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Retraction

Retracted: On Fundamental Algebraic Characterizations of μ -Fuzzy Normal Subgroups

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] I. Masmali, U. Shuaib, A. Razaq, A. Fatima, and G. Alhamzi, "On Fundamental Algebraic Characterizations of μ-Fuzzy Normal Subgroups," *Journal of Function Spaces*, vol. 2022, Article ID 2703489, 10 pages, 2022.

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Research Article

On Fundamental Algebraic Characterizations of μ-Fuzzy Normal Subgroups

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In this article, we present the study of μ-fuzzy subgroups and prove numerous fundamental algebraic attributes of this newly defined notion. We also define the concept of μ-fuzzy normal subgroup and investigate many vital algebraic characteristics of these phenomena. In addition, we characterize the quotient group induced by this particular fuzzy normal subgroup and establish a group isomorphism between the quotient groups G/κ^{μ} and G/κ_*^{μ}. Furthermore, we initiate the study of level subgroup, open level subgroup, and tangible subgroup of a μ-fuzzy subgroup and emphasize the significance of μ-fuzzy normal subgroups by establishing a relationship between these newly defined notions and μ-fuzzy normal subgroup.

1. Introduction

The theory of fuzzy logic is based on the concept of relative graded membership as inspired by the processes of human perception. This logic deals with information that is uncertain, imprecise, vague, partly true, or without clear boundaries. Moreover, this theory provides a mathematical framework within which ambiguous conceptual phenomena can be studied with precision. New computing methods based on fuzzy logic can be used in the development of intelligent system for decision-making, identification, pattern recognition, optimization, and control system. This particular logic is currently being used in the industrial practice of advanced information technology. One of the most important applications of group theory is its key role in geometry and cryptography. Geometry is the study of properties of a space that are invariant under a group of transformations of that space. The theory of groups is used to classify the symmetries of molecules, crystal structures, and regular polyhedral. It is also used to solve the old issues of algebra.

Fuzzy subsets (FSs) have a central position in modern mathematics. In 1965, Zadeh [1] proposed the idea of FSs. The idea of fuzzy subgroup (FSG) was presented by Rosenfeld [2] in the framework of FSs in 1971. Das [3] defined the level subgroups (LSGs) of a FSG in 1981. Mukherjee and Bhattacharya [4] characterized the notions of fuzzy normal subgroup (FNSG) and fuzzy coset in 1984. Mashour et al. [5] described many important properties of FNSGs in 1990. The characterizations of fuzzy conjugate subgroups and fuzzy characteristic subgroups by their level subgroups were presented in [6]. To see more on the development of theory of FSGs, we refer to [7–11]. The fuzzy logic was effectively used in industrial management [12], coding theory [13] and forecasting systems [14]. Rowlands and Wang [15] applied fuzzy logic to present a useful technique to design fuzzy-SPC evaluation and control method in 2000.

Liu [16] developed a new method of constructing quotient groups induced by FNSGs and proved the corresponding isomorphism theorem. In 2005, Mordeson et al. [17] discussed the zero power of a FSG along with various important analytical characteristics of FSGs. Li et al. [18] gave the efficient solutions of linear programming problems based on fuzzy logic. Faraz and Shapiro [19] designed a fuzzy statistical control chart that explains the existence of fuzziness in data in 2010. The authors proposed a new type of FNSGs and fuzzy cosets [20]. Celik et al. [21] introduced the concept of fuzzy soft rings and studied some of their structural properties. Borah and Hazarika [22] innovated the study of mixed fuzzy soft topology with separation axioms along with applications of fuzzy soft sets in chemistry. In addition, useful characteristics of FSGs in various algebraic structures can be viewed in [23–27]. Ullah et al. [28] utilized the idea of complex Pythagorean FS in pattern recognition problems. Alghazzawi et al. [29] introduced ρ anti-intuitionistic fuzzy subgroups and presented a comprehensive study of this concept. In [30] Bejines et al. investigated various properties of aggregation of fuzzy subgroups.

We recommend the readers to study [31–39] to obtain more information on this topic.

The fuzzy sets have ability to play an effective role to solve many physical problems. These sets provide us the meaningful representations of measuring uncertainty. Despite all of these advantages, we still face vast complications to counter various physical situations. This motivates us to define the notion of μ-fuzzy set through which one can have multiple options to investigate a specific real-world situation in much efficient way by choosing appropriate value of the parameter μ.

The μ-fuzzy sets are capable to deal with the uncertainty and vagueness of a physical problem much more effectively than with the theory of the classic fuzzy set, especially in the area of logic programming and decision-making, financial services, psychological examinations, medical diagnosis, career determination, and artificial intelligence. For instance, μ-fuzzy sets are used to improve the resolution of a certain picture. A photograph of a person describes his many biological features like tall or short, heavy or thin, old or young, and male or female. Sometimes this picture becomes corrupted due to many distortions in the lenses. Such distortions are removed under the framework of these particular fuzzy sets by choosing the appropriate value of the parameter μ.

In this article, we initiate the study of μ-FSs and their level sets along with many set theoretical properties of these phenomena. We also propose the concept of μ-FSG based over a μ-FS and investigate some of their various algebraic properties. We extend this ideology by defining the notions of μ-FNSG and μ-fuzzy coset and establish a group isomorphism between the quotient groups of these newly defined ideas. In addition, we characterize the concept of level subgroup (LSG), open level subgroup (OLSG), and tangible subgroup (TSG) of a μ-FSG and establish a relationship between a μ-FNSG and the above stated notions.

After a brief discussion about the historical background and significance of FSG, the rest of the article is organized as follows: the second section contains a brief review of some fundamental definitions of basic notions which are quite useful to understand the novelty of this study. Section 3 is dedicated to introduce the notion of μ-FSG defined over a μ-FS, and many fundamental algebraic characteristics of this notion are investigated. In Section 4, we extend the study of μ-FSGs by defining the notions of μ-FNSG and quotient group induced by this particular FNSG. Furthermore, we establish a group isomorphism between the quotient groups G/κ^{μ} and G/κ^{μ}_{*}. In addition, we define the concepts of LSG, OLSG, and TSG of a μ-FSG and shed light the importance of μ-FNSG by developing an important relationship between these concepts and μ-FNSG.

2. Preliminaries

In this section, we study some fundamental concepts of FSGs which are quite essential to obtain the basic group theoretic outcomes in terms of their respective fuzzy counterparts.

Definition 1 (see [1]). Consider an element u_1 of a universe U. Any function κ from U to the closed unit interval is called a fuzzy set.

Remark 2 (see [17]). Any two FSs κ and ω of a universe U admit the following characteristics. For any $u_1 \in U$.

(1) $\kappa \subseteq \omega$ iff $\kappa(u_1) \leq \omega(u_1)$

(2) $\kappa = \omega$ iff $\kappa \subseteq \omega$ and $\omega \subseteq \kappa$

(3) The complement of the FS κ is κ' and is determined as $\kappa'(u_1) = 1 - \kappa(u_1)$

(4) $(\kappa \cap \omega)(u_1) = \min\{\kappa(u_1), \omega(u_1)\}$

(5) $(\kappa \cup \omega)(u_1) = \max\{\kappa(u_1), \omega(u_1)\}$

Definition 3 (see [17]). The α-cut and strong α-cut of a FS κ are denoted by ${}^{\alpha}\kappa$ and ${}^{\alpha+}\kappa$, respectively, and are defined in the following ways:

(1) ${}^{\alpha}\kappa = \{u_1 \in U : \kappa(u_1) \geq \alpha, 0 \leq \alpha \leq 1\}$

(2) ${}^{\alpha+}\kappa = \{u_1 \in U : \kappa(u_1) > \alpha, 0 \leq \alpha \leq 1\}$

Definition 4 (see [17]). The support of an FS κ is denoted by ${}^{*}\kappa$ and is described as ${}^{*}\kappa = \{u_1 \in U : \kappa(u_1) > 0\}$.

Definition 5 (see [17]). An FS κ of a group G is called FSG of G if it satisfies the following conditions:

(i) $\kappa(u_1 u_2) \geq \min\{\kappa(u_1), \kappa(u_2)\}$

(ii) $\kappa(u_1^{-1}) = \kappa(u_1)$, for all $u_1, u_2 \in G$

Definition 6 (see [4]). The FSG κ of a group G is said to be fuzzy normal subgroup of G if $\kappa(u_1 u_2) = \kappa(u_2 u_1)$, for all $u_1, u_2 \in G$.

Definition 7 (see [4]). Let κ be FSG and u_1 be a fixed element of G. Then, fuzzy left coset of κ in G is denoted by $u_1\kappa$ and is interpreted in the following way: $(u_1\kappa)(u_2) = \kappa(u_1^{-1}u_2)$, for all $u_1, u_2 \in G$. The fuzzy right coset is determined in the same manner.

Definition 8 (see [8]). Let κ and ω be two FSs of groups G_1 and G_2, respectively, and $\eta : G_1 \longrightarrow G_2$ be a mapping. Then, $\eta(\kappa)$ and $\eta^{-1}(\omega)$ are defined as $(\kappa)(u_2) = \begin{cases} \sup\{\kappa(u_1): u_1 \in \eta^{-1}(u_2)\}\,; \text{if } \eta^{-1}(u_2) \neq \varnothing \\ 0\,; \text{Otherwise} \end{cases}$, and $\eta^{-1}(\omega)(u_1) = \omega(\eta(u_1))$ for every $u_1, u_2 \in G_1$.

3. Algebraic Properties of μ -Fuzzy Subgroups

This section is committed to initiate the conception of μ-FSG defined over μ-FS. We study several algebraic aspects of these phenomena.

Definition 9. Let κ be a FS of universe U and $\mu \in [0, 1]$. Then, FS κ^μ is called the μ-fuzzy set of universe U w.r.t. κ and is defined as $\kappa^\mu(u_1) = s_p\{\kappa(u_1), \mu\}$, for all $u_1 \in U$, where the algebraic sum s_p is described as $s_p\{\kappa(u_1), \mu\} = \kappa(u_1) + \mu - (\kappa(u_1)\bullet\mu)$.

Remark 10.

(1) Evidently, we obtain the classical FS from μ-FS for $\mu = 0$ whereas the case becomes crisp set for the value of $\mu = 1$

(2) Let κ^μ and ω^μ be any two μ-FSs of U and V, respectively, and $\eta : U \longrightarrow V$ be a mapping. Then

(3) $\eta(\kappa^\mu) = (\eta(\kappa))^\mu$

(4) $\eta^{-1}(\omega^\mu) = (\eta^{-1}(\omega))^\mu$

Definition 11. For any μ-FS κ^μ and $0 \leq \alpha \leq 1$, we denoted α -cut of κ^μ by ${}^\alpha\kappa^\mu$ such that ${}^\alpha\kappa^\mu = \{u_1 \in U : \kappa^\mu(u_1) \geq \alpha\}$.

Definition 12. Let κ^μ be a μ-FS and $0 \leq \alpha \leq 1$. The strong α -cut of κ^μ is denoted by ${}^{\alpha+}\kappa^\mu$ and is described as ${}^{\alpha+}\kappa^\mu = \{u_1 \in U : \kappa^\mu(u_1) > \alpha\}$.

Example 1.

(1) An important application of μ-FS in the medical decision-making is to determine the amount of the drug dose for patients by setting an appropriate value of the parameter μ related to age and weight of the patients. The radiation therapy can be improved in the framework of μ-FSs by selecting an appropriate value of μ

(2) Another important significance of μ-FSs can be viewed in the formulation of default timings for the traffic lights as it effectively supports the fuzzy controller to adjust the timings depending upon the value of the parameter μ

Definition 13.

(1) The union of μ-FSs κ^μ and ω^μ is denoted by $\kappa^\mu \cup \omega^\mu$ and is defined as follows:

$$(\kappa^\mu \cup \omega^\mu)(u_1) = \max\{\kappa^\mu(u_1), \omega^\mu(u_1)\}. \quad (1)$$

(2) The intersection of μ-FSs κ^μ and ω^μ is denoted by $\kappa^\mu \cap \omega^\mu$ and is interpreted as follows:

$$(\kappa^\mu \cap \omega^\mu)(u_1) = \min\{\kappa^\mu(u_1), \omega^\mu(u_1)\}. \quad (2)$$

(3) The complement of μ-FS κ^μ is denoted by κ^{μ^c} and is describe as follows:

$$\kappa^{\mu^c}(u_1) = 1 - \kappa^\mu(u_1), \text{for all } u_1 \in U. \quad (3)$$

Definition 14. Let κ be an FS of a group G and $\mu \in [0, 1]$. Then, κ is called a μ-fuzzy subgroup of G if κ^μ satisfies the following conditions:

(1) $\kappa^\mu(u_1u_2) \geq \min\{\kappa^\mu(u_1), \kappa^\mu(u_2)\}$

(2) $\kappa^\mu(u_1^{-1}) = \kappa^\mu(u_1)$, for all $u_1, u_2 \in G$

The above conditions can also be defined in the following way.

$$\kappa^\mu\left(u_1u_2^{-1}\right) \geq \min\{\kappa^\mu(u_1), \kappa^\mu(u_2)\}, \text{for all } u_1, u_2 \in G. \quad (4)$$

Remark 15.

Every μ-FSG κ^μ of a group G admits the following characteristics

(a) $\kappa^\mu(u_1) \leq \kappa^\mu(e)$, for all $u_1 \in G$

(b) $\kappa^\mu(u_1u_2^{-1}) = \kappa^\mu(e)$ which implies that $\kappa^\mu(u_1) = \kappa^\mu(u_2)\kappa^\mu|_{N(\kappa^\mu)}, u_1, u_2 \in G$

(2) Every FSG of G is a μ-FSG of G but the converse is not true, in general

(3) The intersection of a family of μ-FSGs of G is also a μ-FSG of G

(4) The union of any two μ-FSGs of G need not be a μ-FSG of G

(5) In the following result, we establish some important relationships between α-cut and strong α-cut of μ-FS

Theorem 16. *Let κ^μ and ω^μ be any two μ-FS; then, the following attributes hold for any $\alpha, \beta \in [0, 1]$.*

(i) ${}^{\alpha+}\kappa^\mu \subseteq {}^{\alpha}\kappa^\mu \alpha \leq \beta$ *implies that* ${}^{\beta}\kappa^\mu \subseteq {}^{\alpha}\kappa^\mu$ *and* ${}^{\beta+}\kappa^\mu \subseteq {}^{\alpha+}\kappa^\mu$

(ii) ${}^{\alpha}\kappa^{\mu'} = {}^{(1-\alpha)+}\kappa^{\mu'}$

(iii) ${}^{\alpha}(\kappa^\mu \cap \omega^\mu) = {}^{\alpha}\kappa^\mu \cap {}^{\alpha}\omega^\mu$ *and* ${}^{\alpha}(\kappa^\mu \cup \omega^\mu) = {}^{\alpha}\kappa^\mu \cup {}^{\alpha}\omega^\mu$

(iv) ${}^{\alpha+}(\kappa^\mu \cap \omega^\mu) = {}^{\alpha+}\kappa^\mu \cap {}^{\alpha+}\omega^\mu$ *and* ${}^{\alpha+}(\kappa^\mu \cup \omega^\mu) = {}^{\alpha+}\kappa^\mu \cup {}^{\alpha+}\omega^\mu$

Proof.

(i) In view of Definition 12, for any element $u_1 \in U$, $\kappa^\mu(u_1) > \alpha$. It means that $\kappa^\mu(u_1) \geq \alpha$. Thus, ${}^{\alpha+}\kappa^\mu \subseteq {}^{\alpha}\kappa^\mu$

(ii) Let $u_1 \in {}^{\beta}\kappa^\mu$, then by using Definition 11, we have $\kappa^\mu(u_1) \geq \beta > \alpha$. This implies that $u_1 \in {}^{\alpha}\kappa^\mu$. Consequently, ${}^{\beta}\kappa^\mu \subseteq {}^{\alpha}\kappa^\mu$

Similarly, the assertion can be proved for strong α – cut of κ^μ.

(iii) Let $u_1 \in {}^{\alpha}\kappa^{\mu'}$, then $1 - \kappa^\mu(u_1) \geq \alpha$. It follows that $\kappa^\mu(u_1) \leq 1 - \alpha$. This means that $u_1 \notin {}^{(1-\alpha)+}\kappa^\mu$. Therefore, $u_1 \in {}^{(1-\alpha)+}\kappa^{\mu'}$. Hence

$$ {}^{\alpha}\kappa^{\mu'} \subseteq {}^{(1-\alpha)+}\kappa^{\mu'}. \tag{5} $$

Now suppose $u_1 \in {}^{(1-\alpha)+}\kappa^{\mu'}$, then $u_1 \notin {}^{(1-\alpha)+}\kappa^\mu$. This implies that $1 - \alpha \geq \kappa^\mu(u_1)$; therefore, $\alpha \leq 1 - \kappa^\mu(u_1)$. It follows that $u_1 \in {}^{\alpha}\kappa^{\mu'}$. Hence

$$ {}^{(1-\alpha)+}\kappa^{\mu'} \subseteq {}^{\alpha}\kappa^{\mu'}. \tag{6} $$

From (5) and (6), we obtain ${}^{\alpha}\kappa^{\mu'} = {}^{(1-\alpha)+}\kappa^{\mu'}$.

(iv) Let $u_1 \in {}^{\alpha}(\kappa^\mu \cap \omega^\mu)$, then $(\kappa^\mu \cap \omega^\mu)(u_1) \geq \alpha$. This implies that $\min\{\kappa^\mu(u_1), \omega^\mu(u_1)\} \geq \alpha$, so $\kappa^\mu(u_1) \geq \alpha$ and $\omega^\mu(u_1) \geq \alpha$. Therefore, $u_1 \in {}^{\alpha}\kappa^\mu \cap {}^{\alpha}\omega^\mu$, and consequently,

$$ {}^{\alpha}(\kappa^\mu \cap \omega^\mu) \subseteq {}^{\alpha}\kappa^\mu \cap {}^{\alpha}\omega^\mu. \tag{7} $$

The application of efinition 11 on $u_1 \in {}^{\alpha}\kappa^\mu \cap {}^{\alpha}\omega^\mu$ yields that $\kappa^\mu(u_1) \geq \alpha$ and $\omega^\mu(u_1) \geq \alpha$. This shows that $\min\{\kappa^\mu(u_1), \omega^\mu(u_1)\} \geq \alpha$. Consequently,

$$ {}^{\alpha}\kappa^\mu \cap {}^{\alpha}\omega^\mu \subseteq {}^{\alpha}(\kappa^\mu \cap \omega^\mu). \tag{8} $$

From (7) and (8), we obtain the required equality.

The second part of (iv) can be proved in the same manner.

(v) For any element $u_1 \in {}^{\alpha+}(\kappa^\mu \cup \omega^\mu)$, we obtain $(\kappa^\mu \cup \omega^\mu)(u_1) > \alpha$. Therefore, $\max\{\kappa^\mu(u_1), \omega^\mu(u_1)\} > \alpha$. This means that $\kappa^\mu(u_1) > \alpha$ or $\omega^\mu(u_1) > \alpha$. Therefore, $u_1 \in {}^{\alpha+}\kappa^\mu \cup {}^{\alpha+}\omega^\mu$, and consequently,

$$ {}^{\alpha+}(\kappa^\mu \cup \omega^\mu) \subseteq {}^{\alpha+}\kappa^\mu \cup {}^{\alpha+}\omega^\mu. \tag{9} $$

The application of Definition 12 on $u_1 \in {}^{\alpha+}\kappa^\mu \cup {}^{\alpha+}\omega^\mu$ yields that $\kappa^\mu(u_1) > \alpha$ or $\omega^\mu(u_1) > \alpha$. This shows that $\max\{\kappa^\mu(u_1), \omega^\mu(u_1)\} > \alpha$. Consequently,

$$ {}^{\alpha+}\kappa^\mu \cup {}^{\alpha+}\omega^\mu \subseteq {}^{\alpha+}(\kappa^\mu \cup \omega^\mu). \tag{10} $$

By comparing the relations (9) and (10), we get the required result.

The other part of (v) can be proved in the same manner.

The following theorem presents a necessary and sufficient condition for a μ-FS to be a μ-FSG. □

Theorem 17. *A μ-FS κ^μ is a μ-FSG of a group G if and only if ${}^{\alpha}\kappa^\mu$ is a subgroup of G for each $0 \leq \alpha \leq \kappa^\mu(e)$, where e is the identity element of G.*

Proof. Suppose κ^μ is a μ-FSG and $u_1, u_2 \in {}^{\alpha}\kappa^\mu$, then in the view of Definition 11, we have $\kappa^\mu(u_1) \geq \alpha$ and $\kappa^\mu(u_2) \geq \alpha$. This implies that $\kappa^\mu(u_1u_2) \geq \min\{\kappa^\mu(u_1), \kappa^\mu(u_2)\} \geq \min\{\alpha, \alpha\} \geq \alpha$. Consequently, $u_1u_2 \in {}^{\alpha}\kappa^\mu$. Moreover, for $u_1 \in {}^{\alpha}\kappa^\mu$, we have $\kappa^\mu(u_1) \geq \alpha$, implying that $\kappa^\mu(u_1^{-1}) = \kappa^\mu(u_1) \geq \alpha$ for all $u_1 \in G$. This means that $u_1^{-1} \in {}^{\alpha}\kappa^\mu$, and hence, ${}^{\alpha}\kappa^\mu$ is a subgroup of G.

Conversely, assume that ${}^{\alpha}\kappa^\mu$ is a subgroup of G. For any $u_1, u_2 \in G$, let $\kappa^\mu(u_1) = \alpha$ and $\kappa^\mu(u_1) = \beta$. Then, clearly, $u_1 \in {}^{\alpha}\kappa^\mu$ and $u_2 \in {}^{\beta}\kappa^\mu$. Suppose $\alpha < \beta$, then ${}^{\beta}\kappa^\mu \subseteq {}^{\alpha}\kappa^\mu$. Since $u_2 \in {}^{\alpha}\kappa^\mu$ and ${}^{\alpha}\kappa^\mu$ is a subgroup of G, therefore, $u_1u_2 \in {}^{\alpha}\kappa^\mu$. Then, $\kappa^\mu(u_1u_2) \geq \alpha$ implying that $\kappa^\mu(u_1u_2) \geq \min\{\kappa^\mu(u_1), \kappa^\mu(u_2)\}$. For any $u_1 \in G$, let $\kappa^\mu(u_1) = \alpha$. Then, $u_1 \in {}^{\alpha}\kappa^\mu$ and $u_1^{-1} \in {}^{\alpha}\kappa^\mu$. This implies that $\kappa^\mu(u_1^{-1}) = \kappa^\mu(u_1)$, and hence, κ^μ is a μ-FSG of G. □

4. Fundamental Algebraic Characterizations of μ-Fuzzy Normal Subgroups

In this section, we categorize the ideas of μ-fuzzy cosets and μ-FNSGs and prove fundamental algebraic characteristics of these phenomena. We also define the quotient group induced by μ-FNSG and obtain a group isomorphism between G/κ^μ and G/κ^μ_*. Furthermore, we innovate the study of the concepts of LSG, OLSG, and TSG of a μ-FSG and develop a relationship of these notions with μ-FNSG.

4.1. Definition. Let κ^μ be a μ-FSG and u_1 be any fixed element of a group G.Then, μ-fuzzy left coset of κ^μ in G is represented by $u_1\kappa^\mu$ and is defined as follows:

$$ u_1\kappa^\mu(u_2) = \kappa^\mu\left(u_1^{-1}u_2\right) + \mu - \left(\kappa^\mu\left(u_1^{-1}u_2\right).\mu\right) \text{ for all } u_1, u_2 \in G. \tag{11} $$

Likewise, the notion of μ-fuzzy right coset $\kappa^{\mu}u_1$ can be established.

4.2. Definition. A μ-FSG κ^{μ} is called μ-fuzzy normal subgroup of ω^{μ} if $u_1\kappa^{\mu} = \kappa^{\mu}u_1$ for all $u_1 \in G$.

In other words, a μ-FNSG can also be defined as $\kappa^{\mu}(u_1 u_2) = \kappa^{\mu}(u_2 u_1)$, for all $u_1, u_2 \in G$.

Theorem 18. *The following statements are equivalent for any μ-FSG κ^{μ} of a group G.*

(i) κ^{μ} *is a μ-FNSG of G*

(ii) $\kappa^{\mu}(u_1u_2u_1^{-1}) = \kappa^{\mu}(u_2)$

(iii) $\kappa^{\mu}(u_1u_2u_1^{-1}) \geq \kappa^{\mu}(u_2)$

(iv) $\kappa^{\mu}(u_1u_2u_1^{-1}) \leq \kappa^{\mu}(u_2)$,*for all* $u_1, u_2 \in G$

Proof. i $\Rightarrow$ ii: For any two elements $u_1, u_2 \in G$, we have $\kappa^{\mu}(u_1u_2u_1^{-1}) = \kappa^{\mu}(u_1^{-1}\bullet u_1u_2) = \kappa^{\mu}(u_2)$.

ii $\Rightarrow$ iii: The required inequality is a straightforward implication of ii.

iii $\Rightarrow$ iv: Consider $\kappa^{\mu}(u_2) = \kappa^{\mu}(u_1^{-1}\bullet u_1u_2u_1^{-1}\bullet(u_1^{-1})^{-1}) \geq \kappa^{\mu}(u_1u_2u_1^{-1})$.

In other words, $\kappa^{\mu}(u_1u_2u_1^{-1}) \leq \kappa^{\mu}(u_2)$.

iv$\Rightarrow$ i: Consider

$$\kappa^{\mu}(u_1u_2) = \kappa^{\mu}\left(u_1u_2\bullet u_1u_1^{-1}\right) = \kappa^{\mu}\left(u_1\bullet u_2u_1\bullet u_1^{-1}\right), \tag{12}$$

$$\kappa^{\mu}(u_1u_2) \leq \kappa^{\mu}(u_2u_1). \tag{13}$$

Moreover,

$$\kappa^{\mu}(u_2u_1) = \kappa^{\mu}\left(u_2u_1\bullet u_2u_2^{-1}\right) = \kappa^{\mu}\left(u_2\bullet u_1u_2\bullet u_2^{-1}\right). \tag{14}$$

The required result can be acquired by using (12) and (13).

In the following theorem, we find a necessary and sufficient condition for a μ-FSG to be μ-FNSG. □

Theorem 19. *A μ-FSG κ^{μ} is a μ-FNSG of a group G if and only if ${}^{\alpha}\kappa^{\mu}$ is a normal subgroup of G for each $0 \leq \alpha \leq \kappa^{\mu}(e)$, where e is the identity element of G.*

Proof. Suppose κ^{μ} is a μ-FNSG. The application of Definition 11 on $u_1, u_2 \in {}^{\alpha}\kappa^{\mu}$, we have $\kappa^{\mu}(u_1) \geq \alpha$ and $\kappa^{\mu}(u_2) \geq \alpha$. Consider $\kappa^{\mu}(u_1u_2u_1^{-1}) = \kappa^{\mu}(u_2) \geq \alpha$. Consequently, $u_1u_2u_1^{-1} \in {}^{\alpha}\kappa^{\mu}$, and hence, ${}^{\alpha}\kappa^{\mu}$ is a normal subgroup of G.

Conversely, assume that ${}^{\alpha}\kappa^{\mu}$ is a normal subgroup of G. Let $u_1, u_2 \in G$ and $\kappa^{\mu}(u_2) = \alpha$, then clearly, $u_2 \in {}^{\alpha}\kappa^{\mu}$. Since ${}^{\alpha}\kappa^{\mu}$ is a normal subgroup of G, therefore, $u_1u_2u_1^{-1} \in {}^{\alpha}\kappa^{\mu}$. This means that $\kappa^{\mu}(u_1u_2u_1^{-1}) \geq \alpha = \kappa^{\mu}(u_2)$. It follows that κ^{μ} satisfies fact iii of Theorem 18 and according to Theorem 20 κ^{μ} is a μ-FNSG of G. □

4.3. Definition. The support of a μ-FS κ^{μ} is denoted by ${}^{*}\kappa^{\mu}$ and is described as ${}^{*}\kappa^{\mu} = \{u_1 \in U : \kappa^{\mu}(u_1) > 0\}$.

In the following result, we show that support of κ^{μ} is a normal subgroup of G.

Theorem 20. *Let κ^{μ} be a μ-FNSG of a group G. Then, ${}^{*}\kappa^{\mu}$ is normal subgroup of G.*

Proof. The application of Definitions (9) and (15) on $u_1, u_2 \in {}^{*}\kappa^{\mu}$, we have $\kappa^{\mu}(u_1u_2^{-1}) > 0$. Consequently, $u_1u_2^{-1} \in {}^{*}\kappa^{\mu}$; therefore, ${}^{*}\kappa^{\mu}$ is a subgroup of G. Moreover, by applying the normality of κ^{μ}, we have $\kappa^{\mu}(u_1u_2u_1^{-1}) = \kappa^{\mu}(u_2) > 0$, for any $u_1 \in G$ and $u_2 \in {}^{*}\kappa^{\mu}$. Consequently, $u_1u_2u_1^{-1} \in {}^{*}\kappa^{\mu}$; thus, ${}^{*}\kappa^{\mu}$ is a normal subgroup of G.

In the following result, we show that κ^{μ}_{*} is normal subgroup of G. □

Theorem 21. *Let κ^{μ} be a μ-FNSG of a group G; then, the set $\kappa^{\mu}_{*} = \{u_1 \in G : \kappa^{\mu}(u_1) = \kappa^{\mu}(e)\}$ is a normal subgroup of G.*

Proof. The application of Definition (9) on $u_1, u_2 \in \kappa^{\mu}_{*}$, we have

$$\kappa^{\mu}\left(u_1u_2^{-1}\right) \geq \kappa^{\mu}(e). \tag{15}$$

We also know that

$$\kappa^{\mu}(e) \geq \kappa^{\mu}\left(u_1u_2^{-1}\right). \tag{16}$$

By the comparison of (15) and (16), we get $\kappa^{\mu}(u_1u_2^{-1}) = \kappa^{\mu}(e)$.

Hence, κ^{μ}_{*} is a subgroup of G.

Moreover, by applying the normality of κ^{μ}, we have $\kappa^{\mu}(u_1u_2u_1^{-1}) = \kappa^{\mu}(u_2) = \kappa^{\mu}(e)$, for any $u_1 \in G$ and $u_2 \in \kappa^{\mu}_{*}$. Consequently, $u_1u_2u_1^{-1} \in \kappa^{\mu}_{*}$; this shows that κ^{μ}_{*} is a normal subgroup of G. □

The following example describes the fact that leads to note that the converses of the Theorems 20 and 21 are not true.

Example 2. Consider a subgroup $H = \{e, ab\}$ of the dihedral group $D_3 = \{e, a, a^2, b, ab, a^2b\}$. Clearly, H is not normal in D_3. The FS κ of D_3 is as follows:

$$\kappa(u_1) = \begin{cases} 1.0, & \text{if } u_1 = e, \\ 0.5, & \text{if } u_1 = ab, \\ 0.4, & \text{otherwise.} \end{cases} \tag{17}$$

The μ-FS κ^{μ} of D_3 correspond to the value $\mu = 0.6$ is given by

$$\kappa^{0.6}(u_1) = \begin{cases} 1.00, & \text{if } u_1 = e, \\ 0.80, & \text{if } u_1 = ab, \\ 0.76, & \text{otherwise.} \end{cases} \tag{18}$$

Then, $\kappa^{0.6}_{*} = \{e\}$ and ${}^{*}\kappa^{0.6} = D_3$. Obviously, $\kappa^{0.6}_{*}$ and ${}^{*}\kappa^{0.6}$ are normal subgroups of D_3. But $\kappa^{0.6}$ is not a $0.6-$FNSG of D_3 because ${}^{0.8}\kappa^{0.6} = H$.

The following result describes another approach to obtain a μ-FNSG.

Theorem 22. *A μ-FSG κ^μ is a μ-FNSG of a group G if and only if κ^μ is constant on the conjugate classes of G.*

Proof. Suppose κ^μ is a μ-FNSG of G. Then

$$\kappa^\mu\left(u_2^{-1}u_1u_2\right) = \kappa^\mu\left(u_1u_2u_2^{-1}\right) = \kappa^\mu\left(u_1\right), \text{for all}\, u_1, u_2 \in G. \tag{19}$$

This means that κ^μ is a constant on the conjugate classes of G.

Conversely, suppose that κ^μ is constant on each conjugate class of G. Then, $\kappa^\mu\left(u_1u_2\right) = \kappa^\mu\left(u_1u_2u_1u_1^{-1}\right) = \kappa^\mu\left(u_1\left(u_2u_1\right)u_1^{-1}\right) = \kappa^\mu\left(u_2u_1\right)$, for all $u_1, u_2 \in G$. Consequently, κ^μ is a μ-FNSG of G. □

Theorem 23. *Every μ-FNSG κ^μ of G admits the following properties for all $u_1, u_2 \in G$:*

(i) $u_1\kappa^\mu = u_2\kappa^\mu \Leftrightarrow u_1\kappa_*^\mu = u_2\kappa_*^\mu$

(ii) $\kappa^\mu u_1 = \kappa^\mu u_2 \Leftrightarrow \kappa_*^\mu u_1 = \kappa_*^\mu u_2$

Proof. (i) Suppose that $u_1\kappa^\mu = u_2\kappa^\mu$, where $u_1, u_2 \in G$. By using Definition (12) in the above relation, we obtain

$$\kappa^\mu\left(u_1^{-1}u_2\right) = \left(u_1\kappa^\mu\right)\left(u_2\right) = \left(u_2\kappa^\mu\right)\left(u_2\right) = \kappa^\mu\left(u_2^{-1}u_2\right) = \kappa^\mu(e). \tag{20}$$

So, $u_1^{-1}u_2 \in \kappa_*^\mu$. This means that $u_2 \in u_1\kappa_*^\mu$, and ultimately, we have

$$u_2\kappa_*^\mu \subseteq u_1\kappa_*^\mu. \tag{21}$$

Similarly

$$u_1\kappa_*^\mu \subseteq u_2\kappa_*^\mu. \tag{22}$$

By comparing the relations (21) and (22), we get the required equality.

Conversely, let $u_1\kappa_*^\mu = u_2\kappa_*^\mu$. Then, $u_1^{-1}u_2 \in \kappa_*^\mu$ and $u_2^{-1}u_1 \in \kappa_*^\mu$. By applying Definition (12) on $u_1\kappa^\mu$, we have

$$\begin{aligned}\left(u_1\kappa^\mu\right)\left(g_1\right) = \kappa^\mu\left(u_1^{-1}g_1\right) &= \kappa^\mu\left(\left(u_1^{-1}u_2\right)\left(u_2^{-1}g_1\right)\right)\\ &\geq \min\left\{\kappa^\mu\left(u_1^{-1}u_2\right), \kappa^\mu\left(u_2^{-1}g_1\right)\right\}\\ &= \min\left\{\kappa^\mu(e), \kappa^\mu\left(u_2^{-1}g_1\right)\right\} \geq \kappa^\mu\left(u_2^{-1}g_1\right),\end{aligned} \tag{23}$$

which implies that

$$\left(u_1\kappa^\mu\right)\left(g_1\right) = \left(u_2\kappa^\mu\right)\left(g_1\right). \tag{24}$$

Similarly

$$\left(u_2\kappa^\mu\right)\left(g_1\right) = \left(u_1\kappa^\mu\right)\left(g_1\right). \tag{25}$$

The required equality can be obtained by using (24) and (25).

(ii) This part proved in the same way. □

Theorem 24. *Let κ^μ be a μ-FNSG of a group G and $\kappa^\mu u_1 = \kappa^\mu u_2$; then, $\kappa^\mu(u_1) = \kappa^\mu(u_2)$, for all $u_1, u_2 \in G$.*

Proof. Let $u_1, u_2 \in G$ and $\kappa^\mu u_1 = \kappa^\mu u_2$; then, $\kappa^\mu u_1(u_2) = \kappa^\mu u_2(u_2)$. This implies that $\kappa^\mu(u_2u_1^{-1}) = \kappa^\mu(e)$. Consider

$$\begin{aligned}\kappa^\mu\left(u_1\right) = \kappa^\mu\left(u_1u_2^{-1}u_2\right) &\geq \min\left\{\kappa^\mu\left(u_1u_2^{-1}\right), \kappa^\mu\left(u_2\right)\right\}\\ &= \min\left\{\kappa^\mu(e), \kappa^\mu\left(u_2\right)\right\}.\end{aligned} \tag{26}$$

This means that

$$\kappa^\mu\left(u_1\right) \geq \kappa^\mu\left(u_2\right). \tag{27}$$

Likewise,

$$\kappa^\mu\left(u_1\right) \geq \kappa^\mu\left(u_2\right). \tag{28}$$

In view of (27) and (28), we obtain $\kappa^\mu\left(u_1\right) = \kappa^\mu\left(u_2\right)$. □

Theorem 25. *Let κ^μ be a μ-FNSG of G; then, for all $u_1, u_2, v_1, v_2 \in G$, $u_1\kappa^\mu = v_1\kappa^\mu$ and $u_2\kappa^\mu = v_2\kappa^\mu$ implies $u_1u_2\kappa^\mu = v_1v_2\kappa^\mu$.*

Proof. The application of Theorem 23 and using the given condition on κ^μ, we have $u_1^{-1}v_1, u_2^{-1}v_2 \in \kappa_*^\mu$. Consider

$$\begin{aligned}\left(u_1u_2\right)^{-1}v_1v_2 &= \left(u_2^{-1}u_1^{-1}\right)v_1v_2 = u_2^{-1}\left(u_1^{-1}v_1\right)v_2\\ &= u_2^{-1}\left(u_1^{-1}v_1\right)\left(u_2^{-1}u_2\right)v_2 = u_2^{-1}\left(u_1^{-1}v_1\right)u_2\left(u_2^{-1}v_2\right),\end{aligned} \tag{29}$$

which implies that $\left(u_1u_2\right)^{-1}v_1v_2 \in \kappa_*^\mu$. It follows that $v_1v_2 \in u_1u_2\kappa_*^\mu$; therefore

$$v_1v_2\kappa_*^\mu \subseteq u_1u_2\kappa_*^\mu. \tag{30}$$

Similarly

$$u_1u_2\kappa_*^\mu \subseteq v_1v_2\kappa_*^\mu. \tag{31}$$

By comparing the relations (30) and (31), we get the required equality. □

Theorem 26. *Suppose that κ^μ is a μ-FNSG of G and G/κ^μ is the set of all μ-fuzzy cosets of κ^μ in G. Define a binary operation $*$ on G/κ^μ in the following way: $\kappa^\mu u_1 * \kappa^\mu u_2 = \kappa^\mu u_1u_2$, where $u_1, u_2 \in G$. Then, $(G/\kappa^\mu,*)$ forms a group.*

Proof. Let $u_0, v_0, u_1, u_2 \in G$ and $\kappa^\mu(u_0), \kappa^\mu(v_0), \kappa^\mu(u_1), \kappa^\mu(u_2) \in G/\kappa^\mu$ such that $\kappa^\mu u_0 = \kappa^\mu u_1$ and $\kappa^\mu v_0 = \kappa^\mu u_2$. Then, for each $g_1 \in G$, we have $\kappa^\mu u_1u_2(g_1) = \kappa^\mu(g_1u_2^{-1}u_1^{-1})$ and $\kappa^\mu u_0v_0(g_1) = \kappa^\mu(g_1v_0^{-1}u_0^{-1})$. Moreover,

$$\begin{aligned}\kappa^\mu\left(g_1u_2^{-1}u_1^{-1}\right) &= \kappa^\mu\left(g_1v_0^{-1}v_0u_2^{-1}u_1^{-1}\right)\\ &= \kappa^\mu\left(g_1v_0^{-1}u_0^{-1}u_0v_0u_2^{-1}u_1^{-1}\right)\\ &\geq \min\left\{\kappa^\mu\left(g_1v_0^{-1}u_0^{-1}\right), \kappa^\mu\left(u_0v_0u_2^{-1}u_1^{-1}\right)\right\}.\end{aligned} \tag{32}$$

Since $\kappa^\mu u_0 = \kappa^\mu u_1$ and $\kappa^\mu v_0 = \kappa^\mu u_2$, therefore $\kappa^\mu(g_1u_1^{-1}) = \kappa^\mu(g_1u_0^{-1})$ and $\kappa^\mu(g_1u_2^{-1}) = \kappa^\mu(g_1v_0^{-1})$. Particularly, we have

$$\kappa^\mu\left(u_0v_0u_2^{-1}u_1^{-1}\right) = \kappa^\mu\left(u_0v_0u_2^{-1}u_0^{-1}\right) = \kappa^\mu\left(v_0u_2^{-1}\right) = \kappa^\mu(e). \tag{33}$$

Thus, $\kappa^\mu(u_0v_0u_2^{-1}u_1^{-1}) = \kappa^\mu(e)$.

Also, we know that $\kappa^\mu(e) \geq \kappa^\mu(g_1v_0^{-1}u_0^{-1})$. This implies that

$$\kappa^\mu\left(g_1u_2^{-1}u_1^{-1}\right) \geq \kappa^\mu\left(g_1v_0^{-1}u_0^{-1}\right). \tag{34}$$

Similarly, we have

$$\kappa^\mu\left(g_1v_0^{-1}u_0^{-1}\right) \geq \kappa^\mu\left(g_1u_2^{-1}u_1^{-1}\right). \tag{35}$$

By comparing (33) and (35), we have

$$\kappa^\mu\left(g_1v_0^{-1}u_0^{-1}\right) = \kappa^\mu\left(g_1u_2^{-1}u_1^{-1}\right). \tag{36}$$

Thus, $\kappa^\mu u_0v_0(g_1) = \kappa^\mu u_1u_2(g_1)$. Hence, $*$ is well defined.

(1) Furthermore, $*$ is associative

(2) $\kappa^\mu u_1^{-1}$ is inverse of $\kappa^\mu u_1$

(3) $\kappa^\mu e = \kappa^\mu$ is identity element in G/κ^μ

Hence, $(G/\kappa^\mu, *)$ is a group.

Note that G/κ^μ is known as the μ-fuzzy quotient group induced by κ^μ.

In the following result, we establish a group isomorphism between G/κ^μ and G/κ^μ_*. □

Theorem 27. *Let $\kappa^\mu \in \mu$-FNSG. Then, $G/\kappa^\mu \cong G/\kappa^\mu_*$.*

Proof. The application of Theorem 19 and using the fact that $\kappa^\mu \in \mu-$FNSG, we have κ^μ_* that is a normal subgroup of G. This ensures the existence of G/κ^μ_*. Define a map $\eta : G/\kappa^\mu \longrightarrow G/\kappa^\mu_*$ by the rule $\eta(u_1\kappa^\mu) = u_1\kappa^\mu_*$, $u_1 \in G$. Consider

$$u_1\kappa^\mu = u_2\kappa^\mu. \tag{37}$$

In view of Theorem 23, the above relation becomes

$$u_1\kappa^\mu_* = u_2\kappa^\mu_*. \tag{38}$$

This implies that

$$\eta(u_1\kappa^\mu) = \eta(u_2\kappa^\mu). \tag{39}$$

This shows that η is a well defined mapping.

The function η is injective because

$$\eta(u_1\kappa^\mu) = \eta(u_2\kappa^\mu), \tag{40}$$

implies that

$$u_1\kappa^\mu_* = u_2\kappa^\mu_* \tag{41}$$

By applying heorem 23, we have

$$u_1\kappa^\mu = u_2\kappa^\mu. \tag{42}$$

Clearly, η is surjective because for each $u_1\kappa^\mu_* \in G/\kappa^\mu_*$, there exists $u_1\kappa^\mu \in G/\kappa^\mu$ such that $\eta(u_1\kappa^\mu) = u_1\kappa^\mu_*$.

Moreover, η is homomorphism as for each $u_1\kappa^\mu, u_2\kappa^\mu \in G/\kappa^\mu$,

$$\begin{aligned}\eta(u_1\kappa^\mu \cdot u_2\kappa^\mu) &= \eta(u_1u_2\kappa^\mu) = u_1u_2\kappa^\mu_* = u_1\kappa^\mu_* \cdot u_2\kappa^\mu_*\\ &= \eta(u_1\kappa^\mu)\cdot\eta(u_2\kappa^\mu).\end{aligned} \tag{43}$$

Consequently, η is a group isomorphism between G/κ^μ and G/κ^μ_*.

In the upcoming theorem, we establish a natural homomorphism between the group G and its quotient group by μ-FNSG κ^μ. □

Theorem 28. *Let $\kappa^\mu \in \mu$-FNSG and $u_1 \in G$. Then, the map $\eta : G \longrightarrow G/\kappa^\mu$ defined by $\eta(u_1) = \kappa^\mu u_1$ is a natural homomorphism with its kernel κ^μ_*.*

Proof. Let $u_1, u_2 \in G$, then

$$\eta(u_1u_2) = \kappa^\mu u_1u_2, = \kappa^\mu u_1 \cdot \kappa^\mu u_2 = \eta(u_1).\eta(u_2). \tag{44}$$

It means that η is a homomorphism.

Also

$$\begin{aligned}\ker(\eta) &= \{u_1 \in G : \eta(u_1) = \kappa^\mu e\} = \{u_1 \in G : \kappa^\mu(u_1) = \kappa^\mu e\}\\ &= \{u_1 \in G : \kappa^\mu u_1(u_1) = \kappa^\mu e(u_1)\}\\ &= \{u_1 \in G : \kappa^\mu(e) = \kappa^\mu(u_1)\}\ker(\eta) = \kappa^\mu_*.\end{aligned} \tag{45}$$

□

Definition 29. Let κ^μ and ω^μ be any two μ-FSGs and u_1 be a fixed element of a group G. Then, κ^μ is μ-fuzzy conjugate to ω^μ if there exists an element $g_1 \in G$ such that $\kappa^\mu(u_1) = \omega^\mu(g_1u_1g_1^{-1})$. If such is the case, then we write $\kappa^\mu = (\omega^\mu)^{g_1}$, where $(\omega^\mu)^{g_1}(u_1) = \omega^\mu(g_1u_1g_1^{-1})$.

Theorem 30. *For any μ-FSG κ^{μ} of G, then, $\bigcap_{g_1\in G}(\kappa^{\mu})^{g_1}$ is a μ-FNSG of G and is the largest μ-FNSG of G that is contained in κ^{μ}.*

Proof. Given that $(\kappa^{\mu})^{g_1}$ is a μ-FSG and g_1 be any element of G, in view of Remark 15, we have $\bigcap_{g_1\in G}(\kappa^{\mu})^{g_1}$ that is a μ-FSG of G. Note that, for all $u_1 \in G$,

$$\{(\kappa^{\mu})^{g_1} \mid g_1 \in G\} = \{(\kappa^{\mu})^{g_1 u_1} \mid g_1 \in G\}. \tag{46}$$

Thus,

$$\begin{aligned}\min_{g_1\in G}(\kappa^{\mu})^{g_1}\left(u_1u_2u_1^{-1}\right) &= \min_{g_1\in G}\kappa^{\mu}\left(g_1u_1u_2u_1^{-1}g_1^{-1}\right)\\ &= \min_{g_1\in G}\kappa^{\mu}\left(g_1u_1u_2(g_1u_1)^{-1}\right)\\ &= \min_{g_1\in G}(\kappa^{\mu})^{g_1u_1}(u_2)\\ &= \min_{g_1\in G}(\kappa^{\mu})^{g_1}(u_2),\end{aligned} \tag{47}$$

for all $u_1, u_2 \in G$. Hence, by Theorem 18, $\bigcap_{g_1\in G}(\kappa^{\mu})^{g_1}$ is a μ-FNSG of G.

Now let ω^{μ} be a μ-FNSG of G with $\omega^{\mu} \subseteq \kappa^{\mu}$. Then, $\omega^{\mu} = (\omega^{\mu})^{g_1} \subseteq (\kappa^{\mu})^{g_1}$ for all $g_1 \in G$. So $\omega^{\mu} \subseteq \bigcap_{g_1\in G}(\kappa^{\mu})^{g_1}$. Therefore, $\bigcap_{g_1\in G}(\kappa^{\mu})^{g_1}$ is the largest μ-FNSG of G that is contained in κ^{μ}. □

Definition 31. Let κ^{μ} be a μ-FSG of G and $\alpha = \kappa^{\mu}(e)$ be the tip of κ^{μ}. Then

(i) The level subgroup of κ^{μ} is denoted by $\kappa^{\mu}_{\tau}[\tau, \alpha]$ and is defined as follows:

$$\kappa^{\mu}_{\tau}[\tau, \alpha] = \{u_1 \in G : \tau \le \kappa^{\mu}(u_1) \le \alpha, 0 \le \tau \le \alpha\}. \tag{48}$$

(ii) The open level subgroup of κ^{μ} is denoted by $\kappa^{\mu}_{\tau}]\tau, \alpha]$ and is defined as follows:

$$\kappa^{\mu}_{\tau}]\tau, \alpha] = \{u_1 \in G : \tau < \kappa^{\mu}(u_1) \le \alpha, 0 \le \tau < \alpha\}. \tag{49}$$

(iii) The tangible subgroup of κ^{μ} is denoted by $\kappa^{\mu}_{\tau\in \mathrm{Im}\,(\kappa^{\mu})}[\tau, \alpha]$ and is defined as follows:

$$\kappa^{\mu}_{\tau\in \mathrm{Im}\,(\kappa^{\mu})}[\tau, \alpha] = \{u_1 \in G : \tau \le \kappa^{\mu}(u_1) \le \alpha, \tau \in \mathrm{Im}\,(\kappa^{\mu}), 0 < \tau \le \alpha\}. \tag{50}$$

Example 3. Consider the quaternion group $Q_8 = \{\pm1, \pm i, \pm j, \pm k\}$ of order 8. The FS κ of Q_8 is defined as follows:

$$\kappa(u_1) = \begin{cases} 0.90, & \text{if } u_1 = 1,\\ 0.72, & \text{if } u_1 \in \{-1, \pm j\},\\ 0.50, & \text{otherwise.}\end{cases} \tag{51}$$

The μ-FS κ^{μ} of Q_8 correspond to the value $\mu = 0.40$ is given by

$$\kappa^{0.4}(u_1) = \begin{cases} 0.94, & \text{if } u_1 = 1,\\ 0.83, & \text{if } u_1 \in \{-1, \pm j\},\\ 0.70, & \text{otherwise.}\end{cases} \tag{52}$$

Note that the tip of κ^{μ} is $\alpha = 0.94$ and $\mathrm{Im}\,(\kappa^{0.40}) = \{0.70, 0.83, 0.94\}$.

(i) The LSG corresponding to the value $\tau = 0.83$ is given by

$$\kappa^{0.40}_{0.83}[0.83, 0.94] = \{\pm1, \pm j\}. \tag{53}$$

(ii) The OLSG corresponding to the value $\tau = 0.90$ is given by $\kappa^{0.40}_{0.90}[0.90, 0.94] = \{1\}$

(iii) The TSG corresponding to the value $\tau = 0.70$ is given by $\kappa^{0.40}_{0.70\in \mathrm{Im}\,(\kappa^{\mu})}[0.70, 0.94] = Q_8$

Clearly, $\kappa^{0.40}_{0.83}[0.83, 0.94]$, $\kappa^{0.40}_{0.90}[0.90, 0.94]$, and $\kappa^{0.40}_{0.70\in \mathrm{Im}\,(\kappa^{\mu})}[0.70, 0.94]$ are the subgroups of Q_8.

The following result describes another approach to obtain a μ-FNSG.

Theorem 32. *For any μ-FSG κ^{μ} of a group G, the following statements are equivalent:*

(i) κ^{μ} is a μ-FNSG in G

(ii) Every LSG of κ^{μ} is normal in G

(iii) Every TSG of κ^{μ} is normal in G

(iv) Every OLSG of κ^{μ} is normal in G

Proof. i ⇒ ii: The application of Theorem 19 ensures the validity of i ⇒ ii.

ii ⇒ iii: Since every TSG of κ^{μ} is a LSG of κ^{μ}, which is normal in G. Thus, every TSG of κ^{μ} is normal in G.

iii ⇒ iv: Let $0 \le \tau \le \alpha$, where α is the tip of κ^{μ}. Then,

$$\kappa^{\mu}_{\tau}]\tau, \alpha] = \bigcup_{\substack{\tau<\lambda\\ \lambda\in Im(\kappa^{\mu})}} \kappa^{\mu}_{\lambda\in Im(\kappa^{\mu})}[\lambda, \alpha], \tag{54}$$

which is normal in G, being the union of an increasing chain of normal subgroups of G.

iv ⇒ i: If possible, let there exist two elements $u_1, u_2 \in G$ such that $\kappa^{\mu}(u_1u_2) > \kappa^{\mu}(u_2u_1) = \tau$. Then, $u_1u_2 \in \kappa^{\mu}_{\tau}]\tau, \alpha]$, since $\kappa_\tau^\mu]\tau, \alpha]$ is normal in G. Therefore, $u_2 u_1 \in \kappa_\tau^\mu]\tau, \alpha]$. This implies that $\tau < \kappa^\mu(u_1 u_2)$, which is not true. □

5. Conclusions

In this work, we introduced the notion of μ-FSG defined over μ-FS and studies various important properties. We also presented the concept of μ-FNSGs and proved some important related results. Furthermore, we have defined the quotient group G/κ^μ and proved the existence of isomorphism between G/κ^μ and G/κ_*^μ. In addition, we have defined the notions of LSG, OLSG, and TSG of a μ-FSG and have developed a relationship between these notions and μ-FNSG. We are hopeful this article provides a base to study many more important group theoretic topics like isomorphism theorems, Cayley's theorem, Lagrange's theorem, Sylow's theorem, and many others inμ-fuzzy environment.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors have contributed equally to this paper in all aspects. All authors have read and agreed to the published version of the manuscript.

References

[1] L. A. Zadeh, "Fuzzy sets. Information and control," *Inform Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] A. Rosenfeld, "Fuzzy groups," *Journal of Mathematical Analysis and Applications*, vol. 35, no. 3, pp. 512–517, 1971.

[3] P. S. Das, "Fuzzy groups and level subgroups," *Journal of Mathematical Analysis and Applications*, vol. 84, no. 1, pp. 264–269, 1981.

[4] N. P. Mukherjee and P. Bhattacharya, "Fuzzy normal subgroups and fuzzy cosets," *Information Sciences*, vol. 34, no. 3, pp. 225–239, 1984.

[5] A. S. Mashour, H. Ghanim, and F. I. Sidky, "Normal fuzzy subgroups," *Information Sciences*, vol. 20, pp. 53–59, 1990.

[6] V. N. Dixit, R. Kumar, and N. Ajmal, "Level subgroups and union of fuzzy subgroups," *Fuzzy Sets and Systems*, vol. 37, no. 3, pp. 359–371, 1990.

[7] D. S. Malik, J. N. Mordeson, and P. S. Nair, "Fuzzy normal subgroups in fuzzy subgroups," *Journal of the Korean Mathematical Society*, vol. 29, no. 1, pp. 1–8, 1992.

[8] S. Ray, "Isomorphic fuzzy groups," *Fuzzy Sets and Systems*, vol. 50, no. 2, pp. 201–207, 1992.

[9] N. Ajmal, "The lattice of fuzzy normal subgroups is modular," *Information Sciences*, vol. 83, no. 3-4, pp. 199–209, 1995.

[10] M. A. Mishref, "Normal fuzzy subgroups and fuzzy normal series of finite groups," *Fuzzy Sets and Systems*, vol. 72, no. 3, pp. 379–383, 1995.

[11] J. Zhou, S. Y. Li, and S. Y. Li, "LP-fuzzy normal subgroups and fuzzy quotient groups," *Journal of Fuzzy Mathematics*, vol. 5, pp. 27–40, 1997.

[12] A. J. Van der Wal, "Application of fuzzy logic control in industry," *Fuzzy Sets and Systems*, vol. 74, no. 1, pp. 33–41, 1995.

[13] S. C. Cheng and J. N. Mordeson, "Applications of fuzzy algebra in automata theory and coding theory," in *Proceedings of IEEE 5th International Fuzzy Systems*, pp. 125–129, New Orleans, LA, USA, Sept. 1996.

[14] M. Y. Chow and H. Tram, "Application of fuzzy logic technology for spatial load forecasting," in *Proceedings of 1996 Transmission and Distribution Conference and Exposition*, pp. 608–614, Los Angeles, CA, USA, Sept. 1996.

[15] H. Rowlands and L. R. Wang, "An approach of fuzzy logic evaluation and control in SPC," *Quality and Reliability Engineering International*, vol. 16, no. 2, pp. 91–98, 2000.

[16] Y. L. Liu, "Quotient groups induced by fuzzy subgroups," *Quasigroups and Related Systems*, vol. 11, pp. 71–78, 2004.

[17] J. N. Mordeson, K. R. Bhutani, and A. Rosenfeld, *Fuzzy Group Theory*, Springer, New York, 2005.

[18] X. Q. Li, B. Zhang, and H. Li, "Computing efficient solutions to fuzzy multiple objective linear programming problems," *Fuzzy Sets and Systems*, vol. 157, no. 10, pp. 1328–1332, 2006.

[19] A. Faraz and A. F. Shapiro, "An application of fuzzy random variables to control charts," *Fuzzy Sets and Systems*, vol. 161, no. 20, pp. 2684–2694, 2010.

[20] S. Abdullah, M. Aslam, T. A. Khan, and M. Naeem, "A new type of fuzzy normal subgroups and fuzzy cosets," *Journal of Intelligent & Fuzzy Systems*, vol. 25, no. 1, pp. 37–47, 2013.

[21] Y. Celik, C. Ekiz, and S. Yamak, "Applications of fuzzy soft sets in ring theory," *Annals Fuzzy Mathematics and Informatics*, vol. 5, no. 3, pp. 451–462, 2013.

[22] M. J. Borah and B. Hazarika, "Some results of mixed fuzzy soft topology and applications in chemistry," *Annals of Fuzzy Mathematics and Informatics*, vol. 12, no. 1, pp. 129–138, 2016.

[23] M. Riaz and M. R. Hashmi, "Fuzzy parameterized fuzzy soft topology with applications," *Annals of Fuzzy Mathematics and Informatics*, vol. 13, no. 5, pp. 593–613, 2017.

[24] A. Khan, M. Izhar, and K. Hila, "On algebraic properties of DFS sets and its application in decision making problems," *Journal of Intelligent & Fuzzy Systems*, vol. 36, no. 6, pp. 6265–6281, 2019.

[25] A. Khan, M. Izhar, and M. M. Khalaf, "Generalised multi-fuzzy bipolar soft sets and its application in decision making," *Journal of Intelligent & Fuzzy Systems*, vol. 37, no. 2, pp. 2713–2725, 2019.

[26] M. Gulistan, N. Yaqoob, Z. Rashid, F. Smarandache, and H. A. Wahab, "A study on neutrosophic cubic graphs with real life applications in industries," *Symmetry*, vol. 10, no. 6, p. 203, 2018.

[27] M. Gulistan, I. Beg, and N. Yaqoob, "A new approach in decision making problems under the environment of neutrosophic cubic soft matrices," *Journal of Intelligent & Fuzzy Systems*, vol. 36, no. 1, pp. 295–307, 2019.

[28] K. Ullah, T. Mahmood, Z. Ali, and N. Jan, "On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition," *Complex & Intelligent Systems*, vol. 6, no. 1, pp. 15–27, 2020.

[29] D. Alghazzawi, U. Shuaib, T. Fatima, A. Razaq, and M. A. Binyamin, "Algebraic characteristics of anti-intuitionistic

fuzzy subgroups over a certain averaging operator," *IEEE Access*, vol. 8, pp. 205014–205021, 2020.

[30] C. Bejines, M. J. Chasco, and J. Elorza, "Aggregation of fuzzy subgroups," *Fuzzy Sets and Systems*, vol. 418, pp. 170–184, 2021.

[31] S. Dogra and M. Pal, "Picture fuzzy subgroup," *Kragujevac Journal of Mathematics*, vol. 47, no. 6, pp. 911–933, 2023.

[32] H. Alolaiyan, U. Shuaib, L. Latif, and A. Razaq, "T-intuitionistic fuzzification of Lagrange's theorem of t-intuitionistic fuzzy subgroup," *IEEE Access*, vol. 7, pp. 158419–158426, 2019.

[33] M. Akilesh and R. Muthuraj, "Properties of multi anti L-fuzzy quotient group A of a group G," *The Journal of Mathematics and Computer Science*, vol. 12, 2022.

[34] S. Bhunia, G. Ghorai, Q. Xin, and F. I. Torshavn, "On the characterization of Pythagorean fuzzy subgroups," *AIMS Mathematics*, vol. 6, no. 1, pp. 962–978, 2021.

[35] H. Alolaiyan, H. A. Alshehri, M. H. Mateen, D. Pamucar, and M. Gulzar, "A novel algebraic structure of (α, β)-complex fuzzy subgroups," *Entropy*, vol. 23, no. 8, p. 992, 2021.

[36] D. A. Kattan, M. Amin, and A. Bariq, "Certain structure of Lagrange's theorem with the application of interval-valued intuitionistic fuzzy subgroups," *Journal of Function Spaces*, vol. 2022, 9 pages, 2022.

[37] A. Imtiaz, U. Shuaib, H. Alolaiyan, A. Razaq, and M. Gulistan, "On structural properties of-complex fuzzy sets and their applications," *Complexity*, vol. 2020, 13 pages, 2020.

[38] K. R. Balasubramanian and R. Revathy, "Product of (λ, ?)-multifuzzy subgroups of a group," *Annals of the Romanian Society for Cell Biology*, vol. 25, no. 6, pp. 2448–2460, 2021.

[39] K. Ahmad, M. Bal, and M. Aswad, "The kernel of fuzzy and anti-fuzzy groups," *Journal of Neutrosophic and Fuzzy Systems*, vol. 1, no. 1, pp. 48–54, 2022.