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Research Article

The Second Hankel Determinant of Logarithmic Coefficients for Starlike and Convex Functions Involving Four-Leaf-Shaped Domain

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In this particular research article, we take an analytic function $Q_{4\mathscr{L}} = 1 + 5/6z + 1/6z^5$, which makes a four-leaf-shaped image domain. Using this specific function, two subclasses, $\mathscr{S}^*_{4\mathscr{L}}$ and $\mathscr{C}_{4\mathscr{L}}$, of starlike and convex functions will be defined. For these classes, our aim is to find some sharp bounds of inequalities that consist of logarithmic coefficients. Among the inequalities to be studied here are Zalcman inequalities, the Fekete-Szegö inequality, and the second-order Hankel determinant.

1. Introduction and Definitions

To properly comprehend the findings provided in the paper, certain important literature on geometric function theory must first be discussed. In this regard, the letters $\mathscr{S}$ and $\mathscr{A}$ stand for the normalized univalent (or schlicht) functions class and the normalized holomorphic (or analytic) functions class, respectively. These primary notions are defined in the disc $\mathbb{U}_d = \{z \in \mathbb{C} : |z| < 1\}$ by

$$\mathscr{A} = \left\{ F \in \mathscr{G}(\mathbb{U}_d) \colon F(z) = \sum_{l=1}^{\infty} b_l z^l \right\}, \tag{1}$$

where $\mathscr{G}(\mathbb{U}_d)$ expresses holomorphic functions class, and

$$\mathscr{S} = \{ F \in \mathscr{A} : F \text{ is Schlicht in } \mathbb{U}_d \}. \tag{2}$$

This class $\mathscr{S}$ evolved as the foundational component of cutting-edge research in this area. In his paper [1], Koebe established the presence of a "covering constant" ζ, demonstrating that if F is holomorphic and Schlicht in $\mathbb{U}_d$ with $F'(0) = 1$ and $F(0) = 0$, then $F(\mathbb{U}_d) = \{w : |w| < \zeta\}$. Many mathematicians were intrigued by this beautiful result. Within a few years, the wonderful article by Bieberbach [2], which gave rise to the renowned coefficient hypothesis, was published.

The below expression provided the coefficients λ_n of logarithmic function $J_F(z)$ for $F \in \mathscr{S}$

$$J_F(z) = \frac{1}{2} \log \left(\frac{F(z)}{z} \right) = \lambda_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots, \forall z \in \mathbb{U}_d. \tag{3}$$

The above coefficients have a considerable impact on the theory of Schlicht functions in many estimations. De Branges [3] achieved that $n \geq 1$ in1985,

$$\sum_{l=1}^{n} l(n-l+1)|\lambda_n|^2 \leq \sum_{l=1}^{n} \frac{n-l+1}{l}, \tag{4}$$

and equality will be achieved if F has the form $z/(1-e^{i\varphi}z)^2$ for some $\varphi \in \mathbb{R}$. It is obvious that this inequality provides the most general version of the well-known Bieberbach-Robertson-Milin conjectures concerning the Taylor coefficients of $F \in \mathcal{S}$. We quote [4–6] for further information on the demonstration of de Brange's conclusion. By taking into account, the logarithmic coefficients, in 2005, Kayumov [7] established Brennan's conjecture for conformal mappings. The major contributions to study the bounds of logarithmic coefficients for various holomorphic univalent functions are due to Alimohammadi et al. [8], Obradović et al. [9], Ye [10], Deng [11], Girela [12], Roth [13], and Andreev and Duren [14].

For the prescribed functions $Q_1, Q_2 \in \mathcal{A}$, the relation of subordination between Q_1 and Q_2 is as follows (mathematically as $Q_1 \prec Q_2$), if an holomorphic function u comes in $\mathbb{U}_d$ with the limitation $|u(z)| < 1$ and $u(0) = 0$ in a manner that $Q_1(z) = Q_2(u(z))$ satisfy. Consequently, the following relation applies if $Q_2 \in \mathcal{S}$ in $\mathbb{U}_d$:

$$Q_1(z) \prec Q_2(z), (z \in \mathbb{U}_d) \tag{5}$$

if and only if

$$Q_1(0) = Q_2(0) \& Q_1(\mathbb{U}_d) \subset Q_2(\mathbb{U}_d). \tag{6}$$

By applying the notion of subordination, Ma and Minda [15] proposed a consolidated version of the set $\mathcal{S}^*(\psi)$ in 1992, and the following is a description of it:

$$\mathcal{S}^*(\psi) = \left\{ g \in \mathcal{S} : \frac{zg'(z)}{g(z)} \prec \psi(z) \text{ for } z \in \mathbb{U}_d \right\}, \tag{7}$$

with the Schlicht function ψ that satisfies

$$\psi'(0) > 0 \& \Re\psi > 0. \tag{8}$$

Various subclasses of the set $\mathcal{S}$ have been examined in the past few years as particular choices for family $\mathcal{S}^*(\psi)$. For instance,

(i) $\mathcal{S}^*_{\cos} \equiv \mathcal{S}^*(\cos z)$ (see [16]) and $\mathcal{S}^*_{\cosh} \equiv \mathcal{S}^*(\cosh z)$ (see [17])

(ii) $\mathcal{S}^*_{\tanh} \equiv \mathcal{S}^*(1 + \tanh z)$ (see [18, 19])

(iii) $\mathcal{S}^*_e \equiv \mathcal{S}^*(e^z)$ (see [20, 21]) and $\mathcal{S}^*_\rho \equiv \mathcal{S}^*(1 + \sinh^{-1} z)$ (see [22])

(iv) $\mathcal{S}\mathcal{S}^*(\xi) \equiv \mathcal{S}^*(\psi(z))$ with $\psi(z) = (1 + z/1 - z)^\xi$ and $0 < \xi \le 1$ (see [23])

(v) $\mathcal{S}^*_{\mathcal{L}} \equiv \mathcal{S}^*(\sqrt{1+z})$ (see [24]) and $\mathcal{S}^*_{\text{car}} \equiv \mathcal{S}^*(1 + 4/3z + 2/3\, z^2)$ (see [25, 26])

For given $q, n \in \mathbb{N} = \{1, 2, \cdots\}$, $b_1 = 1$, and $F \in \mathcal{S}$ with the series representation (1), the Hankel determinant $H_{q,n}(F)$ is expressed by

$$H_{q,n}(F) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+q-1} \\ b_{n+1} & b_{n+2} & \cdots & b_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n+q-1} & b_{n+q} & \cdots & b_{n+2q-2} \end{vmatrix}, \tag{9}$$

and it was established by Pommerenke and Pommerenke [27, 28]. For several subcollections of Schlicht functions, the determinant $H_{q,n}(F)$ has been examined. In specific, the sharp estimate of the functional $|H_{2,2}(F)| = |b_2 b_4 - b_3^2|$ for sets $\mathcal{C}$(convexfunctions), $\mathcal{S}^*$(starlikefunctions), and $\mathcal{R}$ (boundedturningfunctions) were determined in [29, 30]. However, for the class of close-to-convex functions, the exact bounds of this determinant remain open [31]. The researchers were inspired by the works of Babalola [32], Bansal, et al. [33], Zaprawa [34], Kwon et al. [35], Kowalczyk et al. [36], and Lecko et al. [37].

It is easy to deduce from equation (2) that, for $F \in \mathcal{S}$, the logarithmic coefficients are computed by

$$\lambda_1 = \frac{1}{2} b_2, \tag{10}$$

$$\lambda_2 = \frac{1}{2}\left(b_3 - \frac{1}{2} b_2^2\right), \tag{11}$$

$$\lambda_3 = \frac{1}{2}\left(b_4 - b_2 b_3 + \frac{1}{3} b_2^3\right), \tag{12}$$

$$\lambda_4 = \frac{1}{2}\left(b_5 - b_2 b_4 + b_2^2 b_3 - \frac{1}{2} b_3^2 - \frac{1}{4} b_2^4\right). \tag{13}$$

Currently, Lecko and Kowalczyk and Kowalczyk and Lecko [38, 39] studied the following Hankel determinant $H_{q,n}(J_F/2)$ of logarithmic coefficients

$$H_{q,n}\left(\frac{J_F}{2}\right) = \begin{vmatrix} \lambda_n & \lambda_{n+1} & \cdots & \lambda_{n+q-1} \\ \lambda_{n+1} & \lambda_{n+2} & \cdots & \lambda_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_{n+q-1} & \lambda_{n+q} & \cdots & \lambda_{n+2q-2} \end{vmatrix}. \tag{14}$$

It has been noted that

$$H_{2,1}\left(\frac{J_F}{2}\right) = \lambda_1 \lambda_3 - \lambda_2^2, \quad H_{2,2}\left(\frac{J_F}{2}\right) = \lambda_2 \lambda_4 - \lambda_3^2. \tag{15}$$

By the virtue of the function $Q_{4\mathcal{L}} = 1 + 5/6z + 1/6z^5$, we define the following classes:

$$\mathcal{S}^*_{4\mathcal{L}} = \left\{ F \in \mathcal{S} : \frac{zF'(z)}{F(z)} \prec Q_{4\mathcal{L}}, (z \in \mathbb{U}_d) \right\}, \tag{16}$$

$$\mathcal{C}_{4\mathcal{L}} = \left\{ F \in \mathcal{S} : 1 + \frac{zF''(z)}{F'(z)} \prec Q_{4\mathcal{L}}, (z \in \mathbb{U}_d) \right\}. \quad (17)$$

Alternatively, $F \in \mathcal{S}^*_{4\mathcal{L}}$ if and only if an analytic function q occurs that satisfies $q(z) \prec Q_{4\mathcal{L}}$ in such that

$$F(z) = z \ \exp\left(\int_0^z \frac{q(t)-1}{t} dt \right). \quad (18)$$

By taking $q(z) = Q_{4\mathcal{L}}$ in (18), we achieve the following function, which serves as an extremal in many of the class $\mathcal{S}^*_{4\mathcal{L}}$ problems.

$$F_0(z) = z \ \exp\left(\int_0^z \left(\frac{5}{6} + \frac{1}{6} t^4 \right) dt \right) = z + \frac{5}{6} z^2 + \cdots. \quad (19)$$

The following Alexander-type connection-related two classes were mentioned above. The above two families are interlinked by the following Alexander-type relation

$$F \in \mathcal{C}_{4\mathcal{L}} \Leftrightarrow zF' \in \mathcal{S}^*_{4\mathcal{L}}. \quad (20)$$

From (19) and (20), we easily obtain the following extremal functions in various problems of the class $\mathcal{C}_{4\mathcal{L}}$

$$g_0(z) = z + \frac{5}{12} z^2 + \cdots. \quad (21)$$

Clearly, $g_0(z)$, $g_0(z^2)$, $g_0(z^3)$, and $g_0(z^4)$ belong to the class $\mathcal{C}_{4\mathcal{L}}$. That is,

$$\begin{aligned} g_1(z) &= g_0(z) = z + \frac{5}{12} z^2 + \cdots, \\ g_2(z) &= g_0(z^2) = z + \frac{5}{36} z^3 + \cdots, \\ g_3(z) &= g_0(z^3) = z + \frac{5}{72} z^4 + \cdots, \\ g_4(z) &= g_0(z^4) = z + \frac{5}{120} z^5 + \cdots. \end{aligned} \quad (22)$$

In the present paper, our core objective is to find the sharp coefficient type problems of logarithmic functions for the families $\mathcal{S}^*_{4\mathcal{L}}$ and $\mathcal{C}_{4\mathcal{L}}$. Among the inequalities to be studied here are Zalcman inequalities, the Fekete-Szegö inequality, and the second-order Hankel determinant $H_{2,1}(J_F/2)$.

2. A Set of Lemmas

We must first create the class $\mathcal{P}$ in the below set-builder form in order to declare the Lemmas that are employed in our primary findings.

$$\mathcal{P} = \{ q \in \mathcal{G}(\mathbb{U}_d) \colon q(0) = 1 \& \mathfrak{R}eq > 0, (z \in \mathbb{U}_d) \}. \quad (23)$$

That is, if $q \in \mathcal{P}$, then q has the below series expansion

$$q(z) = \sum_{n=0}^{\infty} e_n z^n, (z \in \mathbb{U}_d). \quad (24)$$

The following Lemma consists of the widely used e_2 formula [40], the e_3 formula [41], and the e_4 formula illustrated in [42].

Lemma 1. *Let $q \in \mathcal{P}$ be given in the form (24), then for , ρ, $\delta \in \bar{\mathbb{U}}_d = \mathbb{U}_d \cup \{1\}$.*

Lemma 2. *Let $q \in \mathcal{P}$ be of the form (24), then for $x, \delta, \rho \in \bar{\mathbb{U}}_d = \mathbb{U}_d \cup \{1\}$*

$$2e_2 = e_1^2 - \left(e_1^2 - 4\right)x, \quad (25)$$

$$4e_3 = e_1^3 - 2\left(e_1^2 - 4\right)e_1 x + e_1\left(e_1^2 - 4\right)x^2 - 2\left(e_1^2 - 4\right)\left(1 - |x|^2\right)\rho, \quad (26)$$

$$\begin{aligned} 8e_4 = e_1^4 &- \left(e_1^2 - 4\right)x\left[e_1^2\left(x^2 - 3x + 3\right) + 4x\right] \\ &+ 4\left(e_1^2 - 4\right)\left(1 - |x|^2\right)\left[e(x-1)\rho + \bar{x}\rho^2 - \left(1 - |\rho|^2\right)\delta\right]. \end{aligned} \quad (27)$$

Lemma 3. *Let $q \in \mathcal{P}$ and has the expansion (24). Then,*

$$|e_{n+l} - \mu e_n e_l| \le 2 \max(1, |2\mu - 1|), \quad (28)$$

$$|e_n| \le 2 \, for \, n \ge 1, \quad (29)$$

$$|e_{n+l} - \mu e_n e_l| \le 2, 0 \le \mu \le 1. \quad (30)$$

The inequalities (28)–(30) are taken from [40, 43] and [26, 44, 45], respectively.

Lemma 4 (see [40]). *If $q \in \mathcal{P}$ has the representation (24), then*

$$\frac{1}{2}\left|Je_1^3 - Ke_1e_2 + Le_3\right| \le (|J| + |K - 2J| + |K - J + L|). \quad (31)$$

Lemma 5 [46]. *Let γ, τ, ψ and ς satisfy that $\tau, \varsigma \in (0, 1)$ and*

$$\begin{aligned} 8(1-\varsigma)\varsigma&\left[(\tau(\varsigma+\tau) - \psi)^2 + (\tau\psi - 2\gamma)^2\right] \\ &+ \tau(\psi - 2\varsigma\tau)^2(1-\tau) \le 4\tau^2\varsigma(1-\varsigma)(1-\tau)^2. \end{aligned} \quad (32)$$

If $q \in \mathcal{P}$ has the expansion (24), then

$$\left| \gamma e_1^4 + \varsigma e_2^2 + 2\tau e_1 e_3 - \frac{3}{2}\psi e_1^2 e_2 - e_4 \right| \le 2. \quad (33)$$

3. Coefficient Inequalities for the Class $\mathcal{S}^*_{4\mathcal{L}}$

We start by establishing out the class $\mathcal{S}^*_{4\mathcal{L}}$'s initial coefficient bounds.

Theorem 6. *Let F be the series form (1) and if $F \in \mathcal{S}^{*}_{4\mathcal{L}}$, then*

$$\begin{aligned} |\lambda_1| &\le \frac{5}{12}, \\ |\lambda_2| &\le \frac{5}{24}, \\ |\lambda_3| &\le \frac{5}{36}, \\ |\lambda_4| &\le \frac{5}{48}. \end{aligned} \tag{34}$$

These bounds are sharp.

Proof. Let $F \in \mathcal{S}^{*}_{4\mathcal{L}}$. Then, Schwarz function u may therefore be used to express (16) as

$$\frac{zF'(z)}{F(z)} = 1 + \frac{5}{6}u(z) + \frac{1}{6}(u(z))^5 = \alpha(z). \tag{35}$$

From the use of Schwarz function u and if $q \in \mathcal{P}$, we have

$$q(z) = \frac{1 + (u(z))}{1 - (u(z))} := 1 + e_1 z + e_2 z^2 + \cdots, \tag{36}$$

and by simple computation, we get

$$\begin{aligned} u(z) = {} & \frac{1}{2}e_1 z + \left(\frac{1}{2}e_2 - \frac{1}{4}e_1^2\right)z^2 + \left(\frac{1}{8}e_1^3 - \frac{1}{2}e_1 e_2 + \frac{1}{2}e_3\right)z^3 \\ & + \left(\frac{1}{2}e_4 - \frac{1}{2}e_1 e_3 - \frac{1}{4}e_2^2 - \frac{1}{16}e_1^4 + \frac{3}{8}e_1^2 e_2\right)z^4 + \cdots. \end{aligned} \tag{37}$$

Using (1), we attain

$$\begin{aligned} \frac{zF'(z)}{F(z)} := {} & 1 + b_2 z + \left(-b_2^2 + 2b_3\right)z^2 + \left(-3b_2 b_3 + 3b_4 + b_2^3\right)z^3 \\ & + \left(-2b_3^2 + 4b_5 - 4b_2 b_4 + 4b_2^2 b_3 - b_2^4\right)z^4 + \cdots. \end{aligned} \tag{38}$$

By some calculation and using the series expansion of (37), we get

$$\begin{aligned} \alpha(z) = {} & 1 + \frac{5}{12}e_1 z + \left(\frac{5}{12}e_2 - \frac{5}{24}e_1^2\right)z^2 \\ & + \left(\frac{5}{48}e_1^3 - \frac{5}{12}e_1 e_2 + \frac{5}{12}e_3\right)z^3 \\ & + \left(\frac{5}{12}e_4 - \frac{5}{96}e_1^4 + \frac{5}{16}e_1^2 e_2 - \frac{5}{12}e_1 e_3 - \frac{5}{24}e_2^2\right)z^4 + \cdots. \end{aligned} \tag{39}$$

Now, by comparing (38) and (39), we get

$$b_2 = \frac{5}{12}e_1, \tag{40}$$

$$b_3 = \frac{5}{24}e_2 - \frac{5}{288}e_1^2, \tag{41}$$

$$b_4 = \frac{5}{36}e_3 + \frac{35}{10368}e_1^3 - \frac{5}{96}e_1 e_2, \tag{42}$$

$$b_5 = \frac{5}{48}e_4 - \frac{455}{497664}e_1^4 + \frac{115}{6912}e_1^2 e_2 - \frac{35}{1152}e_2^2 - \frac{5}{108}e_1 e_3. \tag{43}$$

Utilizing (40) and (10), (11), (12), and (13), we have

$$\lambda_1 = \frac{5}{24}e_1, \tag{44}$$

$$\lambda_2 = \frac{5}{48}e_2 - \frac{5}{96}e_1^2, \tag{45}$$

$$\lambda_3 = \frac{5}{288}e_1^3 - \frac{5}{72}e_1 e_2 + \frac{5}{72}e_3, \tag{46}$$

$$\lambda_4 = \frac{5}{96}e_4 - \frac{5}{768}e_1^4 + \frac{5}{128}e_1^2 e_2 - \frac{5}{96}e_1 e_3 - \frac{5}{192}e_2^2. \tag{47}$$

From (44), using triangle inequality and (29), we get

$$|\lambda_1| \le \frac{5}{12}. \tag{48}$$

Also, from (45), application (30), and triangle inequality, we get

$$|\lambda_2| \le \frac{5}{24}. \tag{49}$$

By rearranging (46), we have

$$|\lambda_3| = \frac{5}{288}\left|e_1^3 - 4e_1 e_2 + 4e_3\right|. \tag{50}$$

By Lemma 4 and triangle inequality, we obtain

$$|\lambda_3| \le \frac{5}{36}. \tag{51}$$

By rearranging (47), we have

$$\lambda_4 = -\frac{5}{96}\left(\left(\frac{1}{2}\right)e_2^2 + \left(\frac{1}{8}\right)e_1^4 - \left(\frac{3}{4}\right)e_2 e_1^2 + e_1 e_3 - e_4\right). \tag{52}$$

Comparing the equation of (52) right side with

$$\left|\gamma e_1^4 + \varsigma e_2^2 + 2\tau e_1 e_3 - \frac{3}{2}\psi e_1^2 e_2 - e_4\right|, \tag{53}$$

we get $\gamma = 1/8$, $\varsigma = 1/2$, $\tau = 1/2$, $\psi = 1/2$, and

$$\begin{aligned} & 8(1-\varsigma)\varsigma\left[(\tau(\varsigma+\tau)-\psi)^2 + (\tau\psi - 2\gamma)^2\right] \\ & \quad + \tau(\psi - 2\varsigma\tau)^2(1-\tau) \le 4\tau^2\varsigma(1-\varsigma)(1-\tau)^2. \end{aligned} \tag{54}$$

Thus, Lemma 5's requirements are all met. Hence,

$$|\lambda_4| \leq \frac{5}{96}(2) = \frac{5}{48}. \tag{55}$$

These are sharp outcomes. Equality is determined by using (10)–(13) and

$$\begin{aligned} F_1(z) &= z\ \exp\left(\int_0^z \left(\frac{5}{6} + \frac{1}{6}t^4\right)dt\right) = z + \frac{5}{6}z^2 + \cdots, \\ F_2(z) &= z\ \exp\left(\int_0^z \left(\frac{5}{6}t + \frac{1}{6}t^9\right)dt\right) = z + \frac{5}{12}z^3 + \cdots, \\ F_3(z) &= z\ \exp\left(\int_0^z \left(\frac{5}{6}t^2 + \frac{1}{6}t^{14}\right)dt\right) = z + \frac{5}{18}z^4 + \cdots, \\ F_4(z) &= z\ \exp\left(\int_0^z \left(\frac{5}{6}t^3 + \frac{1}{6}t^{19}\right)dt\right) = z + \frac{5}{24}z^5 + \cdots. \end{aligned} \tag{56}$$

□

Theorem 7. *If $F \in \mathcal{S}^*_{4\mathcal{L}}$, then*

$$\left|\lambda_2 - \mu\lambda_1^2\right| \leq \max\left\{\frac{5}{24}, \frac{5}{48}\left|\frac{5\mu}{3}\right|\right\}. \tag{57}$$

The above stated inequality is best possible.

Proof. By utilizing (44) and (45), we have

$$\left|\lambda_2 - \mu\lambda_1^2\right| = \frac{5}{48}\left|e_2 - \frac{e_1^2}{2}\left(\frac{6+5\mu}{6}\right)\right|. \tag{58}$$

Implementation of (28) and triangle inequality, we get

$$\left|\lambda_2 - \mu\lambda_1^2\right| \leq \max\left\{\frac{5}{24}, \frac{5}{48}\left|\frac{5\mu}{3}\right|\right\}. \tag{59}$$

Equality is determined by using (10), (11), and

$$F_2(z) = z\ \exp\left(\int_0^z \left(\frac{5}{6}t + \frac{1}{6}t^9\right)dt\right) = z + \frac{5}{12}z^3 + \cdots. \tag{60}$$

□

Corollary 8. *If $F \in \mathcal{S}^*_{4\mathcal{L}}$, then*

$$\left|\lambda_2 - \lambda_1^2\right| \leq \frac{5}{24}. \tag{61}$$

This inequality is sharp and can be obtained by using (10), (11), and

$$F_2(z) = z\ \exp\left(\int_0^z \left(\frac{5}{6}t + \frac{1}{6}t^9\right)dt\right) = z + \frac{5}{12}z^3 + \cdots. \tag{62}$$

Theorem 9. *Let F be the expansion (1) and if $F \in \mathcal{S}^*_{4\mathcal{L}}$, then*

$$|\lambda_1\lambda_2 - \lambda_3| \leq \frac{5}{36}. \tag{63}$$

The above stated result is the best possible.

Proof. From (44)–(46), we easily attain

$$|\lambda_1\lambda_2 - \lambda_3| = \frac{65}{2304}\left|-e_1^3 + \frac{42}{13}e_1e_2 - \frac{32}{13}e_3\right|. \tag{64}$$

By using Lemma 4 and triangle inequality, we obtain

$$|\lambda_1\lambda_2 - \lambda_3| \leq \frac{65}{2304}\left(\frac{64}{13}\right) = \frac{5}{36}. \tag{65}$$

Equality is determined by using (10), (11), (12), and

$$F_3(z) = z\ \exp\left(\int_0^z \left(\frac{5}{6}t^2 + \frac{1}{6}t^{14}\right)dt\right) = z + \frac{5}{18}z^4 + \cdots. \tag{66}$$

□

Theorem 10. *Let F be the expansion (1) and if $F \in \mathcal{S}^*_{4\mathcal{L}}$, then*

$$\left|\lambda_4 - \lambda_2^2\right| \leq \frac{5}{48}. \tag{67}$$

The last stated inequality is the finest.

Proof. From the use (45) and (47), we get

$$\left|\lambda_4 - \lambda_2^2\right| = -\frac{5}{96}\left|\left(\frac{17}{24}\right)e_2^2 - \left(\frac{23}{44}\right)e_2e_1^2 + \left(\frac{17}{96}\right)e_1^4 + e_1e_3 - e_4\right|. \tag{68}$$

Comparing the right side of (68) with

$$\left|\gamma e_1^4 + \varsigma e_2^2 + 2\tau e_1e_3 - \frac{3}{2}\psi e_1^2e_2 - e_4\right|, \tag{69}$$

we get $\gamma = 17/96$, $\varsigma = 17/24$, $\tau = 1/2$, $\psi = 23/36$, and

$$\begin{aligned} 8(1-\varsigma)\varsigma&\left[(\tau(\varsigma+\tau)-\psi)^2 + (\tau\psi - 2\gamma)^2\right] \\ &+ \tau(\psi - 2\varsigma\tau)^2(1-\tau) = 0.0051909, \\ 4\tau^2\varsigma&(1-\varsigma)(1-\tau)^2 = 0.051649. \end{aligned} \tag{70}$$

Thus, Lemma 5's requirements are all met. Hence,

$$\left|\lambda_4 - \lambda_2^2\right| \leq \frac{5}{96}(2) = \frac{5}{48}. \tag{71}$$

Equality is determined by using (11), (13), and

$$F_4(z) = z\ \exp\left(\int_0^z \left(\frac{5}{6}t^3 + \frac{1}{6}t^{19}\right)dt\right) = z + \frac{5}{24}z^2 + \cdots. \tag{72}$$

□

Theorem 11. *Let $F \in \mathcal{S}^*_{4\mathcal{L}}$ be the representation (1). Then,*

$$\left|H_{2,1}(J_F/2)\right| \le \frac{25}{576}. \tag{73}$$

This result is sharp.

Proof. We can write the $H_{2,1}(J_F/2)$ as

$$H_{2,1}(J_F/2) = \left|\lambda_1\lambda_3 - \lambda_2^2\right|. \tag{74}$$

From (44)–(46), we have

$$\left|\lambda_1\lambda_3 - \lambda_2^2\right| = \frac{25}{27648}\left|e_1^4 - 4e_1^2e_2 + 16e_1e_3 - 12e_2^2\right|. \tag{75}$$

Using (25) and (26) to express e_2 and e_3 in terms of e_1 and also $e_1 = e$, with $0 \le e \le 2$, we obtain

$$\begin{aligned}\left|\lambda_1\lambda_3 - \lambda_2^2\right| = \frac{25}{27648}\Big|&-4e^2x^2\left(4-e^2\right) + 8e\left(1-|x|^2\right)\left(4-e^2\right)\delta \\ &- 3x^2\left(4-e^2\right)^2\Big|.\end{aligned} \tag{76}$$

By changing $|\delta| \le 1$ and $|x| = c$, where $c \le 1$ and utilizing triangle inequality and pickings $e \in [0, 2]$, so

$$\begin{aligned}\left|\lambda_1\lambda_3 - \lambda_2^2\right| \le \frac{25}{27648}\Big\{&4e^2c^2\left(4-e^2\right) + 8e\left(1-c^2\right)\left(4-e^2\right) \\ &+ 3c^2\left(4-e^2\right)^2\Big\} := \Xi(e,c).\end{aligned} \tag{77}$$

Differentiate with respect to c, we have

$$\frac{\partial \Xi(e,c)}{\partial c} = \frac{25}{27648}\left(-2ce^4 + 16ce^3 - 16ce^2 - 64ce + 96c\right). \tag{78}$$

It is easy exercise to show that $\Xi'(e,c) \ge 0$ on $[0, 1]$, so that $\Xi(e,c) \le \Xi(e,1)$. Putting $c = 1$, we get

$$\left|\lambda_1\lambda_3 - \lambda_2^2\right| \le \frac{25}{27648}\left(4e^2\left(4-e^2\right) + 3\left(4-e^2\right)^2\right) := \Theta(e). \tag{79}$$

As $\Theta'(e) \le 0$, so $\Theta(e)$ is a decreasing function, so that it gives a maximum value at $e = 0$

$$\left|H_{2,1}\left(\frac{J_F}{2}\right)\right| \le \frac{25}{27648}(48) = \frac{25}{576}. \tag{80}$$

Equality is determined by using (10), (11), (12), and

$$F_2(z) = z\ \exp\left(\int_0^z\left(\frac{5}{6}t + \frac{1}{6}t^9\right)dt\right) = z + \frac{5}{12}z^3 + \cdots. \tag{81}$$

□

4. Coefficient Inequalities for the Class $\mathcal{C}_{4\mathcal{L}}$

For the function of class $\mathcal{C}_{4\mathcal{L}}$, we start this portion by determining the absolute values of the first four initial logarithmic coefficients.

Theorem 12. *Let F be given by (1) and if $F \in \mathcal{C}_{4\mathcal{L}}$, then*

$$\begin{aligned}|\lambda_1| &\le \frac{5}{24}, \\ |\lambda_2| &\le \frac{5}{72}, \\ |\lambda_3| &\le \frac{5}{144}, \\ |\lambda_4| &\le \frac{1}{48}.\end{aligned} \tag{82}$$

These bounds are sharp.

Proof. Let $F \in \mathcal{C}_{4\mathcal{L}}$. Then, (17) can be written in the form of Schwarz function as

$$1 + \frac{zF''(z)}{F'(z)} = 1 + \frac{5}{6}u(z) + \frac{1}{6}(u(z))^5 = \psi(z). \tag{83}$$

Using (1), we obtain

$$\begin{aligned}1 + \frac{zF''(z)}{F'(z)} := 1 + 2b_2z + \left(6b_3 - 4b_2^2\right)z^2 + \left(8b_2^3 - 18b_2b_3 + 12b_4\right)z^3 \\ + \left(20b_5 - 16b_2^4 + 48b_2^2b_3 - 32b_2b_4 - 18b_3^2\right)z^4 + \cdots.\end{aligned} \tag{84}$$

Now, by comparing (84) and (39), we get

$$\begin{aligned}b_2 &= \frac{5}{24}e_1, \\ b_3 &= \frac{5}{72}e_2 - \frac{5}{864}e_1^2, \\ b_4 &= \frac{5}{144}e_3 + \frac{35}{41472}e_1^3 - \frac{5}{384}e_1e_2, \\ b_5 &= \frac{1}{48}e_4 - \frac{91}{497664}e_1^4 + \frac{23}{6912}e_1^2e_2 - \frac{7}{1152}e_2^2 - \frac{1}{108}e_1e_3.\end{aligned} \tag{85}$$

Utilizing (85) and (10), (11), (12), and (13) we have

$$\lambda_1 = \frac{5}{24}e_1, \tag{86}$$

$$\lambda_2 = \frac{5}{48}e_2 - \frac{5}{96}e_1^2, \tag{87}$$

$$\lambda_3 = \frac{5}{288}e_1^3 - \frac{5}{72}e_1e_2 + \frac{5}{72}e_3, \tag{88}$$

$$\lambda_4 = \frac{5}{96}e_4 - \frac{5}{768}e_1^4 + \frac{5}{128}e_1^2e_2 - \frac{5}{96}e_1e_3 - \frac{5}{192}e_2^2. \tag{89}$$

From (86), using triangle inequality and (29), we get

$$|\lambda_1| \le \frac{5}{24}. \tag{90}$$

Also, from (87), application (30), and triangle inequality, we get

$$|\lambda_2| \le \frac{5}{72}. \tag{91}$$

By rearranging (88), we have

$$|\lambda_3| = \frac{5}{288}\left|\frac{7}{48}e_1^3 - \frac{19}{24}e_1e_2 + e_3\right|. \tag{92}$$

By Lemma 4 and triangle inequality, we obtain

$$|\lambda_3| \le \frac{5}{144}. \tag{93}$$

By rearranging (89), we have

$$\lambda_4 = -\frac{1}{96}\left(\frac{11}{27}e_2^2 + \frac{13109}{248832}e_1^4 - \frac{2353}{5184}e_2e_1^2 + \frac{19}{24}e_1e_3 - e_4\right). \tag{94}$$

Comparing the right side of (94) with

$$\left|\gamma e_1^4 + \varsigma e_2^2 + 2\tau e_1 e_3 - \frac{3}{2}\psi e_1^2 e_2 - e_4\right|, \tag{95}$$

we get $\gamma = 13109/248832$, $\varsigma = 11/27$, $\tau = 19/48$, and $\psi = 2353/7776$. Thus, all the conditions of Lemma 5 are satisfied. Hence, we have

$$|\lambda_4| \le \frac{1}{96}(2) = \frac{1}{48}. \tag{96}$$

These are sharp outcomes. Equality is determined by using (10), (11), (12), and (13) along with (22). □

Theorem 13. *Let $F \in \mathscr{C}_{4\mathscr{L}}$ be the series form (1). Then,*

$$\left|\lambda_2 - \mu\lambda_1^2\right| \le \max\left\{\frac{5}{72}, \frac{5}{72}\left|\frac{7+15\mu}{12}\right|\right\}, \text{for } \mu \in \mathbb{C}. \tag{97}$$

This inequality is sharp.

Proof. By utilizing (86) and (87), we have

$$\left|\lambda_2 - \mu\lambda_1^2\right| = \frac{5}{144}\left|e_2 - \frac{e_1^2}{2}\left(\frac{19+15\mu}{24}\right)\right|. \tag{98}$$

Implementation of (28) and triangle inequality, we get

$$\left|\lambda_2 - \mu\lambda_1^2\right| \le \max\left\{\frac{5}{72}, \frac{5}{72}\left|\frac{7+15\mu}{12}\right|\right\}. \tag{99}$$

Equality is determined by using (10), (11), and (22). □

For $\lambda = 1$, we get the below corollary.

Corollary 14. *Let $F \in \mathscr{C}_{4\mathscr{L}}$, and it has the form (1). Then,*

$$\left|\lambda_2 - \lambda_1^2\right| \le \frac{5}{72}. \tag{100}$$

This inequality is sharp and can be obtained by using (10), (11), and (22).

Theorem 15. *Let F be the form (1) and if $F \in \mathscr{C}_{4\mathscr{L}}$, then*

$$|\lambda_1\lambda_2 - \lambda_3| \le \frac{5}{144}. \tag{101}$$

This result is sharp.

Proof. By using (86)–(88), we obtain

$$|\lambda_1\lambda_2 - \lambda_3| = \frac{5}{288}\left|-\frac{263}{1152}e_1^3 + e_1e_2 - e_3\right|. \tag{102}$$

By using Lemma 4 and triangle inequality, we obtain

$$|\lambda_1\lambda_2 - \lambda_3| \le \frac{5}{288}(2) = \frac{5}{144}. \tag{103}$$

Equality is determined by using (10), (11), (12), and (22). □

Theorem 16. *Let F be the form (1) and $F \in \mathscr{C}_{4\mathscr{L}}$. Then,*

$$\left|\lambda_4 - \lambda_2^2\right| \le \frac{1}{48}. \tag{104}$$

This result is sharp.

Proof. By using (87) and (89), we obtain

$$\left|\lambda_4 - \lambda_2^2\right| = -\frac{1}{96}\left|\frac{113}{216}e_2^2 - \frac{707}{1296}e_2e_1^2 + \frac{35243}{497664}e_1^4 + \frac{19}{24}e_1e_3 - e_4\right|. \tag{105}$$

Comparing the right side of (68) with

$$\left|\gamma e_1^4 + \varsigma e_2^2 + 2\tau e_1 e_3 - \frac{3}{2}\psi e_1^2 e_2 - e_4\right|, \tag{106}$$

we get $\gamma = 35243/497664$, $\varsigma = 113/216$, $\tau = 19/24$, $\psi = 707/1944$, and

$$8(1-\varsigma)\varsigma\left[(\tau(\varsigma+\tau)-\psi)^2+(\tau\psi-2\gamma)^2\right]+\tau(\psi-2\varsigma\tau)^2(1-\tau)=0.00062010, \tag{107}$$
$$4\tau^2\varsigma(1-\varsigma)(1-\tau)^2=0.057070.$$

Thus, all the conditions of Lemma 5 are satisfied. Hence, we have

$$\left|\lambda_4-\lambda_2^2\right|\le\frac{1}{96}(2)=\frac{1}{48}. \tag{108}$$

Equality is determined by using (11), (13), and (22). □

Theorem 17. *Let F be given the form (1) and* $F\in\mathscr{C}_{4\mathscr{L}}$. *Then,*

$$\left|H_{2,1}\left(\frac{J_F}{2}\right)\right|\le\frac{25}{576}. \tag{109}$$

This result is sharp.

Proof. We can write the $H_{2,1}(F_F/2)$ as;

$$H_{2,1}\left(\frac{J_F}{2}\right)=\left|\lambda_1\lambda_3-\lambda_2^2\right|. \tag{110}$$

From (86)–(88), we have

$$\left|\lambda_1\lambda_3-\lambda_2^2\right|=\frac{1}{47775744}\left|3575e_1^4-22800e_1^2e_2+86400e_1e_3-57600e_2^2\right|. \tag{111}$$

Using (25) and (26) to express e_2 and e_3 in terms of e_1 and also $e_1=e$, with $0\le e\le 2$, we obtain

$$\left|\lambda_1\lambda_3-\lambda_2^2\right|=\frac{1}{47775744}\left|-21600e^2x^2\left(4-e^2\right)+43200e\left(1-|x|^2\right)\left(4-e^2\right)\delta-14400x^2\left(4-e^2\right)^2+3000e^2x\left(4-e^2\right)-625e^4\right|. \tag{112}$$

By replacing $|\delta|\le 1$ and $|x|=c$, where $c\le 1$ and using triangle inequality and taking $e\in[0,2]$, so

$$\left|\lambda_1\lambda_3-\lambda_2^2\right|\le\frac{1}{47775744}\left\{21600e^2c^2\left(4-e^2\right)+43200e\left(1-c^2\right)\left(4-e^2\right)+14400c^2\left(4-e^2\right)^2+3000e^2c\left(4-e^2\right)+625e^4\right\}:=\Omega(e,c). \tag{113}$$

Differentiate with respect to c, we have

$$\frac{\partial\Omega(e,c)}{\partial c}=\frac{1}{47775744}\left(-600(e-2)(e+2)\cdot\left(24ce^2-144ce+5e^2+192c\right)\right). \tag{114}$$

It is a simple exercise to show that $\Omega'(e,c)\ge 0$ on $[0,1]$, so that $\Omega(e,c)\le\Omega(e,1)$. Putting $c=1$ gives

$$\left|\lambda_1\lambda_3-\lambda_2^2\right|\le\frac{1}{47775744}\left(24600e^2\left(4-e^2\right)+625e^4+14400\left(4-e^2\right)^2\right):=\Theta(e). \tag{115}$$

As $\Theta'(e)\le 0$, so $\Theta(e)$ is a decreasing function, so that it gives a maximum value at $e=0$

$$\left|H_{2,1}\left(\frac{J_F}{2}\right)\right|\le\frac{1}{47775744}(230400)=\frac{25}{5184}. \tag{116}$$

Equality is determined by using (10), (11), (12), and (22). □

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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