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Research Article

Kaleva-Seikkala's Type Fuzzy b-Metric Spaces and Several Contraction Mappings

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In this paper, we introduce the concept of Kaleva-Seikkala's type fuzzy b-metric spaces as a generalization of the notion of b-metric spaces and fuzzy metric spaces. In such spaces, we establish Banach type, Reich type, and Chatterjea type fixed-point theorems, which improve the relevant results in fuzzy metric spaces. Two technical lemmas are employed to ensure that a Picard sequence is a Cauchy sequence. Finally, various applications are given to testify the fact that our main theorems extend the cases of b-metric spaces.

1. Introduction

In 1965, the theory of fuzzy sets was introduced by Zadeh in [1]. Henceforth, several researchers have discussed and developed this theory and applied the results to various different areas, such as mathematical programming, modeling theory, cybernetics, neural networks, statistics, construction machinery, and image processing (see, e.g., [2–4]). After this pioneering work, some types of fuzzy metric spaces (briefly, $\mathscr{FMS}$) were presented by numerous authors (refer to [5–7]). In particular, Kramosil and Michalek [7], in 1975, gave the notion of $\mathscr{FMS}$ as a modification of the notion of probabilistic metric space initiated by Menger [8]. More detailed information about such spaces and various fixed-point theorems in these $\mathscr{FMS}$ can be seen in [9–16]. In 1984, another type of fuzzy metric spaces called Kaleva-Seikkala's type fuzzy metric space (briefly, $\mathscr{KS}$-$\mathscr{FMS}$) was initiated by Kaleva and Seikkala [17], which generalized the metric space by defining a nonnegative fuzzy number as the distance between two points and applying a new triangle inequality which is analogous to the common triangle inequality. Drawing inspiration from [17], much work has been done in $\mathscr{KS}$-$\mathscr{FMS}$ (see, e.g., [18–22] and the references therein). Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{N}^+$, and $\mathbb{R}^+$, the sets of natural numbers, positive integer numbers, and positive real numbers, respectively. All the concepts about $\mathscr{KS}$-$\mathscr{FMS}$ not given in this paper are the same as in [23, 24].

As a prevalent generalization of the metric spaces, Bakhtin [25] in 1989 introduced the notion of b-metric spaces (briefly, b-$\mathscr{MS}$), which was formally defined by Czerwik [26] in 1993. In the last decades, many authors investigated the existence and uniqueness of the fixed point for various contractions in b-$\mathscr{MS}$ (see, e.g., [27–33]). Furthermore, Aghajani et al. [34] generalized the concept of the b-$\mathscr{MS}$ to the G_b-metric space and established several fixed-point theorems in such spaces. Very recently, Gupta et al. [35] extended various existing results in G_b-metric spaces.

Regarding the concepts of the b-$\mathscr{MS}$ and several classical contractions in b-$\mathscr{MS}$, we suggest refer to [25, 26, 32, 33].

In 2012, Sedghi and Shobe [36] initiated the definition of b-$\mathscr{FMS}$ as a generalization of $\mathscr{FMS}$ presented by George and Veeramani in [11]. There are some results in such spaces (see, for example, [36, 37]). Following this trend, Chauhan and Gupta [38] introduced the notion of George and Veeramani's type fuzzy cone b-$\mathscr{MS}$ and established new version of Banach contraction principle. As far as we know, there is no paper devoted to propose Kaleva-

Seikkala's type fuzzy b-metric spaces. Due to the existing results mentioned above and application potential, it is significant to focus on this research topic.

In this paper, we introduce the concept of Kaleva-Seikkala's type fuzzy b-metric spaces (briefly, $\mathscr{KS}$-$\mathscr{F}b\mathscr{MS}$) which generalizes the notions of $\mathscr{KS}$-$\mathscr{FMS}$ and b-$\mathscr{MS}$. In Section 2, some basic properties and lemmas of $\mathscr{KS}$-$\mathscr{F}b\mathscr{MS}$ were presented, which will be used later. In Sections 3–5, we establish and prove the fixed-point theorems concerning Banach type contractions, Reich type contractions, and Chatterjea type contractions in such spaces, respectively. It is worth mentioning that the range of all contraction constants in our main results are independent of the space coefficient $\mathfrak{b}$. These results improve and generalize the corresponding results in $\mathscr{KS}$-$\mathscr{F}b\mathscr{MS}$. Moreover, two techinical lemmas for the proof of Cauchy sequence play a pivotal role in the above theorems. In the final section, we give a lemma to show that a b-$\mathscr{MS}$ is a special $\mathscr{F}b\mathscr{MS}$. Applying this lemma, some applications are presented to illustrate the fact that our main results extend the cases of b-$\mathscr{MS}$.

2. Kaleva-Seikkala's Type Fuzzy b-Metric Spaces

In this section, we introduce the concept of $\mathscr{KS}$-$\mathscr{F}b\mathscr{MS}$ and present some elementary lemmas which will be applied in later sections.

Definition 1. Let $\mathfrak{S}$ be a nonempty set, $\mathfrak{d} : \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathfrak{F}^+$ be a mapping, and $\mathfrak{b} \geq 1$ be a real number. Suppose that $\mathfrak{L}, \mathfrak{R}$: $[0,1] \times [0,1] \longrightarrow [0,1]$ be two nondecreasing and symmetric functions such that

$$\mathfrak{L}(0,0) = 0, \mathfrak{R}(1,1) = 1. \tag{1}$$

For $\imath \in (0,1]$, define

$$[\mathfrak{d}(x,y)]_\imath = [\mathfrak{I}_\imath(x,y), \wp_\imath(x,y)], \quad \forall x, y \in \mathfrak{S}. \tag{2}$$

The following conditions are satisfied:
(BM1) $\mathfrak{d}(x,y) = \bar{0} \Leftrightarrow x = y$;
(BM2) for each $x, y \in \mathfrak{S}$, $\mathfrak{d}(x,y) = \mathfrak{d}(y,x)$;
(BM3) for each $x, y, z \in \mathfrak{S}$:
(BM3$\mathfrak{L}$) $\mathfrak{d}(x,y)(\mathfrak{b}(\zeta + \eta)) \geq \mathfrak{L}(\mathfrak{d}(x,z)(\zeta), \mathfrak{d}(z,y)(\eta))$, whenever $\zeta \leq \mathfrak{I}_1(x,z), \eta \leq \mathfrak{I}_1(z,y)$ and $\mathfrak{b}(\zeta + \eta) \leq \mathfrak{I}_1(x,y)$;
(BM3$\mathfrak{R}$) $\mathfrak{d}(x,y)(\mathfrak{b}(\zeta + \eta)) \leq \mathfrak{R}(\mathfrak{d}(x,z)(\zeta), \mathfrak{d}(z,y)(\eta))$, whenever $\zeta \geq \mathfrak{I}_1(x,z), \eta \geq \mathfrak{I}_1(z,y)$ and $\mathfrak{b}(\zeta + \eta) \geq \mathfrak{I}_1(x,y)$.

Then, $\mathfrak{d}$ is called a fuzzy b-metric, and the quintuple $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is called a fuzzy b-metric space with the coefficient $\mathfrak{b}$. If $\mathfrak{d} : \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathfrak{F}^+_\infty$ and $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ satisfies (BM1)-(BM3), then $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is called a generalized fuzzy b-metric space (briefly, $\mathscr{GF}b\mathscr{MS}$).

Lemma 2. *Let $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ be a $\mathscr{F}b\mathscr{MS}$. For each $\imath \in (0,1]$ and $x, y \in \mathfrak{S}$,*

$$[\mathfrak{d}(x,y)]_\imath = [\mathfrak{I}_\imath(x,y), \wp_\imath(x,y)]. \tag{3}$$

Then,

(1) $\lim_{\eta \longrightarrow +\infty} \mathfrak{d}(x,y)(\eta) = 0$ and $\lim_{\eta \longrightarrow -\infty} \mathfrak{d}(x,y)(\eta) = 0$

(2) For each $\imath \in (0,1]$, $\wp_\imath(x,y)$ is a nonincreasing and left continuous function

(3) $\mathfrak{d}(x,y)(\eta)$ is a nonincreasing and left continuous function for $\eta \in (\mathfrak{I}_1(x,y), +\infty)$

Lemma 3. *Suppose that $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ be a $\mathscr{F}b\mathscr{MS}$, and if*
($\mathfrak{R}$-1) $\mathfrak{R}(x,y) \leq \max\{x,y\}$;
($\mathfrak{R}$-2) $\forall \imath \in (0,1]$, $\exists \jmath \in (0,\imath]$ s.t. $\mathfrak{R}(\jmath, \mathfrak{r}) < \imath$ for all $\mathfrak{r} \in (0,\imath)$;
($\mathfrak{R}$-3) $\lim_{\imath \longrightarrow 0^+} \mathfrak{R}(\imath, \imath) = 0$.
Then, ($\mathfrak{R}$-1)$\Rightarrow$($\mathfrak{R}$-2)$\Rightarrow$($\mathfrak{R}$-3).

Lemma 4. *Let $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ be a $\mathscr{F}b\mathscr{MS}$. Then,*
($\mathfrak{R}$
-1)-
$\Rightarrow \wp_\imath(x,y) \leq \mathfrak{b}[\wp_\imath(x,z) + \wp_\imath(z,y)]$ for all $\imath \in (0,1]$ and $x, y, z \in \mathfrak{S}$.

($\mathfrak{R}$-2)$\Rightarrow$ for each $\imath \in (0,1]$, there exists $\jmath = \jmath(\imath) \in (0,\imath]$ such that for all $x, y, z \in \mathfrak{S}$,

$$\wp_\imath(x,y) \leq \mathfrak{b}\left[\wp_\imath(x,z) + \wp_\jmath(z,y)\right]. \tag{4}$$

($\mathfrak{R}$-3)$\Rightarrow$ for each $\imath \in (0,1]$, there exists $\jmath = \jmath(\imath) \in (0,\imath]$ such that for all $x, y, z \in \mathfrak{S}$

$$\wp_\imath(x,y) \leq \mathfrak{b}\left[\wp_\jmath(x,z) + \wp_\jmath(z,y)\right]. \tag{5}$$

Proof.

(1) Suppose that, on the contrary, for some $\imath \in (0,1]$ and $x_0, y_0, z_0 \in \mathfrak{S}$, $\wp_\imath(x_0,y_0) > \mathfrak{b}[\wp_\imath(x_0,z_0) + \wp_\imath(z_0,y_0)]$. We can find $\zeta, \eta \in \mathbb{R}^+$ such that $\mathfrak{b}(\zeta + \eta) = \wp_\imath(x_0,y_0) \geq \mathfrak{I}_1(x_0,y_0)$, $\zeta > \wp_\imath(x_0,z_0) \geq \mathfrak{I}_1(x_0,z_0)$, and $\eta > \wp_\imath(z_0,y_0) \geq \mathfrak{I}_1(z_0,y_0)$, which implies that

$$\mathfrak{d}(x_0,z_0)(\zeta) < \imath \text{ and } \mathfrak{d}(z_0,y_0)(\eta) < \imath. \tag{6}$$

From (BM3$\mathfrak{R}$) and the condition ($\mathfrak{R}$-1), we obtain that

$$\begin{aligned}\imath &\leq \mathfrak{d}(x_0,y_0)(\wp_\imath(x_0,y_0)) = \mathfrak{d}(x_0,y_0)(\mathfrak{b}(\zeta + \eta)) \\ &\leq \mathfrak{R}(\mathfrak{d}(x_0,z_0)(\zeta), \mathfrak{d}(z_0,y_0)(\eta)) \\ &\leq \max\{\mathfrak{d}(x_0,z_0)(\zeta), \mathfrak{d}(z_0,y_0)(\eta)\} < \imath,\end{aligned} \tag{7}$$

which is a contradiction.

(2) Assume that ($\mathfrak{R}$-2) holds, i.e., for every $\imath \in (0,1]$, there is $\jmath = \jmath(\imath) \in (0,\imath]$ such that $\mathfrak{R}(\jmath, \mathfrak{r}) < \imath$ for all $\mathfrak{r} \in (0,\imath)$. Since $\wp_\imath$ is left continuous and nonincreasing, it is sufficient to prove that $\wp_\imath(x,y) \leq \mathfrak{b}[\wp_\jmath(x,z) + \wp_\mathfrak{r}(z,y)]$ for all $\mathfrak{r} \in (0,\imath)$. If for some $\imath \in (0,1]$ and $\mathfrak{r} \in (0,\imath)$, we have $\wp_\imath(x_0,y_0) > \mathfrak{b}[\wp_\jmath(x_0,z_0) + \wp_\mathfrak{r}(z_0, y_0)]$ for some $x_0, y_0, z_0 \in \mathfrak{S}$. Then, we can find ζ and η such that $\mathfrak{b}(\zeta + \eta) = \wp_\imath(x_0,y_0) \geq \mathfrak{I}_1(x_0,y_0)$, ζ

$> \wp_\jmath(x_0, z_0) \geq \mathfrak{I}_1(x_0, z_0)$ and $\eta > \wp_\mathfrak{r}(z_0, y_0) \geq \mathfrak{I}_1(z_0, y_0)$. It follows that

$$\mathfrak{d}(x_0, z_0)(\zeta) < \jmath \text{ and } \mathfrak{d}(z_0, y_0)(\eta) < \mathfrak{r}. \tag{8}$$

By means of (BM3$\mathfrak{R}$), we have

$$\begin{aligned} \imath &\leq \mathfrak{d}(x_0, y_0)(\wp_\imath(x_0, y_0)) = \mathfrak{d}(x_0, y_0)(\mathfrak{b}(\zeta+\eta)) \\ &\leq \mathscr{R}(\mathfrak{d}(x_0, z_0)(\zeta), \mathfrak{d}(z_0, y_0)(\eta)) \leq \mathscr{R}(\jmath, \mathfrak{r}) < \imath, \end{aligned} \tag{9}$$

which is a contradiction.

(3) Suppose that ($\mathfrak{R}$-3) holds, i.e., for each $\imath \in (0, 1]$, we have $\lim_{i \longrightarrow 0^+} \mathfrak{R}(\imath, \imath) = 0$. Then, there is $\jmath_0 \in (0, \imath]$ such that $\mathfrak{R}(\jmath, \jmath) < \imath$ for all $\jmath < \jmath_0$. Assume that, on the contrary, for some $\imath \in (0, 1]$ and $\jmath \in (0, \imath)$, we have $\wp_\imath(x_0, y_0) > \mathfrak{b}[\wp_\jmath(x_0, z_0) + \wp_\jmath(z_0, y_0)]$ for some $x_0, y_0, z_0 \in \mathfrak{S}$. Then, we can find $\zeta, \eta \in \mathbb{N}^+$ such that $\mathfrak{b}(\zeta+\eta) = \wp_\imath(x_0, y_0) \geq \mathfrak{I}_1(x_0, y_0)$, $\zeta > \wp_\jmath(x_0, z_0) \geq \mathfrak{I}_1(x_0, z_0)$ and $\eta > \wp_\jmath(z_0, y_0) \geq \mathfrak{I}_1(z_0, y_0)$, which deduces that

$$\mathfrak{d}(x_0, z_0)(\zeta) < \jmath \text{ and } \mathfrak{d}(z_0, y_0)(\eta) < \jmath. \tag{10}$$

On account of (BM3$\mathfrak{R}$) and the condition ($\mathfrak{R}$-3),

$$\begin{aligned} \imath &\leq \mathfrak{d}(x_0, y_0)(\wp_\imath(x_0, y_0)) = \mathfrak{d}(x_0, y_0)(\mathfrak{b}(\zeta+\eta)) \\ &\leq \mathfrak{R}(\mathfrak{d}(x_0, z_0)(\zeta), \mathfrak{d}(z_0, y_0)(\eta)) \leq \mathfrak{R}(\jmath, \jmath) < \imath, \end{aligned} \tag{11}$$

which is a contradiction. □

Definition 5. Let $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ be a $\mathscr{F}b\mathscr{M}\mathscr{S}$ and $\{x_n\}$ be a suquence in $\mathfrak{S}$.

(1) If $\lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, x) = \bar{0}$, i.e., $\lim_{n \longrightarrow \infty} \wp_\imath(x_n, x) = 0$ for each $\imath \in (0, 1]$, $\{x_n\}$ is said to converge to $x \in \mathfrak{S}$ ($x_n \longrightarrow x$ as $n \longrightarrow \infty$ or $\lim_{n \longrightarrow \infty} x_n = x$)

(2) If $\lim_{n,m \longrightarrow \infty} \mathfrak{d}(x_n, x_m) = \bar{0}$, equivalently, for any given $\epsilon > 0$ and $\imath \in (0, 1]$, there exists $N = N(\epsilon, \imath) \in \mathbb{N}^+$ such that $\wp_\imath(x_n, x_m) < \epsilon$, whenever $n, m \geq N$, $\{x_n\}$ is said to be a Cauchy sequence

(3) If every Cauchy sequence in $\mathfrak{S}$ converges, $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is called complete

Lemma 6. *Let $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ be a $\mathscr{F}b\mathscr{M}\mathscr{S}$ with ($\mathfrak{R}$-2) and $\{x_n\} \subseteq \mathfrak{S}$ be a sequence. If the sequence $\{x_n\}$ converges to both $x \in \mathfrak{S}$ and $y \in \mathfrak{S}$, then $x = y$.*

We end this section by giving an example to illustrate that a $\mathscr{F}b\mathscr{M}\mathscr{S}$ is obviously not a $\mathscr{F}\mathscr{M}\mathscr{S}$ or a b-$\mathscr{M}\mathscr{S}$.

Example 1. Let $\mathfrak{S} = [0,+\infty)$ and $\mathfrak{d}(x, y): \mathbb{R} \longrightarrow \mathbb{R}$ a mapping. If $x = y \in \mathfrak{S}$, we define $\mathfrak{d}(x, y)(\eta) = \bar{0}(\eta)$ for all $\eta \in \mathbb{R}$. If $x, y \in \mathfrak{S}$ with $x \neq y$, $\mathfrak{d}(x, y)$ is defined by

$$\mathfrak{d}(x, y)(\eta) = \begin{cases} 0, & \eta < 0, \\ \dfrac{1}{1 + \left(\eta/(x-y)^4\right)}, & \eta \geq 0. \end{cases} \tag{12}$$

Define that

$$\begin{cases} \mathfrak{L}(a, b) = \min\{a, b\}, \\ \mathfrak{R}(a, b) = \max\{a, b\}. \end{cases} \tag{13}$$

Then, the assertions hold:

(1) $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is complete, and the coefficient is $\mathfrak{b} = 8$;

(2) $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is not a $\mathscr{F}\mathscr{M}\mathscr{S}$.

Proof.

(1) First, we show that (BM1) in Definition 1 is satisfied for all $x, y \in \mathfrak{S}$. From the definition of $\mathfrak{d}(x, y)$, it is sufficient to verify that $\mathfrak{d}(x, y)(\eta) = \bar{0}(\eta)$ implies $x = y$. Note that

$$\bar{0}(\eta) = \begin{cases} 1, & \eta = 0, \\ 0, & \eta \neq 0. \end{cases} \tag{14}$$

Suppose that there exist $x_0 \neq y_0$ such that $\mathfrak{d}(x_0, y_0)(\eta) = \bar{0}(\eta)$. Taking $\eta = 1$, we have $\mathfrak{d}(x_0, y_0)(\eta) = 1/(1 + (1/((x_0 - y_0)^4))) \neq 0$, which is a contradiction. It is easy to verify that for each $x, y \in \mathfrak{S}$, (BM2) in Definition 1 holds. Next, for each $x, y, z \in \mathfrak{S}$, we prove that the condition (BM3) is satisfied. By a simple calculation, we get $\mathfrak{I}_1(x, y) = \wp_1(x, y) = 0$ for all $x, y \in \mathfrak{S}$.

(i) We prove (BM3$\mathfrak{L}$) with $\mathfrak{b} = 8$. Let $\zeta, \eta \in \mathbb{R}$ satisfying

$$\begin{cases} \zeta \leq \mathfrak{I}_1(x, z), \\ \eta \leq \mathfrak{I}_1(z, y), \\ 8(\zeta+\eta) \leq \mathfrak{I}_1(x, y). \end{cases} \tag{15}$$

Since $\mathfrak{L}(a, b) = \min\{a, b\}$, if $\zeta < \mathfrak{I}_1(x, z) = 0$ or $\eta < \mathfrak{I}_1(z, y) = 0$, then

$$\min\{\mathfrak{d}(x, z)(\zeta), \mathfrak{d}(z, y)(\eta)\} = 0 \leq \mathfrak{d}(x, y)(8(\zeta+\eta)). \tag{16}$$

Assume that $\zeta = \mathfrak{I}_1(x, z) = 0$ and $\eta = \mathfrak{I}_1(z, y) = 0$, we obtain that

$$8(\zeta+\eta) = 8(0+0) = 0 = \mathfrak{I}_1(x, y). \tag{17}$$

Thus, we conclude that

$$\mathfrak{d}(x,y)(8(\zeta+\eta)) = 1 = \min\{\mathfrak{d}(x,z)(\zeta), \mathfrak{d}(z,y)(\eta)\} = \mathfrak{L}(\mathfrak{d}(x,z)(\zeta), \mathfrak{d}(z,y)(\eta)). \tag{18}$$

That completes the proof of (BM3$\mathfrak{L}$). □

We prove (BM3$\mathfrak{R}$) with $\mathfrak{b}=8$. Let $\zeta, \eta \in \mathbb{R}$ satisfying

$$\begin{cases} \zeta \geq \mathfrak{I}_1(x,z), \\ \eta \geq \mathfrak{I}_1(z,y), \\ 8(\zeta+\eta) \geq \mathfrak{I}_1(x,y). \end{cases} \tag{19}$$

Now, we consider the following three cases.

Case 1. Assume that

$$\begin{cases} \zeta = \mathfrak{I}_1(x,z) = 0, \\ \eta = \mathfrak{I}_1(z,y) = 0, \end{cases} \tag{20}$$

we have

$$\mathfrak{d}(x,y)(8(\zeta+\eta)) = \mathfrak{d}(x,y)(8(0+0)) = 1 = \max\{\mathfrak{d}(x,z)(0), \mathfrak{d}(z,y)(0)\} = \mathfrak{R}(\mathfrak{d}(x,z)(\zeta), \mathfrak{d}(z,y)(\eta)). \tag{21}$$

Case 2. If

$$\begin{cases} \zeta > \mathfrak{I}_1(x,z) = 0, \\ \eta = \mathfrak{I}_1(z,y) = 0, \end{cases} \tag{22}$$

or

$$\begin{cases} \zeta = \mathfrak{I}_1(x,z) = 0, \\ \eta > \mathfrak{I}_1(z,y) = 0, \end{cases} \tag{23}$$

(Here, we discuss the previous assumption), then

$$\begin{aligned} &8(\zeta+\eta) > 0 = \mathfrak{I}_1(x,y), \\ &\mathfrak{d}(x,y)(8(\zeta+0)) \leq \max\{\mathfrak{d}(x,z)(\zeta), 1\} = \max\{\mathfrak{d}(x,z)(\zeta), \mathfrak{d}(z,y)(0)\} = \mathscr{R}(\mathfrak{d}(x,z)(\zeta), \mathfrak{d}(z,y)(0)). \end{aligned} \tag{24}$$

Case 3. Suppose that

$$\begin{cases} \zeta > \mathfrak{I}_1(x,z) = 0, \\ \eta > \mathfrak{I}_1(z,y) = 0. \end{cases} \tag{25}$$

For each $x, y, z \in \mathscr{X}$,

(a) If $x=y$ and $z \in \mathfrak{S}$, then

$$\mathfrak{d}(x,y)(8(\zeta+\eta)) = \bar{0}(8(\zeta+\eta)) = 0 \leq \max\{\mathfrak{d}(x,z)(\zeta), \mathfrak{d}(z,y)(\eta)\} = \mathfrak{R}(\mathfrak{d}(x,z)(\zeta), \mathfrak{d}(z,y)(\eta)). \tag{26}$$

(b) If $x=z\neq y$ or $x\neq y=z$, without loss of generality, let $x=z\neq y$, then

$$\begin{aligned} \mathfrak{d}(x,y)(8(\zeta+\eta)) &= \frac{1}{1+\left(8(\zeta+\eta)/(x-y)^4\right)} = \frac{1}{1+\left(8(\zeta+\eta)/(z-y)^4\right)} \\ &\leq \frac{1}{1+\left(\eta/(z-y)^4\right)} = \mathfrak{d}(z,y)(\eta) = \max\{0, \mathfrak{d}(z,y)(\eta)\} \\ &= \max\left\{\bar{0}(\zeta), \mathfrak{d}(z,y)(\eta)\right\} = \mathfrak{R}(\mathfrak{d}(x,z)(\zeta), \mathfrak{d}(z,y)(\eta)). \end{aligned} \tag{27}$$

(c) If $x\neq y\neq z$. Note that

$$\min\left\{\frac{\zeta}{(x-z)^4}, \frac{\eta}{(z-y)^4}\right\} \leq \frac{\zeta+\eta}{(x-z)^4+(z-y)^4}. \tag{28}$$

Might as well set $\min\{\zeta/(x-z)^4, \eta/(z-y)^4\} = \zeta/(x-z)^4$, which implies that $\zeta/(x-z)^4 \leq (\zeta+\eta)/((x-z)^4+(z-y)^4)$ and

$$\begin{aligned} \max\{\mathfrak{d}(x,z)(\zeta), \mathfrak{d}(z,y)(\eta)\} &= \max\left\{\frac{1}{1+\left(\zeta/(x-z)^4\right)}, \frac{1}{1+\left(\eta/(z-y)^4\right)}\right\} \\ &= \frac{1}{1+\zeta/(x-z)^4} = \mathfrak{d}(x,z)(\zeta). \end{aligned} \tag{29}$$

Thus, we conclude that

$$\begin{aligned} \mathfrak{d}(x,y)(8(\zeta+\eta)) &= \frac{1}{1+\left(8(\zeta+\eta)/(x-y)^4\right)} = \frac{1}{1+\left(8(\zeta+\eta)/(x-z+z-y)^4\right)} \\ &\leq \frac{1}{1+\left(8(\zeta+\eta)/\left[2(x-z)^2+2(z-y)^2\right]^2\right)} \leq \frac{1}{1+\left(8(\zeta+\eta)/8\left[(x-z)^4+(z-y)^4\right]\right)} \\ &= \frac{1}{1+\left(\zeta+\eta/(x-z)^4+(z-y)^4\right)} \leq \frac{1}{1+\left(\zeta/(x-z)^4\right)} = \mathfrak{d}(x,z)(\zeta) \\ &= \max\{\mathfrak{d}(x,z)(\zeta), \mathfrak{d}(z,y)(\eta)\} = \mathfrak{R}(\mathfrak{d}(x,z)(\zeta), \mathfrak{d}(z,y)(\eta)). \end{aligned} \tag{30}$$

The part of (BM3$\mathscr{R}$) is completed. Therefore, $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is a $\mathscr{F}b\mathscr{M}\mathscr{S}$.

Finally, we prove $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is complete. Let $\{x_n\} \subseteq \mathfrak{S}$ be a Cauchy sequence in $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$. For any $\varepsilon > 0$, there exists $N_1 > 0$ such that

$$\wp_{1/2}(x_m, x_n) = \frac{(1-(1/2))(x_m - x_n)^4}{1/2} = (x_m - x_n)^4 < \epsilon^4, \tag{31}$$

for all $m, n > N_1$. It implies that $|x_m - x_n| < \epsilon$. Thus, $\{x_n\}$ is a Cauchy sequence in $([0,+\infty), |\cdot|)$. Due to $([0,+\infty), |\cdot|)$ is complete, we can find $x^* \in [0,+\infty)$ such that $\lim_{n\longrightarrow\infty} x_n = x^*$. For any $\epsilon > 0$ and $\imath \in (0,1]$, by virtue of $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$, there is $N_2 > 0$ such that

$$|x_n - x^*| < \left(\frac{\imath\epsilon}{1-\imath}\right)^{1/4}, \quad \text{for all } n > N_2. \tag{32}$$

Then,

$$\wp_\imath(x_n, x^*) = \frac{(1-\imath)(x_n - x^*)^4}{\imath} < \frac{1-\imath}{\imath} \cdot \left(\left(\frac{\imath\epsilon}{1-\imath}\right)^{1/4}\right)^4 = \epsilon. \tag{33}$$

Thus, $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$ in $\mathscr{F}b\mathscr{M}\mathscr{S}$. Therefore, $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is complete.

Let $x = 3$, $y = 1$, $z = 2$, and $0 < \zeta = \eta$. Obviously, $\zeta > \mathfrak{I}_1(x, z) = 0$, $\eta > \mathfrak{I}_1(z, y) = 0$ and $\zeta + \eta > \mathfrak{I}_1(x, y) = 0$. By the definition of $\mathfrak{R}$,

$$\begin{aligned}\mathfrak{R}(\mathfrak{d}(3,2)(\zeta), \mathfrak{d}(2,1)(\zeta)) &= \max\{\mathfrak{d}(3,2)(\zeta), \mathfrak{d}(2,1)(\zeta)\} = \max\left\{\frac{1}{1+\left(\zeta/(3-2)^4\right)}, \frac{1}{1+\left(\zeta/(2-1)^4\right)}\right\} = \frac{1}{1+\zeta}.\\ \mathfrak{d}(3,1)(2\zeta) &= \frac{1}{1+\left(2\zeta/(3-1)^4\right)} = \frac{1}{1+(\zeta/8)}.\end{aligned} \tag{34}$$

Since $0 < (\zeta/8) < \zeta$, we have $(1/(1+(\zeta/8))) > (1/(1+\zeta))$. Thus,

$$\mathfrak{d}(3,1)(\zeta+\zeta) > \mathfrak{R}(\mathfrak{d}(3,2)(\zeta), \mathfrak{d}(2,1)(\zeta)). \tag{35}$$

Therefore, $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is not a $\mathscr{F}\mathscr{M}\mathscr{S}$.

3. Banach Type Contractions in $\mathscr{K}\mathscr{S}$-$\mathscr{F}b\mathscr{M}\mathscr{S}$

In this section, we will state and prove a fixed-point theorem for Banach type contractions in $\mathscr{K}\mathscr{S}$-$\mathscr{F}b\mathscr{M}\mathscr{S}$. This theorem extends Banach's results in $\mathscr{F}\mathscr{M}\mathscr{S}$.

Theorem 7. *Suppose that $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ be a complete $\mathscr{F}b\mathscr{M}\mathscr{S}(\mathfrak{b} \geq 1)$ with ($\mathfrak{R}$-2). Let $\mathscr{F}: \mathfrak{S} \longrightarrow \mathfrak{S}$ be a mapping. If there exists $k \in [0,1)$ such that*

$$\wp_\imath(\mathscr{F}x, \mathscr{F}y) \leq k\wp_\imath(x, y), \forall x, y \in \mathfrak{S}, \tag{36}$$

for all $\imath \in (0,1]$, then $\mathscr{F}$ admits a unique fixed point in $\mathfrak{S}$.

Proof. Since $k \in [0,1)$, there exists $\mathfrak{l} \in \mathbb{N}$ such that $k^{\mathfrak{l}} < (1/\mathfrak{b})$. Suppose that $\imath \in (0,1]$, applying (36), we derive

$$\wp_\imath(\mathscr{F}x, \mathscr{F}y) \leq k\wp_\imath(x, y), \tag{37}$$

for all $x, y \in \mathfrak{S}$. Clearly,

$$\wp_\imath(\mathscr{F}^2x, \mathscr{F}^2y) \leq k\wp_\imath(\mathscr{F}x, \mathscr{F}y) \leq k^2\wp_\imath(x, y). \tag{38}$$

Continuing this process, we deduce

$$\wp_\imath\left(\mathscr{F}^{\mathfrak{l}}x, \mathscr{F}^{\mathfrak{l}}y\right) \leq k^{\mathfrak{l}}\wp_\imath(x, y). \tag{39}$$

Let $g = \mathscr{F}^{\mathfrak{l}}$, $\alpha = k^{\mathfrak{l}} \in [0, 1/\mathfrak{b})$. From the above inequality, for each $\imath \in (0,1]$ we obtain that

$$\wp_\imath(gx, gy) \leq \alpha\wp_\imath(x, y), \tag{40}$$

for all $x, y \in \mathfrak{S}$.

Now, we shall prove that g admits a unique fixed point in $\mathfrak{S}$. Taking $x_0 \in \mathfrak{S}$, we construct a sequence $\{x_n\}$ by $x_n = gx_{n-1} = g^n x_0$ ($\forall n \in \mathbb{N}$). For each $\imath \in (0,1]$, $n \in \mathbb{N}$, by (40), we deduce

$$\begin{aligned}\wp_\imath(x_n, x_{n+1}) &= \wp_\imath(gx_{n-1}, gx_n) \leq \alpha\wp_\imath(x_{n-1}, x_n)\\ &\leq \alpha^2\wp_\imath(x_{n-2}, x_{n-1}) \leq \cdots \leq \alpha^n\wp_\imath(x_0, x_1).\end{aligned} \tag{41}$$

Thus, for each $\imath \in (0,1]$, due to $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ satisfies ($\mathfrak{R}$-2), it follows that we can find $\jmath = \jmath(\imath) \in (0, \imath]$ such that

$$\wp_\imath(x, y) \leq \mathfrak{b}\left[\wp_\jmath(x, z) + \wp_\imath(z, y)\right], \forall x, y, z \in \mathfrak{S}. \tag{42}$$

For $m, n \in \mathbb{N}$ with $m < n$, using (41) and (42), we derive

$$\begin{aligned}
\wp_\imath(x_m, x_n) &\le \mathfrak{b}\left[\wp_\jmath(x_m, x_{m+1}) + \wp_\imath(x_{m+1}, x_n)\right] \\
&\le \mathfrak{b}\wp_\jmath(x_m, x_{m+1}) + \mathfrak{b}^2\wp_\jmath(x_{m+1}, x_{m+2}) + \mathfrak{b}^2\wp_\imath(x_{m+2}, x_n) \\
&\le \cdots \le \mathfrak{b}\wp_\jmath(x_m, x_{m+1}) + \mathfrak{b}^2\wp_\jmath(x_{m+1}, x_{m+2}) \\
&\quad + \cdots + \mathfrak{b}^{n-m-1}\wp_\jmath(x_{n-2}, x_{n-1}) + \mathfrak{b}^{n-m-1}\wp_\imath(x_{n-1}, x_n) \\
&\le \mathfrak{b}\wp_\jmath(x_m, x_{m+1}) + \mathfrak{b}^2\wp_\jmath(x_{m+1}, x_{m+2}) \\
&\quad + \cdots + \mathfrak{b}^{n-m-1}\wp_\jmath(x_{n-2}, x_{n-1}) + \mathfrak{b}^{n-m}\wp_\jmath(x_{n-1}, x_n) \\
&\le \mathfrak{b}\alpha^m\wp_\jmath(x_0, x_1) + \mathfrak{b}^2\alpha^{m+1}\wp_\jmath(x_0, x_1) \\
&\quad + \cdots + \mathfrak{b}^{n-m}\alpha^{n-1}\wp_\jmath(x_0, x_1) \\
&\le \mathfrak{b}\alpha^m\left(1 + \mathfrak{b}\alpha + \mathfrak{b}^2\alpha^2 + \cdots\right)\wp_\jmath(x_0, x_1) = \frac{\mathfrak{b}\alpha^m}{1 - \mathfrak{b}\alpha}\wp_\jmath(x_0, x_1).
\end{aligned} \tag{43}$$

Due to $\alpha \in [0, 1/\mathfrak{b})$, we can conclude that $\{x_n\} \subseteq \mathfrak{S}$ is a Cauchy sequence. Note that $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is a complete $\mathcal{F}b\mathcal{M}\mathcal{S}$. Thus, there exists $v \in \mathfrak{S}$ such that $\lim_{n\longrightarrow\infty} x_n = v$, equivalently, $\lim_{n\longrightarrow\infty}\wp_\imath(x_n, v) = 0$ for all $\imath \in (0, 1]$.

For each $\imath \in (0, 1]$ and $n \in \mathbb{N}$, by virtue of (40), we obtain that

$$\wp_\imath(x_{n+1}, gv) = \wp_\imath(gx_n, gv) \le \alpha\wp_\imath(x_n, v). \tag{44}$$

Thus, $\wp_\imath(x_{n+1}, gv) \longrightarrow 0$ as $n \longrightarrow \infty$, that is, $x_{n+1} \longrightarrow gv$ as $n \longrightarrow \infty$. From Lemma 6, we deduce that $gv = v$. Therefore, $v \in \mathfrak{S}$ is a fixed point of g. This completes the proof of the existence of the fixed point.

If g has another fixed point $\varpi \in \mathfrak{S}$, i.e., $g\varpi = \varpi$ and $\varpi \neq v$, then $\wp_{\imath_0}(\varpi, v) > 0$ for some $i_0 \in (0, 1]$. Using (40), we have

$$\wp_{\imath_0}(\varpi, v) = \wp_{\imath_0}(g\varpi, gv) \le \alpha\wp_{\imath_0}(\varpi, v) < \wp_{\imath_0}(\varpi, v), \tag{45}$$

a contradiction. Therefore, $\varpi = v$, and the fixed point of g is unique. Hence, $\mathcal{F}^{\mathfrak{l}} v = v$. Then, $\mathcal{F}^{\mathfrak{l}}(\mathcal{F}v) = \mathcal{F}(\mathcal{F}^{\mathfrak{l}} v) = \mathcal{F}v$, which deduce that $\mathcal{F}v$ is a fixed point of $\mathcal{F}^{\mathfrak{l}}$. By the uniqueness of fixed point of $\mathcal{F}^{\mathfrak{l}} = g$, we get $\mathcal{F}v = v$. Furthermore, if there exists $\bar{v} \in \mathfrak{S}$ such that $\mathcal{F}\bar{v} = \bar{v}$, then $\bar{v}$ is a fixed point of $\mathcal{F}^{\mathfrak{l}}$. Again, by the uniqueness of the fixed point of $\mathcal{F}^{\mathfrak{l}}$, we obtain that $\bar{v} = v$. Thus, $\mathcal{F}$ admits a unique fixed point in $\mathfrak{S}$. □

The following corollary is an immediate consequence of Theorem 7.

Corollary 8. *Suppose that $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ be a complete $\mathcal{F}\mathcal{M}\mathcal{S}$ with ($\mathfrak{R}$-2), $\mathcal{F} : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a mapping. For each $\imath \in (0, 1]$ and $x, y \in \mathfrak{S}$,*

$$\wp_\imath(\mathcal{F}x, \mathcal{F}y) \le k\wp_\imath(x, y), \tag{46}$$

where $k \in [0, 1)$. Then, $\mathcal{F}$ has a unique fixed point in $\mathfrak{S}$.

Proof. Obviously, a $\mathcal{F}b\mathcal{M}\mathcal{S}$ is a generalized $\mathcal{F}\mathcal{M}\mathcal{S}$, and thus, taking $\mathfrak{b} = 1$ in Theorem 7, we obtain the result. □

To support our results, we give an illustrative example in $\mathcal{F}b\mathcal{M}\mathcal{S}$.

Example 2. Assume that $\mathfrak{S} = [0, +\infty)$, $\mathfrak{d}(x, y)$: $\mathbb{R} \longrightarrow \mathbb{R}$ a mapping. If $x = y \in \mathfrak{S}$, we define $\mathfrak{d}(x, y)(\eta) = \bar{0}(\eta)$, $\forall\eta \in \mathbb{R}$. If $x, y \in \mathfrak{S}$ with $x \neq y$, $\mathfrak{d}(x, y)$ is defined by

$$\mathfrak{d}(x, y)(\eta) = \begin{cases} 0, & \eta < 0, \\ \dfrac{1}{1 + \left(\eta/(x - y)^2\right)}, & \eta \ge 0. \end{cases} \tag{47}$$

Define $\mathfrak{L}, \mathfrak{R}$ by

$$\begin{cases} \mathfrak{L}(a, b) = \min\{a, b\}, \\ \mathfrak{R}(a, b) = \max\{a, b\}. \end{cases} \tag{48}$$

Let $Y : \mathfrak{S} \longrightarrow \mathfrak{S}$ defined by

$$Yx = \begin{cases} \dfrac{3x}{4}, & x \in [0, 1), \\ \dfrac{x}{2} + \dfrac{1}{4}, & x \in [1, +\infty). \end{cases} \tag{49}$$

Then, Y is a Banach type contraction with the contraction constant $k = 5/8$.

Proof. We can obtain that $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is a complete $\mathcal{F}b\mathcal{M}\mathcal{S}$ with the coefficient $\mathfrak{b} = 2$, which is similar to Example 1. By the definition of $\mathfrak{d}$, it is easy to prove that Y has the unique fixed point 0. Note that

$$\mathfrak{d}(x, y)(\wp_\imath(x, y)) = \imath, \tag{50}$$

for all $\imath \in (0, 1]$. Then,

$$\wp_\imath(x, y) = \frac{(\imath - 1)(x - y)^2}{\imath}. \tag{51}$$

Now, we divide it into three cases. □

Case 1. If $x, y \in [0, 1)$, we can see that

$$\begin{aligned}
\wp_\imath(Yx, Yy) &= \frac{(\imath - 1)(Yx - Yy)^2}{\imath} = \frac{(\imath - 1)((3x/4) - (3y/4))^2}{\imath} \\
&= \frac{\imath - 1}{\imath} \cdot \frac{9}{16}(x - y)^2 = \frac{9}{16} \cdot \frac{(\imath - 1)(x - y)^2}{\imath} \\
&= \frac{9}{16}\wp_\imath(x, y) \le \frac{5}{8}\wp_\imath(x, y).
\end{aligned} \tag{52}$$

Case 2. Suppose that $x, y \in [1,+\infty)$, we can derive that

$$\begin{aligned}\wp_\imath(Yx, Yy) &= \frac{(\imath-1)(Yx-Yy)^2}{\imath} = \frac{(\imath-1)[((x/2)+(1/4))-((y/2)+(1/4))]^2}{\imath} \\ &= \frac{\imath-1}{\imath}\cdot\frac{1}{4}(x-y)^2 = \frac{1}{4}\cdot\frac{(\imath-1)(x-y)^2}{\imath} \\ &= \frac{1}{4}\wp_\imath(x,y) \le \frac{5}{8}\wp_\imath(x,y).\end{aligned} \tag{53}$$

Case 3. Assume that $x \in [0,1)$ and $y \in [1,+\infty)$, we get

$$\begin{aligned}\wp_\imath(Yx, Yy) &= \frac{(\imath-1)(Yx-Yy)^2}{\imath} = \frac{(\imath-1)((3x/4)-(y/2)-(1/4))^2}{\imath} \\ &= \frac{\imath-1}{\imath}\cdot\frac{1}{16}(3x-2y-1)^2.\end{aligned} \tag{54}$$

Note that $(x-1)^2 \le (x-y)^2$. Thus, we deduce

$$\begin{aligned}(3x-2y-1)^2 &= [(x-1)+2(x-y)]^2 \\ &\le 2\left[(x-1)^2+4(x-y)^2\right] \le 10(x-y)^2,\end{aligned} \tag{55}$$

then

$$\wp_\imath(Yx, Yy) \le \frac{\imath-1}{\imath}\cdot\frac{10}{16}(x-y)^2 = \frac{5}{8}\cdot\frac{(\imath-1)(x-y)^2}{\imath} = \frac{5}{8}\wp_\imath(x,y). \tag{56}$$

In conclusion, Y is a Banach type contraction, and the contraction constant is $k = 5/8$.

4. Reich Type Contractions in $\mathscr{KS}$-$\mathscr{F}b\mathscr{MS}$

In this section, our main contribution is to establish a fixed-point theorem concerning Reich type contraction. Firstly, we introduce a crucial lemma to show that a Picard sequence is a Cauchy sequence.

Lemma 9. *Suppose that $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ be a $\mathscr{F}b\mathscr{MS}$ ($\mathfrak{b} \ge 1$) with ($\mathfrak{R}$-2) and $\{x_n\} \subseteq \mathfrak{S}$ a sequence. Assume that there exists $\mathfrak{Q} \in [0,1)$ such that for each $\imath \in (0,1]$, we can find $\mathfrak{P}_\imath \ge 0$ satisfying*

$$\wp_\imath(x_n, x_{n+1}) \le \mathfrak{P}_\imath \mathfrak{Q}^{n+1}, \forall n \in \mathbb{N}, \tag{57}$$

then $\{x_n\} \subseteq \mathfrak{S}$ is a Cauchy sequence.

Proof. Owing to $\mathfrak{Q} < 1$, we can find $\mathfrak{l} \in \mathbb{N}^+$ such that $\mathfrak{Q}^{\mathfrak{l}} < 1/\mathfrak{b}$. For each $\imath \in (0,1]$ and $m, n \in \mathbb{N}^+$ with $m < n$, we will show that

$$\lim_{m,n\longrightarrow\infty} \wp_\imath(x_m, x_n) = 0. \tag{58}$$

Since $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is with ($\mathfrak{R}$-2), there exists $\jmath = \jmath(\imath) \in (0,\imath]$ such that

$$\wp_\imath(x,y) \le \mathfrak{b}\left[\wp_\jmath(x,z) + \wp_\imath(z,y)\right], \ \forall x,y,z \in \mathfrak{S}. \tag{59}$$

(i) If $n - m \le \mathfrak{l}$, then taking account of (57) and (59), we get

$$\begin{aligned}\wp_\imath(x_m, x_n) &\le \mathfrak{b}\left[\wp_\jmath(x_m, x_{m+1}) + \wp_\imath(x_{m+1}, x_n)\right] \le \mathfrak{b}\wp_\jmath(x_m, x_{m+1}) \\ &\quad + \mathfrak{b}^2\wp_\jmath(x_{m+1}, x_{m+2}) + \mathfrak{b}^2\wp_\imath(x_{m+2}, x_n) \\ &\le \cdots \le \mathfrak{b}\wp_\jmath(x_m, x_{m+1}) + \mathfrak{b}^2\wp_\jmath(x_{m+1}, x_{m+2}) \\ &\quad + \cdots + \mathfrak{b}^{n-m-1}\wp_\jmath(x_{n-2}, x_{n-1}) + \mathfrak{b}^{n-m-1}\wp_\imath(x_{n-1}, x_n) \\ &\le \mathfrak{b}\wp_\jmath(x_m, x_{m+1}) + \mathfrak{b}^2\wp_\jmath(x_{m+1}, x_{m+2}) \\ &\quad + \cdots + \mathfrak{b}^{n-m-1}\wp_\jmath(x_{n-2}, x_{n-1}) + \mathfrak{b}^{n-m}\wp_\jmath(x_{n-1}, x_n) \\ &= \sum_{i=1}^{n-m}\mathfrak{b}^i\wp_\jmath(x_{m+i-1}, x_{m+i}) \le \sum_{i=1}^{n-m}\mathfrak{b}^i\mathfrak{P}_\jmath\mathfrak{Q}^{m+i} \\ &\le \sum_{i=1}^{\mathfrak{l}}\mathfrak{b}^i\mathfrak{P}_\jmath\mathfrak{Q}^{m+i} = \mathfrak{Q}^m\mathfrak{C}_\imath,\end{aligned} \tag{60}$$

where $\mathfrak{C}_\imath = \sum_{i=1}^{\mathfrak{l}}(\mathfrak{b}\mathfrak{Q})^i\mathfrak{P}_\jmath$.

(ii) If $n - m > \mathfrak{l}$, take $\theta = \lceil (n-m)/\mathfrak{l} \rceil - 1$, where $\lceil n \rceil = \min\{\kappa \ge n : \kappa \in \mathbb{N}^+\}$, then $\theta < ((n-m)/\mathfrak{l}) \le \theta + 1$. Thus, $0 < n - (m + \theta\mathfrak{l}) \le \mathfrak{l}$. Then, making the most of (5) and (6), we get

$$\begin{aligned}\wp_\imath(x_m, x_n) &\le \mathfrak{b}\left[\wp_\jmath(x_m, x_{m+\mathfrak{l}}) + \wp_\imath(x_{m+\mathfrak{l}}, x_n)\right] \\ &\le \mathfrak{b}\wp_\jmath(x_m, x_{m+\mathfrak{l}}) + \mathfrak{b}^2\wp_\jmath(x_{m+\mathfrak{l}}, x_{m+2\mathfrak{l}}) \\ &\quad + \mathfrak{b}^2\wp_\imath(x_{m+2\mathfrak{l}}, x_n) \le \mathfrak{b}\wp_\jmath(x_m, x_{m+\mathfrak{l}}) \\ &\quad + \mathfrak{b}^2\wp_\jmath(x_{m+\mathfrak{l}}, x_{m+2\mathfrak{l}}) + \cdots + \mathfrak{b}^\theta\wp_\jmath\left(x_{m+(\theta-1)\mathfrak{l}}, x_{m+\theta\mathfrak{l}}\right) \\ &\quad + \mathfrak{b}^\theta\wp_\imath(x_{m+\theta\mathfrak{l}}, x_n) \le \mathfrak{b}\wp_\jmath(x_m, x_{m+\mathfrak{l}}) \\ &\quad + \mathfrak{b}^2\wp_\jmath(x_{m+\mathfrak{l}}, x_{m+2\mathfrak{l}}) + \cdots + \mathfrak{b}^\theta\wp_\jmath\left(x_{m+(\theta-1)\mathfrak{l}}, x_{m+\theta\mathfrak{l}}\right) \\ &\quad + \mathfrak{b}^{\theta+1}\wp_\jmath(x_{m+\theta\mathfrak{l}}, x_n) \le \mathfrak{b}\mathfrak{Q}^m\mathfrak{C}_\jmath + \mathfrak{b}^2\mathfrak{Q}^{m+\mathfrak{l}}\mathfrak{C}_\jmath \\ &\quad + \cdots + \mathfrak{b}^\theta\mathfrak{Q}^{m+(\theta-1)\mathfrak{l}}\mathfrak{C}_\jmath + \mathfrak{b}^{\theta+1}\mathfrak{Q}^{m+\theta\mathfrak{l}}\mathfrak{C}_\jmath \\ &= \mathfrak{b}\mathfrak{Q}^m\mathfrak{C}_\jmath\sum_{i=0}^{\theta}\left(\mathfrak{b}\mathfrak{Q}^{\mathfrak{l}}\right)^i \le \mathfrak{b}\mathfrak{Q}^m\mathfrak{C}_\jmath\sum_{i=0}^{\infty}\left(\mathfrak{b}\mathfrak{Q}^{\mathfrak{l}}\right)^i = \frac{\mathfrak{b}\mathfrak{Q}^m\mathfrak{C}_\jmath}{1-\mathfrak{b}\mathfrak{Q}^{\mathfrak{l}}}.\end{aligned} \tag{61}$$

Therefore, for each $\imath \in (0,1]$ and $m, n \in \mathbb{N}^+$ $(m < n)$, we conclude that

$$\wp_\imath(x_m, x_n) \le \mathfrak{Q}^m \max\left\{\mathfrak{C}_\imath, \frac{\mathfrak{b}\mathfrak{C}_\jmath}{1-\mathfrak{b}\mathfrak{Q}^\mathfrak{l}}\right\} \longrightarrow 0 \text{ as } m \longrightarrow \infty. \tag{62}$$

The proof is completed. □

Remark 10. In fact, we cannot prove the uniqueness of the fixed point concerning the Reich type contractions without additional conditions. To figure this out, we consider the assumption that $\mathfrak{d}$ has the Fatou property.

Let us review the definition of the Fatou property.

Definition 11. Suppose that $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ be a $\mathscr{F}b\mathscr{M}\mathscr{S}$ $(\mathfrak{b} \ge 1)$ with ($\mathfrak{R}$-2) and $\{x_n\} \subseteq \mathfrak{S}$ be a sequence. We say that $\mathfrak{d}$ has the Fatou property if, for each $\imath \in (0, 1]$,

$$\wp_\imath(x, y) \le \liminf_{n\longrightarrow\infty} \wp_\imath(x_n, y), \tag{63}$$

whenever $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and any $y \in \mathfrak{S}$.

Remark 12. It is obvious that $\mathfrak{d}$ has Fatou property if $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is a $\mathscr{K}\mathscr{S}$ - $\mathscr{F}\mathscr{M}\mathscr{S}$.

Now, we establish and prove our main contents of this section.

Theorem 13. *Suppose that $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ be a complete $\mathscr{F}b\mathscr{M}\mathscr{S}$ $(\mathfrak{b} \ge 1)$ with ($\mathfrak{R}$-2). Assume that $\mathscr{F} : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a mapping. If there exist $a_1, a_2, a_3 \ge 0$ and $a_1 + a_2 + a_3 < 1$ such that*

$$\wp_\imath(\mathscr{F}x, \mathscr{F}y) \le a_1\wp_\imath(x, y) + a_2\wp_\imath(x, \mathscr{F}x) + a_3\wp_\imath(y, \mathscr{F}y), \ \forall x, y \in \mathfrak{S}, \tag{64}$$

for all $i \in (0, 1]$. If $\mathfrak{d}$ has the Fatou property, then $\mathscr{F}$ admits a unique fixed point in $\mathfrak{S}$.

Proof. Let $z_0 \in \mathfrak{S}$, we can construct a sequence $\{z_n\}_{n=0}^\infty$ by $z_n = \mathscr{F}z_{n-1} = \mathscr{F}^n z_0$. Assume that $\imath \in (0, 1]$. By (64), we have

$$\begin{aligned}\wp_\imath(z_n, z_{n+1}) &\le a_1\wp_\imath(z_{n-1}, z_n) + a_2\wp_\imath(z_{n-1}, z_n) + a_3\wp_\imath(z_n, z_{n+1}) \\ &= (a_1 + a_2)\wp_\imath(z_{n-1}, z_n) + a_3\wp_\imath(z_n, z_{n+1}).\end{aligned} \tag{65}$$

It immediately follows that

$$\wp_\imath(z_n, z_{n+1}) \le \frac{a_1 + a_2}{1 - a_3}\wp_\imath(z_{n-1}, z_n) = \beta\wp_\imath(z_{n-1}, z_n), \tag{66}$$

where $\beta = (a_1 + a_2)/(1 - a_3)$. Since $a_1 + a_2 + a_3 < 1$, we obtain that $\beta = (a_1 + a_2)/(1 - a_3) \in [0, 1)$. Again by (64), we get

$$\begin{aligned}\wp_\imath(z_n, z_{n+1}) &\le \beta\wp_\imath(z_{n-1}, z_n) \le \beta^2\wp_\imath(z_{n-2}, z_{n-1}) \\ &\le \cdots \le \beta^n\wp_\imath(z_0, z_1) = \beta^{n+1}\mathfrak{P}_\imath,\end{aligned} \tag{67}$$

where $\mathfrak{P}_\imath = (1/\beta)\wp_\imath(z_0, z_1) \ge 0$. Using Lemma 9, we can conclude that $\{z_n\} \subseteq \mathfrak{S}$ is a Cauchy sequence. Since $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is complete, we can find $u \in \mathfrak{S}$ such that $\lim_{n\longrightarrow\infty} z_n = u$. Next, we prove that u is a fixed point of $\mathscr{F}$. Suppose that, on the contrary, $\mathscr{F}u \ne u$, that is, $\wp_{\imath_0}(u, \mathscr{F}u) > 0$ for some $\imath_0 \in (0, 1]$. By means of (64), we deduce that

$$\begin{aligned}\wp_{\imath_0}(z_{n+1}, \mathscr{F}u) = \wp_{\imath_0}(\mathscr{F}z_n, \mathscr{F}u) &\le a_1\wp_{\imath_0}(z_n, u) \\ &+ a_2\wp_{\imath_0}(z_n, \mathscr{F}z_n) + a_3\wp_{\imath_0}(u, \mathscr{F}u).\end{aligned} \tag{68}$$

Using the fact that $z_n \longrightarrow u$ as $n \longrightarrow \infty$ and $\mathfrak{d}$ has the Fatou property, we get

$$\wp_{\imath_0}(u, \mathscr{F}u) \le \liminf_{n\longrightarrow\infty} \wp_{\imath_0}(z_n, \mathscr{F}u) = \liminf_{n\longrightarrow\infty} \wp_{\imath_0}(z_{n+1}, \mathscr{F}u). \tag{69}$$

Note that $\lim_{n\longrightarrow\infty}\wp_{\imath_0}(z_n, u) = \lim_{n\longrightarrow\infty}\wp_{\imath_0}(z_n, z_{n+1}) = 0$. Combining (68) and (69), we obtain that

$$\begin{aligned}\wp_{\imath_0}(u, \mathscr{F}u) &\le \liminf_{n\longrightarrow\infty} \wp_{\imath_0}(z_{n+1}, \mathscr{F}u) \\ &\le \liminf_{n\longrightarrow\infty} \left[a_1\wp_{\imath_0}(z_n, u) + a_2\wp_{\imath_0}(z_n, \mathscr{F}z_n) + a_3\wp_{\imath_0}(u, \mathscr{F}u)\right] \\ &= a_1 \lim_{n\longrightarrow\infty} \wp_{\imath_0}(z_n, u) + a_2 \lim_{n\longrightarrow\infty} \wp_{\imath_0}(z_n, z_{n+1}) + a_3\wp_{\imath_0}(u, \mathscr{F}u) \\ &= a_3\wp_{i_0}(u, \mathscr{F}u) < \wp_{\imath_0}(u, \mathscr{F}u),\end{aligned} \tag{70}$$

which is a contradiction. Thus, $\mathscr{F}u = u$.

If $v \in \mathfrak{S}$ is a fixed point of $\mathscr{F}$ and $v \ne u$, that is, $fv = v$ and $\wp_{\imath_1}(v, u) > 0$ for some $\imath_1 \in (0, 1]$, by virtue of (64), we derive

$$\begin{aligned}\wp_{\imath_1}(v, u) = \wp_{\imath_1}(\mathscr{F}v, \mathscr{F}u) &\le a_1\wp_{\imath_1}(v, u) + a_2\wp_{\imath_1}(v, \mathscr{F}v) \\ &+ a_3\wp_{\imath_1}(u, \mathscr{F}u) = a_1\wp_{\imath_1}(v, u) + a_2\wp_{\imath_1}(v, v) \\ &+ a_3\wp_{\imath_1}(u, u) = a_1\wp_{\imath_1}(v, u) < \wp_{\imath_1}(v, u),\end{aligned} \tag{71}$$

which contradicts the assumption that $v \ne u$. Therefore, $v = u$ and $\mathscr{F}$ has a unique fixed point in $\mathfrak{S}$. □

Theorem 13 can deduce the following corollary in $\mathscr{F}\mathscr{M}\mathscr{S}$.

Corollary 14. *Suppose that $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ be a complete $\mathscr{F}\mathscr{M}\mathscr{S}$ with ($\mathfrak{R}$-2). Assume that $\mathscr{F} : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a mapping. If there exist $a_1, a_2, a_3 \ge 0$ and $a_1 + a_2 + a_3 < 1$ such that*

$$\wp_\imath(\mathscr{F}x, \mathscr{F}y) \le a_1\wp_\imath(x, y) + a_2\wp_\imath(x, \mathscr{F}x) + a_3\wp_\imath(y, \mathscr{F}y), \ \forall x, y \in \mathfrak{S}, \tag{72}$$

for all $\imath \in (0, 1]$. Then, $\mathscr{F}$ has a unique fixed point in $\mathfrak{S}$.

Remark 15. Taking $a_2 = a_3 = 0$ in Theorem 13, we can obtain a result for Banach contractions in $\mathscr{F}b\mathscr{M}\mathscr{S}$. If we take $a_1 = 0$ and $a_2 = a_3$ in Theorem 13, the fixed-point theorem for Kannan type contractions is derived in the same context.

Corollary 16. *Let* $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ *be a complete* $\mathscr{F}b\mathscr{M}\mathscr{S}$ $(\mathfrak{b} \geq 1)$ *with* $(\mathfrak{R}\text{-}2)$ *and* $\mathscr{F} : \mathfrak{S} \longrightarrow \mathfrak{S}$. *If there exists* $\gamma \in [0, 1)$ *such that*

$$\wp_\imath(\mathscr{F}x, \mathscr{F}y) \leq \gamma[\wp_\imath(x, \mathscr{F}x) + \wp_\imath(y, \mathscr{F}y)], \quad \forall x, y \in \mathfrak{S}, \quad (73)$$

for all $\imath \in (0, 1]$. *If* $\mathfrak{d}$ *has the Fatou property, then* $\mathscr{F}$ *has a unique fixed point in* $\mathfrak{S}$.

5. Chatterjea Type Contractions in $\mathscr{K}\mathscr{S}$-$\mathscr{F}b\mathscr{M}\mathscr{S}$

The following lemma is crucial in order to prove a Picard sequence is a Cauchy sequence.

Lemma 17. *Let* $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ *be a complete* $\mathscr{F}b\mathscr{M}\mathscr{S}$ $(\mathfrak{b} \geq 1)$ *with* $(\mathfrak{R}\text{-}2)$. *Assume that* $\mathscr{F} : \mathfrak{S} \longrightarrow \mathfrak{S}$ *be a mapping. Define the sequence* $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ *by* $x_n = \mathscr{F}x_{n-1} = \mathscr{F}^n x_0$. *Suppose that for each* $\imath \in (0, 1]$

$$\wp_\imath(\mathscr{F}x, \mathscr{F}y) \leq \frac{c}{2}[\wp_\imath(x, \mathscr{F}y) + \wp_\imath(y, \mathscr{F}x)], \forall x, y \in \mathfrak{S}, \quad (74)$$

where $c \in [0, 1)$. *Let* $q \in \mathbb{N}$ *such that* $\mathfrak{b}c^q < 1$. *Then, the following assertions hold:*

(i) For each $\imath \in (0, 1]$ *there exists* $H_\imath > 0$ *such that* $\wp_\imath(x_k, x_n) \leq H_\imath$, *for all* $k \in \{0, 1, \cdots, q\}$, $n \in \mathbb{N}$

(ii) For each $\imath \in (0, 1]$, *there exists* $\jmath \in (0, \imath]$ *and* $H_\jmath > 0$ *such that* $\wp_\imath(x_m, x_n) \leq 2\mathfrak{b}H_\jmath$, *for all* $m, n \in \mathbb{N}$

(iii) The sequence $\{x_n\} \subseteq \mathfrak{S}$ *is a Cauchy sequence*

Proof.

(i) Firstly, we show by induction that, for each $\imath \in (0, 1]$,

$$\wp_\imath(x_k, x_n) \leq H_\imath = \frac{\mathfrak{b}}{1 - \mathfrak{b}c^q} \max\left\{\wp_\jmath(x_\imath, x_\jmath) : 0 \leq \imath, \jmath \leq q\right\}, \quad (75)$$

for all $n \in \mathbb{N}$, $k \in \{0, 1, \cdots, q\}$ and for some $\jmath \in (0, \imath]$.

Clearly, (75) holds for all $n \leq q$. Assume that (75) holds for all $n \leq m$ $(m \in \mathbb{N})$. Notice that $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ satisfies the condition $(\mathfrak{R}\text{-}2)$. By Lemma 4 (2), we can find $\jmath = \jmath(\imath) \in (0, \imath]$ such that

$$\wp_\imath(x, y) \leq \mathfrak{b}\left[\wp_\jmath(x, z) + \wp_\imath(z, y)\right], \forall x, y, z \in \mathfrak{S}. \quad (76)$$

Next, in order to prove (75) for $n = m + 1$, we divide it into two cases.

Case 1. Assume that $k = 0$. Then, by (74) and (76), we have

$$\begin{aligned}
\wp_\imath(x_0, x_{n+1}) &\leq \mathfrak{b}\left[\wp_\jmath(x_0, x_q) + \wp_\imath(x_q, x_{n+1})\right] \leq \mathfrak{b}\wp_\jmath(x_0, x_q) + \mathfrak{b} \cdot \frac{c}{2}\left[\wp_\imath(x_{q-1}, x_{n+1}) + \wp_\imath(x_q, x_n)\right] \leq \mathfrak{b}\wp_\jmath(x_0, x_q) \\
&+ \mathfrak{b} \cdot \left(\frac{c}{2}\right)^2 \left[\wp_\imath(x_{q-2}, x_{n+1}) + 2\wp_\imath(x_{q-1}, x_n) + \wp_\imath(x_q, x_{n-1})\right] \leq \cdots \leq \mathfrak{b}\wp_\jmath(x_0, x_q) \\
&+ \mathfrak{b} \cdot \left(\frac{c}{2}\right)^q \left[C_q^0\wp_\imath(x_0, x_{n+1}) + C_q^1\rho_\mathfrak{t}(x_1, x_n) + \cdots + C_q^{q-1}\wp_\imath(x_{q-1}, x_{n-q+2}) + C_q^q\wp_\imath(x_q, x_{n-q+1})\right] \\
&\leq \mathfrak{b}\wp_\jmath(x_0, x_q) + \mathfrak{b} \cdot \left(\frac{c}{2}\right)^q \wp_\imath(x_0, x_{n+1}) + \mathfrak{b} \cdot \left(\frac{c}{2}\right)^q \left[C_q^1\wp_\jmath(x_1, x_n) + \cdots + C_q^{q-1}\wp_\jmath(x_{q-1}, x_{n-q+2}) + C_q^q\wp_\jmath(x_q, x_{n-q+1})\right],
\end{aligned} \quad (77)$$

which implies that

$$\begin{aligned}
\wp_\imath(x_0, x_{n+1}) &\leq \frac{\mathfrak{b}\wp_\jmath(x_0, x_q)}{1 - \mathfrak{b}(c/2)^q} + \frac{\mathfrak{b} \cdot (c/2)^q}{1 - \mathfrak{b}(c/2)^q}\left[C_q^1\wp_\jmath(x_1, x_n) + \cdots + C_q^{q-1}\wp_\jmath(x_{q-1}, x_{n-q+2}) + C_q^q\wp_\jmath(x_q, x_{n-q+1})\right] \\
&\leq \frac{\mathfrak{b}\wp_\jmath(x_0, x_q)}{1 - \mathfrak{b}(c/2)^q} + \frac{\mathfrak{b} \cdot (c/2)^q}{1 - \mathfrak{b}(c/2)^q}\left[C_q^1 + C_q^2 + \cdots + C_q^q\right]H_\imath \leq \frac{\mathfrak{b}}{1 - \mathfrak{b}(c/2)^q} \max\left\{\wp_\jmath(x_\imath, x_\jmath) : 0 \leq i, j \leq q\right\} \\
&+ \frac{\mathfrak{b} \cdot (c/2)^q}{1 - \mathfrak{b}(c/2)^q}\left[C_q^1 + C_q^2 + \cdots + C_q^q\right]H_\imath = \frac{1 - \mathfrak{b}c^q}{1 - \mathfrak{b}(c/2)^q}H_i + \frac{\mathfrak{b} \cdot (c/2)^q}{1 - \mathfrak{b}(c/2)^q}(2^q - 1)H_\imath = \frac{H_i}{1 - \mathfrak{b}(c/2)^q}\left[1 - \mathfrak{b}c^q + \mathfrak{b}c^q - b\left(\frac{c}{2}\right)^q\right] = H_\imath.
\end{aligned} \quad (78)$$

Case 2. Suppose that $k \in \{1, 2, \cdots, q\}$. In this case, taking account of (74), we deduce that

$$\begin{aligned}\wp_\imath(x_k, x_{n+1}) &\le \frac{c}{2}\wp_\imath(x_{k-1}, x_{n+1}) + \frac{c}{2}\wp_\imath(x_k, x_n) \\ &\le \left[\left(\frac{c}{2}\right)^2 \wp_\imath(x_{k-2}, x_{n+1}) + \left(\frac{c}{2}\right)^2 \wp_\imath(x_{k-1}, x_n)\right] \\ &\quad + \frac{c}{2}H_\imath \le \cdots \le \left[\left(\frac{c}{2}\right)^k \rho_\mathfrak{t}(x_0, x_{n+1}) + \left(\frac{c}{2}\right)^k \wp_\imath(x_1, x_n)\right] \\ &\quad + \left[\left(\frac{c}{2}\right)^{k-1} + \left(\frac{c}{2}\right)^{k-2} + \cdots + \frac{c}{2}\right] H_\imath \le \left(\frac{c}{2}\right)^k \wp_\imath(x_0, x_{n+1}) \\ &\quad + \left(\frac{c}{2}\right)^k H_\imath + \left[\left(\frac{c}{2}\right)^{k-1} + \left(\frac{c}{2}\right)^{k-2} + \cdots + \frac{c}{2}\right] H_\imath. \end{aligned} \tag{79}$$

From case 1, we derive

$$\begin{aligned}\wp_\imath(x_k, x_{n+1}) &\le \left(\frac{c}{2}\right)^k H_\imath + \left(\frac{c}{2}\right)^k H_\imath + \left[\left(\frac{c}{2}\right)^{k-1} + \left(\frac{c}{2}\right)^{k-2} + \cdots + \frac{c}{2}\right] H_\imath \\ &\le \left[\left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^{k-1} + \cdots + \frac{1}{2}\right] H_\imath = H_\imath. \end{aligned} \tag{80}$$

Therefore, from the above two cases, we prove that (75) holds for all $n \in \mathbb{N}$.

(ii) For each $\imath \in (0, 1]$ and $m, n \in \mathbb{N}$, by means of (74) and (76), we obtain that

$$\wp_\imath(x_m, x_n) \le \mathfrak{b}\left[\wp_\jmath(x_m, x_q) + \wp_\imath(x_q, x_n)\right] \le \mathfrak{b}\left[H_\jmath + H_\imath\right] \le 2\mathfrak{b}H_\jmath, \tag{81}$$

where $H_\jmath = (\mathfrak{b}/(1 - \mathfrak{b}c^q)) \max\{\wp_{\jmath_0}(x_i, x_j) : 0 \le i, j \le q\}$, for some $\jmath_0 \in (0, \jmath]$.

(iii) For each $\imath \in (0, 1]$ and $m, n \in \mathbb{N}$ $(m < n)$, from (74) and (ii), we have

$$\begin{aligned}\wp_\imath(x_m, x_n) &\le \frac{c}{2}[\wp_\imath(x_{m-1}, x_n) + \wp_\imath(x_m, x_{n-1})] \\ &\le \left(\frac{c}{2}\right)^2 [\wp_\imath(x_{m-2}, x_n) + 2\wp_\imath(x_{m-1}, x_{n-1}) + \wp_\imath(x_m, x_{n-2})] \\ &\le \cdots \le \left(\frac{c}{2}\right)^m \left[C_m^0 \wp_\imath(x_0, x_n) + C_m^1 \wp_\imath(x_1, x_{n-1}) + \cdots + C_m^m \wp_\imath(x_m, x_{n-m})\right] \\ &\le \left(\frac{c}{2}\right)^m \left[C_m^0 + C_m^1 + \cdots + C_m^m\right] \cdot 2\mathfrak{b}H_\jmath \le \left(\frac{c}{2}\right)^m \cdot 2^m \cdot 2\mathfrak{b}H_\jmath = 2\mathfrak{b}H_\jmath c^m. \end{aligned} \tag{82}$$

Thus, we can obtain that $\lim_{m,n\longrightarrow\infty} \wp_\imath(x_m, x_n) = 0$. Therefore, we conclude that $\{x_n\} \subseteq \mathfrak{S}$ is a Cauchy sequence. □

By Lemma 17, we establish a fixed-point theorem for Chatterjea type contraction in $\mathscr{F}b\mathscr{M}\mathscr{S}$.

Theorem 18. *Suppose that $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ be a complete $\mathscr{F}b\mathscr{M}\mathscr{S}$ ($\mathfrak{b} \ge 1$) with ($\mathfrak{R}$-2). Assume that $\mathscr{F} : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a mapping. For each $\imath \in (0, 1]$,*

$$\wp_\imath(\mathscr{F}x, \mathscr{F}y) \le \frac{c}{2}[\wp_\imath(x, \mathscr{F}y) + \wp_\imath(y, \mathscr{F}x)], \ \forall x, y \in \mathfrak{S}, \tag{83}$$

where $c \in [0, 1)$. Then, $\mathscr{F}$ has a unique fixed point $u \in \mathfrak{S}$, and for any $x \in \mathfrak{S}$, the sequence of iterates $\{\mathscr{F}^n x\}$ converges to u.

Proof. Let $x_0 \in \mathfrak{S}$, we construct a sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ by $x_n = \mathscr{F}x_{n-1} = \mathscr{F}^n x_0$. Owing to $c \in [0, 1)$, there exists $q \in \mathbb{N}$ such that $\mathfrak{b}c^q < 1$. Using the Lemma 17, we can obtain that $\{x_n\} \subseteq \mathfrak{S}$ is a Cauchy sequence. Due to $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is complete, there exists $u \in \mathfrak{S}$ such that $\lim_{n\longrightarrow\infty} x_n = u$. Next, we show that u is a fixed point of $\mathscr{F}$. For each $\imath \in (0, 1]$, from Lemma 4 (40), we can find $\jmath \in (0, \imath]$ such that

$$\wp_\imath(x, y) \le \mathfrak{b}\left[\wp_\jmath(x, z) + \wp_\imath(z, y)\right], \ \forall x, y, z \in \mathfrak{S}. \tag{84}$$

By virtue of (83) and (84), we have

$$\begin{aligned}\wp_\imath(x_n, \mathscr{F}u) &\le \frac{c}{2}\wp_\imath(x_{n-1}, \mathscr{F}u) + \frac{c}{2}\wp_\imath(x_n, u) \\ &\le \left[\left(\frac{c}{2}\right)^2 \wp_\imath(x_{n-2}, \mathscr{F}u) + \left(\frac{c}{2}\right)^2 \wp_\imath(x_{n-1}, u)\right] \\ &\quad + \frac{c}{2}\wp_\imath(x_n, u) \le \cdots \le \left(\frac{c}{2}\right)^q \wp_\imath(x_{n-q}, \mathscr{F}u) \\ &\quad + \left(\frac{c}{2}\right)^q \wp_\imath(x_{n-q+1}, u) + \left(\frac{c}{2}\right)^{q-1} \wp_\imath(x_{n-q+2}, u) \\ &\quad + \cdots + \frac{c}{2}\wp_\imath(x_n, u) \le \mathfrak{b}\left(\frac{c}{2}\right)^q \left[\wp_\jmath(x_{n-q}, x_n) + \wp_\imath(x_n, \mathscr{F}u)\right] \\ &\quad + \left(\frac{c}{2}\right)^q \wp_\imath(x_{n-q+1}, u) + \left(\frac{c}{2}\right)^{q-1} \wp_\imath(x_{n-q+2}, u) + \cdots + \frac{c}{2}\wp_\imath(x_n, u). \end{aligned} \tag{85}$$

It follows that

$$\wp_\imath(x_n, \mathscr{F}u) \le \frac{\mathfrak{b}(c/2)^q}{1 - \mathfrak{b}(c/2)^q}\wp_\jmath(x_{n-q}, x_n) + \frac{1}{1 - \mathfrak{b}(c/2)^q}\left[\left(\frac{c}{2}\right)^q \wp_\imath(x_{n-q+1}, u) + \left(\frac{c}{2}\right)^{q-1} \wp_\imath(x_{n-q+2}, u) + \cdots + \frac{c}{2}\wp_\imath(x_n, u)\right]. \tag{86}$$

Note that $\lim_{n\longrightarrow\infty}\wp_\jmath(x_{n-q}, x_n) = 0$ and $\lim_{n\longrightarrow\infty}\wp_\imath(x_{n-i}, u) = 0$, $i = 0, 1, \cdots, q-1$. Therefore, we obtain that $\lim_{n\longrightarrow\infty}\wp_\imath(x_n, \mathcal{F}u) = 0$, that is, x_n converges to $\mathcal{F}u$, by Lemma 6, we have $\mathcal{F}u = u$. This completes the proof of the existence of the fixed point of $\mathcal{F}$.

If there exists $v \in \mathfrak{S}$ such that $\mathcal{F}v = v$ and $v \neq u$, that is, $\wp_{\imath_0}(v, u) > 0$ for some $\imath_0 \in (0, 1]$. Taking (83) into account, we have

$$\begin{aligned}\wp_{\imath_0}(v, u) &= \wp_{\imath_0}(\mathcal{F}v, \mathcal{F}u) \le \frac{c}{2}\left[\wp_{\imath_0}(v, \mathcal{F}u) + \wp_{\imath_0}(u, \mathcal{F}v)\right] \\ &= \frac{c}{2}\cdot 2\wp_{\imath_0}(v, u) = c\wp_{\imath_0}(v, u) < \wp_{\imath_0}(v, u),\end{aligned} \tag{87}$$

which contradicts the fact that $v \neq u$. Therefore, $v = u$ and $\mathcal{F}$ has a unique fixed point in $\mathfrak{S}$. □

Corollary 19. *Suppose that $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ be a complete $\mathcal{F}\mathcal{MS}$ with ($\mathfrak{R}$-2), $\mathcal{F} : \mathfrak{S} \longrightarrow \mathfrak{S}$. If there exists $c \in [0, 1)$ such that*

$$\wp_\imath(\mathcal{F}x, \mathcal{F}y) \le \frac{c}{2}[\wp_\imath(x, \mathcal{F}y) + \wp_\imath(y, \mathcal{F}x)], \quad \forall x, y \in \mathfrak{S}, \tag{88}$$

for all $\imath \in (0, 1]$. Then, $\mathcal{F}$ has a unique fixed point $u \in \mathfrak{S}$, and for any $x \in \mathfrak{S}$, the sequence of iterates $\{\mathcal{F}^n x\}$ converges to u.

Proof. Taking $\mathfrak{b} = 1$ in Theorem 18, the desired result is obtained immediately. □

6. Applications

The aim of the following lemma is to prove that a b-$\mathcal{MS}$ is a special $\mathcal{F}b\mathcal{MS}$. And then, we can establish the relevant fixed-point theorems in b-$\mathcal{MS}$ as corollaries of our main results presented in Sections 3–5.

Lemma 20. *Let $(\mathfrak{S}, \mathfrak{D})$ be a b - $\mathcal{MS}$ and $\mathfrak{d}(x, y)$: $\mathbb{R} \longrightarrow \mathbb{R}$ a mapping defined by*

$$\mathfrak{d}(x, y)(\eta) = \bar{\mathfrak{D}}(x, y)(\eta) = \bar{0}(\eta - \mathfrak{D}(x, y)), \ \forall x, y \in \mathfrak{S}. \tag{89}$$

Then, $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ is a $\mathcal{F}b\mathcal{MS}$, where

$$\begin{cases} \mathfrak{L}(a, b) = \min\{a, b\}, \\ \mathfrak{R}(a, b) = \max\{a, b\}. \end{cases} \tag{90}$$

Proof. In view of (89), we have

$$\mathfrak{d}(x, y)(\eta) = \begin{cases} 1, & \eta = \mathfrak{D}(x, y), \\ 0, & \eta \neq \mathfrak{D}(x, y). \end{cases} \tag{91}$$

It is easy to verify (BM1) and (BM2) in Definition 1. Next, for every $x, y, z \in \mathfrak{S}$, we prove that (BM3) holds.

First, we prove (BM3$\mathfrak{L}$). Let ζ, $\eta \in \mathbb{R}$ satisfying

$$\begin{cases} \zeta \le \mathfrak{I}_1(x, z), \\ \eta \le \mathfrak{I}_1(z, y), \\ \mathfrak{b}(\zeta + \eta) \le \mathfrak{I}_1(x, y). \end{cases} \tag{92}$$

Note that $\mathfrak{L}(a, b) = \min\{a, b\}$. Suppose that $\zeta < \mathfrak{I}_1(x, z)$ or $\eta < \mathfrak{I}_1(z, y)$, we can obtain that

$$\min\{\mathfrak{d}(x, z)(\zeta), \mathfrak{d}(z, y)(\eta)\} = 0 \le \mathfrak{d}(x, y)(\mathfrak{b}(\zeta + \eta)). \tag{93}$$

Assume that $\zeta = \mathfrak{I}_1(x, z)$ and $\eta = \mathfrak{I}_1(z, y)$, we deduce

$$\mathfrak{b}(\zeta + \eta) = \mathfrak{b}(\mathfrak{D}(x, z) + \mathfrak{D}(z, y)) \ge \mathfrak{D}(x, y). \tag{94}$$

Note that $\mathfrak{b}(\zeta + \eta) \le \mathfrak{I}_1(x, y) = \mathfrak{D}(x, y)$. Thus, we conclude that

$$\begin{aligned}\mathfrak{d}(x, y)(\mathfrak{b}(\zeta + \eta)) &= \mathfrak{d}(x, y)(\mathfrak{D}(x, y)) = 1 \\ &= \min\{\mathfrak{d}(x, z)(\zeta), \mathfrak{d}(z, y)(\eta)\} \\ &= \mathfrak{L}(\mathfrak{d}(x, z)(\zeta), \mathfrak{d}(z, y)(\eta)).\end{aligned} \tag{95}$$

That completes the proof of (BM3$\mathfrak{L}$).

Now, we verify (BM3$\mathfrak{R}$). Let $\zeta, \eta \in \mathbb{R}$ satisfying

$$\begin{cases} \zeta \ge \mathfrak{I}_1(x, z), \\ \eta \ge \mathfrak{I}_1(z, y), \\ \mathfrak{b}(\zeta + \eta) \ge \mathfrak{I}_1(x, y). \end{cases} \tag{96}$$

Assume that $\zeta = \mathfrak{I}_1(x, z)$ or $\eta = \mathfrak{I}_1(z, y)$, we get

$$\max\{\mathfrak{d}(x, z)(\zeta), \mathfrak{d}(z, y)(\eta)\} = 1 \ge \mathfrak{d}(x, y)(\mathfrak{b}(\zeta + \eta)). \tag{97}$$

If $\zeta > \mathfrak{I}_1(x, z)$ and $\eta > \mathfrak{I}_1(z, y)$, we derive

$$\mathfrak{b}(\zeta + \eta) > \mathfrak{b}(\mathfrak{D}(x, z) + \mathfrak{D}(z, y)) \ge \mathfrak{D}(x, y). \tag{98}$$

Thus, we conclude that

$$\begin{aligned}\mathfrak{d}(x, y)(\mathfrak{b}(\zeta + \eta)) &= 0 = \max\{\mathfrak{d}(x, z)(\zeta), \mathfrak{d}(z, y)(\eta)\} \\ &= \mathfrak{R}(\mathfrak{d}(x, z)(\zeta), \mathfrak{d}(z, y)(\eta)).\end{aligned} \tag{99}$$

The part of (BM3$\mathfrak{R}$) is completed. □

Remark 21. From Lemma 20, it is obvious that $(\mathfrak{S}, \mathfrak{D})$ and $(\mathfrak{S}, \mathfrak{d}, \mathfrak{L}, \mathfrak{R}, \mathfrak{b})$ are homeomorphic, and for each $\imath \in (0, 1]$, $\wp_\imath(x, y) = \mathfrak{D}(x, y)$. Therefore, we give the following three results as corollaries of Theorem 7, Theorem 13, and Theorem 18, respectively.

Corollary 22 (see [32]). *Suppose that $(\mathfrak{S}, \mathfrak{d})$ be a complete b - $\mathcal{MS}$ $(\mathfrak{b} \geq 1)$, $\mathcal{F} : \mathfrak{S} \longrightarrow \mathfrak{S}$. If $x, y \in \mathfrak{S}$,*

$$\mathfrak{d}(\mathcal{F}x, \mathcal{F}y) \leq k\mathfrak{d}(x, y), \tag{100}$$

where $k \in [0, 1)$. Then, $\mathcal{F}$ has a unique fixed point in $\mathfrak{S}$.

Corollary 23. *Let $(\mathfrak{S}, \mathfrak{d})$ be a complete b - $\mathcal{MS}$ $(\mathfrak{b} \geq 1)$, $\mathcal{F} : \mathfrak{S} \longrightarrow \mathfrak{S}$. Suppose that there exist $a_1, a_2, a_3 \geq 0$ and $a_1 + a_2 + a_3 < 1$ such that*

$$\mathfrak{d}(\mathcal{F}x, \mathcal{F}y) \leq a_1\mathfrak{d}(x, y) + a_2\mathfrak{d}(x, \mathcal{F}x) + a_3\mathfrak{d}(y, \mathcal{F}y), \quad \forall x, y \in \mathfrak{S}. \tag{101}$$

If $\mathfrak{d}$ has the Fatou property, then $\mathcal{F}$ has a unique fixed point in $\mathfrak{S}$.

Corollary 24 (see [33]). *Let $(\mathfrak{S}, \mathfrak{d})$ be a complete b - $\mathcal{MS}$ $(\mathfrak{b} \geq 1)$, $\mathcal{F} : \mathfrak{S} \longrightarrow \mathfrak{S}$. If*

$$\mathfrak{d}(\mathcal{F}x, \mathcal{F}y) \leq \frac{c}{2}[\mathfrak{d}(x, \mathcal{F}y) + \mathfrak{d}(y, \mathcal{F}x)], \quad \forall x, y \in \mathfrak{S}, \tag{102}$$

where $c \in [0, 1)$, then $\mathcal{F}$ has a unique fixed point x^*, and for any $x \in \mathfrak{S}$, the sequence of iterates $\{\mathcal{F}^n x\}$ converges to x^*.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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