Research Article

Product-Type Operators on the Space of Fractional Cauchy Transforms

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The space of fractional Cauchy transforms plays a central role in classical complex analysis, harmonic analysis, and geometric measure theory. In this paper, we study the boundedness and compactness of product-type operators from the space of fractional Cauchy transforms to the Zygmund-type space in terms of the function theoretic characterization of Julia–Carathéodory type.

1. Introduction

Let \( \mathcal{S} = \mathcal{S}(D) \) be the class of all holomorphic self-maps of the unit disk \( D \) of the complex plane \( C \), \( T \) be the boundary of \( D \), \( \mathbb{N}_0 \) be the set of all nonnegative integers, and \( \mathbb{N} \) be the set of all positive integers. Denote by \( H(D) \) the space of all holomorphic functions on \( D \).

We first recall the spaces we work on. Let \( \alpha > 0 \) be a real number and \( \mathcal{M} \) be the space of all complex Borel measures on \( T \) endowed with the total variation norm. The family \( \mathcal{F}_\alpha \) of fractional Cauchy transforms is the collection of holomorphic functions \( f \) in \( D \) for which

\[
    f(z) = \int_\mathcal{M} \frac{d\mu(\xi)}{1 - \xi z}^\alpha, \quad (z \in D),
\]

for some \( \mu \in \mathcal{M} \). The space \( \mathcal{F}_\alpha \) is a Banach space, with respect to the norm given by

\[
    \|f\|_{\mathcal{F}_\alpha} = \inf \{\|\mu\|\},
\]

where the infimum extends over all measures \( \mu \).

The fractional Cauchy transforms space \( \mathcal{F}_\alpha \) plays a central role in classical complex analysis, harmonic analysis, and geometric measure theory which has phenomenal development in connection with the Calderon–Zygmund-type singular integral theory. The space \( \mathcal{F}_\alpha \) may be identified with \( \mathcal{M}/\mathcal{H}^1_0 \), the quotient of the Banach space \( \mathcal{M} \) by \( \mathcal{H}^1_0 \), and the subspace of \( L^1 \) consisting of functions with mean value 0 whose conjugate belongs to the Hardy space \( H^1 \). Hence, \( \mathcal{F}_\alpha \) is isometrically isomorphic to \( \mathcal{M}/\mathcal{H}^1_0 \). Furthermore, \( \mathcal{M} \) admits a decomposition \( \mathcal{M} = L^1 \oplus \mathcal{M}_s \), where \( \mathcal{M}_s \) is the space of Borel measures, which are singular with respect to Lebesgue measure, and \( \mathcal{H}^1_0 \subset L^1 \). According to the Lebesgue decomposition theorem, each \( \mu \in \mathcal{M} \) can be written as \( \mu = \mu_a \oplus \mu_s \), where \( \mu_a \) is absolutely continuous with respect to the Lebesgue measure and \( \mu_s \) is singular with respect to the Lebesgue measure \( (\mu_a \perp \mu_s) \). Consequently, \( \mathcal{F}_\alpha \) is isometrically isomorphic to \( L^1/\mathcal{H}^1_0 \). Hence, \( \mathcal{F}_\alpha \) can be written as \( \mathcal{F}_\alpha = \mathcal{F}_\alpha a \oplus \mathcal{F}_\alpha s \), where \( \mathcal{F}_\alpha a \) is isomorphic to \( L^1/\mathcal{H}^1_0 \), the closed subspace of \( \mathcal{M} \) of absolutely continuous measures, and \( \mathcal{F}_\alpha s \) is isomorphic to \( \mathcal{M}_s \), the subspace of \( \mathcal{M} \) of singular measures. For further results about the space of fractional Cauchy transforms, we refer to [1–12] and references therein.

Let \( \vartheta \) be a weight, that is, \( \vartheta \) is a positive continuous function on \( D \). A positive continuous function \( \nu \) on the interval \( [0, 1] \) is said to be normal if there are \( \delta \in [0, 1) \) and \( \tau \) and \( \eta \), \( 0 < \tau < \eta \) such that
\[
\frac{\nu(r)}{1 - r} \text{ is decreasing on } \delta, 1 \text{ and } \lim_{r \to 1} \frac{\nu(r)}{1 - r} = 0, \\
\frac{\nu(r)}{1 - r} \text{ is increasing on } \delta, 1 \text{ and } \lim_{r \to 1} \frac{\nu(r)}{1 - r} = \infty.
\]

(3)

In this paper, we assume the normal weighted function \( \nu: \mathbb{D} \to [0, \infty) \) is also radial, i.e., \( \nu(z) = \nu(|z|), z \in \mathbb{D} \).

Now, the Zygmund-type space \( Z \) consists of all \( f \in H(\mathbb{D}) \) such that
\[
b_r(f) = \sup_{z \in \mathbb{D}} \nu(z) |f''(z)| < \infty.
\]

(4)

With the norm \( \|f\|_Z = |f(0)| + |f'(0)| + b_r(f) \), the Zygmund-type space is a Banach space.

For \( \phi \in \mathcal{S} \) and \( u \in H(\mathbb{D}) \), the weighted composition operator, which plays an important role in the isometry theory of Banach spaces, induced by \( u \) and \( \phi \) is given by
\[
W_{u, \phi}(f) = u \cdot f \circ \phi, \quad f \in H(\mathbb{D}).
\]

(5)

We can regard this operator as a generalization for a multiplication operator \( M_u \) induced by \( u \) and a composition operator \( C_\phi \) induced by \( \phi \), where \( M_u f = u \cdot f \) and \( C_\phi f = f \circ \phi \).

An extensive study concerning the theory of (weighted) composition operators has been established during the past four decades on various settings. We refer to standard references [13–15] for various aspects about the theory of composition operators acting on holomorphic function spaces, especially the problems of relating operator-theoretic properties of \( C_\phi \) to function theoretic properties of \( \phi \). The differentiation operator, on \( H(\mathbb{D}) \), is defined by
\[
Df(z) = f'(z), \quad z \in \mathbb{D}.
\]

(6)

Note that \( D \) is typically unbounded on many familiar spaces of holomorphic functions. The differential operator plays an important role in various fields such as dynamical system theory and operator theory.

The products of any two of \( C_\phi, M_u, \) and \( D \) can be obtained in six ways, i.e., \( M_u C_\phi, C_\phi M_u, M_u D, D M_u, C_\phi D, \) and \( D C_\phi \). Similarly, the products of all of \( C_\phi, M_u, \) and \( D \) can also be obtained in six ways, i.e., \( M_u C_\phi D, C_\phi M_u D, M_u D C_\phi, C_\phi D M_u, D M_u C_\phi, \) and \( D C_\phi M_u \). In order to treat above product-type operators in a unified manner, Stević et al. [16], for the first time, introduced the so-called Stević–Sharma operator:
\[
T_{u_1, u_2, \phi} f(z) = u_1(z) f(\phi(z)) + u_2(z) f'(\phi(z)), \quad f \in H(\mathbb{D}),
\]

(7)

for \( \phi \in \mathcal{S} \), \( u_1, u_2 \in H(\mathbb{D}) \). This operator is related to the various products of multiplication, composition, and differentiation operators. It is clear that all products of multiplication, composition, and differentiation operator in the following six ways can be obtained from the operator \( T_{u_1, u_2, \phi} \) by choosing different \( u_1 \) and \( u_2 \). More specially, we have
\[
M_u D C_\phi = T_{0, u_2, \phi}, \\
C_\phi M_u D = T_{0, u_1, \phi}, \\
M_u C_\phi D = T_{0, u_1, \phi}, \\
D M_u C_\phi = T_{u_2, 0, \phi}, \\
C_\phi D M_u = T_{u_1, 0, \phi}, \\
D C_\phi M_u = T_{u_1, u_2, \phi}.
\]

(8)

Recently, product-type operators on some spaces of holomorphic functions on the unit disk have become a subject of increasing interest (see [17–19] and references therein). Hibschweiler et al. [20] first characterized the boundedness and compactness of \( DC_\phi \) between Bergman spaces and Hardy spaces. Liu and Yu [21] investigated the boundedness and compactness of the operator \( DC_\phi \) from \( H^{\infty} \) and Bloch spaces to Zygmund spaces. Ohno [22] considered the boundedness and compactness of \( C_\phi D \) on Hardy space \( H^2 \). Zhu [23] studied the boundedness and compactness of linear operators which are obtained by taking products of multiplication, composition, and differentiation operators from Bergman-type spaces to Bers-type spaces. Quite recently, Zhang and Liu [24] presented the boundedness and compactness of the operator \( T_{u_1, u_2, \phi} \) from Hardy spaces to Zygmund-type spaces. Liu et al. [26] investigated the compactness of the operator \( T_{u_1, u_2, \phi} \) on logarithmic Bloch spaces. Yu and Liu [27] gave the complete characterizations for the boundedness and compactness of the operator \( T_{u_1, u_2, \phi} \) from Hardy spaces to the logarithmic Bloch spaces. Liu et al. [28] presented the boundedness and compactness of the operator \( T_{u_1, u_2, \phi} \) from \( H^{\infty} \) space to the logarithmic Bloch spaces. Jiang [29] characterized the boundedness and compactness of the operator \( T_{u_1, u_2, \phi} \) from the Zygmund spaces to the Bloch–Orlicz spaces. Li and Guo [29] studied the boundedness and compactness of the operator \( T_{u_1, u_2, \phi} \) from Zygmund-type spaces to Bloch–Orlicz spaces.

Inspired by the above results, the purpose of the paper is devoted to the boundedness and compactness of the operator \( T_{u_1, u_2, \phi} \) from the fractional Cauchy transforms’ spaces to the Zygmund-type spaces over the unit disk in terms of the function theoretic characterization of Julia–Carathéodory type. As the applications of our main results, readers easily can obtain the boundedness and compactness characterizations of all six product-type operators:
\[
M_u D C_\phi, \\
C_\phi M_u D, \\
M_u C_\phi D, \\
D M_u C_\phi, \\
C_\phi D M_u, \\
D C_\phi M_u.
\]

(9)
from the space of fractional Cauchy transforms to the Zygmund-type spaces.

2. Preliminaries

In this section, we recall some basic facts and preliminary results to be used in the sequel.

Suppose X and Y are two Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively. Recall that a linear operator $T$ from $X$ to $Y$ is bounded if there is a positive constant $C$ such that $\|T(f)\|_Y \leq C \|f\|_X$, for all $f \in X$. The bounded operator $T$: $X \rightarrow Y$ is said to be compact if the image of every bounded set of $X$ is relatively compact in $Y$. Equivalently, $T$: $X \rightarrow Y$ is compact if and only if the image of every bounded sequence in $X$ has a subsequence that converges in $Y$.

The following lemma gives a convenient compactness criterion for the Stević–Sharma operator $T_{u_1,u_2,\varphi}f(z) = u_1(z)f(\varphi(z)) + u_2(z)f'(\varphi(z))$ acting from the space of fractional Cauchy transforms $\mathcal{F}_a$ to the Zygmund-type spaces $\mathcal{L}_\nu$.

**Lemma 1.** Suppose $u_1, u_2 \in H(D), \varphi \in \mathcal{S}$. Then, the operator $T_{u_1,u_2,\varphi}: \mathcal{F}_a \rightarrow \mathcal{L}_\nu$ is compact if and only if $T_{u_1,u_2,\varphi}f_n \rightarrow 0$ in $\mathcal{L}_\nu$, for any bounded sequence $\{f_n\}$ in $\mathcal{F}_a$, such that $\{f_n\} \rightarrow 0$ uniformly on compact subsets of $D$.

A proof can be found in Proposition 3.11 of [13] for a single composition operator over the unit disk, and it can be easily modified for the operator $T_{u_1,u_2,\varphi}$ on $\mathcal{F}_a$.

The following lemma is taken from [30] which is vital to construct the test functions on the space of fractional Cauchy transforms.

**Lemma 2.** Let $\alpha, \beta > 0, f \in H(D)$

1. If $f \in \mathcal{F}_a, z \in D$, then there exists a $C > 0$ such that
   \[ |f(z)| \leq C \|f\|_{\mathcal{F}_a}/(1 - |z|^2)^\alpha \]
2. If $f \in \mathcal{F}_a$, then $f \in \mathcal{F}_{a+1}$ and $\|f\|_{\mathcal{F}_{a+1}} \leq C \|f\|_{\mathcal{F}_a}$
3. If $f' \in \mathcal{F}_{a+1}$, then $f \in \mathcal{F}_a$, and there exists a $C > 0$ such that $\|f\|_{\mathcal{F}_a} \leq C \|f(0)\| + C \|f'\|_{\mathcal{F}_{a+1}}$
4. If $f \in \mathcal{F}_a$ and $g \in \mathcal{F}_\beta$, then $fg \in \mathcal{F}_{a+\beta}$ and $\|fg\|_{\mathcal{F}_{a+\beta}} \leq \|f\|_{\mathcal{F}_a} \|g\|_{\mathcal{F}_\beta}$.

Based on Lemma 2, we can obtain the following lemma, see Lemma 2 of [31], for the detailed proof.

**Lemma 3.** Let $\alpha > 0, s \in \mathbb{N}_0, z \in D$. Put
\[ r_z^s(w) = \left(1 - |z|^2\right)^s \left(1 - |w|^2\right)^{-s} , \quad w \in D. \]

Then, $r_z^s \in \mathcal{F}_a$ and $\sup_{z \in D} \|r_z^s\|_{\mathcal{F}_a} < \infty$.

Furthermore, we need the following lemma to prove our main results.

**Lemma 4.** Let $\alpha > 0$. Suppose that $f \in \mathcal{F}_a$ and $n \in \mathbb{N}_0$. Then, there is a positive constant $C$ independent of $f$ such that
\[ |f^{(n)}(z)| \leq C \|f\|_{\mathcal{F}_a}/(1 - |z|^2)^{\alpha n} \]

**Proof.** For $f \in \mathcal{F}_a$, there is a $\mu \in \mathcal{M}$ such that (1) holds. Then, we have
\[ f^{(n)}(z) = n! \sum_{j=0}^{n} \frac{\mu(z^j)}{(1 - |z|^2)^{\alpha j}} \]

Thus, we have
\[ |f^{(n)}(z)| \leq C \int \frac{\mu(z)}{(1 - |z|^2)^{\alpha n}} \leq C \int \frac{\mu(z)}{(1 - |z|^2)^{\alpha n}} = C \|f\|_{\mathcal{F}_a}/(1 - |z|^2)^{\alpha n} \]

Taking infimum over all measures, $\mu \in \mathcal{M}$, for which (1) holds; the proof is complete.

3. Main Results and Proofs

In this section, we devote to investigating the boundedness and compactness of the operator $T_{u_1,u_2,\varphi}$ acting from the spaces of fractional Cauchy transforms to the Zygmund-type spaces in terms of the function theoretic characterization of Julia–Carathéodory type.

**Theorem 1.** Let $\alpha > 0$. Suppose $u_1, u_2 \in H(D), \varphi \in \mathcal{S}$. Then, $T_{u_1,u_2,\varphi}: \mathcal{F}_a \rightarrow \mathcal{L}_\nu$ is bounded if and only if the following conditions are satisfied:

\[ \sup_{z \in D} \frac{\nu(z)\|u_1''(z)\|}{(1 - |\varphi(z)|^2)^{\alpha + 1}} < \infty, \]
\[ \sup_{z \in D} \frac{\nu(z)|2u_1'(z)\varphi'(z) + u_1(z)\varphi''(z) + u_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\alpha + 1}} < \infty, \]
\[ \sup_{z \in D} \frac{\nu(z)|u_1(z)(\varphi'(z))^2 + 2u_2(z)\varphi'(z) + u_2(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^{\alpha + 2}} < \infty, \]
\[ \sup_{z \in D} \frac{\nu(z)|u_2(z)(\varphi'(z))|}{(1 - |\varphi(z)|^2)^{\alpha + 3}} < \infty. \]

**Proof.** Suppose that (14)–(17) hold. Let $f \in \mathcal{F}_a$ with $\|f\|_{\mathcal{F}_a} \leq 1$. Using Lemma 4, we have
Applying (14)–(17), it follows from the last above inequality that $T_{u_1,u_2,\varphi}F_{\mathcal{A}} \to F_{\mathcal{S}}$ is bounded.

Conversely, assume that $T_{u_1,u_2,\varphi}F_{\mathcal{A}} \to F_{\mathcal{S}}$ is bounded. Then, there exists a constant $C$ such that

$$\left\| T_{u_1,u_2,\varphi}f \right\|_{F_{\mathcal{S}}} \leq C \left\| f \right\|_{F_{\mathcal{A}}},$$

for all $f \in F_{\mathcal{A}}$. It is elementary to deduce that $z^n \in F_{\mathcal{A}}$, for $n \in \mathbb{N}_0$. First, take the function $f(z) = 1$, we obtain

$$\sup_{z \in \mathbb{D}} \nu(z) |u_1^n(z)| < \infty.$$  

Then, put $f(z) = z$, and we apply (20) to have

$$\sup_{z \in \mathbb{D}} \nu(z) |2u_1(z)\varphi(z) + u_1(z)\varphi''(z) + u_1''(z)| < \infty.$$  

Next, taking $f(z) = (z^2/2)$, (20) and (21) yield that

$$\nu(z) |u_1(z)(\varphi'(z))^2 + 2u_1'(z)\varphi'(z) + u_2(z)\varphi''(z)| \leq \left\| T_{u_1,u_2,\varphi}f \right\|_{F_{\mathcal{S}}} < \infty.$$  

Furthermore, putting $f(z) = (z^3/6)$, we deduce that

$$\begin{align*}
\infty > & \left\| T_{u_1,u_2,\varphi}f \right\|_{F_{\mathcal{S}}} \\
& = \nu(z) |u_2''(z) f(\varphi(z)) + 2u_1'(z)\varphi'(z) + u_1(z)\varphi''(z) + u_1''(z) f'(\varphi(z))| \\
& + (u_1(z)\varphi'(z))^2 + 2u_1'(z)\varphi'(z) + u_2(z)\varphi''(z) f''(\varphi(z)) + u_2(z)(\varphi'(z))^2 f^3(\varphi(z))| \\
& = \nu(z) |u_1'(z)\varphi'(z) + 2u_1'(z)\varphi'(z) + u_2(z)\varphi''(z) f''(\varphi(z)) + u_2(z)(\varphi'(z))^2 f^3(\varphi(z))| \\
& + (u_1(z)\varphi'(z))^2 + 2u_1'(z)\varphi'(z) + u_2(z)\varphi''(z) \phi(\varphi(z)) + u_2(z)(\varphi'(z))^2|.
\end{align*}$$
Applying (20)–(22) gives that
\[ \sup_{z \in D} v(z) a_z(z) (\varphi'(z))^2 < \infty. \] (24)

Fix \( w \in D \) and \( a, b, c \in \mathbb{R} \). Consider the following test function:
\[
    f_w(z) = \frac{a(1-\|w\|^2)}{(1-\|w\|^2)^{a+1}} - \frac{(1-\|w\|^2)^2}{(1-\|w\|^2)^{a+2}} \\
    + \frac{b(1-\|w\|^2)^3}{(1-\|w\|^2)^{a+3}} + \frac{c(1-\|w\|^2)^4}{(1-\|w\|^2)^{a+4}}
\] (25)

where
\[
    f'_w(z) = \frac{a(1-\|w\|^2)}{(1-\|w\|^2)^{a+2}} - \frac{(1-\|w\|^2)^2}{(1-\|w\|^2)^{a+3}} \\
    + \frac{(a+3)b(1-\|w\|^2)^3}{(1-\|w\|^2)^{a+4}} + \frac{(a+4)c(1-\|w\|^2)^4}{(1-\|w\|^2)^{a+5}}
\]

Hence,
\[
    \sup_{z \in D} \nu(z) a_z(z) (\varphi'(z))^2 \leq C. \]

Thus, we have
\[
C \geq \| T_{u_1, u_2, f \varphi; \varphi(w)} \| \geq \nu(w) \| (T_{u_1, u_2, f \varphi; \varphi(w)})'' (w) \| \\
= \nu(w) \| u''_1(w) f \varphi(w) \varphi'(w) \| \\
= \frac{(a+b+c-1)\nu(w) |u''_1(w)|}{(1-|\varphi(w)|^2)^a}
\] (28)

Lemma 3 gives that \( f_w \in \mathcal{F}_a \) and \( \sup_{z \in D} f_w \leq C \). A straightforward calculation shows that
\[
    \sup_{z \in D} \nu(z) a_z(z) (\varphi'(z))^2 < \infty.
\] (29)

Put \( a = \frac{(a+2)/3a+3}{(a+2)/3a+12} \), \( b = \frac{(a+2)/a+3}{(a+2)/a+12} \), and \( c = -\frac{(a+2)/3a+3}{(a+2)/3a+12} \) in (25) such that
\[
f_w(w) = \frac{a + b + c - 1}{(1-\|w||^2)^a},
\] (27)

\[
f'_w(w) = f''_w(w) = f'''_w(w) = 0.
\]

Thus, we have
\[
    s \geq \| T_{u_1, u_2, f \varphi; \varphi(w)} \| \geq \nu(w) \| (T_{u_1, u_2, f \varphi; \varphi(w)})'' (w) \| \\
= \nu(w) \| u''_1(w) f \varphi(w) \varphi'(w) \| \\
= \frac{(a+b+c-1)\nu(w) |u''_1(w)|}{(1-|\varphi(w)|^2)^a}.
\] (28)

It follows from Lemma 3 that \( g_w \in \mathcal{F}_a \) and \( \sup_{z \in D} g_w \leq C \). In addition,
\[ g'_w(z) = -\frac{(\alpha + 1)\overline{w}(1 - |w|^2)}{(1 - |w|^2)^{\alpha+1}} + \frac{(\alpha + 2)\overline{w}(1 - |w|^2)^2}{(1 - |w|^2)^{\alpha+2}} + \frac{(\alpha + 3)\overline{w}(1 - |w|^2)^3}{(1 - |w|^2)^{\alpha+3}} + \frac{(\alpha + 4)\overline{w}(1 - |w|^2)^4}{(1 - |w|^2)^{\alpha+4}} \]

\[ g''_w(z) = -\frac{(\alpha + 1)(\alpha + 2)\overline{w}(1 - |w|^2)}{(1 - |w|^2)^{\alpha+3}} + \frac{(\alpha + 2)(\alpha + 3)\overline{w}(1 - |w|^2)^2}{(1 - |w|^2)^{\alpha+4}} + \frac{(\alpha + 3)(\alpha + 4)\overline{w}(1 - |w|^2)^3}{(1 - |w|^2)^{\alpha+5}} + \frac{(\alpha + 4)(\alpha + 5)\overline{w}(1 - |w|^2)^4}{(1 - |w|^2)^{\alpha+6}} \]

\[ g^{(3)}_w(z) = -\frac{(\alpha + 1)(\alpha + 2)(\alpha + 3)\overline{w}(1 - |w|^2)}{(1 - |w|^2)^{\alpha+4}} + \frac{(\alpha + 2)(\alpha + 3)(\alpha + 4)\overline{w}(1 - |w|^2)^2}{(1 - |w|^2)^{\alpha+5}} + \frac{(\alpha + 3)(\alpha + 4)(\alpha + 5)\overline{w}(1 - |w|^2)^3}{(1 - |w|^2)^{\alpha+6}} + \frac{(\alpha + 4)(\alpha + 5)(\alpha + 6)\overline{w}(1 - |w|^2)^4}{(1 - |w|^2)^{\alpha+7}} \]

Put \( d = (3\alpha + 7)/(\alpha + 3), e = -(\alpha + 2)(3\alpha + 11)/(\alpha + 3) \)
\( (\alpha + 4), \) and \( h = ((\alpha + 2)/(\alpha + 4))^{(30)} \) \( (\alpha + 4) \) such that
\[
g_w(w) = g'_w(w) = g''_w(w) = 0, \]
\[
g''_w(w) = \frac{(-\alpha + 1) + d(\alpha + 2) + e(\alpha + 3) + h(\alpha + 4))\overline{w}}{(1 - |w|^2)^{\alpha+1}} \]

(32)

\[ C \geq \left\| T_{u_1, u_2, \varphi} g_{\varphi(w)} \right\|_{\mathcal{F}} \]
\[ \geq \nu(w)\left\| T''_{u_1, u_2, \varphi} g_{\varphi(w)} \right\| (w) \]
\[ = \nu(w)\left\| 2u'_1(z)\varphi'(w) + u_1(w)\varphi''(w) + u_2''(w) \right\| g_{\varphi(w)}(\varphi(w)) \]
\[ (-\alpha + 1) + d(\alpha + 2) + e(\alpha + 3) + h(\alpha + 4) \cdot \frac{\nu(w)\left\| 2u'_1(w)\varphi'(w) + u_1(w)\varphi''(w) + u_2''(w) \right\| |\varphi(w)|}{(1 - |\varphi(w)|^2)^{\alpha+1}} \]

(33)

Hence,
\[ \sup_{\mathbb{D}} \frac{\nu(z)\left\| u'_1(z)\varphi'(z) + u_1(z)\varphi''(z) + u_2''(z) \right\| |\varphi(z)|}{(1 - |\varphi(z)|^2)^{\alpha+1}} < \infty, \]

(34)

\[ \sup_{|\varphi(z)|} \frac{1}{2} \frac{\nu(z)\left\| 2u'_1(z)\varphi'(z) + u_1(z)\varphi''(z) + u_2''(z) \right\|}{(1 - |\varphi(z)|^2)^{\alpha+1}} \]
\[ \leq \sup_{|\varphi(z)|} \frac{1}{2} \frac{\nu(z)\left\| 2u'_1(z)\varphi'(z) + u_1(z)\varphi''(z) + u_2''(z) \right\| |\varphi(z)|}{(1 - |\varphi(z)|^2)^{\alpha+1}} < \infty. \]

(35)
Applying (21) yields that

\[ \sup_{|p(z)|} \left| \frac{\nu(z)\left[2\mu'_1(z)\varphi'(z) + \mu_1(z)\varphi''(z) + \mu''_2(z)\right]}{(1 - |\varphi(z)|^2)^{\alpha+1}} \right| \leq C \sup_{|p(z)|} \frac{\nu(z)\left[2\mu'_1(z)\varphi'(z) + \mu_1(z)\varphi''(z) + \mu''_2(z)\right]}{(1 - |\varphi(z)|^2)^{\alpha+1}} < \infty, \quad (36) \]

Combining (20) and (36), we can obtain

\[ \sup_{z \in \mathbb{D}} \frac{\nu(z)\left[2\mu'_1(z)\varphi'(z) + \mu_1(z)\varphi''(z) + \mu''_2(z)\right]}{(1 - |\varphi(z)|^2)^{\alpha+1}} < \infty, \quad (37) \]

which means that (15) holds.

Fix \( w \in \mathbb{D} \) and \( p, q, i \in \mathbb{R} \). Consider the following test function:

\[ h_w(z) = \frac{p(1 - |w|^2)}{(1 - \overline{w}z)^{\alpha+1}} + \frac{i(1 - |w|^2)^2}{(1 - \overline{w}z)^{\alpha+2}} - \frac{(1 - |w|^2)^3}{(1 - \overline{w}z)^{\alpha+3}} + \frac{q(1 - |w|^2)^4}{(1 - \overline{w}z)^{\alpha+4}} \]

Lemma 3 yields that \( h_w \in \mathcal{F}_\alpha \) and \( \sup_{w \in \mathbb{D}} \|h_w\|_{\mathcal{F}_\alpha} \leq C \). Furthermore, we obtain that

\[ h_w'(z) = \frac{\nu(z)\left[2\mu'_1(z)\varphi'(z) + \mu_1(z)\varphi''(z) + \mu''_2(z)\right]}{(1 - |\varphi(z)|^2)^{\alpha+1}} \leq C \sup_{|p(z)|} \frac{\nu(z)\left[2\mu'_1(z)\varphi'(z) + \mu_1(z)\varphi''(z) + \mu''_2(z)\right]}{(1 - |\varphi(z)|^2)^{\alpha+1}} \]

Put \( p = \frac{(-\alpha - 4)/3\alpha + 10}{i = (3\alpha + 11)/3\alpha + 10} \), and \( q = (\alpha + 3/(3\alpha + 10)) \) in (38) such that

\[ h_w(w) = h_w'(w) = h_w^{(3)}(w) = 0, \]

\[ h_w^*(w) = p(\alpha + 1)(\alpha + 2) + i(\alpha + 2)(\alpha + 3) + q(\alpha + 4)(\alpha + 5) - (\alpha + 3)(\alpha + 4) \frac{(\overline{w})^2}{(1 - |w|^2)^{\alpha+3}}, \quad (40) \]
Thus, we have
\[
C \geq \left\| T_{u_1,u_2}J_{\varphi(w)} \right\|_{\mathcal{F}_s} \\
\geq \nu(w) \left( T_{u_1,u_2}h_{\varphi(w)} \right)^\alpha (w) \\
= \nu(w) \left| u_1(w)(\varphi'(w))^2 + 2u_2(w)\varphi'(w) + u_2(z)\varphi''(w) \right| \cdot \left| h_{\varphi(w)} \right| \cdot |\varphi(w)|^2 \\
= (p(\alpha + 1)(\alpha + 2) + i(\alpha + 2)(\alpha + 3) + q(\alpha + 4)(\alpha + 5) - (\alpha + 3)(\alpha + 4)) \\
\cdot \frac{\nu(w)\left| u_1(w)(\varphi'(w))^2 + 2u_2(w)\varphi'(w) + u_2(z)\varphi''(w) \right| |\varphi(w)|^2}{\left(1 - |\varphi(w)|^2\right)^{\alpha + 2}}.
\]

Hence,
\[
\sup_{z \in D} \frac{\nu(z)\left| u_1(z)(\varphi'(z))^2 + 2u_2(z)\varphi'(z) + u_2(z)\varphi''(z) \right| |\varphi(z)|^2}{\left(1 - |\varphi(z)|^2\right)^{\alpha + 2}} < \infty,
\]
which implies that
\[
\sup_{|\varphi(z)|} > \frac{1}{2} \frac{\nu(z)\left| u_1(z)(\varphi'(z))^2 + 2u_2(z)\varphi'(z) + u_2(z)\varphi''(z) \right| |\varphi(z)|^2}{\left(1 - |\varphi(z)|^2\right)^{\alpha + 2}} \leq C \sup_{|\varphi(z)|} > \frac{1}{2} \frac{\nu(z)\left| u_1(z)(\varphi'(z))^2 + 2u_2(z)\varphi'(z) + u_2(z)\varphi''(z) \right| |\varphi(z)|^2}{\left(1 - |\varphi(z)|^2\right)^{\alpha + 2}} < \infty.
\]

Applying (22) yields that
\[
\sup_{|\varphi(z)|} \leq \frac{1}{2} \frac{\nu(z)\left| u_1(z)(\varphi'(z))^2 + 2u_2(z)\varphi'(z) + u_2(z)\varphi''(z) \right| |\varphi(z)|^2}{\left(1 - |\varphi(z)|^2\right)^{\alpha + 2}} \leq C \sup_{|\varphi(z)|} \leq \frac{1}{2} \frac{\nu(z)\left| u_1(z)(\varphi'(z))^2 + 2u_2(z)\varphi'(z) + u_2(z)\varphi''(z) \right| |\varphi(z)|^2}{\left(1 - |\varphi(z)|^2\right)^{\alpha + 2}} < \infty.
\]

Together (43) with (44), we can obtain
\[
\left( \nu(z)\left| u_1(z)(\varphi'(z))^2 + 2u_2(z)\varphi'(z) + u_2(z)\varphi''(z) \right| \right) < \infty\sup_{z \in D} \frac{\left| \left| u_1(z)(\varphi'(z))^2 + 2u_2(z)\varphi'(z) + u_2(z)\varphi''(z) \right| \right|}{\left(1 - |\varphi(z)|^2\right)^{\alpha + 2}} \leq C. \quad \text{(45)}
\]
which means that (16) holds.

Fix \( w \in D \) and \( l,m,n \in \mathbb{R} \). Consider the following test function:
\[
J_w(z) = \frac{m(1 - |w|^2)}{(1 - w)z} + \frac{l(1 - |w|^2)^2}{(1 - w)z} + \frac{n(1 - |w|^2)^3}{(1 - w)z} + \frac{(1 - |w|^2)^4}{(1 - w)z}.
\]

Applying Lemma 3 yields that \( J_w \in \mathcal{F}_a \) and \( \sup_{w \in D} \left\| J_w \right\|_{\mathcal{F}_s} \leq C \). A straightforward calculation shows that
\[ J'_w(z) = \frac{(\alpha + 1)n\bar{w}(1-|w|^2)}{(1 - \bar{w}^2)^{\alpha + 2}} + \frac{(\alpha + 2)n\bar{w}(1-|w|^2)^2}{(1 - \bar{w}^2)^{\alpha + 3}} \\
+ \frac{(\alpha + 3)n\bar{w}(1-|w|^2)^3}{(1 - \bar{w}^2)^{\alpha + 4}} - \frac{(\alpha + 4)n\bar{w}(1-|w|^2)^4}{(1 - \bar{w}^2)^{\alpha + 5}}, \]

\[ J''_w(z) = \frac{(\alpha + 1)(\alpha + 2)n\bar{w}^2(1-|w|^2)}{(1 - \bar{w}^2)^{\alpha + 3}} + \frac{(\alpha + 2)(\alpha + 3)n\bar{w}^3(1-|w|^2)^2}{(1 - \bar{w}^2)^{\alpha + 4}} \\
+ \frac{(\alpha + 3)(\alpha + 4)n\bar{w}^4(1-|w|^2)^3}{(1 - \bar{w}^2)^{\alpha + 5}} - \frac{(\alpha + 4)(\alpha + 5)n\bar{w}^5(1-|w|^2)^4}{(1 - \bar{w}^2)^{\alpha + 6}}, \]

\[ J^{(3)}_w(z) = \frac{(\alpha + 1)(\alpha + 2)(\alpha + 3)n\bar{w}^3(1-|w|^2)}{(1 - \bar{w}^2)^{\alpha + 4}} + \frac{(\alpha + 2)(\alpha + 3)(\alpha + 4)n\bar{w}^4(1-|w|^2)^2}{(1 - \bar{w}^2)^{\alpha + 5}} \\
+ \frac{(\alpha + 3)(\alpha + 4)(\alpha + 5)n\bar{w}^5(1-|w|^2)^3}{(1 - \bar{w}^2)^{\alpha + 6}} - \frac{(\alpha + 4)(\alpha + 5)(\alpha + 6)n\bar{w}^6(1-|w|^2)^4}{(1 - \bar{w}^2)^{\alpha + 7}}. \]

Put \( l = -3, m = 3, \) and \( n = 1 \) in (46) such that

\[ J_w(w) = J'_w(w) = J''_w(w) = 0, \]

\[ J^{(3)}_w(w) = (l(\alpha + 2)(\alpha + 3)(\alpha + 4) + m(\alpha + 3)(\alpha + 4)(\alpha + 5) \\
+ n(\alpha + 1)(\alpha + 2)(\alpha + 3) - (\alpha + 4)\cdot (\alpha + 5)\cdot (\alpha + 6)) \cdot \frac{\bar{w}^3}{(1 - |w|^2)^{\alpha + 3}}. \]

Thus, we have

\[ C \geq \| T_{u_1, u_2, \varphi} J_{\varphi(w)} \|_{\mathcal{F}_\varphi}, \]

\[ \geq \nu(w) \left| T_{u_1, u_2, \varphi} J_{\varphi(w)} n(w) \right| \]

\[ = (l(\alpha + 2)(\alpha + 3)(\alpha + 4) + m(\alpha + 3)(\alpha + 4)(\alpha + 5) + n(\alpha + 1)(\alpha + 2)(\alpha + 3) - (\alpha + 4)\cdot (\alpha + 5)\cdot (\alpha + 6)) \]

\[ \cdot \frac{\nu(w) \cdot |u_2(w) (\varphi(w))^2| |\varphi(w)|^3}{(1 - |\varphi(w)|^2)^{\alpha + 3}}. \]

Hence,

\[ \sup_{z \in \mathcal{B}} \frac{\nu(z)|u_2(z) (\varphi'(z))^2| |\varphi(z)|^3}{(1 - |\varphi(z)|^2)^{\alpha + 3}} < \infty, \]

\[ \sup_{|\varphi(z)|} > \frac{1}{2} \frac{\nu(z)|u_2(z) (\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^{\alpha + 3}} \]

\[ \leq C \sup_{|\varphi(z)|} > \frac{\nu(z)|u_2(z) (\varphi'(z))^2| |\varphi(z)|^3}{(1 - |\varphi(z)|^2)^{\alpha + 3}} \]

which implies that

\[ < \infty. \]
Applying (24) yields that
\[
\sup_{|\varphi(z)| \leq 1/2} \frac{\nu(z)\|u_2(z)(\varphi'(z))^2\|}{(1 - |\varphi(z)|^{2})^{\alpha+3}} \leq C \sup_{|\varphi(z)| \leq 1/2} \nu(z)\|u_2(z)(\varphi'(z))^2\| < \infty.
\]
Together (51) with (52), we can obtain
\[
\sup_{z \in \mathbb{D}} \frac{\nu(z)\|u_2(z)(\varphi'(z))^2\|}{(1 - |\varphi(z)|^{2})^{\alpha+3}} < \infty,
\]
which means that (17) holds. The proof is complete. \(\square\)

**Theorem 2.** Let \(\alpha > 0\). Suppose \(u_1, u_2 \in H(D), \varphi \in \mathcal{S}\). Then, \(T_{u_1, u_2, \varphi} : \mathcal{F}_\alpha \longrightarrow \mathcal{L}_\varphi\) is compact if and only if \(T_{u_1, u_2, \varphi} : \mathcal{F}_\alpha \longrightarrow \mathcal{L}_\varphi\) is bounded, and the following conditions are satisfied:

\[
\begin{align*}
\lim_{|\varphi(z)| \longrightarrow 1} \frac{\nu(z)\|u_1''(z)\|}{(1 - |\varphi(z)|^{2})^{\alpha}} &= 0, \\
\lim_{|\varphi(z)| \longrightarrow 1} \frac{\nu(z)\|2u_1'(z)\varphi'(z) + u_1(z)\varphi''(z) + u_2'(z)\|}{(1 - |\varphi(z)|^{2})^{\alpha+1}} &= 0, \\
\lim_{|\varphi(z)| \longrightarrow 1} \frac{\nu(z)\|u_1(z)(\varphi'(z))^2 + 2u_1'(z)\varphi'(z) + u_2(z)\varphi''(z)\|}{(1 - |\varphi(z)|^{2})^{\alpha+2}} &= 0, \\
\lim_{|\varphi(z)| \longrightarrow 1} \frac{\nu(z)\|u_2(z)(\varphi'(z))^2\|}{(1 - |\varphi(z)|^{2})^{\alpha+3}} &= 0.
\end{align*}
\]

**Proof.** Assume that \(T_{u_1, u_2, \varphi} : \mathcal{F}_\alpha \longrightarrow \mathcal{L}_\varphi\) is bounded and (54)–(57) hold. Due to Lemma 1, in order to prove that \(T_{u_1, u_2, \varphi} : \mathcal{F}_\alpha \longrightarrow \mathcal{L}_\varphi\) is compact, it suffices to show that, for any bounded sequence \(\{f_k\}\) in \(\mathcal{F}_\alpha\) with \(f_k \longrightarrow 0\) uniformly on compact subsets of \(D\), \(\|T_{u_1, u_2, \varphi}f_k\|_{\mathcal{L}_\varphi} \longrightarrow 0\) as \(k \longrightarrow \infty\).

We may assume that \(\|f_k\|_{\mathcal{F}_\alpha} \leq 1\), for all \(k \in \mathbb{N}\). By (54)–(57), we have that, for any \(\epsilon > 0\), there exists \(r \in (0, 1)\) such that

\[
\begin{align*}
\frac{\nu(z)\|u_1''(z)\|}{(1 - |\varphi(z)|^{2})^{\alpha}} &< \epsilon, \\
\frac{\nu(z)\|2u_1'(z)\varphi'(z) + u_1(z)\varphi''(z) + u_2'(z)\|}{(1 - |\varphi(z)|^{2})^{\alpha+1}} &< \epsilon, \\
\frac{\nu(z)\|u_1(z)(\varphi'(z))^2 + 2u_1'(z)\varphi'(z) + u_2(z)\varphi''(z)\|}{(1 - |\varphi(z)|^{2})^{\alpha+2}} &< \epsilon, \\
\frac{\nu(z)\|u_2(z)(\varphi'(z))^2\|}{(1 - |\varphi(z)|^{2})^{\alpha+3}} &< \epsilon.
\end{align*}
\]

for \(r < |\varphi(z)| < 1\). From the boundedness of the operator \(T_{u_1, u_2, \varphi} : \mathcal{F}_\alpha \longrightarrow \mathcal{L}_\varphi\) and the proof of Theorem 1, we obtain that (20)–(24) hold. Since \(f_k \longrightarrow 0\) uniformly on compact subsets of \(D\), Cauchy estimates show that \(f^{(n)}_k\) converges to 0 uniformly on compact subsets of \(D\). Then, there exists \(K_0 \in \mathbb{N}\), for \(k > K_0\), such that
From (58)–(61) and Lemma 4, we have

\[
\begin{align*}
\|T_{u_1, u_2, \varphi f_k}\|_{L^r} = & \|T_{u_1, u_2, \varphi f_k}^r (0)\| \\
+ & \left|T_{u_1, u_2, \varphi f_k}^r (0)\right| + \sup_{z \in \mathbb{D}} v(z) \left|T_{u_1, u_2, \varphi}^r f_k (0)\right| \leq v(0) \left|u_1 (0) f_k (\varphi (0)) + u_2 (0) f_k^r (\varphi (0))\right| \\
+ & v(0) \left|u_1 (0) f_k^r (\varphi (0)) + (u_1 (0) \varphi' (0) + u_2 (0) \varphi' (0)) f_k^r (\varphi (0)) + u_2 (0) \varphi' (0) f_k^r (\varphi (0))\right| \\
+ & \sup_{z \in \mathbb{D}} v(z) \left|u_1'' (z)\right| \left|f_k (\varphi (z))\right| \\
+ & \sup_{z \in \mathbb{D}} v(z) \left|u_1 (z) \varphi' (z) + u_1 (z) \varphi'' (z) + u_2'' (z)\right| \left|f_k (\varphi (z))\right| \\
+ & \sup_{z \in \mathbb{D}} v(z) \left|u_2 (z) \varphi' (z) + u_2 (z) \varphi'' (z)\right| \left|f_k^r (\varphi (0))\right| \left|f_k^r (\varphi (z))\right| \\
\leq & C \varepsilon + \sup_{r < |\varphi(z)| < 1} v(z) \left|u_1'' (z)\right| \left|f_k (\varphi (z))\right| \\
+ & \sup_{r < |\varphi(z)| < 1} v(z) \left|2u_1' (z) \varphi' (z) + u_1 (z) \varphi'' (z) + u_2'' (z)\right| \left|f_k (\varphi (z))\right| \\
+ & \sup_{r < |\varphi(z)| < 1} v(z) \left|u_1 (z) \varphi' (z) + u_2 (z) \varphi'' (z)\right| \left|f_k^r (\varphi (0))\right| \left|f_k^r (\varphi (z))\right| \\
\leq & C \varepsilon + C \sup_{r < |\varphi(z)| < 1} \left( \frac{v(z) \left|u_1'' (z)\right|}{(1 - |\varphi(z)|^2)^{\frac{a}{2}}} + \frac{v(z) \left|2u_1' (z) \varphi' (z) + u_1 (z) \varphi'' (z) + u_2'' (z)\right|}{(1 - |\varphi(z)|^2)^{\frac{a}{2} + 1}} \right. \\
+ & \left. \frac{v(z) \left|u_1 (z) \varphi' (z) + u_2 (z) \varphi'' (z)\right|}{(1 - |\varphi(z)|^2)^{\frac{a}{2}}} \right) \leq C \varepsilon,
\end{align*}
\]
which implies that $T_{u_{1}, u_{2}}: F_{a} \rightarrow L_{y}$ is compact.

Conversely, it is clear that the compactness of $T_{u_{1}, u_{2}}: F_{a} \rightarrow L_{y}$ implies that the boundedness of $T_{u_{1}, u_{2}}: F_{a} \rightarrow L_{y}$, if $\|\phi\|_{\infty} < 1$, it is clear that (54)–(57) is vacuous and obviously holds. Hence, assume that $\|\phi\|_{\infty} = 1$. Let $\{z_{k}\}$ be a sequence in $D$ such that $|\phi(z_{k})| \to 1$ as $k \to \infty$. Take the test functions

$$f_{k}(z) = f_{\phi(z_{k})}(z),$$

where $f_{w}$ is defined in (25). From the proof of Theorem 1, we have that $\sup_{x \in \mathbb{N}} \|f_{k}\|_{F_{X}} \leq C$ and

$$f_{k}(\phi(z_{k})) = \frac{a + b + c - 1}{(1 - |\phi(z_{k})|^{2})^{\alpha+1}},$$

$$f_{k}^{'}(\phi(z_{k})) = f_{k}^{'}(\phi(z_{k})) = f_{k}^{(3)}(\phi(z_{k})) = 0.$$ 

It is obvious that $f_{k}$ converges to 0 uniformly on compact subsets of $D$. From Lemma 1 and the compactness of $T_{u_{1}, u_{2}}: F_{a} \rightarrow L_{y}$, we have

$$
\frac{(a + b + c - 1)|\phi(z_{k})| |u_{\alpha}^{\prime}(z_{k})|}{(1 - |\phi(z_{k})|^{2})^{\alpha+1}} \leq \|T_{u_{1}, u_{2}}f_{k}\|_{L_{y}} \to 0, \tag{66}
$$

as $k \to \infty$, which implies that (54) holds.

Take the test functions:

$$g_{k}(z) = g_{\phi(z_{k})}(z),$$

where $g_{w}$ is defined in (30). From the proof of Theorem 1, we have that $\sup_{x \in \mathbb{N}} \|g_{k}\|_{F_{X}} \leq C$ and

$$g_{k}(\phi(z_{k})) = g_{k}^{'}(\phi(z_{k})) = g_{k}^{(3)}(\phi(z_{k})) = 0,$$

$$g_{k}^{'}(\phi(z_{k})) = \frac{(-a + 1 + d(a + 2) + c(a + 3) + h(a + 4))\phi(z_{k})}{(1 - |\phi(z_{k})|^{2})^{\alpha+1}}.$$ 

It is obvious that $g_{k}$ converges to 0 uniformly on compact subsets of $D$. From Lemma 1 and the compactness of $T_{u_{1}, u_{2}}: F_{a} \rightarrow L_{y}$, we have

$$\frac{v(z_{k})|2u_{1}^{\prime}(z_{k})\phi^{\prime}(z_{k}) + u_{1}(z_{k})\phi^{\prime\prime}(z_{k}) + u_{2}^{\prime}(z_{k})|\phi(z_{k})}{(1 - |\phi(z_{k})|^{2})^{\alpha+1}} \leq \|T_{u_{1}, u_{2}}g_{k}\|_{L_{y}} \to 0, \tag{69}
$$

as $k \to \infty$. Thus, for $|\phi(z_{k})| \to 1$, we have

$$\lim_{k \to \infty} \frac{v(z_{k})|2u_{1}^{\prime}(z_{k})\phi^{\prime}(z_{k}) + u_{1}(z_{k})\phi^{\prime\prime}(z_{k}) + u_{2}^{\prime}(z_{k})|}{(1 - |\phi(z_{k})|^{2})^{\alpha+1}} = 0, \tag{70}
$$

which implies (55) holds.

Analogously, choosing the test functions $h_{k}(z) = h_{\phi(z_{k})}(z)$ and $f_{k}(z) = f_{\phi(z_{k})}(z)$ which is defined in (38) and (46), same to the above approach, we can deduce (56) and (57). Consequently, the proof is complete.

**Remark 1.** Due to Theorems 1 and 2, we may easily obtain the characterizations for the boundedness and compactness of all six product-type operators:

$$M_{u}DC_{\phi},$$
$$C_{\phi}M_{u}D,$$
$$M_{u}C_{\phi}D,$$
$$DM_{u}C_{\phi},$$
$$C_{\phi}DM_{u},$$
$$DC_{\phi}M_{u},$$

from the space of fractional Cauchy transforms to the Zygmund-type spaces. We leave the details to the readers interested in this research area.

**Data Availability**

No data were used to support the findings of the study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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