{m-1}, w_m)\left(\sqrt{k}\right)^{m-1}\Big]. \end{aligned} \tag{53}$$

Consider

$$S_n = p_\theta(w_0, w_1) \sum_{j=1}^{n} \left(\sqrt{k}\right)^j \prod_{i=1}^{j} \theta(w_i, w_{i+1}, w_m), \tag{54}$$

For $m > n$, we conclude that

$$p_\theta(w_n, w_m) \le p_\theta(w_0, w_1)[S_{m-1} - S_{n-1}]. \tag{55}$$

Consider the series

$$p_\theta(w_0, w_1) \sum_{n=1}^{\infty} \left(\sqrt{k}\right)^n \prod_{i=1}^{n} \theta(w_i, w_{i+1}, w_m). \tag{56}$$

Substitute

$$v_n = \left(\sqrt{k}\right)^n \prod_{i=1}^{n} \theta(w_i, w_{i+1}, w_m). \tag{57}$$

Then, we have

$$\lim_{n \longrightarrow \infty} \frac{v_{n+1}}{v_n} = \sqrt{k}\theta(w_{n+1}, w_{n+2}, w_m) < 1. \tag{58}$$

Therefore, by ratio test, the series is convergent. So we conclude that $\lim_{m,n\longrightarrow\infty}[S_{m-1} - S_{n-1}] = 0$. By using (55), $\{w_n\}$ is a Cauchy sequence in M. Since M is complete, $w_n \longrightarrow w^* \in M$ with respect to p_θ, that is,

$$\lim_{n\longrightarrow\infty} p_\theta(w_n, w^*) = p_\theta(w^*, w^*) = \lim_{m,n\longrightarrow\infty} p_\theta(w_n, w_m) = 0. \tag{59}$$

By (42), we get $H_{p_\theta}(Tw_n, Tw^*) \le kp_\theta(w_n, w^*)$. Therefore, $\lim_{n\longrightarrow\infty} H_{p_\theta}(Tw_n, Tw^*) = 0$. Now, $w_{n+1} \in Tw_n$ gives that

$$p_\theta(w_{n+1}, Tw^*) \le \Omega_{p_\theta}(Tw_n, Tw^*) \le H_{p_\theta}(Tw_n, Tw^*). \tag{60}$$

By using continuity of p_θ, we get

$$p_\theta(w^*, Tw^*) = \lim_{n\longrightarrow\infty} p_\theta(w_{n+1}, Tw^*) = 0. \tag{61}$$

By (59), $p_\theta(w^*, w^*) = 0$. Thus we have

$$p_\theta(w^*, w^*) = p_\theta(w^*, Tw^*) = 0, \tag{62}$$

which implies $w^* \in Tw^*$.

The following corollary follows immediately from Theorem 20 by setting $\theta(w_1, w_2, w_3) = b$ for all $w_1, w_2, w_3 \in M$. □

Corollary 21. *Let (M, p_b) be a complete partial b-metric space and p_b be a continuous functional. Suppose that $T : M \longrightarrow CB^{p_b}(M)$ is a multivalued mapping such that for all $w_1, w_2 \in M$, we have*

$$H_{p_b}(Tw_1, Tw_2) \le kp_b(w_1, w_2), \tag{63}$$

where $k \in (0, 1)$ and $b < 1/\sqrt{k}$. Then, T has a fixed point.

Remark 22. By setting $\theta(w_1, w_2, w_3) = 1$ for all w_1, w_2, w_3, the main result of [16] becomes a special case of Theorem 20.

4. Conclusion

In this article, the notion of a partial extended b-metric space was introduced with suitable examples. Fixed point results for single-valued mappings endowed on partial extended b-metric spaces are established. These results are extensions of many existing results in literature.

The idea of a partial Hausdorff distance in partial extended b-metric spaces was presented. Furthermore, some fixed point theorems involving multivalued mappings have been proved. In the future, one can prove the above results under the platform of dislocated partial extended b-metric spaces.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper.

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