Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2746942, 20 pages
https://doi.org/10.1155/2022/2746942

Research Article

On the Nonlinearity of Extended *s*-Type Weighted Nakano Sequence Spaces of Fuzzy Functions with Some Applications

Awad A. Bakery [1,2] **and Elsayed A. E. Mohamed** [1,3]

[1] *College of Science and Arts at Khulis, Department of Mathematics, University of Jeddah, Jeddah, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Egypt*
[3] *Department of Mathematics, Faculty of Education, Alzaiem Alazhari University, Khartoum, Sudan*

Correspondence should be addressed to Elsayed A. E. Mohamed; elsayed.bayen2020@yahoo.com

Received 29 January 2022; Revised 22 February 2022; Accepted 28 February 2022; Published 17 March 2022

Academic Editor: Azhar Hussain

We have defined and studied the weighted Nakano sequence spaces of fuzzy functions. We have constructed the ideal generated by extended s-fuzzy functions and the sequence spaces of fuzzy functions. We present some topological and geometric structures of this class of ideal and multiplication mappings acting on this sequence space of fuzzy functions. Moreover, the existence of Caristi's fixed point is examined. To show how the work is done, some examples and applications to the existence of solutions for a class of nonlinear summable and matrix equations are also talked about.

1. Introduction

The mathematical description of the hydrodynamics of non-Newtonian fluids provided additional impetus to the learning about variable exponent Lebesgue spaces (see [1, 2]). Electric rheological fluids have many applications, including military technology, civil engineering, and orthopedics. In cybernetics, artificial intelligence, and fuzzy control, the concept of fuzziness was widely embraced after Zadeh [3] introduced fuzzy sets and fuzzy set operations. Javed et al. [4] investigated the Banach contraction in R-fuzzy b-metric spaces and discussed some related fixed point results to ensure a fixed point's existence and uniqueness. A nontrivial example is given to illustrate the feasibility of the proposed methods. They offered an application to solve the first kind of Fredholm-type integral equation. In [5], Rehman and Aydi proved some common fixed point theorems for mappings involving generalized rational-type fuzzy cone-contraction conditions in fuzzy cone metric spaces. They gave a common solution of two definite Fredholm integral equations. The concept of orthogonal partial b-metric spaces was pioneered by Javed et al. [6]. They presented a unique fixed point for some orthogonal contractive-type mappings with some examples and an application. Humaira et al. [7] discussed the existence theorem for a unique solution to a coupled system of impulsive fractional differential equations in complex-valued fuzzy metric spaces and the fuzzy version of some fixed point results by using the definition and presented some properties of a complex-valued fuzzy metric space with some applications. In this study, Sarwar and Rodríguez-López [8] looked into the concept of extended fuzzy rectangular b-metric space. They explained that some fixed point results in the literature could be generalized by α-admittance in this space. They used this to show solutions for a group of integral equations. Many researchers in sequence spaces and summability theory were active in studying fuzzy sequence spaces and their properties. Different classes of sequences of fuzzy real numbers have been discussed by Nanda [9], Nuray and Savas [10], Matloka [11], Altinok et al. [12], Colak et al. [13], Hazarika and Savas [14], and many others. In [10], the Nakano sequences of fuzzy integers were defined and analyzed. The mappings' ideal theory is well regarded in functional analysis. Using s-numbers is an essential technique. Pietsch [15–18] developed and studied the theory of s-numbers of linear bounded mappings between Banach spaces. He offered and explained some topological and geometric structures of the quasi-ideals of ℓ_p-type mappings. Then, Constantin [19] generalized the class of ℓ_p-type mappings to the class of ces_p-type mappings. Makarov and Faried [20] showed some inclusion relations of ℓ_p-type mappings. As a generalization of ℓ_p-type mappings, Stolz mappings and mappings' ideal were examined by Tita [21, 22]. In [23], Maji and Srivastava studied the class $A_p^{(s)}$ of s-type ces_p mappings using s-number sequence and Cesàro sequence spaces and they introduced a new class $A_{p,q}^{(s)}$ of s-type $ces(p, q)$ mappings by weighted ces_p with $1 < p < \infty$. In [24], the class of s-type $Z(u, v; \ell_p)$ mappings was defined and some of their properties were explained. Yaying et al. [25] defined and studiedχ_r^η, with$r$-Cesàro matrix inℓ_η, with$r \in (0, 1]$and$1 \leq \eta \leq \infty$. They explained the quasi-Banach ideal of type χ_r^η, with $r \in (0, 1]$ and $1 < \eta < \infty$. Komal et al. [26] explained the multiplication mappings defined on ces_p equipped with the Luxemburg norm. The multiplication mappings acting on Cesàro second-order function spaces discussed by Ilkhan et al. [27]. Many fixed point theorems in a particular space work by either expanding the self-mapping acting on it or expanding the space itself. In this paper, we have introduced the concept of premodular spaces of fuzzy numbers, which are important extensions of the concept of modular spaces. We also extended s-fuzzy numbers to build large spaces of solutions to many nonlinear summable and matrix equations of fuzzy numbers. It is the first attempt to examine Caristi's fixed point in certain premodular vector spaces. This work is aimed at introducing the particular space of sequences of fuzzy numbers, in short (pssf), under a particular function to be pre-quasi (pssf). We have defined and analyzed weighted Nakano sequence spaces of fuzzy functions. Extended s-fuzzy functions and weighted Nakano sequence spaces of fuzzy functions have been used to create the mappings' ideal. The topological and geometric characteristics of mappings' ideal and multiplication mappings acting on this sequence space of fuzzy functions are offered. Caristi's fixed point is also discussed in this paper. Some supporting examples and applications to the existence of solutions for a class of nonlinear summable and matrix equations are also explored to provide a better understanding of the work that has been done.

2. Definitions and Preliminaries

As a reminder, Matloka [11] presented the concept of ordinary convergence of sequences of fuzzy numbers, where he introduced bounded and convergent fuzzy numbers, explored some of their features, and proved that any convergent fuzzy number sequence is bounded. Nanda [9] explained the sequences of fuzzy numbers and proved the set of all convergent sequences of fuzzy numbers from a complete metric space. Kumar et al. [28] investigated the limit points and cluster points of sequences of fuzzy numbers. Assume Ω is the set of all closed and bounded intervals on the real-line $\mathfrak{R}$. Let $f = [f_1, f_2]$ and $g = [g_1, g_2]$ in Ω; suppose

$$f \leq g \quad \text{if and only if } f_1 \leq g_1 \text{ and } f_2 \leq g_2. \tag{1}$$

Define a metric ρ on Ω by

$$\rho(f, g) = \max\{|f_1 - g_1|, |f_2 - g_2|\}. \tag{2}$$

Matloka [11] showed that ρ is a metric on Ω, (Ω, ρ) is a complete metric space, and the relation $\leq$ is a partial order on Ω.

Definition 1. A fuzzy number g is a mapping $g : \mathfrak{R} \longrightarrow [0, 1]$ which verifies the following four settings:

(a) g is fuzzy convex; i.e., for $x, y \in \mathfrak{R}$ and $\alpha \in [0, 1]$, $g(\alpha x + (1 - \alpha)y) \geq \min\{g(x), g(y)\}$

(b) g is normal; i.e., one has $y_0 \in \mathfrak{R}$ such that $g(y_0) = 1$

(c) $g$is upper semicontinuous; i.e., for all$\alpha > 0$, $g^{-1}([0, x + \alpha))$and for all$x \in [0, 1]$, which is open in the usual topology of$\mathfrak{R}$

(d) The closure of $g^0 \coloneqq \{y \in \mathfrak{R} : g(y) > 0\}$ is compact

The β-level set of $g, 0 < \beta < 1$ indicated by g^β is defined as

$$g^\beta = \{y \in \mathfrak{R} : g(y) \geq \beta\}. \tag{3}$$

The set of every upper semicontinuous, normal, convex fuzzy number, and g^β is compact and is denoted by $\mathfrak{R}([0, 1])$. The set $\mathfrak{R}$ can be embedded in $\mathfrak{R}([0, 1])$, when we define $k \in \mathfrak{R}([0, 1])$ by

$$\bar{k}(x) = \begin{cases} 1, & x = k, \\ 0, & x \neq k. \end{cases} \tag{4}$$

$\bar{0}$ and $\bar{1}$ denote the additive identity and multiplicative identity in $\mathfrak{R}[0, 1]$ in $\mathfrak{R}[0, 1]$, respectively.

The arithmetic operations on $\mathfrak{R}[0, 1]$ are defined as

$$\begin{aligned} (f \oplus g)(q) &= \sup_{q \in \mathfrak{R}} \min\{f(p), g(q - p)\}, \\ (f \ominus g)(q) &= \sup_{q \in \mathfrak{R}} \min\{f(p), g(p - q)\}, \\ (f \otimes g)(q) &= \sup_{q \in \mathfrak{R}} \min\left\{f(p), g\left(\frac{q}{p}\right)\right\}, \\ \left(\frac{f}{g}\right)(q) &= \sup_{q \in \mathfrak{R}} \min\{f(pq), g(p)\}, \\ pf(q) &= \begin{cases} f(p^{-1}q), & p \neq 0 \\ 0, & p = 0. \end{cases} \end{aligned} \tag{5}$$

The absolute value $|f|$ of $f \in \mathfrak{R}[0,1]$ is defined as

$$|f|(\mathrm{q}) = \begin{cases} \max\{f(q), f(-q)\}, & \text{if } q \geq 0, \\ 0, & \text{if } q < 0. \end{cases} \quad (6)$$

Suppose $f, g \in \mathfrak{R}[0,1]$ and the β-level sets are $[f]^\beta = [f_1^\beta, f_2^\beta]$ and $[g]^\beta = [g_1^\beta, g_2^\beta]$, $\beta \in [0,1]$. A partial ordering for any $f, g \in \mathfrak{R}[0,1]$ as follows: $f \preceq g$ if and only if $f^\beta \leq g^\beta$, for all $\beta \in [0,1]$. Hence, the above operations can be defined by β-level sets as

$$\begin{aligned}
[f \oplus g]^\beta &= \left[f_1^\beta + g_1^\beta, f_2^\beta + g_2^\beta\right], \\
[f \ominus g]^\beta &= \left[f_1^\beta - g_2^\beta, f_2^\beta - g_1^\beta\right], \\
[f \otimes g]^\beta &= \left[\min_{j\in\{1,2\}} f_j^\beta g_j^\beta, \max_{j\in\{1,2\}} f_j^\beta g_j^\beta\right], \\
\left[f^{-1}\right]^\beta &= \left[\left(f_2^\beta\right)^{-1}, \left(f_1^\beta\right)^{-1}\right], \quad f_j^\beta > 0, \text{for every } \beta \in (0,1], \\
[xf]^\beta &= \begin{cases} \left[xf_1^\beta, xf_2^\beta\right], & x \geq 0, \\ \left[xf_2^\beta, xf_1^\beta\right], & x < 0. \end{cases}
\end{aligned} \quad (7)$$

Assume $\bar{\rho} : \mathfrak{R}[0,1] \times \mathfrak{R}[0,1] \longrightarrow \mathfrak{R}^+ \cup \{0\}$ is defined by $\bar{\rho}(f, g) = \sup_{0\leq\beta\leq1} \rho(f^\beta, g^\beta)$.

Recall that

(1) $(\mathfrak{R}[0,1], \bar{\rho})$ is a complete metric space

(2) $\bar{\rho}(f + k, g + k) = \bar{\rho}(f, g)$ for all $f, g, k \in \mathfrak{R}[0,1]$

(3) $\bar{\rho}(f + k, g + l) \leq \bar{\rho}(f, g) + \bar{\rho}(k, l)$

(4) $\bar{\rho}(\xi f, \xi g) = |\xi| \bar{\rho}(f, g)$, for all $\xi \in \mathfrak{R}$

Definition 2. A sequence of fuzzy numbers $f = (f_j)$ is called

(a) bounded if there are two fuzzy numbers g, l such that $g \leq f_j \leq l$

(b) convergent to a fuzzy real number f_0 if for all $\varepsilon > 0$, one has $n_0 \in \mathcal{N}$ such that $\bar{\rho}(f_j, f_0) < \varepsilon$, for all $j \geq j_0$

By ℓ_∞ and ℓ_r, we denote the spaces of bounded and r-absolutely summable sequences of $\mathfrak{R}$, respectively.

Lemma 3 (see [29]). *Suppose $\tau_q > 0$, $K = \max\{1, \sup_q \tau_q\}$, and $Y_q, Z_q \in \mathfrak{R}$ with $q \in \mathcal{N}$; hence,*

$$\left|Y_q + Z_q\right|^{\tau_q} \leq 2^{K-1}\left(\left|Y_q\right|^{\tau_q} + \left|Z_q\right|^{\tau_q}\right). \quad (8)$$

We will explain our main results.

3. Some Properties of $\ell_{\tau(.)}^F$

This section introduces the particular space of sequences of fuzzy functions (pssf), under definite function to be pre-quasi (pssf). We investigate sufficient setup of $\ell_{\tau(.)}^F$ equipped with definite function h to be pre-quasi closed and Banach (pssf). We also present the Fatou property of various h on $\ell_{\tau(.)}^F$.

Let $\omega(F)$ and $\ell_\infty(F)$ mark the classes of all and bounded sequence spaces of fuzzy functions, respectively. Suppose $\tau = (\tau_q) \in \mathfrak{R}^{+\mathcal{N}}$, where $\mathfrak{R}^{+\mathcal{N}}$ is the space of positive real sequences. The variable exponent sequence space of fuzzy functions is denoted by

$$\begin{aligned}
\ell_{\tau(.)}^F &= \left\{Z = \left(Z_q\right) \in \omega(F) : h(rZ) < \infty, \text{for some } r > 0\right\}, \\
&\text{when } h(Z) = \sum_{q=0}^{\infty} \frac{1}{\tau_q}\left[\bar{\rho}\left(Z_q, \bar{0}\right)\right]^{\tau_q}.
\end{aligned} \quad (9)$$

Theorem 4. *If $(\tau_q) \in \ell_\infty$, then*

$$\ell_{\tau(.)}^F = \left\{Z = \left(Z_q\right) \in \omega(F) : h(rZ) < \infty, \text{for any } r > 0\right\}. \quad (10)$$

Proof.

$$\begin{aligned}
\ell_{\tau(.)}^F &= \left\{Z = \left(Z_q\right) \in \omega(F) : h(rZ) < \infty, \text{for some } r > 0\right\} \\
&= \Bigg\{Z = \left(Z_q\right) \in \omega(F) : \inf_q |r|^{\tau_q} \sum_{q=0}^{\infty} \frac{1}{\tau_q}\left[\bar{\rho}\left(Z_q, \bar{0}\right)\right]^{\tau_q} \\
&\quad \leq \sum_{q=0}^{\infty} \frac{1}{\tau_q}\left[\bar{\rho}\left(rZ_q, \bar{0}\right)\right]^{\tau_q} < \infty, \text{for some } r > 0\Bigg\} \\
&= \left\{Z = \left(Z_q\right) \in \omega(F) : \sum_{q=0}^{\infty} \frac{1}{\tau_q}\left[\bar{\rho}\left(Z_q, \bar{0}\right)\right]^{\tau_q} < \infty\right\} \\
&= \left\{Z = \left(Z_q\right) \in \omega(F) : h(rZ) < \infty, \text{for any } r > 0\right\}.
\end{aligned} \quad (11)$$

□

By $[0,\infty)^{\mathbf{U}}$, we denote the space of all functions $h : \mathbf{U} \longrightarrow [0,\infty)$. Nakano [30] introduced the concept of modular vector spaces.

Definition 5. Suppose $\mathbf{U}$ is a vector space. A function $h \in [0,\infty]^U$ is called modular if the next conditions hold:

(a) If $Y \in \mathbf{U}$, $Y = \bar{\vartheta} \Leftrightarrow h(Y) = 0$ with $h(Y) \geq 0$, where $\bar{\vartheta} = (\bar{0}, \bar{0}, \bar{0}, \cdots)$

(b) $h(\eta Z) = h(Z)$ holds, for all $Z \in \mathbf{U}$ and $|\eta| = 1$

(c) The inequality $h(\alpha Y + (1-\alpha)Z) \leq h(Y) + h(Z)$ satisfies, for all $Y, Z \in \mathbf{U}$ and $\alpha \in [0,1]$

Definition 6. The linear space $\mathbf{U}$ is said to be a particular space of sequences of fuzzy functions (pssf), if

(a) $\{\bar{\mathfrak{b}}_q\}_{q\in\mathcal{N}} \subseteq \mathbf{U}$, where $\bar{\mathfrak{b}}_q = \{\bar{0}, \bar{0}, \cdots, \bar{1}, \bar{0}, \bar{0}, \cdots\}$, while $\bar{1}$ displays at the q^{th} place

(b) $\mathbf{U}$ is solid; i.e., suppose $Y = (Y_q) \in \omega(F)$, $Z = (Z_q) \in \mathbf{U}$ and $|Y_q| \leq |Z_q|$, for all $q \in \mathcal{N}$, then $Y \in \mathbf{U}$

(c) $(Y_{[q/2]})_{q=0}^{\infty} \in \mathbf{U}$, where $[q/2]$ marks the integral part of $q/2$, if $(Y_q)_{q=0}^{\infty} \in \mathbf{U}$

Definition 7. A subclass$\mathbf{U}_h$of$\mathbf{U}$is said to be a premodular (pssf), if one has$h \in [0,\infty)^{\mathbf{U}}$ which satisfies the next settings:

(i) If $Y \in \mathbf{U}$, $Y = \bar{\vartheta} \Leftrightarrow h(Y) = 0$ with $h(Y) \geq 0$

(ii) There is $Q \geq 1$; the inequality $h(\alpha Y) \leq Q|\alpha| h(Y)$ holds, for every $Y \in \mathbf{U}$ and $\alpha \in \mathfrak{R}$

(iii) There is $P \geq 1$; the inequality $h(Y + Z) \leq P(h(Y) + h(Z))$ holds, for every $Y, Z \in \mathbf{U}$

(iv) If $|Y_q| \leq |Z_q|$, for every $q \in \mathcal{N}$, one has $h((Y_q)) \leq h((Z_q))$

(v) The inequality $h((Y_q)) \leq h((Y_{[q/2]})) \leq P_0 h((Y_q))$ holds, for some $P_0 \geq 1$

(vi) The closure of $E = \mathbf{U}_h$, where E is the space of finite sequences of fuzzy functions

(vii) There is $\sigma > 0$ with $h(\bar{\alpha}, \bar{0}, \bar{0}, \bar{0}, \cdots) \geq \sigma |\alpha| h(\bar{1}, \bar{0}, \bar{0}, \bar{0}, \cdots)$, where

$$\bar{\alpha}(y) = \begin{cases} 1, & y = \alpha, \\ 0, & y \neq \alpha \end{cases} \tag{12}$$

Clearly, the concept of premodular vector spaces is more general than modular vector spaces. Some examples of premodular vector spaces but not modular vector spaces are shown.

Example 1. The function $h(Z) = (\sum_{q=0}^{\infty}((3q+4)/(q+1))[\bar{\rho}(Z_q, \bar{0})]^{(q+1)/(3q+4)})^4$ is a premodular (not a modular) on the vector space $\ell^F(((q+1)/(3q+4))_{q=0}^{\infty})$. As for every $Z, Y \in \ell^F(((q+1)/(3q+4))_{q=0}^{\infty})$, one has

$$h\left(\frac{Z+Y}{2}\right) = \left(\sum_{q=0}^{\infty} \frac{3q+4}{q+1}\left[\bar{\rho}\left(\frac{Z_q + Y_q}{2}, \bar{0}\right)\right]^{(q+1)/(3q+4)}\right)^4 \leq 4(h(Z) + h(Y)). \tag{13}$$

Example 2. The function $h(Z) = \sum_{q=0}^{\infty}((q+4)/(2q+3))[\bar{\rho}(Z_q, \bar{0})]^{(2q+3)/(q+4)}$ is a premodular (not a modular) on the vector space $\ell^F(((2q+3)/(q+4))_{q=0}^{\infty})$. As for every $Z, Y \in \ell^F(((2q+3)/(q+4))_{q=0}^{\infty})$, one has

$$h\left(\frac{Z+Y}{2}\right) = \sum_{q=0}^{\infty} \frac{q+4}{2q+3}\left[\bar{\rho}\left(\frac{Z_q + Y_q}{2}, \bar{0}\right)\right]^{(2q+3)/(q+4)} \leq \frac{2}{\sqrt[4]{8}}(h(Z) + h(Y)), \tag{14}$$

an example of premodular vector space and modular vector space.

Example 3. The function $h(Y) = \inf\{\alpha > 0 : \sum_{q=0}^{\infty}((q+2)/(2q+3))[\bar{\rho}(Y_q/\alpha, \bar{0})]^{(2q+3)/(q+2)} \leq 1\}$ is a premodular (modular) on the vector space $\ell^F(((2q+3)/(q+2))_{q=0}^{\infty})$.

Definition 8. Suppose $\mathbf{U}$ is a (pssf). The function $h \in [0,\infty)^{U}$ is said to be a pre-quasi-norm on $\mathbf{U}$, if the following setups are verified:

(i) If $Y \in \mathbf{U}$, $Y = \bar{\vartheta} \Leftrightarrow h(Y) = 0$ with $h(Y) \geq 0$, where $\bar{\vartheta} = (\bar{0}, \bar{0}, \bar{0},)$

(ii) There is $Q \geq 1$; the inequality $h(\alpha Y) \leq Q|\alpha| h(Y)$ satisfies, for every $Y \in \mathbf{U}$ and $\alpha \in \mathfrak{R}$

(iii) There is $P \geq 1$; the inequality $h(Y + Z) \leq P(h(Y) + h(Z))$ holds, for each $Y, Z \in \mathbf{U}$.

Clearly, from the last two definitions, we conclude the following two theorems.

Theorem 9. *Every premodular (pssf) is a pre-quasi-normed (pssf).*

Theorem 10. *Every quasi-normed (pssf) is a pre-quasi-normed (pssf).*

Definition 11.

(a) The function h on $\ell_{\tau(.)}^F$ is named h-convex, if

$$h(\alpha Y + (1-\alpha)Z) \leq \alpha h(Y) + (1-\alpha)h(Z), \tag{15}$$

for every $\alpha \in [0,1]$ and $Y, Z \in \ell_{\tau(.)}^F$.

(b) When$\lim_{q\longrightarrow\infty} h(Y_q - Y) = 0$,we called$\{Y_q\}_{q\in\mathcal{N}} \subseteq (\ell_{\tau(.)}^F)_h$ h-convergent to$Y \in (\ell_{\tau(.)}^F)_h$

(c) $\{Y_q\}_{q\in\mathcal{N}} \subseteq (\ell_{\tau(.)}^F)_h$ is h-Cauchy, if $\lim_{q,r\longrightarrow\infty} h(Y_q - Y_r) = 0$

(d) $\Gamma \subset (\ell_{\tau(.)}^F)_h$ is h-closed, when for all h-converges $\{Y_q\}_{q\in\mathcal{N}} \subset \Gamma$ to Y, then $Y \in \Gamma$

(e) $\Gamma \subset (\ell^F_{\tau(.)})_h$ is h-bounded, if $\delta_h(\Gamma) = \sup \{h(Y - Z): Y, Z \in \Gamma\} < \infty$

(f) A pre-quasi-norm h on $\ell^F_{\tau(.)}$ verifies the Fatou property, when for all $\{Z^q\} \subseteq (\ell^F_{\tau(.)})_h$ under $\lim_{q\longrightarrow\infty} h(Z^q - Z) = 0$ and $Y \in (\ell^F_{\tau(.)})_h$, one has $h(Y - Z) \le \sup_r \inf_{q\ge r} h(Y - Z^q)$

Theorem 12. *$(\ell^F_{\tau(.)})_h$, where $h(Y) = [\sum_{q=0}^{\infty}(1/\tau_q)\,[\bar{\rho}(Y_q, \bar{0})]^{\tau_q}]^{1/K}$, for all $Y \in \ell^F_{\tau(.)}$, is a premodular (pssf), when $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, where $\mathbf{I}$ is the space of every increasing sequences of reals.*

Proof.

(i) Evidently, $h(Y) \ge 0$ and $h(Y) = 0 \Leftrightarrow Y = \bar{\vartheta}$.

(1-i) Let $Y, Z \in \ell^F_{\tau(.)}$. One has

$$h(Y+Z) = \left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(Y_q + Z_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} \le \left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(Y_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} + \left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(Z_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} = h(Y) + h(Z) < \infty, \tag{16}$$

then $Y + Z \in \ell^F_{\tau(.)}$

(ii) One gets $P \ge 1$ with $h(Y + Z) \le P(h(Y) + h(Z))$, for all $Y, Z \in \ell^F_{\tau(.)}$.

(1-ii) Assuming $\alpha \in \mathfrak{R}$ and $Y \in \ell^F_{\tau(.)}$, we obtain

$$\begin{aligned} h(\alpha Y) &= \left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(\alpha Y_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} \\ &\le \sup_q |\alpha|^{\tau_q/K}\left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(Y_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} \\ &\le Q|\alpha|h(v) < \infty \end{aligned} \tag{17}$$

As $\alpha Y \in \ell^F_{\tau(.)}$, hence, from setups (1-i) and (1-ii), we get $\ell^F_{\tau(.)}$ is linear. Also, $\bar{\mathfrak{b}}_p \in \ell^F_{\tau(.)}$, for all $p \in \mathcal{N}$, since $h(\bar{\mathfrak{b}}_p) = [\sum_{q=0}^{\infty}(1/\tau_q)[\bar{\rho}(\bar{\mathfrak{b}}_p, \bar{0})]^{\tau_q}]^{1/K} = 1/\tau_b$.

(iii) There is $Q = \max\{1, \sup_q |\alpha|^{(\tau_q/K)-1}\} \ge 1$ with $h(\alpha Y) \le Q|\alpha|h(Y)$, for all $Y \in \ell^F_{\tau(.)}$ and $\alpha \in \mathfrak{R}$

(1) Assume $|Y_q| \le |Z_q|$, for all $q \in \mathcal{N}$ and $Z \in \ell^F_{\tau(.)}$. One finds

$$h(Y) = \left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(Y_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} \le \left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(Z_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} = h(Z) < \infty, \tag{18}$$

then $Y \in \ell^F_{\tau(.)}$

(iv) Obviously, from (2)

(1) Let $(Y_q) \in \ell^F_{\tau(.)}$; we get

$$\begin{aligned} h\left(\left(Y_{[q/2]}\right)\right) &= \left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(Y_{[q/2]}, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} \\ &= \left[\sum_{q=0}^{\infty}\frac{1}{\tau_{2q}}\left[\bar{\rho}\left(Y_q, \bar{0}\right)\right]^{\tau_{2q}} + \sum_{q=0}^{\infty}\frac{1}{\tau_{2q+1}}\left[\bar{\rho}\left(Y_q, \bar{0}\right)\right]^{\tau_{2q+1}}\right]^{1/K} \\ &\le 2^{\frac{1}{K}}\left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(Y_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} = 2^{1/K}h\left(\left(Y_q\right)\right), \end{aligned} \tag{19}$$

then $(Y_{[q/2]}) \in \ell^F_{\tau(.)}$

(v) From (3), we obtain $P_0 = 2^{1/K} \ge 1$

(vi) Evidently, the closure of $E = \ell^F_{\tau(.)}$

(vii) There is $0 < \sigma \le |\alpha|^{(\tau_0/K)-1}$, for $\alpha \ne 0$ or $\sigma > 0$, for $\alpha = 0$ with

$$h(\bar{\alpha}, \bar{0}, \bar{0}, \bar{0}, \cdots) \ge \sigma|\alpha| h(\bar{1}, \bar{0}, \bar{0}, \bar{0}, \cdots) \tag{20}$$

□

Example 4. For $(\tau_q) \in [1, \infty)^{\mathcal{N}}$, the function $h(Y) = \inf\{\alpha > 0 : \sum_{q\in\mathcal{N}}(1/\tau_q)[\bar{\rho}(Y_q/\alpha, \bar{0})]^{\tau_q} \le 1\}$ is a norm on $\ell^F_{\tau(.)}$.

Example 5. The function $h(Y) = \sqrt[3]{\sum_{q\in\mathcal{N}}((q+1)/(3q+2))[\bar{\rho}(Y_q, \bar{0})]^{(3q+2)/(q+1)}}$ is a pre-quasi-norm (not a norm) on $\ell^F(((3q+2)/(q+1))_{q=0}^{\infty})$.

Example 6. The function $h(Y) = \sum_{q\in\mathcal{N}}((q+1)/(3q+2))[\bar{\rho}(Y_q, \bar{0})]^{(3q+2)/(q+1)}$ is a pre-quasi=norm (not a quasinorm) on $\ell^F(((3q+2)/(q+1))_{q=0}^{\infty})$.

Example 7. The function $h(Y) = \sqrt[d]{\sum_{q\in\mathcal{N}}(1/d)[\bar{\rho}(Y_q, \bar{0})]^d}$ is a pre-quasi-norm, quasinorm, and not a norm on ℓ^F_d, with $0 < d < 1$.

Theorem 13. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, then $(\ell^F_{\tau(.)})_h$ is a pre-quasi-Banach (pssf), where $h(Y) = [\sum_{q=0}^{\infty}(1/\tau_q)[\bar{\rho}(Y_q, \bar{0})]^{\tau_q}]^{1/K}$, for every $Y \in \ell^F_{\tau(.)}$.*

Proof. By Theorem 9 and Theorem 12, one obtains$(\ell^F_{\tau(.)})_h$ which is a pre-quasi-normed (pssf). If $Y^l = (Y^l_q)_{q=0}^{\infty}$ is a Cauchy sequence in $(\ell^F_{\tau(.)})_h$. Then, for every $\varepsilon \in (0, 1)$, one has $l_0 \in \mathcal{N}$ such that for all $l, m \geq l_0$, one gets

$$h\left(Y^l - Y^m\right) = \left[\sum_{q=0}^{\infty} \frac{1}{\tau_q}\left[\bar{\rho}\left(Y^l_q - Y^m_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} < \varepsilon. \quad (21)$$

This implies $\bar{\rho}(Y^l_q - Y^m_q, \bar{0}) < \varepsilon$.As$(\mathfrak{R}[0, 1], \bar{\rho})$is a complete metric space, then$(Y^m_q)$is a Cauchy sequence in $\mathfrak{R}[0, 1]$, for fixed $q \in \mathcal{N}$. Then, $\lim_{m\longrightarrow\infty} Y^m_q = Y^0_q$, for constant $q \in \mathcal{N}$. Hence, $h(Y^l - Y^0) < \varepsilon$, for every $l \geq l_0$. Since $h(Y^0) = h(Y^0 - Y^l + Y^l) \leq h(Y^l - Y^0) + h(Y^l) < \infty$, so $Y^0 \in \ell^F_{\tau(.)}$. □

Theorem 14. *Suppose $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 0$; then, $(\ell^F_{\tau(.)})_h$ is a pre-quasi-closed (pssf), where $h(Y) = [\sum_{q=0}^{\infty}(1/\tau_q)[\bar{\rho}(Y_q, \bar{0})]^{\tau_q}]^{1/K}$, for every $Y \in \ell^F_{\tau(.)}$.*

Proof. In view of Theorem 12 and Theorem 9, the space $(\ell^F_{\tau(.)})_h$ is a pre-quasi-normed (pssf). Assume $Y^l = (Y^l_q)_{q=0}^{\infty} \in (\ell^F_{\tau(.)})_h$ and $\lim_{l\longrightarrow\infty} h(Y^l - Y^0) = 0$; then, for all $\varepsilon \in (0, 1)$, there is $l_0 \in \mathcal{N}$ such that for all $l \geq l_0$, we obtain

$$\varepsilon > h\left(Y^l - Y^0\right) = \left[\sum_{q=0}^{\infty} \frac{1}{\tau_q}\left[\bar{\rho}\left(Y^l_q - Y^0_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K}. \quad (22)$$

This implies $\bar{\rho}(Y^l_q - Y^0_q, \bar{0}) < \varepsilon$. As $(\mathfrak{R}[0, 1], \bar{\rho})$is a complete metric space, therefore,$(Y^l_q)$is a convergent sequence in $\mathfrak{R}[0, 1]$, for fixed$q \in \mathcal{N}$. So, $\lim_{l\longrightarrow\infty} Y^l_q = Y^0_q$, for fixed $q \in \mathcal{N}$. Since $h(Y^0) = h(Y^0 - Y^l + Y^l) \leq h(Y^l - Y^0) + h(Y^l) < \infty$, one has $Y^0 \in \ell^F_{\tau(.)}$. □

Theorem 15. *The function $h(Y) = [\sum_{q=0}^{\infty}(1/\tau_q)[\bar{\rho}(Y_q, \bar{0})]^{\tau_q}]^{1/K}$ holds the Fatou property, if $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, for all $Y \in \ell^F_{\tau(.)}$.*

Proof. Let $\{Z^r\} \subseteq (\ell^F_{\tau(.)})_h$ such that $\lim_{r\longrightarrow\infty} h(Z^r - Z) = 0$. Since $(\ell^F_{\tau(.)})_h$ is a pre-quasi-closed space, one has $Z \in (\ell^F_{\tau(.)})_h$. For all $Y \in (\ell^F_{\tau(.)})_h$, one gets

$$\begin{aligned} h(Y - Z) &= \left[\sum_{q=0}^{\infty} \frac{1}{\tau_q}\left[\bar{\rho}(Y_q - Z_q, \bar{0})\right]^{\tau_q}\right]^{1/K} \leq \left[\sum_{q=0}^{\infty} \frac{1}{\tau_q}\left[\bar{\rho}\left(Y_q - Z^r_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} \\ &+ \left[\sum_{q=0}^{\infty} \frac{1}{\tau_q}\left[\bar{\rho}\left(Z^r_q - Z_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} \leq \sup_m \inf_{r\geq m} h(Y - Z^r). \end{aligned} \quad (23)$$

□

Theorem 16. *The function $h(Y) = \sum_{q=0}^{\infty}(1/\tau_q)[\bar{\rho}(Y_q, \bar{0})]^{\tau_q}$ does not hold the Fatou property, for all $Y \in \ell^F_{\tau(.)}$, when $(\tau_q) \in \ell_\infty$ and $\tau_q > 1$ with $q \in \mathcal{N}$.*

Proof. Let $\{Z^r\} \subseteq (\ell^F_{\tau(.)})_h$ so that $\lim_{r\longrightarrow\infty} h(Z^r - Z) = 0$. Since $(\ell^F_{\tau(.)})_h$ is a pre-quasi-closed space, one gets $Z \in (\ell^F_{\tau(.)})_h$. For every $Z \in (\ell^F_{\tau(.)})_h$, we obtain

$$\begin{aligned} h(Y - Z) &= \sum_{q=0}^{\infty} \frac{1}{\tau_q}\left[\bar{\rho}(Y_q - Z_q, \bar{0})\right]^{\tau_q} \leq 2^{\sup_q \tau_q - 1} \\ &\cdot \left(\sum_{q=0}^{\infty} \frac{1}{\tau_q}\left[\bar{\rho}\left(Y_q - Z^r_q, \bar{0}\right)\right]^{\tau_q} + \sum_{q=0}^{\infty} \frac{1}{\tau_q}\left[\bar{\rho}\left(Z^r_q - Z_q, \bar{0}\right)\right]^{\tau_q}\right) \\ &\leq 2^{\sup_q \tau_q - 1} \sup_m \inf_{r\geq m} h(Y - Z^r). \end{aligned} \quad (24)$$

□

4. Caristi's Fixed Point Theorem in $(\ell^F_{\tau(.)})_h$

In this section, the existence of Caristi's fixed point in $(\ell^F_{\tau(.)})_h$ is presented according to Farkas [31], where $h(Y) = [\sum_{q=0}^{\infty}(1/\tau_q)[\bar{\rho}(Y_q, \bar{0})]^{\tau_q}]^{1/K}$, for all $Y \in \ell^F_{\tau(.)}$.

Definition 17. The function $\Psi_1 : (\ell^F_{\tau(.)})_h \longrightarrow (-\infty,\infty]$ is said to be lower semicontinuous at $G^{(0)} \in (\ell^F_{\tau(.)})_h$ if $\liminf_{G\longrightarrow G^{(0)}} \Psi_1(G) = \Psi_1(G^{(0)})$, where $\liminf_{G\longrightarrow G^{(0)}} \Psi_1(G) = \sup_{V\in\mathscr{V}(G^{(0)})} \inf_{G\in V} \Psi_1(G)$, where $\mathscr{V}(G^{(0)})$ is a neighborhood system of $G^{(0)}$.

Definition 18. The function $\Psi_1 : (\ell^F_{\tau(.)})_h \longrightarrow (-\infty,\infty]$ is called proper, when

$$\mathscr{D}(\Psi_1) = \left\{G \in \left(\ell^F_{\tau(.)}\right)_h : \Psi_1(G) < \infty\right\} \neq \varnothing. \quad (25)$$

Theorem 19. *Assume $\Xi \neq \varnothing$ and Ξ is a h-closed subset of $(\ell^F_{\tau(.)})_h$ and $\Psi_1 : \Xi \longrightarrow (-\infty,\infty]$ is a proper, h -lower semicontinuous function with $\inf_{G\in\Xi}\Psi_1(G) > -\infty$. Suppose $\gamma > 0$, $\{\varpi_q\} \subset (0,\infty)$, and $G^{(0)} \in \Xi$ with $\Psi_1(G^{(0)}) \leq \inf_{G\in\Xi}\Psi_1(G) +$*

γ. Therefore, one has $\{G^{(q)}\} \in \Xi$ which h-converges to some $G^{(\gamma)}$, and

(i) $h(G^{(\gamma)} - G^{(q)}) \leq \gamma/2^q \varpi_0$, with $q \in \mathcal{N}$

(ii) $\Psi_1(G^{(\gamma)}) + \sum_{q=0}^{\infty} \varpi_q h(G^{(\gamma)} - G^{(q)}) \leq \Psi_1(G^{(0)})$

(iii) when $G \neq G^{(\gamma)}$, we have

$$\Psi_1\left(G^{(\gamma)}\right) + \sum_{q=0}^{\infty} \varpi_q h\left(G^{(\gamma)} - G^{(q)}\right) < \Psi_1(G) + \sum_{q=0}^{\infty} \varpi_q h\left(G - G^{(q)}\right) \quad (26)$$

Proof. Let $S(G^{(0)}) = \{G \in \Xi : \Psi_1(G) + \varpi_0 h(G - G^{(0)}) \leq \Psi_1(G^{(0)})\}$. As $G^{(0)} \in S(G^{(0)})$, hence, $S(G^{(0)}) \neq \varnothing$. Since Ψ_1 is h-lower semicontinuous, h verifies the Fatou property, and Ξ is h-closed, one has $S(G^{(0)})$ which is h-closed. Choose $G^{(1)} \in S(G^{(0)})$ and

$$\Psi_1\left(G^{(1)}\right) + \varpi_0 h\left(G^{(1)} - G^{(0)}\right) \leq \inf_{G \in S\left(G^{(0)}\right)} \left\{\Psi_1(G) + \varpi_0 h\left(G - G^{(0)}\right)\right\} + \frac{\gamma \varpi_1}{2\varpi_0}. \quad (27)$$

Take

$$S\left(G^{(1)}\right) = \left\{G \in S\left(G^{(0)}\right) : \Psi_1(G) + \sum_{j=0}^{1} \varpi_j h\left(G - G^{(j)}\right) \leq \Psi_1\left(G^{(1)}\right) + \varpi_0 h\left(G^{(1)} - G^{(0)}\right)\right\}. \quad (28)$$

As $S(G^{(0)})$, we get $S(G^{(1)}) \neq \varnothing$ and h-closed. Assume that one has built $\{G^{(0)}, G^{(1)}, G^{(2)}, ,G^{(q)}\}$ and $\{S(G^{(0)}), S(G^{(1)}), S(G^{(2)}), \cdots, S(G^{(q)})\}$. Next, choose $G^{(q+1)} \in S(G^{(q)})$ and

$$\Psi_1\left(G^{(q+1)}\right) + \sum_{j=0}^{q} \varpi_j h\left(G^{(q+1)} - G^{(j)}\right) \leq \inf_{G \in S\left(G^{(q)}\right)} \left\{\Psi_1(G) + \sum_{j=0}^{q} \varpi_j h\left(G - G^{(j)}\right)\right\} + \frac{\gamma \varpi_q}{2^q \varpi_0}. \quad (29)$$

Let

$$S\left(G^{(q+1)}\right) \coloneqq \left\{G \in S\left(G^{(q)}\right) : \Psi_1(G) + \sum_{j=0}^{q+1} \varpi_j h\left(G - G^{(j)}\right) \leq \Psi_1\left(G^{(q+1)}\right) + \sum_{j=0}^{q} \varpi_j h\left(G^{(q+1)} - G^{(j)}\right)\right\}. \quad (30)$$

Hence, we form by induction the sequences $\{G^{(q)}\}$ and $\{S(G^{(q)})\}$. Fix $q \in \mathcal{N}$. Suppose $W \in S(G^{(q)})$. One obtains

$$\Psi_1(W) + \sum_{j=0}^{q} \varpi_j h\left(W - G^{(j)}\right) \leq \Psi_1\left(G^{(q)}\right) + \sum_{j=0}^{q-1} \varpi_j h\left(G^{(q)} - G^{(j)}\right). \quad (31)$$

Hence,

$$\begin{aligned} \varpi_q h\left(W - G^{(q)}\right) &\leq \Psi_1\left(G^{(q)}\right) + \sum_{j=0}^{q-1} \varpi_j h\left(G^{(q)} - G^{(j)}\right) - \left[\Psi_1(W) + \sum_{j=0}^{q-1} \varpi_j h\left(W - G^{(j)}\right)\right] \leq \Psi_1\left(G^{(q)}\right) \\ &+ \sum_{j=0}^{q-1} \varpi_j h\left(G^{(q)} - G^{(j)}\right) - \inf_{G \in S\left(G^{(q-1)}\right)} \left[\Psi_1(G) + \sum_{j=0}^{q-1} \varpi_j h\left(G - G^{(j)}\right)\right] \leq \frac{\gamma \varpi_q}{2^q \varpi_0}. \end{aligned} \quad (32)$$

As $\{S(G^{(q)})\}$ is decreasing with $G^{(q)} \in S(G^{(q)})$, for every $q \in \mathcal{N}$, we obtain

$$h\left(G^{(q+p)} - G^{(q)}\right) \leq \frac{\gamma}{2^q \varpi_0}, \quad (33)$$

with $q, p \in \mathcal{N}$. This implies $\{G^{(q)}\}$ is h-Cauchy. Since $(\ell^F_{\tau(.)})_h$ is a h-Banach space, hence, $\{G^{(q)}\}$ has h-limits $G^{(\gamma)}$ and $\bigcap_{q \in \mathcal{N}} S(G^{(q)}) = \{G^{(\gamma)}\}$. Since $G^{(q+1)} \in S(G^{(q)})$, we can see

$$\Psi_1\left(G^{(q+1)}\right) + \sum_{j=0}^{q} \varpi_j h\left(G^{(q+1)} - G^{(j)}\right) \leq \Psi_1\left(G^{(q)}\right) + \sum_{j=0}^{q-1} \varpi_j h\left(G^{(q)} - G^{(j)}\right). \quad (34)$$

Hence, $\{\Psi_1(G^{(q)}) + \sum_{j=0}^{q-1} \varpi_j h(G^{(q)} - G^{(j)})\}$ is decreasing. After, let $G \neq G^{(\gamma)}$. One gets $m \in \mathcal{N}$ with $G \notin S(G^{(q)})$, with $q \geq m$, i.e.,

$$\Psi_1\left(G^{(q)}\right) + \sum_{j=0}^{q-1} \varpi_j h\left(G^{(q)} - G^{(j)}\right) < \Psi_1(G) + \sum_{j=0}^{q} \varpi_j h\left(G - G^{(j)}\right). \quad (35)$$

Since $G^{(\gamma)} \in S(G^{(q)})$, with $q \geq m$, we get

$$\Psi_1\left(G^{(\gamma)}\right) + \sum_{j=0}^{q} \varpi_j h\left(G^{(\gamma)} - G^{(j)}\right) \leq \Psi_1\left(G^{(q)}\right) + \sum_{j=0}^{q-1} \varpi_j h\left(G^{(q)} - G^{(j)}\right) \leq \Psi_1\left(G^{(m)}\right) + \sum_{j=0}^{m-1} \varpi_j h\left(G^{(m)} - G^{(j)}\right). \quad (36)$$

Putting $q \longrightarrow \infty$ in the previous inequality, one can see

$$\begin{aligned}\Psi_1\left(G^{(\gamma)}\right) + \sum_{j=0}^{\infty} \varpi_j h\left(G^{(\gamma)} - G^{(j)}\right) \le \Psi_1(x_m) \\ + \sum_{j=0}^{m-1} \varpi_j h\left(G^{(m)} - G^{(j)}\right) < \Psi_1(G) + \sum_{j=0}^{m} \varpi_j h\left(G - G^{(j)}\right) \\ \le \Psi_1(G) + \sum_{j=0}^{\infty} \varpi_j h\left(G - G^{(j)}\right).\end{aligned} \tag{37}$$

This gives

$$\Psi_1\left(G^{(\gamma)}\right) + \sum_{q=0}^{\infty} \varpi_q h\left(G^{(\gamma)} - G^{(q)}\right) < \Psi_1(G) + \sum_{q=0}^{\infty} \varpi_q h\left(G - G^{(q)}\right). \tag{38}$$

□

Example 8. Suppose $\Xi = \{Y \in (\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h : Y_0 = \bar{0}\}$ and $h(Y) = \sqrt{\sum_{q\in\mathcal{N}}((q+2)/(2q+3))(\bar{\rho}(Y_q, \bar{0}))^{(2q+3)/(q+2)}}$, for every $Y \in (\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h$. Suppose $\gamma > 0$, $\{\varpi_q\} \subset (0,\infty)$, and $G^{(0)} \in \Xi$ with $\sup_{G\in\Xi} \ln h(G) \le \ln h(G^{(0)}) + \gamma$.Since$\Psi_1 : \Xi \longrightarrow (-\infty,\infty]$, where$\Psi_1(G) = -\ln(h(G))$, clearly,$\Xi \neq \varnothing$and$\Xi$is a$h$-closed subset of$(\ell^F_{\tau(.)})_h$, and$\Psi_1$is a proper,$h$-lower semicontinuous function with$\inf_{G\in\Xi}\Psi_1(G) > -\infty$. From Theorem 19, one has $\{G^{(q)}\} \in \Xi$ which h-converges to some $G^{(\gamma)}$, and

(i) $h(G^{(\gamma)} - G^{(q)}) \le \gamma/2^q \varpi_0$, with $q \in \mathcal{N}$

(ii) $\ln(h(G^{(0)})) + \sum_{q=0}^{\infty} \varpi_q h(G^{(\gamma)} - G^{(q)}) \le \ln(h(G^{(\gamma)}))$

(iii) when $G \neq G^{(\gamma)}$, we have

$$\begin{aligned}\ln(h(G)) + \sum_{q=0}^{\infty} \varpi_q h\left(G^{(\gamma)} - G^{(q)}\right) < \ln\left(h\left(G^{(\gamma)}\right)\right) \\ + \sum_{q=0}^{\infty} \varpi_q h\left(G - G^{(q)}\right)\end{aligned} \tag{39}$$

Example 9. Suppose $\Xi = \{Y \in (\ell^F(((q+1)/(3q+4))_{q=0}^{\infty}))_h : Y_0 = \bar{0}\}$ and $h(Y) = \sum_{q\in\mathcal{N}}((3q+4)/(q+1))(\bar{\rho}(Y_q, \bar{0}))^{(q+1)/(3q+4)}$, for every $Y \in (\ell^F(((q+1)/(3q+4))_{q=0}^{\infty}))_h$. Suppose $\gamma > 0$, $\{\varpi_q\} \subset (0,\infty)$, and $G^{(0)} \in \Xi$ with $h(G^{(0)}) \le \gamma$. Since $\Psi_1 : \Xi \longrightarrow (-\infty,\infty]$, where $\Psi_1(G) = h(G)$, clearly, $\Xi \neq \varnothing$ and Ξ is a h-closed subset of $(\ell^F_{\tau(.)})_h$, and Ψ_1 is a proper, h-lower semicontinuous function with $\inf_{G\in\Xi}\Psi_1(G) > -\infty$. From Theorem 19, one has $\{G^{(q)}\} \in \Xi$ which h-converges to some $G^{(\gamma)}$, and

(i) $h(G^{(\gamma)} - G^{(q)}) \le \gamma/2^q \varpi_0$, with $q \in \mathcal{N}$

(ii) $h(G^{(\gamma)}) + \sum_{q=0}^{\infty} \varpi_q h(G^{(\gamma)} - G^{(q)}) \le h(G^{(0)})$

(iii) when $G \neq G^{(\gamma)}$, we have

$$h\left(G^{(\gamma)}\right) + \sum_{q=0}^{\infty} \varpi_q h\left(G^{(\gamma)} - G^{(q)}\right) < h(G) + \sum_{q=0}^{\infty} \varpi_q h\left(G - G^{(q)}\right) \tag{40}$$

Theorem 20. *If $\Xi \neq \varnothing$ and Ξ is a h-closed subset of $(\ell^F_{\tau(.)})_h$, by choosing $\gamma > 0$ and $\{\varpi_n\}$ and $0 < \omega = \sum_{n=0}^{\infty} \varpi_n < \infty$, suppose $H : \Xi \longrightarrow \Xi$ is a mapping and there is a function $\Psi_1 : \Xi \longrightarrow (-\infty,\infty]$ that holds a proper andh-lower semicontinuous with $\inf_{G\in\Xi}\Psi_1(G) > -\infty$ and*

(1) $h(H(G) - Y) - h(G - Y) \le h(H(G) - G)$, for any $G, Y \in \Xi$

(2) $h(H(G) - G) \le \Psi_1(G) - \Psi_1(H(G))$, with $G \in \Xi$

Hence, H has a fixed point in Ξ.

Proof. Since $0 < \omega = \sum_{n=0}^{\infty} \varpi_n < \infty$, we get $\Psi_2 := \omega\Psi_1$ which is also proper, h-lower semicontinuous, and bounded from below. If $G \in \Xi$, one gets

$$\omega h(H(G) - G) \le \Psi_2(G) - \Psi_2(H(G)). \tag{41}$$

As $\inf_{G\in\Xi}\Psi_2(G) > -\infty$, one obtains $G^{(0)} \in \Xi$ with $\Psi_2(G^{(0)}) < \inf_{G\in\Xi}\Psi_2(G) + \gamma$. From Theorem 19, there is $\{G^{(q)}\}$ which h-converges to some $G^{(\gamma)} \in \Xi$, and

$$\Psi_2\left(G^{(\gamma)}\right) + \sum_{q=0}^{\infty} \varpi_q h\left(G^{(\gamma)} - G^{(q)}\right) < \Psi_2(G) + \sum_{q=0}^{\infty} \varpi_q h\left(G - G^{(q)}\right), \tag{42}$$

for every $G \neq G^{(\gamma)}$. Assume that $H(G^{(\gamma)}) \neq G^{(\gamma)}$; we have

$$\begin{aligned}\Psi_2\left(G^{(\gamma)}\right) + \sum_{q=0}^{\infty} \varpi_q h\left(G^{(\gamma)} - G^{(q)}\right) < \Psi_2\left(H\left(G^{(\gamma)}\right)\right) \\ + \sum_{q=0}^{\infty} \varpi_q h\left(H\left(G^{(\gamma)}\right) - G^{(q)}\right).\end{aligned} \tag{43}$$

Then,

$$\begin{aligned}\Psi_2\left(G^{(\gamma)}\right) - \Psi_2\left(H\left(G^{(\gamma)}\right)\right) < \sum_{q=0}^{\infty} \varpi_q h\left(H\left(G^{(\gamma)}\right) - G^{(q)}\right) \\ - \sum_{q=0}^{\infty} \varpi_q h\left(G^{(\gamma)} - G^{(q)}\right) = \sum_{q=0}^{\infty} \varpi_q \left(h\left(H\left(G^{(\gamma)}\right) - G^{(q)}\right) \right.\\ \left. - h\left(G^{(\gamma)} - G^{(q)}\right)\right).\end{aligned} \tag{44}$$

From condition (41), one can see

$$\Psi_2\left(G^{(\gamma)}\right)-\Psi_2\left(H\left(G^{(\gamma)}\right)\right)<\sum_{q=0}^{\infty}\varpi_q h\left(H\left(G^{(\gamma)}\right)-G^{(\gamma)}\right)$$
$$=\omega h\left(H\left(G^{(\gamma)}\right)-G^{(\gamma)}\right). \tag{45}$$

The inequality (1) implies that

$$\omega h\left(H\left(G^{(\gamma)}\right)-G^{(\gamma)}\right)\le\Psi_2\left(G^{(\gamma)}\right)-\Psi_2\left(H\left(G^{(\gamma)}\right)\right) < \omega h\left(H\left(G^{(\gamma)}\right)-G^{(\gamma)}\right). \tag{46}$$

This is a disagreement. Therefore, $H(G^{(\gamma)})=G^{(\gamma)}$. □

Example 10. Suppose $\Xi=\{Y\in(\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h : Y_0=\bar{0}\}$ and $h(Y)=\sqrt{\sum_{q\in\mathcal{N}}((q+2)/(2q+3))(\bar{\rho}(Y_q,\bar{0}))^{(2q+3)/(q+2)}}$, for every $Y\in(\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h$. Suppose $H:\Xi\longrightarrow\Xi$ is a mapping and $h(H(G)-G)\le\ln h(H(G))-\ln h(G)$, with $G\in\Xi$. From Theorem 20, H has a fixed point in Ξ.

Example 11. Suppose $\Xi=\{Y\in(\ell^F(((q+1)/(3q+4))_{q=0}^{\infty}))_h : Y_0=\bar{0}\}$ and $h(Y)=\sum_{q\in\mathcal{N}}((3q+4)/(q+1))(\bar{\rho}(Y_q,\bar{0}))^{(q+1)/(3q+4)}$, for every $Y\in(\ell^F(((q+1)/(3q+4))_{q=0}^{\infty}))_h$. Suppose $H:\Xi\longrightarrow\Xi$ is a mapping and $h(H(G)-G)\le h(G)-h(H(G))$, with $G\in\Xi$. From Theorem 20, H has a fixed point in Ξ.

Definition 21. Pick up$\boldsymbol{U}_h$as a pre-quasi-normed (pssf),$V:\boldsymbol{U}_h\longrightarrow\boldsymbol{U}_handZ\in\boldsymbol{U}_h$. The operator V is called h-sequentially continuous at Z, if and only if, when $\lim_{q\longrightarrow\infty}h(Y_q-Z)=0$, then $\lim_{q\longrightarrow\infty}h(VY_q-VZ)=0$.

Example 12. Suppose $V:(\ell^F(((q+1)/(2q+4))_{q=0}^{\infty}))_h\longrightarrow(\ell^F(((q+1)/(2q+4))_{q=0}^{\infty}))_h$, where $h(Z)=[\sum_{q=0}^{\infty}((2q+4)/(q+1))(\bar{\rho}(Z_q,\bar{0}))^{(q+1)/(2q+4)}]^4$, for every $Z\in\ell^F(((q+1)/(2q+4))_{q=0}^{\infty})$ and

$$V(Z)=\begin{cases}\dfrac{1}{18}\left(\bar{\mathfrak{b}}_0+Z\right), & Z_0(y)\in\left[0,\dfrac{1}{17}\right),\\[2mm] \dfrac{1}{17}\bar{\mathfrak{b}}_0, & Z_0(y)=\dfrac{1}{17},\\[2mm] \dfrac{1}{18}\bar{\mathfrak{b}}_0, & Z_0(y)\in\left(\dfrac{1}{17},1\right].\end{cases} \tag{47}$$

V is clearly both h-sequentially continuous and discontinuous at $(1/17)\bar{\mathfrak{b}}_0\in(\ell^F(((\mathrm{q}+1)/(2\mathrm{q}+4))_{\mathrm{q}=0}^{\infty}))_h$.

Example 13. Assume $V:(\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h\longrightarrow(\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h$, where $h(g)=\sqrt{\sum_{q=0}^{\infty}((q+2)/(2q+3))(\bar{\rho}(g_q,\bar{0}))^{(2q+3)/(q+2)}}$, for every $g\in\ell^F(((2q+3)/(q+2))_{q=0}^{\infty})$ and

$$V(g)=\begin{cases}\dfrac{g}{4}, & h(g)\in[0,1),\\[2mm] \dfrac{g}{5}, & h(g)\in[1,\infty).\end{cases} \tag{48}$$

Suppose $\{Z^{(n)}\}\subseteq(\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h$ is such that $\lim_{n\longrightarrow\infty}h(Z^{(n)}-Z^{(0)})=0$, where $Z^{(0)}\in(\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h$ with $h(Z^{(0)})=1$.

As the pre-quasi-norm h is continuous, we have

$$\lim_{n\longrightarrow\infty}h\left(VZ^{(n)}-VZ^{(0)}\right)=\lim_{n\longrightarrow\infty}h\left(\frac{Z^{(n)}}{4}-\frac{Z^{(0)}}{5}\right)=h\left(\frac{Z^{(0)}}{20}\right)>0. \tag{49}$$

Therefore, V is not h-sequentially continuous at $Z^{(0)}$.

Theorem 22. *If $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ with $\tau_0>1$ and $V:(\ell^F_{\tau(.)})_h\longrightarrow(\ell^F_{\tau(.)})_h$, where $h(Y)=\sum_{q=0}^{\infty}(1/\tau_q)[\bar{\rho}(Y_q,\bar{0})]^{\tau_q}$, for all $Y\in\ell^F(\tau)$, suppose*

(1) $V:(\ell^F_{\tau(.)})_h\longrightarrow(\ell^F_{\tau(.)})_h$ is a mapping, and there is a function $\Psi_1:(\ell^F_{\tau(.)})_h\longrightarrow(-\infty,\infty]$ that holds a proper and h-lower semicontinuous with $\inf_{G\in(\ell^F_{\tau(.)})_h}\Psi_1(G)>-\infty$, and there is $\alpha\in[0,1)$ so that $h(V^{l+1}G-V^lG)\le\alpha^l(\Psi_1(G)-\Psi_1(V(G)))$, with $G\in(\ell^F_{\tau(.)})_h$

(2) V is h-sequentially continuous at $Z\in(\ell^F_{\tau(.)})_h$

(3) there is $Y\in(\ell^F_{\tau(.)})_h$ with $\{V^lY\}$ which has $\{V^{l_j}Y\}$ converging to Z

Then, $Z\in(\ell^F_{\tau(.)})_h$ is a fixed point of V.

Proof. Assume Z is not a fixed point of V; one has $VZ\ne Z$. From parts (2) and (3), we get

$$\lim_{l_j\longrightarrow\infty}h\left(V^{l_j}Y-Z\right)=0,\qquad \lim_{l_j\longrightarrow\infty}h\left(V^{l_j+1}Y-VZ\right)=0. \tag{50}$$

From part (1), one obtains

$$\begin{aligned}0<h(VZ-Z)&=h\left(\left(VZ-V^{l_j+1}Y\right)+\left(V^{l_j}Y-Z\right)+\left(V^{l_j+1}Y-V^{l_j}Y\right)\right)\\&\le 2^{2\sup_i\tau_i-2}h\left(V^{l_j+1}Y-VZ\right)+2^{2\sup_i\tau_i-2}h\left(V^{l_j}Y-Z\right)\\&\quad+2^{\sup_i\tau_i-1}\alpha^{l_j}(\Psi_1(Y)-\Psi_1(VY)).\end{aligned} \tag{51}$$

As $l_j \longrightarrow \infty$, one has a contradiction. Then, Z is a fixed point of V. □

5. Multiplication Mappings on $\ell^F_{\tau(.)}$

In this section, we examine the sufficient conditions on $(\ell^F_{\tau(.)})_h$ such that the multiplication mapping defined on it is bounded, isometry, approximable, compact, closed range, invertible, and Fredholm, where $h(Y) = [\sum_{q=0}^{\infty}(1/\tau_q)[\bar{\rho}(Y_q, \bar{0})]^{\tau_q}]^{1/K}$, for every $Y \in \ell^F_{\tau(.)}$.

The space of approximable and compact bounded linear mappings from a Banach space Δ into a Banach space Λ will be marked by $Y(\Delta, \Lambda)$ and $\mathscr{L}_c(\Delta, \Lambda)$, and if $\Delta = \Lambda$, we mark $Y(\Delta)$ and $\mathscr{L}_c(\Delta)$, respectively.

Definition 23. Let $\kappa \in \mathbb{C}^{\mathscr{N}} \cap \ell_\infty$ and $\mathbf{U}_h$ be a pre-quasi-normed (pssf). A mapping $V_\kappa : \mathbf{U}_h \longrightarrow \mathbf{U}_h$ is called multiplication mapping if $V_\kappa Y = \kappa Y = (\kappa_r Y_r)_{r=0}^{\infty} \in \mathbf{U}$, with $Y \in \mathbf{U}$. When $V_\kappa \in \mathscr{L}(\mathbf{U})$, we call it a multiplication mapping generated by κ.

Theorem 24. *Suppose $\kappa \in \mathbb{C}^{\mathscr{N}}$ and $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$; then, $\kappa \in \ell_\infty$, if and only if $V_\kappa \in \mathscr{L}((\ell^F_{\tau(.)})_h)$.*

Proof. Let the conditions be satisfied. Assume $\kappa \in \ell_\infty$; hence, one gets $\varepsilon > 0$ with $|\kappa_r| \le \varepsilon$, with $r \in \mathscr{N}$. If $x \in (\ell^F_{\tau(.)})_h$, since $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, then

$$\begin{aligned} h(V_\kappa Y) = h(\kappa Y) = h\left((\kappa_r Y_r)_{r=0}^{\infty}\right) &= \left[\sum_{r=0}^{\infty}\frac{1}{\tau_r}[\bar{\rho}(\kappa_r Y_r, \bar{0})]^{\tau_r}\right]^{1/K} \\ &\le \sup_r \varepsilon^{\tau_r/K}\left[\sum_{r=0}^{\infty}\frac{1}{\tau_r}[\bar{\rho}(Y_r, \bar{0})]^{\tau_r}\right]^{1/K} = \sup_r \varepsilon^{\tau_r/K} h(Y). \end{aligned} \tag{52}$$

One has $V_\kappa \in \mathscr{L}((\ell^F_{\tau(.)})_h)$. Conversely, assume that $V_\kappa \in \mathscr{L}((\ell^F_{\tau(.)})_h)$. Suppose $\kappa \notin \ell_\infty$, then, for each $j \in \mathscr{N}$, there is $i_j \in \mathscr{N}$ such that $\kappa_{i_j} > j$. Since $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, we have

$$\begin{aligned} h\left(V_\kappa \bar{\mathfrak{b}}_{i_j}\right) = h\left(\kappa \bar{\mathfrak{b}}_{i_j}\right) &= h\left(\left(\kappa_r\left(\bar{\mathfrak{b}}_{i_j}\right)_r\right)_{r=0}^{\infty}\right) \\ &= \left[\sum_{r=0}^{\infty}\frac{1}{\tau_r}\left[\bar{\rho}\left(\kappa_r\left(\bar{\mathfrak{b}}_{i_j}\right)_r, \bar{0}\right)\right]^{\tau_r}\right]^{1/K} = \left[\frac{1}{\tau_{i_j}}\left|\kappa_{i_j}\right|^{\tau_{i_j}}\right]^{1/K} \\ &> \left[\frac{1}{\tau_{i_j}} j^{\tau_{i_j}}\right]^{1/K} h\left(\bar{\mathfrak{b}}_{i_j}\right). \end{aligned} \tag{53}$$

This shows that $V_\kappa \notin \mathscr{L}((\ell^F_{\tau(.)})_h)$. Therefore, $\kappa \in \ell_\infty$. □

Theorem 25. *Pick up$\kappa \in \mathbb{C}^{\mathscr{N}}$ and$(\ell^F_{\tau(.)})_h$as a pre-quasi-normed (pssf). Then, $|\kappa_q| = 1$, for every $q \in \mathscr{N}$, if and only if V_κ is an isometry.*

Proof. If $|\kappa_r| = 1$, for all $r \in \mathscr{N}$, one gets

$$\begin{aligned} h(V_\kappa Y) = h(\kappa Y) = h\left((\kappa_r Y_r)_{r=0}^{\infty}\right) &= \left[\sum_{r=0}^{\infty}\frac{1}{\tau_r}[\bar{\rho}(\kappa_r Y_r, \bar{0})]^{\tau_r}\right]^{1/K} \\ &= \left[\sum_{r=0}^{\infty}\frac{1}{\tau_r}[\bar{\rho}(Y_r, \bar{0})]^{\tau_r}\right]^{1/K} = h(Y), \end{aligned} \tag{54}$$

for all $Y \in (\ell^F_{\tau(.)})_h$. Therefore, V_κ is an isometry. Conversely, Assume that $|\kappa_i| < 1$ for some $i = i_0$. We obtain

$$\begin{aligned} h\left(V_\kappa \bar{\mathfrak{b}}_{i_0}\right) = h\left(\kappa \bar{\mathfrak{b}}_{i_0}\right) &= h\left(\left(\kappa_r\left(\bar{\mathfrak{b}}_{i_0}\right)_r\right)_{r=0}^{\infty}\right) \\ &= \left[\sum_{r=0}^{\infty}\frac{1}{\tau_r}\left[\bar{\rho}\left(\kappa_r\left(e_{i_0}\right)_r, \bar{0}\right)\right]^{\tau_r}\right]^{1/K} \\ &< \left[\sum_{r=0}^{\infty}\frac{1}{\tau_r}\left[\bar{\rho}\left(\left(e_{i_0}\right)_r, \bar{0}\right)\right]^{\tau_r}\right]^{1/K} = h\left(\bar{\mathfrak{b}}_{i_0}\right). \end{aligned} \tag{55}$$

When $|\kappa_{i_0}| > 1$, we can prove that $h(V_\kappa \bar{\mathfrak{b}}_{i_0}) > h(\bar{\mathfrak{b}}_{i_0})$. One gets a contradiction. So, $|\kappa_r| = 1$, with $r \in \mathscr{N}$.

By card (A), we indicate the cardinality of the set A. □

Theorem 26. *Suppose $\kappa \in \mathbb{C}^{\mathscr{N}}$ and $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$. Then, $V_\kappa \in Y((\ell^F_{\tau(.)})_h)$ if and only if $(\kappa_r)_{r=0}^{\infty} \in c_0$.*

Proof. Assume that $V_\kappa \in Y((\ell^F_{\tau(.)})_h)$. One gets $V_\kappa \in \mathscr{L}_c((\ell^F_{\tau(.)})_h)$. To show$(\kappa_r)_{r=0}^{\infty}$belongs to$c_0$, suppose$(\kappa_r)_{r=0}^{\infty} \notin c_0$. One gets $\nu > 0$ such that $\Lambda_\nu = \{r \in \mathscr{N} : |\kappa_r| \ge \nu\}$ verifies card $(\Lambda_\nu) = \infty$. If $z_i \in \Lambda_\nu$, with $i \in \mathscr{N}$, then $\{\bar{\mathfrak{b}}_{z_i} : z_i \in \Lambda_\nu\}$ is an infinite bounded in $(\ell^F_{\tau(.)})_h$. Let

$$\begin{aligned} h\left(V_\kappa \bar{\mathfrak{b}}_{z_i} - V_\kappa \bar{\mathfrak{b}}_{z_j}\right) &= h\left(\kappa \bar{\mathfrak{b}}_{z_i} - \kappa \bar{\mathfrak{b}}_{z_j}\right) \\ &= h\left(\left(\kappa_r\left(\left(\bar{\mathfrak{b}}_{z_i}\right)_r - \left(\bar{\mathfrak{b}}_{z_j}\right)_r\right)\right)_{r=0}^{\infty}\right) \\ &= \left[\sum_{r=0}^{\infty}\frac{1}{\tau_r}\left[\bar{\rho}\left(\kappa_r\left(\left(\bar{\mathfrak{b}}_{z_i}\right)_r - \left(\bar{\mathfrak{b}}_{z_j}\right)_r\right), \bar{0}\right)\right]^{\tau_r}\right]^{1/K} \\ &\ge \inf_r \nu^{\tau_r/K} h\left(\bar{\mathfrak{b}}_{z_i} - \bar{\mathfrak{b}}_{z_j}\right), \end{aligned} \tag{56}$$

for all $z_i, z_j \in \Lambda_\nu$. One obtains $\{\bar{\mathfrak{b}}_{z_i} : z_i \in \Lambda_\nu\} \in \ell_\infty(F)$ which cannot have a convergent subsequence under V_κ, which gives $V_\kappa \notin \mathscr{L}_c((\ell^F_{\tau(.)})_h)$. Then, $V_\kappa \notin Y((\ell^F_{\tau(.)})_h)$; this gives a contradiction. Then, $\lim_{i\longrightarrow\infty}\kappa_i = 0$. On the other hand, if $\lim_{i\longrightarrow\infty}\kappa_i = 0$, hence, for each $\nu > 0$, the set $\Lambda_\nu = \{i \in \mathscr{N} : |\kappa_i| \ge \nu\}$ holds card $(\Lambda_\nu) < \infty$. So, for all $\nu > 0$, the space

$((\ell_{\tau(.)}^F)_h)_{\Lambda_v} = \{Y = (Y_i) \in \mathbb{C}^{\Lambda_v}\}$ is finite dimensional. Then, $V_\kappa|((\ell_{\tau(.)}^F)_h)_{\Lambda_v}$ is a finite rank mapping. For every $i \in \mathcal{N}$, define $\kappa_i \in \mathbb{C}^{\mathcal{N}}$ by

$$(\kappa_i)_j = \begin{cases} \kappa_j, & j \in \Lambda_{1/i}, \\ 0, & \text{otherwise.} \end{cases} \tag{57}$$

Obviously, V_{κ_i} has $\text{rank}(V_{\kappa_i}) < \infty$ as $\dim((\ell_{\tau(.)}^F)_h)_{\Lambda_{1/i}} < \infty$, with $i \in \mathcal{N}$; hence, as $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, we get

$$\begin{aligned} h((V_\kappa - V_{\kappa_i})Y) &= h\left(\left(\left(\kappa_j - (\kappa_i)_j\right)Y_j\right)_{j=0}^{\infty}\right) \\ &= \left[\sum_{j=0}^{\infty}\frac{1}{\tau_j}\left[\bar{\rho}\left(\left(\kappa_j - (\kappa_i)_j\right)Y_j, \bar{0}\right)\right]^{\tau_j}\right]^{1/K} \\ &= \left[\sum_{j=0, j\in\Lambda_{1/i}}^{\infty}\frac{1}{\tau_j}\left[\bar{\rho}\left(\left(\kappa_j - (\kappa_i)_j\right)Y_j, \bar{0}\right)\right]^{\tau_j}\right. \\ &\quad \left. + \sum_{j=0, j\notin\Lambda_{1/i}}^{\infty}\frac{1}{\tau_j}\left[\bar{\rho}\left(\left(\kappa_j - (\kappa_i)_j\right)Y_j, \bar{0}\right)\right]^{\tau_j}\right]^{1/K} \\ &= \left[\sum_{j=0, j\notin\Lambda_{1/i}}^{\infty}\frac{1}{\tau_j}\left[\bar{\rho}\left(\kappa_j Y_j, \bar{0}\right)\right]^{\tau_j}\right]^{1/K} \\ &\le \sup_j\left(\frac{1}{i}\right)^{\tau_j/K}\left[\sum_{j=0, j\notin\Lambda_{1/i}}^{\infty}\frac{1}{\tau_j}\left[\bar{\rho}\left(Y_j, \bar{0}\right)\right]^{\tau_j}\right]^{1/K} < \left(\frac{1}{i}\right)^{\tau_0/K} h(Y). \end{aligned} \tag{58}$$

This implies that $\|V_\kappa - V_{\kappa_i}\| \le (1/i)^{\tau_0/K}$; hence, V_κ is an approximable mapping. □

Theorem 27. *If $\kappa \in \mathbb{C}^{\mathcal{N}}$ and $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, then, $V_\kappa \in \mathscr{L}_c((\ell_{\tau(.)}^F)_h)$, if and only if $(\kappa_i)_{i=0}^{\infty} \in c_0$.*

Proof. Easy. □

Corollary 28. *Assume $\kappa \in \mathbb{C}^{\mathcal{N}}$ and $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$; then, $\mathscr{L}_c((\ell_{\tau(.)}^F)_h) \subsetneq \mathscr{L}((\ell_{\tau(.)}^F)_h)$.*

Proof. As I is a multiplication mapping on $(\ell_{\tau(.)}^F)_h$ generated by $\kappa = (1, 1, \cdots)$, therefore, $I \notin \mathscr{L}_c((\ell_{\tau(.)}^F)_h)$ and $I \in \mathscr{L}((\ell_{\tau(.)}^F)_h)$. □

Definition 29 (see [32]). A mapping $D \in \mathscr{L}(V)$ is called Fredholm if it satisfies that $\dim(\ker D) < \infty$, $\dim(R(D))^c < \infty$ and D has closed range, where $(R(D))^c$ denotes the complement of the range D.

Theorem 30. *If $\kappa \in \mathbb{C}^{\mathcal{N}}$ and $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, then κ is bounded away from zero on $(\ker(\kappa))^c$, if and only if $R(V_\kappa)$ is closed., where $V_\kappa \in \mathscr{L}((\ell_{\tau(.)}^F)_h)$.*

Proof. Let the sufficient condition be satisfied. Therefore, there is $\varepsilon > 0$ with $|\kappa_i| \ge \varepsilon$, for all $i \in (\ker(\kappa))^c$. To explain$R(V_\kappa)$is closed, assume$Z$is a limit point of$R(V_\kappa)$. One has $V_\kappa X_i$ in $(\ell_{\tau(.)}^F)_h$, with $i \in \mathcal{N}$ such that $\lim_{i\longrightarrow\infty} V_\kappa X_i = Z$. Clearly, $(V_\kappa X_i)$ is a Cauchy sequence. Since $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, one has

$$\begin{aligned} h(V_\kappa X_i - V_\kappa X_j) &= \left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(\kappa_q(X_i)_q - \kappa_q(X_j)_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} \\ &= \left[\sum_{q=0, q\in(\ker(\kappa))^c}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(\kappa_q(X_i)_q - \kappa_q(X_j)_q, \bar{0}\right)\right]^{\tau_q}\right. \\ &\quad \left. + \sum_{q=0, q\notin(\ker(\kappa))^c}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(\kappa_q(X_i)_q - \kappa_q(X_j)_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} \\ &\ge \left[\sum_{q=0, q\in(\ker(\kappa))^c}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(\kappa_q(X_i)_q - \kappa_q(X_j)_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} \\ &= \left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(\kappa_q(Y_i)_q - \kappa_q(Y_j)_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} \\ &> \inf_q \varepsilon^{\frac{\tau_q}{K}}\left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left((Y_i)_q - (Y_j)_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K}, \end{aligned} \tag{59}$$

where

$$(Y_i)_q = \begin{cases} (X_i)_q, & q \in (\ker(\kappa))^c, \\ 0, & q \notin (\ker(\kappa))^c. \end{cases} \tag{60}$$

So (Y_i) is a Cauchy sequence in $(\ell_{\tau(.)}^F)_h$. Since $(\ell_{\tau(.)}^F)_h$ is complete, one has $X \in (\ell_{\tau(.)}^F)_h$ such that $\lim_{i\longrightarrow\infty} Y_i = X$. Since V_κ is continuous, $\lim_{i\longrightarrow\infty} V_\kappa Y_i = V_\kappa X$. But $\lim_{i\longrightarrow\infty} V_\kappa X_i = \lim_{i\longrightarrow\infty} V_\kappa Y_i = Z$; hence, $V_\kappa X = Z$. So, $Z \in R(V_\kappa)$. One obtains$R(V_\kappa)$which is closed. On the other hand, assume $R(V_\kappa)$ is closed; hence, V_κ is bounded away from zero on $((\ell_{\tau(.)}^F)_h)_{(\ker(\kappa))^c}$. One gets $\varepsilon > 0$ such that $h(V_\kappa X) \ge \varepsilon h(X)$, with $X \in ((\ell_{\tau(.)}^F)_h)_{(\ker(\kappa))^c}$.

Assume $\Lambda = \{q \in (\ker(\kappa))^c : |\kappa_q| < \varepsilon\}$. If $\Lambda \ne \phi$, then for $i_0 \in \Lambda$, we obtain

$$\begin{aligned} h(V_\kappa \bar{\mathfrak{b}}_{i_0}) &= h\left(\left(\kappa_q(\bar{\mathfrak{b}}_{i_0})_q\right)_{q=0}^{\infty}\right) = \left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(\kappa_q(\bar{\mathfrak{b}}_{n_0})_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} \\ &< \left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left[\bar{\rho}\left(\varepsilon(\bar{\mathfrak{b}}_{n_0})_q, \bar{0}\right)\right]^{\tau_q}\right]^{1/K} \le \sup_q \varepsilon^{\tau_q/K} h(\bar{\mathfrak{b}}_{n_0}). \end{aligned} \tag{61}$$

One gets a contradiction. Then, $\Lambda = \phi$ with $|\kappa_q| \geq \varepsilon$, and $q \in (\ker(\kappa))^c$. □

Theorem 31. *If $\kappa \in \mathbb{C}^{\mathcal{N}}$ and and $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, one has $d > 0$ and $D > 0$ with $d < |\kappa_q| < D$, and $q \in \mathcal{N}$, if and only if $V_\kappa \in \mathscr{L}((\ell^F_{\tau(.)})_h)$ is invertible.*

Proof. Let the conditions be verified. Define $\gamma \in \mathbb{C}^{\mathcal{N}}$ by $\gamma_q = 1/\kappa_q$. From Theorem 7.5, we have $V_\kappa, V_\gamma \in \mathscr{L}((\ell^F_{\tau(.)})_h)$ and $V_\kappa . V_\gamma = V_\gamma . V_\kappa = I$; hence, V_γ is the inverse of V_κ. On the other side, assume V_κ is invertible. Hence, $R(V_\kappa) = ((\ell^F_{\tau(.)})_h)_{\mathcal{N}}$. One gets, $R(V_\kappa)$ which is closed. By Theorem 30, we have $d > 0$ with $|\kappa_q| \geq d$, and $q \in (\ker(\kappa))^c$. We have $\ker(\kappa) = \phi$, if $\kappa_{q_0} = 0$; for several $q_0 \in \mathcal{N}$, one has $e_{q_0} \in \ker(V_\kappa)$, which is a contradiction, as $\ker(V_\kappa)$ is trivial. Therefore, $|\kappa_q| \geq d$, for all $q \in \mathcal{N}$. As V_κ is bounded, by Theorem 7.5, one gets $D > 0$ such that $|\kappa_q| \leq D$, for all $q \in \mathcal{N}$. □

Theorem 32. *If $\kappa \in \mathbb{C}^{\mathcal{N}}$ and $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, then $V_\kappa \in \mathscr{L}((\ell^F_{\tau(.)})_h)$ is Fredholm mapping, if and only if (a) card$(\ker(\kappa)) < \infty$ and (b) $|\kappa_q| \geq \varepsilon$, with $q \in (\ker(\kappa))^c$.*

Proof. Assume V_κ is Fredholm. When card$(\ker(\kappa)) = \infty$, one has $\bar{\mathfrak{b}}_n \in \ker(V_\kappa)$, with $n \in \ker(\kappa)$. As $\bar{\mathfrak{b}}_n$'s are linearly independent, one obtains card$(\ker(V_\kappa) = \infty$, which implies a contradiction. Hence, card$(\ker(\kappa)) < \infty$. In view of Theorem 30, the condition (b) is confirmed. After, when the necessary conditions hold, prove that V_κ is Fredholm. According to Theorem 30, the condition (b) proves that $R(V_\kappa)$ is closed. The condition (a) gives that $\dim(\ker(V_\kappa)) < \infty$ and $\dim((R(V_\kappa))^c) < \infty$. So, V_κ is Fredholm. □

6. Mappings' Ideal

The structure of the mappings' ideal by $(\ell^F_{\tau(.)})_h$, where $h(g) = [\sum_{m=0}^{\infty}(1/\tau_m)(\bar{\rho}(g_m, \bar{0}))^{\tau_m}]^{1/K}$, for every $g \in \ell^F_{\tau(.)}$, and extended s-fuzzy functions has been explained. We study the enough setups on $(\ell^F_{\tau(.)})_h$ such that the class $\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}$ is complete and closed. We investigate the enough setups (not necessary) on $(\ell^F_{\tau(.)})_h$ such that the closure of $\mathfrak{F} = \bar{\mathfrak{F}}^{\alpha}_{(\ell^F_{\tau(.)})_h}$. This gives a negative answer of Rhoades [33] open problem about the linearity of s-type $(\ell^F_{\tau(.)})_h$ spaces. We explain the enough setups on $(\ell^F_{\tau(.)})_h$ such that $\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}$ is strictly contained for different powers, $\bar{\mathfrak{F}}^{\alpha}_{(\ell^F_{\tau(.)})_h}$ is minimum, the class $\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}$ is simple, and the space of every bounded linear mappings in which the sequence of eigenvalues in $(\ell^F_{\tau(.)})_h$ equals $\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}$.

We indicate the space of all bounded, finite rank linear mappings from an infinite dimensional Banach space Δ into an infinite dimensional Banach space Λ by $\mathscr{L}(\Delta, \Lambda)$ and $\mathfrak{F}(\Delta, \Lambda)$, and when $\Delta = \Lambda$, we inscribe $\mathscr{L}(\Delta)$ and $\mathfrak{F}(\Delta)$.

Definition 33. (see [34]). An s-number function is $s : \mathscr{L}(\Delta, \Lambda) \longrightarrow \mathfrak{R}^{+\mathcal{N}}$ which sorts every $V \in \mathscr{L}(\Delta, \Lambda)$ and $(s_d(V))_{d=0}^{\infty}$ which verifies the following settings:

(a) $\|V\| = s_0(V) \geq s_1(V) \geq s_2(V) \geq \cdots \geq 0$, for all $V \in \mathscr{L}(\Delta, \Lambda)$

(b) $s_{l+d-1}(V_1 + V_2) \leq s_l(V_1) + s_d(V_2)$, for all $V_1, V_2 \in \mathscr{L}(\Delta, \Lambda)$ and $l, d \in \mathcal{N}$

(c) $s_d(VYW) \leq \|V\| s_d(Y) \|W\|$, for all $W \in \mathscr{L}(\Delta_0, \Delta)$, $Y \in \mathscr{L}(\Delta, \Lambda)$ and $V \in \mathscr{L}(\Lambda, \Lambda_0)$, where Δ_0 and Λ_0 are arbitrary Banach spaces

(d) When $V \in \mathscr{L}(\Delta, \Lambda)$ and $\gamma \in \mathfrak{R}$, then $s_d(\gamma V) = |\gamma| s_d(V)$

(e) Suppose rank $(V) \leq d$; then, $s_d(V) = 0$, for each $V \in \mathscr{L}(\Delta, \Lambda)$

(f) $s_{l\geq q}(I_q) = 0$ or $s_{l<q}(I_q) = 1$, where I_q denotes the unit map on the q-dimensional Hilbert space ℓ_2^q

We mention here some examples of s-numbers:

(1) The qth Kolmogorov number, described by $d_q(X)$, is marked by

$$d_q(X) = \inf_{\dim J \leq q} \sup_{\|f\| \leq 1} \inf_{g \in J} \|Xf - g\| \tag{62}$$

(2) The qth approximation number, described by $\alpha_q(X)$, is marked by

$$\alpha_q(X) = \inf \{\|X - Y\| : Y \in \mathscr{L}(\Delta, \Lambda), \text{rank } (Y) \leq q\} \tag{63}$$

Definition 34 (see [17]). Assume $\mathscr{L}$ is the class of all bounded linear mappings within any two arbitrary Banach spaces. A subclass $\mathscr{U}$ of $\mathscr{L}$ is said to be a mappings' ideal, if all $\mathscr{U}(\Delta, \Lambda) = \mathscr{U} \cap \mathscr{L}(\Delta, \Lambda)$ verifies the following conditions:

(i) $I_\Gamma \in \mathscr{U}$, where Γ marks the Banach space of one dimension

(ii) The space $\mathscr{U}(\Delta, \Lambda)$ is linear over $\mathfrak{R}$

(iii) If $W \in \mathscr{L}(\Delta_0, \Delta)$, $X \in \mathscr{U}(\Delta, \Lambda)$ and $Y \in \mathscr{L}(\Lambda, \Lambda_0)$ then, $YXW \in \mathscr{U}(\Delta_0, \Lambda_0)$

Notations 1.

$$\begin{aligned}
&\bar{\mathfrak{F}}_{\mathbf{U}} := \{\bar{\mathfrak{F}}_{\mathbf{U}}(\Delta, \Lambda)\}, \text{ where } \bar{\mathfrak{F}}_{\mathbf{U}}(\Delta, \Lambda) := \left\{V \in \mathscr{L}(\Delta, \Lambda) : \left((s_j(\bar{V}))_{j=0}^{\infty} \in \mathbf{U}\right\},\right.\\
&\bar{\mathfrak{F}}^{\alpha}_{\mathbf{U}} := \{\bar{\mathfrak{F}}^{\alpha}_{\mathbf{U}}(\Delta, \Lambda)\}, \text{ where } \bar{\mathfrak{F}}^{\alpha}_{\mathbf{U}}(\Delta, \Lambda) := \left\{V \in \mathscr{L}(\Delta, \Lambda) : \left((\alpha_j(\bar{V}))_{j=0}^{\infty} \in \mathbf{U}\right\},\right.\\
&\bar{\mathfrak{F}}^{d}_{\mathbf{U}} := \left\{\bar{\mathfrak{F}}^{d}_{\mathbf{U}}(\Delta, \Lambda)\right\}, \text{ where } \bar{\mathfrak{F}}^{d}_{\mathbf{U}}(\Delta, \Lambda) := \left\{V \in \mathscr{L}(\Delta, \Lambda) : \left((d_j(\bar{V}))_{j=0}^{\infty} \in \mathbf{U}\right\},\right.
\end{aligned} \tag{64}$$

where

$$s_j\bar{(V)}(x) = \begin{cases} 1, & x = s_j(V), \\ 0, & x \neq s_j(V). \end{cases} \quad (65)$$

Theorem 35. *Suppose* $\mathbf{U}$ *is a (pssf); then,* $\bar{\mathrm{F}}_U$ *is a mappings' ideal.*

Proof.

(i) Assume $V \in \mathfrak{F}(\Delta, \Lambda)$ and rank $(V) = n$ with $n \in \mathcal{N}$, as $\bar{\mathfrak{b}}_i \in \mathbf{U}$ for all $i \in \mathcal{N}$ and $\mathbf{U}$ is a linear space, one has- $(s_i\bar{(V)})_{i=0}^{\infty} = (s_0\bar{(V)}, s_1\bar{(V)}, \cdots, s_{n1}\bar{(V)}, \bar{0}, \bar{0}, \bar{0}, \cdots) = \sum_{i=0}^{n-1} s_i\bar{(V)}\bar{\mathfrak{b}}_i \in \mathbf{U}$, for $V \in \bar{\mathrm{F}}_{\mathbf{U}}(\Delta, \Lambda)$, then $\mathfrak{F}(\Delta, \Lambda) \subseteq \bar{\mathrm{F}}_E(\Delta, \Lambda)$

(ii) Suppose $V_1, V_2 \in \bar{\mathrm{F}}_{\mathbf{U}}(\Delta, \Lambda)$ and $\beta_1, \beta_2 \in \mathfrak{R}$; then, by Definition 6 condition (78), one has $(s_{[i/2]}\bar{(V_1)})_{i=0}^{\infty} \in \mathbf{U}$ and $(s_{[i/2]}\bar{(V_1)})_{i=0}^{\infty} \in \mathbf{U}$, as $i \geq 2[i/2]$; by the definition of s-numbers and $s_i(P)$ which is a decreasing sequence, we have- $s_i(\beta_1 V_1 \bar{+} \beta_2 V_2) \leq s_{2[i/2]}(\beta_1 \bar{V_1} + \beta_2 V_2) \leq s_{[i/2]}(\beta_1 V_1) \bar{+} s_{[i/2]}(\beta_2 V_2) = |\beta_1| s_{[i/2]}\bar{(V_1)} + |\beta_2| s_{[i/2]}\bar{(V_2)}$ for each $i \in \mathcal{N}$. In view of Definition 6 condition (77) and $\mathbf{U}$ whcih is a linear space, one obtains $(s_i(\beta_1 V_1 \bar{+} \beta_2 V_2))_{i=0}^{\infty} \in \mathbf{U}$; hence, $\beta_1 V_1 + \beta_2 V_2 \in \bar{\mathrm{F}}_{\mathbf{U}}(\Delta, \Lambda)$

(iii) Suppose $P \in \mathscr{L}(\Delta_0, \Delta)$, $T \in \bar{\mathrm{F}}_{\mathbf{U}}(\Delta, \Lambda)$, and $R \in \mathscr{L}(\Lambda, \Lambda_0)$; one has $(s_i\bar{(T)})_{i=0}^{\infty} \in \mathbf{U}$ and as $s_i(\bar{\mathrm{RTP}}) \leq \|R\| s_i\bar{(T)}\|P\|$; by Definition 6 conditions (41) and (77), one gets $(s_i(\bar{\mathrm{RTP}}))_{i=0}^{\infty} \in \mathbf{U}$, then $\mathrm{RTP} \in \bar{\mathrm{F}}_{\mathbf{U}}(\Delta_0, \Lambda_0)$

According to Theorem 12 and Theorem 35, one concludes the next theorem. □

Theorem 36. *Let* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$*; one has* $\bar{\mathrm{F}}_{(\ell^F_{\tau(.)})_h}$ *which is a mappings' ideal.*

Definition 37 (see [35]). A function $H \in [0,\infty)^{\mathcal{U}}$ is called a pre-quasi-norm on the ideal $\mathcal{U}$ if the next conditions hold:

(1) Let $V \in \mathcal{U}(\Delta, \Lambda)$, $H(V) \geq 0$ and $H(V) = 0$, if and only if $V = 0$

(2) We have $Q \geq 1$ so as to $H(\alpha V) \leq D|\alpha| H(V)$, for every $V \in \mathcal{U}(\Delta, \Lambda)$ and $\alpha \in \mathfrak{R}$

(3) We have $\mathrm{P} \geq 1$ so that $H(V_1 + V_2) \leq P[H(V_1) + H(V_2)]$, for each $V_1, V_2 \in \mathcal{U}(\Delta, \Lambda)$

(4) We have $\sigma \geq 1$, when $V \in \mathscr{L}(\Delta_0, \Delta)$, $X \in \mathcal{U}(\Delta, \Lambda)$ and $Y \in \mathscr{L}(\Lambda, \Lambda_0)$ then $H(YXV) \leq \sigma \|Y\| H(X) \|V\|$

Theorem 38 (see [36]). *Every quasinorm on the ideal* $\mathcal{U}$ *is a pre-quasi-norm on the same ideal.*

Theorem 39. *If* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$*, then* H *is a pre-quasi-norm on* $\bar{\mathrm{F}}_{(\ell^F_{\tau(.)})_h}$*, with* $H(Z) = h(s_q\bar{(Z)})_{q=0}^{\infty}$*, for all* $Z \in \bar{\mathrm{F}}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$.

Proof.

(1) When $X \in \bar{\mathrm{F}}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$, $H(X) = h(s_q\bar{(X)})_{q=0}^{\infty} \geq 0$ and $H(X) = h(s_q\bar{(X)})_{q=0}^{\infty} = 0$, if and only if $s_q\bar{(X)} = \bar{0}$, for all $q \in \mathcal{N}$, if and only if $X = 0$

(2) There is $Q \geq 1$ with $H(\alpha X) = h(s_q(\bar{\alpha X}))_{q=0}^{\infty} \leq Q|\alpha| H(X)$, for all $X \in \bar{\mathrm{F}}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$ and $\alpha \in \mathfrak{R}$

(3) One has $PP_0 \geq 1$ so that for $X_1, X_2 \in \bar{\mathrm{F}}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$, one can see

$$\begin{aligned} H(X_1 + X_2) &= h\left(s_q(X_1 \bar{+} X_2)\right)_{q=0}^{\infty} \\ &\leq P\left(h\left(s_{[q/2]}\bar{(X_1)}\right)_{q=0}^{\infty} + h\left(s_{[q/2]}\bar{(X_2)}\right)_{q=0}^{\infty}\right) \\ &\leq PP_0\left(h\left(s_q(\bar{X}_1)\right)_{q=0}^{\infty} + h\left(s_q(\bar{X}_2)\right)_{q=0}^{\infty}\right) \end{aligned} \quad (66)$$

(4) We have $\rho \geq 1$; if $X \in \mathscr{L}(\Delta_0, \Delta)$, $Y \in \bar{\mathrm{F}}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$, and $Z \in \mathscr{L}(\Lambda, \Lambda_0)$, then $H(ZYX) = h(s_q(\bar{ZYX}))_{q=0}^{\infty} \leq h(\|X\|\|Z\| s_q\bar{(Y)})_{q=0}^{\infty} \leq \rho \|X\| H(Y) \|Z\|$.

In the next theorems, we will use the notation $(\bar{\mathrm{F}}_{(\ell^F_{\tau(.)})_h}, H)$, where $H(V) = h((s_q\bar{(V)})_{q=0}^{\infty})$, for all $V \in \bar{\mathrm{F}}_{(\ell^F_{\tau(.)})_h}$. □

Theorem 40. *Suppose* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$*; one has* $(\bar{\mathrm{F}}_{(\ell^F_{\tau(.)})_h}, H)$ *which is a pre-quasi-Banach ideal.*

Proof. Suppose $(V_a)_{a\in\mathcal{N}}$ is a Cauchy sequence in $\bar{\mathrm{F}}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$. Since $\mathscr{L}(\Delta, \Lambda) \supseteq S_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$, one has

$$\begin{aligned} H(V_r - V_a) &= h\left(\left(s_q(\bar{V_r V_a})\right)_{q=0}^{\infty}\right) \geq h\left(s_0(\bar{V_r}\mathrm{V}_a), \bar{0}, \bar{0}, \bar{0}, \cdots\right) \\ &= \frac{1}{\tau_0}\|V_r - V_a\|^{\tau_0}. \end{aligned} \quad (67)$$

Hence, $(V_a)_{a\in\mathcal{N}}$ is a Cauchy sequence in $\mathscr{L}(\Delta, \Lambda)$. As $\mathscr{L}(\Delta, \Lambda)$ is a Banach space, so there exists $V \in \mathscr{L}(\Delta, \Lambda)$ so that $\lim_{a\to\infty}\|V_a - V\| = 0$ and since $(s_q(\bar{V}_a))_{q=0}^{\infty} \in (\ell^F_{\tau(.)})_h$, for all $a \in \mathcal{N}$, and $(\ell^F_{\tau(.)})_h$ is a premodular (pssf); hence, one can see

$$\begin{aligned} H(V) &= h\left(\left(s_q(\bar{V}\infty)\right)_{q=0}^{\infty}\right) \leq h\left(\left(s_{[q/2]}(\bar{V}V_a)\right)_{q=0}^{\infty}\right) \\ &+ h\left(\left(s_{[q/2]}(\bar{V}_a)_{q=0}^{\infty}\right)\right) \leq h\left((\|V_a - V\|\bar{1})_{q=0}^{\infty}\right) \\ &+ (2)^{1/K} h\left(\left(s_q(\bar{V}_a)\right)_{q=0}^{\infty}\right) < \varepsilon. \end{aligned} \tag{68}$$

We obtain $(s_q(\bar{V}))_{q=0}^{\infty} \in (\ell_{\tau(.)}^F)_h$; hence, $V \in \bar{\mathrm{F}}_{(\ell_{\tau(.)}^F)_h}(\Delta, \Lambda)$. □

Theorem 41. *If* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$*, one has (* $\bar{\mathrm{F}}_{(\ell_{\tau(.)}^F)_h}, H)$ *which is a pre-quasi-closed ideal.*

Proof. Suppose $V_a \in \bar{\mathrm{F}}_{(\ell_{\tau(.)}^F)_h}(\Delta, \Lambda)$, for all $a \in \mathcal{N}$ and $\lim_{a\longrightarrow\infty} H(V_a - V) = 0$. Therefore, there is $\varsigma > 0$ and as $\mathscr{L}(\Delta, \Lambda) \supseteq S_{(\ell_{\tau(.)}^F)_h}(\Delta, \Lambda)$, one has

$$\begin{aligned} H(V_a - V) &= h\left(\left(s_q(\bar{V}_a V)\right)_{q=0}^{\infty}\right) \geq h\left(s_0(\bar{V}_a V), \bar{0}, \bar{0}, \bar{0}, \cdots\right) \\ &= \frac{1}{\tau_0}\|V_r - V_a\|^{\tau_0}. \end{aligned} \tag{69}$$

So $(V_a)_{a\in\mathcal{N}}$ is convergent in $\mathscr{L}(\Delta, \Lambda)$, i.e., $\lim_{a\longrightarrow\infty}\|V_a - V\| = 0$, and since $(s_q(\bar{V}_a))_{q=0}^{\infty} \in (\ell_{\tau(.)}^F)_h$, for all $q \in \mathcal{N}$, and $(\ell_{\tau(.)}^F)_h$ is a premodular (pssf), hence, one can see

$$\begin{aligned} H(V) &= h\left(\left(s_q(\bar{V})\right)_{q=0}^{\infty}\right) \leq h\left(\left(s_{[q/2]}(\bar{V}V_a)\right)_{q=0}^{\infty}\right) \\ &+ h\left(\left(s_{[q/2]}(\bar{V}_a)_{q=0}^{\infty}\right)\right) \leq h\left((\|V_a - V\|\bar{1})_{q=0}^{\infty}\right) \\ &+ (2)^{1/K} h\left(\left(s_q(\bar{V}_a)\right)_{q=0}^{\infty}\right) < \varepsilon. \end{aligned} \tag{70}$$

We obtain $(s_q(\bar{V}))_{q=0}^{\infty} \in (\ell_{\tau(.)}^F)_h$; hence, $V \in \bar{\mathrm{F}}_{(\ell_{\tau(.)}^F)_h}(\Delta, \Lambda)$. □

Definition 42. A pre-quasi-norm H on the ideal $\bar{\mathrm{F}}_{U_h}$ verifies the Fatou property if for every $\{T_q\}_{q\in\mathcal{N}} \subseteq \bar{\mathrm{F}}_{U_h}(\Delta, \Lambda)$ so that $\lim_{q\longrightarrow\infty} H(T_q - T) = 0$ and $M \in \bar{\mathrm{F}}_{U_h}(\Delta, \Lambda)$, one gets

$$H(M - T) \leq \sup_q \inf_{j\geq q} H(M - T_j). \tag{71}$$

Theorem 43. *Suppose* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$*; then,* $(\bar{\mathrm{F}}_{(\ell_{\tau(.)}^F)_h}, H)$ *does not satisfy the Fatou property.*

Proof. If $\{T_q\}_{q\in\mathcal{N}} \subseteq \bar{\mathrm{F}}_{(\ell_{\tau(.)}^F)_h}(\Delta, \Lambda)$ with $\lim_{q\longrightarrow\infty} H(T_q - T) = 0$, since $\bar{\mathrm{F}}_{(\ell_{\tau(.)}^F)_h}$ is a pre-quasi-closed ideal, hence, $T \in \bar{\mathrm{F}}_{(\ell_{\tau(.)}^F)_h}(\Delta, \Lambda)$. So with $M \in \bar{\mathrm{F}}_{(\ell_{\tau(.)}^F)_h}(\Delta, \Lambda)$, one has

$$\begin{aligned} H(M - T) &= \left[\sum_{q=0}^{\infty} \frac{1}{\tau_q}\left(\bar{\rho}\left(s_q(\bar{M}T), \bar{0}\right)\right)^{\tau_q}\right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty} \frac{1}{\tau_q}\left(\bar{\rho}\left(s_{[q/2]}(\bar{M}T_i), \bar{0}\right)\right)^{\tau_q}\right]^{1/K} \\ &+ \left[\sum_{q=0}^{\infty} \frac{1}{\tau_q}\left(\bar{\rho}\left(s_{[q/2]}(\bar{T}_i T), \bar{0}\right)\right)^{\tau_q}\right]^{1/K} \\ &\leq (2)^{\frac{1}{K}} \sup_r \inf_{j\geq r}\left[\sum_{q=0}^{\infty} \frac{1}{\tau_q}\left(\bar{\rho}\left(s_q(\bar{M}T_j), \bar{0}\right)\right)^{\tau_q}\right]^{1/K}. \end{aligned} \tag{72}$$

□

Definition 44. An operator $V : \bar{\mathrm{F}}_{U_h}(\Delta, \Lambda) \longrightarrow \bar{\mathrm{F}}_{U_h}(\Delta, \Lambda)$ is said to be H-sequentially continuous at M, where $M \in \bar{\mathrm{F}}_{U_h}(\Delta, \Lambda)$, if and only if $\lim_{r\longrightarrow\infty} H(T_r - M) = 0 \Rightarrow \lim_{r\longrightarrow\infty} H(VT_r - VM) = 0$.

Example 14. If $V : \bar{\mathrm{F}}_{(\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h}(\Delta, \Lambda) \longrightarrow \bar{\mathrm{F}}_{(\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h}(\Delta, \Lambda)$, where $H(T) = \sqrt{\sum_{q=0}^{\infty}((q+2)/(2q+3))(\bar{\rho}(s_q(\bar{T}), \bar{0}))^{(2q+3)/(q+2)}}$, for every $T \in \bar{\mathrm{F}}_{(\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h}(\Delta, \Lambda)$ and

$$V(T) = \begin{cases} \dfrac{T}{6}, & H(T) \in [0, 1), \\ \dfrac{T}{7}, & H(T) \in [1, \infty), \end{cases} \tag{73}$$

evidently, V is H-sequentially continuous at the zero operator $\Theta \in \bar{\mathrm{F}}_{(\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h}$. Let $\{T^{(j)}\} \subseteq \bar{\mathrm{F}}_{(\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h}$ be such that $\lim_{j\longrightarrow\infty} H(T^{(j)} - T^{(0)}) = 0$, where $T^{(0)} \in \bar{\mathrm{F}}_{(\ell^F(((2q+3)/(q+2))_{q=0}^{\infty}))_h}$ with $H(T^{(0)}) = 1$. Since the pre-quasi-norm H is continuous, one gets

$$\begin{aligned} \lim_{j\longrightarrow\infty} H\left(VT^{(j)} - VT^{(0)}\right) &= \lim_{j\longrightarrow\infty} H\left(\frac{T^{(0)}}{6} - \frac{T^{(0)}}{7}\right) \\ &= H\left(\frac{T^{(0)}}{42}\right) > 0. \end{aligned} \tag{74}$$

Therefore, V is not H-sequentially continuous at $T^{(0)}$.

Theorem 45. *Pick up* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ *with* $\tau_0 > 0$ *and* $V : \bar{\mathrm{F}}_{(\ell_{\tau(.)}^F)_h}(\Delta, \Lambda) \longrightarrow \bar{\mathrm{F}}_{(\ell_{\tau(.)}^F)_h}(\Delta, \Lambda)$*. Assume*

(i) *there is a function* $\Psi_1 : \bar{\mathrm{F}}_{(\ell_{\tau(.)}^F)_h}(\Delta, \Lambda) \longrightarrow (-\infty, \infty]$ *that holds a proper and h-lower semicontinuous with* $\inf_{G\in\bar{\mathrm{F}}_{(\ell_{\tau(.)}^F)_h}(\Delta,\Lambda)} \Psi_1(G) > -\infty$ *and there is* $\alpha \in [0, 1)$ *so* *that* $H(V^{l+1}G - V^l G) \le \alpha^l(\Psi_1(G) - \Psi_1(V(G)))$, *with* $G \in \bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$

(ii) V is H-sequentially continuous at an element $M \in \bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$

(iii) there are $G \in \bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$ *such that the sequence of iterates* $\{V^r G\}$ *has a* $\{V^{r_m} G\}$ *converging to M*

Then, $M \in \bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$ *is a fixed point of V.*

Proof. Let M be not a fixed point of V; hence, $VM \neq M$. By using parts (ii) and (iii), we get

$$\lim_{r_m \longrightarrow \infty} H(V^{r_m} G - M) = 0 \text{ and } \lim_{r_m \longrightarrow \infty} H\left(V^{r_m+1} G - VM\right) = 0. \tag{75}$$

By using part (i), one obtains

$$\begin{aligned} 0 < H(VM - M) &= H\left(\left(VM - V^{r_m+1}G\right) + \left(V^{r_m}G - M\right) + \left(V^{r_m+1}G - V^{r_m}G\right)\right) \le (2)^{1/K} H\left(V^{r_m+1}G - VM\right) \\ &+ (2)^{2/K} H(V^{r_m}G - M) + (2)^{2/K}\alpha^{r_m}(\Psi_1(G) - \Psi_1(VG)). \end{aligned} \tag{76}$$

As $r_m \longrightarrow \infty$, there is a contradiction. Hence, M is a fixed point of V. □

Theorem 46. $\bar{\mathfrak{F}}^{\alpha}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda) =$ *the closure of* $\mathfrak{F}(\Delta, \Lambda)$, *if* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ *with* $\tau_0 > 0$. *But the converse is not necessarily true.*

Proof. As $\mathfrak{b}_x \in (\ell^F_{\tau(.)})_h$, for every $x \in \mathcal{N}$, and $(\ell^F_{\tau(.)})_h$ is a linear space, suppose $Z \in \mathfrak{F}(\Delta, \Lambda)$; one has $(\alpha_x(\bar{Z}))_{x=0}^{\infty} \in E$. Therefore, the closure of $\mathfrak{F}(\Delta, \Lambda) \subseteq \bar{\mathfrak{F}}^{\alpha}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$. Assume $Z \in \bar{\mathfrak{F}}^{\alpha}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$; we have $(\alpha_x(\bar{Z}))_{x=0}^{\infty} \in (\ell^F_{\tau(.)})_h$. As $h(\alpha_x(\bar{Z}))_{x=0}^{\infty} < \infty$, assume $\rho \in (0,1)$; then, there is $x_0 \in \mathcal{N} - \{0\}$ with $h((\alpha_x(\bar{Z}))_{x=x_0}^{\infty}) < \rho/4$. Since $(\alpha_x(\bar{Z}))_{x=0}^{\infty}$ is decreasing, we have

$$\begin{aligned} \sum_{x=x_0+1}^{2x_0} \frac{1}{\tau_x}\left[\bar{\rho}\left(\alpha_{2x_0}(\bar{Z}), \bar{0}\right)\right]^{\tau_x} &\le \sum_{x=x_0+1}^{2x_0} \frac{1}{\tau_x}\left[\bar{\rho}\left(\alpha_x(\bar{Z}), \bar{0}\right)\right]^{\tau_x} \\ &\le \sum_{x=x_0}^{\infty} \frac{1}{\tau_x}\left[\bar{\rho}\left(\alpha_x(\bar{Z}), \bar{0}\right)\right]^{\tau_x} < \frac{\rho}{4}. \end{aligned} \tag{77}$$

Hence, there is $Y \in \mathfrak{F}_{2x_0}(\Delta, \Lambda)$ so that $\operatorname{rank}(Y) \le 2x_0$ and

$$\sum_{x=2x_0+1}^{3x_0} \frac{1}{\tau_x}\left[\bar{\rho}\left(\|\bar{Z}Y\|, \bar{0}\right)\right]^{\tau_x} \le \sum_{x=x_0+1}^{2x_0} \frac{1}{\tau_x}\left[\bar{\rho}\left(\|\bar{Z}Y\|, \bar{0}\right)\right]^{\tau_x} < \frac{\rho}{4}. \tag{78}$$

Since $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, we can choose

$$\sum_{x=0}^{x_0} \frac{1}{\tau_x}\left[\bar{\rho}\left(\|\bar{Z}Y\|, \bar{0}\right)\right]^{\tau_x} < \frac{\rho}{4}. \tag{79}$$

In view of inequalities (2)-(4), one has

$$\begin{aligned} d(Z, Y) = h\left(\alpha_x(\bar{Z}Y)\right)_{x=0}^{\infty} &= \sum_{x=0}^{3x_0-1} \frac{1}{\tau_x}\left[\bar{\rho}\left(\alpha_x(\bar{Z}Y), \bar{0}\right)\right]^{\tau_x} \\ &+ \sum_{x=3x_0}^{\infty} \frac{1}{\tau_x}\left[\bar{\rho}\left(\alpha_x(\bar{Z}Y), \bar{0}\right)\right]^{\tau_x} \le \sum_{x=0}^{3x_0} \frac{1}{\tau_x}\left[\bar{\rho}\left(\|\bar{Z}Y\|, \bar{0}\right)\right]^{\tau_x} \\ &+ \sum_{x=x_0}^{\infty} \frac{1}{\tau_{x+2x_0}}\left[\bar{\rho}\left(\alpha_{x+2x_0}(\bar{ZY}), \bar{0}\right)\right]^{\tau_{x+2x_0}} \\ &\le \sum_{x=0}^{3x_0} \frac{1}{\tau_x}\left[\bar{\rho}\left(\|\bar{Z}Y\|, \bar{0}\right)\right]^{\tau_x} + \sum_{x=x_0}^{\infty} \frac{1}{\tau_x}\left[\bar{\rho}\left(\alpha_x(\bar{Z}), \bar{0}\right)\right]^{\tau_x} \\ &\le 3\sum_{x=0}^{x_0} \frac{1}{\tau_x}\left[\bar{\rho}\left(\|\bar{Z}Y\|, \bar{0}\right)\right]^{\tau_x} + \sum_{x=x_0}^{\infty} \frac{1}{\tau_x}\left[\bar{\rho}\left(\alpha_x(\bar{Z}), \bar{0}\right)\right]^{\tau_x} < \rho. \end{aligned} \tag{80}$$

Therefore, $\bar{\mathfrak{F}}^{\alpha}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda) \subseteq$ the closure of $\mathfrak{F}(\Delta, \Lambda)$. Contrarily, one has a counter example as $I_6 \in \bar{\mathfrak{F}}^{\alpha}_{(\ell^F((0,0,1,1,1,)))_h}(\Delta, \Lambda)$, but $\eta_0 > 0$ is not verified. □

Theorem 47. *Suppose* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $0 < \tau_x^{(1)} < \tau_x^{(2)}$, *for all* $x \in \mathcal{N}$; *hence,*

$$\bar{\mathfrak{F}}_{\left(\ell^F\left(\left(\tau_x^{(1)}\right)\right)\right)_h}(\Delta, \Lambda) \not\supseteq \bar{\mathfrak{F}}_{\left(\ell^F\left(\left(\tau_x^{(2)}\right)\right)\right)_h}(\Delta, \Lambda) U \mathscr{L}(\Delta, \Lambda). \tag{81}$$

Proof. Let $Z \in \bar{\mathfrak{F}}_{(\ell^F((\tau_x^{(1)})))_h}(\Delta, \Lambda)$; hence, $(s_x(\bar{Z})) \in (\ell^F((\tau_x^{(1)})))_h$. One gets

$$\sum_{x=0}^{\infty} \frac{1}{\tau_x^{(2)}}\left[\bar{\rho}\left(s_x(\bar{Z}), \bar{0}\right)\right]^{\tau_x^{(2)}} < \sum_{x=0}^{\infty} \frac{1}{\tau_x^{(1)}}\left[\bar{\rho}\left(s_x(\bar{Z}), \bar{0}\right)\right]^{\tau_x^{(1)}} < \infty, \tag{82}$$

then $Z \in \bar{\mathfrak{F}}_{(\ell^F((\tau_x^{(2)})))_h}(\Delta, \Lambda)$. After, if we choose $(s_x(\bar{Z}))_{x=0}^{\infty}$ with $\bar{\rho}(s_x(\bar{Z}), \bar{0}) = \sqrt[\tau x1]{\tau_x^{(1)}/(x+1)}$, we have $Z \in \mathscr{L}(\Delta, \Lambda)$ such that

$$\sum_{x=0}^{\infty} \frac{1}{\tau_x^{(1)}}\left[\bar{\rho}\left(s_x(\bar{Z}), \bar{0}\right)\right]^{\tau_x^{(1)}} = \sum_{x=0}^{\infty} \frac{1}{x+1} = \infty, \tag{83}$$

$$\sum_{x=0}^{\infty}\frac{1}{\tau_x^{(2)}}\left[\bar{\rho}\left(s_x(\bar{Z}),\bar{0}\right)\right]^{\tau_x^{(2)}} \le \sum_{x=0}^{\infty}\frac{1}{\tau_x^{(1)}}\left(\frac{\tau_x^{(1)}}{x+1}\right)^{\tau_x^{(2)}/\tau_x^{(1)}} \le \sup_x\left(\tau_x^{(1)}\right)^{\left(\tau_x^{(2)}/\tau_x^{(1)}\right)-1}\sum_{x=0}^{\infty}\left(\frac{1}{x+1}\right)^{\tau_x^{(2)}/\tau_x^{(1)}}<\infty. \tag{84}$$

Then, $Z \notin \bar{\Psi}_{(\ell^F((\tau_x^{(1)})))_h}(\Delta, \Lambda)$ and $Z \in \bar{\Psi}_{(\ell^F((\tau_x^{(2)})))_h}(\Delta, \Lambda)$.

Clearly, $\bar{\Psi}_{(\ell^F((\tau_x^{(2)})))_h}(\Delta, \Lambda) \subset \mathscr{L}(\Delta, \Lambda)$. Next, if we put $(s_x(\bar{Z}))_{x=0}^{\infty}$ such that $\bar{\rho}(s_x(\bar{Z}), \bar{0}) = \sqrt[\tau x2]{\tau_x^{(2)}/(x+1)}$, we have $Z \in \mathscr{L}(\Delta, \Lambda)$ such that $Z \notin \bar{\Psi}_{(\ell^F((\tau_x^{(2)})))_h}(\Delta, \Lambda)$. □

Theorem 48. *Assume $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$; hence, $\bar{\Psi}^{\alpha}_{(\ell^F_{\tau(.)})_h}$ is minimum.*

Proof. Let $\bar{\Psi}^{\alpha}_{\ell^F_{\tau(.)}}(\Delta, \Lambda) = \mathscr{L}(\Delta, \Lambda)$; then, there is $\eta > 0$ with $H(Z) \le \eta\|Z\|$, where $H(Z) = \sum_{q=0}^{\infty}(1/\tau_q)[\bar{\rho}(\alpha_q(\bar{Z}), \bar{0})]^{\tau_q}$, for every $Z \in \mathscr{L}(\Delta, \Lambda)$. By using Dvoretzky's theorem [37], with $r \in \mathcal{N}$, we get quotient spaces Δ/Y_r and subspaces M_r of Λ which can be mapped onto ℓ_2^r by isomorphisms V_r and X_r with $\|V_r\|\|V_r^{-1}\| \le 2$ and $\|X_r\|\|X_r^{-1}\| \le 2$. If I_r is the identity map on ℓ_2^r, T_r is the quotient map from Δ onto Δ/Y_r and J_r is the natural embedding map from M_r into Λ. Assume m_q is the Bernstein numbers [16]; then,

$$\begin{aligned}1 = m_q(I_r) &= m_q\left(X_rX_r^{-1}I_rV_rV_r^{-1}\right) \le \|X_r\|m_q\left(X_r^{-1}I_rV_r\right)\left\|V_r^{-1}\right\| \\ &= \|X_r\|m_q\left(J_rX_r^{-1}I_rV_r\right)\left\|V_r^{-1}\right\| \le \|X_r\|d_q\left(J_rX_r^{-1}I_rV_r\right)\left\|V_r^{-1}\right\| \\ &= \|X_r\|d_q\left(J_rX_r^{-1}I_rV_rT_r\right)\left\|V_r^{-1}\right\| \le \|X_r\|\alpha_q\left(J_rX_r^{-1}I_rV_rT_r\right)\left\|V_r^{-1}\right\|,\end{aligned} \tag{85}$$

for $0 \le x \le r$. Then, we have

$$1 \le \left(\|X_r\|\left\|V_r^{-1}\right\|\right)^{\tau_q}\bar{\rho}\left(\alpha_q\left(J_rX_r^{\bar{1}}I_rV_rT_r\right),\bar{0}\right)^{\tau_q}. \tag{86}$$

So, there are $\rho \ge 1$; we obtain

$$\begin{aligned}\sum_{q=0}^{r}\frac{1}{\tau_q} &\le \rho\|X_r\|\left\|V_r^{-1}\right\|\sum_{q=0}^{r}\frac{1}{\tau_q}\left[\bar{\rho}\left(\alpha_q\left(J_rX_r^{\bar{1}}I_rV_rT_r\right),\bar{0}\right)\right]^{\tau_q} \\ \Rightarrow \sum_{q=0}^{r}\frac{1}{\tau_q} &\le \rho\|X_r\|\left\|V_r^{-1}\right\|H\left(J_rX_r^{-1}I_rV_rT_r\right) \\ \Rightarrow \sum_{q=0}^{r}\frac{1}{\tau_q} &\le \rho\eta\|X_r\|\left\|V_r^{-1}\right\|\left\|J_rX_r^{-1}I_rV_rT_r\right\| \\ \Rightarrow \sum_{q=0}^{r}\frac{1}{\tau_q} &\le \rho\eta\|X_r\|\left\|V_r^{-1}\right\|\left\|J_rX_r^{-1}\right\|\|I_r\|\|V_rT_r\| \\ &= \rho\eta\|X_r\|\left\|V_r^{-1}\right\|\left\|X_r^{-1}\right\|\|I_r\|\|V_r\| \le 4\rho\eta.\end{aligned} \tag{87}$$

So there is a contradiction, if $r \longrightarrow \infty$. Therefore, Δ and Λ both cannot be infinite dimensional if $\bar{\Psi}^{\alpha}_{\ell^F_{\tau(.)}}(\Delta, \Lambda) = \mathscr{L}(\Delta, \Lambda)$.

As with the previous theorem, we can easily prove the next theorem. □

Theorem 49. *Suppose $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$; hence, $\bar{\Psi}^{d}_{\ell^F_{\tau(.)}}$ is minimum.*

Lemma 50 (see [17]). *If $B \in \mathscr{L}(\Delta, \Lambda)$ and $B \notin Y(\Delta, \Lambda)$, then $D \in \mathscr{L}(\Delta)$ and $M \in \mathscr{L}(\Lambda)$ with $MBDe_b = e_b$, with $b \in \mathcal{N}$.*

Theorem 51 (see [17]). *In general, we have*

$$\mathfrak{F}(\Delta) Þ Y(\Delta) Þ \mathscr{L}_c(\Delta) Þ \mathscr{L}(\Delta). \tag{88}$$

Theorem 52. *Let $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $0 < \tau_x^{(1)} < \tau_x^{(2)}$, for all $x \in \mathcal{N}$; hence,*

$$\mathscr{L}\left(\bar{\Psi}_{\left(\ell^F\left(\left(\tau_x^{(2)}\right)\right)\right)_h}(\Delta,\Lambda), \bar{\Psi}_{\left(\ell^F\left(\left(\tau_x^{(1)}\right)\right)\right)_h}(\Delta,\Lambda)\right) = Y\left(\bar{\Psi}_{\left(\ell^F\left(\left(\tau_x^{(2)}\right)\right)\right)_h}(\Delta,\Lambda), \bar{\Psi}_{\left(\ell^F\left(\left(\tau_x^{(1)}\right)\right)\right)_h}(\Delta,\Lambda)\right). \tag{89}$$

Proof. Assume $X \in \mathscr{L}(\bar{\Psi}_{(\ell^F((\tau_x^{(2)})))_h}(\Delta, \Lambda), \bar{\Psi}_{(\ell^F((\tau_x^{(1)})))_h}(\Delta, \Lambda))$ and $X \notin Y(\bar{\Psi}_{(\ell^F((\tau_x^{(2)})))_h}(\Delta, \Lambda), \bar{\Psi}_{(\ell^F((\tau_x^{(1)})))_h}(\Delta, \Lambda))$. By using Lemma 50, we have $Y \in \mathscr{L}(\bar{\Psi}_{(\ell^F((\tau_x^{(2)})))_h}(\Delta, \Lambda))$ and $Z \in \mathscr{L}(\bar{\Psi}_{(\ell^F((\tau_x^{(1)})))_h}(\Delta, \Lambda))$ so that $ZXYI_b = I_b$; hence, with $b \in \mathcal{N}$, one has

$$\|I_b\|_{\bar{\Psi}_{\left(\ell^F\left(\left(\tau_x^{(1)}\right)\right)\right)_h}(\Delta,\Lambda)} = \sum_{x=0}^{\infty}\frac{1}{\tau_x^{(1)}}\left[\bar{\rho}\left(s_x(\bar{I}_b),\bar{0}\right)\right]^{\tau_x^{(1)}} \le \|ZXY\|\|I_b\|_{\bar{\Psi}_{\left(\ell^F\left(\left(\tau_x^{(2)}\right)\right)\right)_h}(\Delta,\Lambda)} \le \sum_{x=0}^{\infty}\frac{1}{\tau_x^{(2)}}\left[\bar{\rho}\left(s_x(\bar{I}_b),\bar{0}\right)\right]^{\tau_x^{(2)}}. \tag{90}$$

This fails Theorem 47. So $X \in Y(\bar{\Psi}_{(\ell^F((\tau_x^{(2)})))_h}(\Delta, \Lambda), \bar{\Psi}_{(\ell^F((\tau_x^{(1)})))_h}(\Delta, \Lambda))$. □

Corollary 53. *Assume $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $0 < \tau_x^{(1)} < \tau_x^{(2)}$, for all $x \in \mathcal{N}$; hence,*

$$\mathscr{L}\left(\bar{\Psi}_{\left(\ell^F\left(\left(\tau_x^{(2)}\right)\right)\right)_h}(\Delta,\Lambda), \bar{\Psi}_{\left(\ell^F\left(\left(\tau_x^{(1)}\right)\right)\right)_h}(\Delta,\Lambda)\right) = \mathscr{L}_c\left(\bar{\Psi}_{\left(\ell^F\left(\left(\tau_x^{(2)}\right)\right)\right)_h}(\Delta,\Lambda), \bar{\Psi}_{\left(\ell^F\left(\left(\tau_x^{(1)}\right)\right)\right)_h}(\Delta,\Lambda)\right). \tag{91}$$

Proof. Evidently, as $Y \subset \mathscr{L}_c$. □

Definition 54 (see [17]). A Banach space Δ is called simple, if there is only one nontrivial closed ideal in $\mathscr{L}(\Delta)$.

Theorem 55. *Let $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 0$; hence, $\bar{\Psi}_{(\ell^F_{\tau(.)})_h}$ is simple.*

Proof. Let $X \in \mathscr{L}_c(\bar{\Psi}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda))$ and $X \notin Y(\bar{\Psi}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda))$. From Lemma 50, there exist $Y, Z \in \mathscr{L}(\bar{\Psi}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda))$ with $ZXYI_b = I_b$, which gives that $I_{\bar{\Psi}_{(\ell^F_{\tau(.)})_h}(\Delta,\Lambda)} \in \mathscr{L}_c(\bar{\Psi}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda))$. Then, $\mathscr{L}(\bar{\Psi}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)) = \mathscr{L}_c(\bar{\Psi}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda))$; hence, $\bar{\Psi}_{(\ell^F_{\tau(.)})_h}$ is a simple Banach space. □

Notations 2.

$$(\bar{\Psi}_{\mathbf{U}})^\lambda := \left\{(\bar{\Psi}_{\mathbf{U}})^\lambda(\Delta, \Lambda)\,;\, \Delta \text{ and } \Lambda \text{ are Banach spaces}\right\},$$
$$\text{where } (\bar{\Psi}_{\mathbf{U}})^\lambda(\Delta, \Lambda) := \left\{X \in \mathscr{L}(\Delta, \Lambda)\colon \left((\lambda_x(X))_{x=0}^\infty \in \mathbf{U}\right.\right.$$
$$\left.\text{and } \|X - \bar{\rho}(\lambda_x(X), \bar{0})I\| \text{ is not invertible, with } x \in \mathcal{N}\right\}. \quad (92)$$

Theorem 56. *Assume $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap I$ with $\tau_0 > 0$; hence,*

$$\left(\bar{\Psi}_{(\ell^F_{\tau(.)})_h}\right)^\lambda(\Delta, \Lambda) = \bar{\Psi}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda). \quad (93)$$

Proof. Suppose $X \in (\bar{\Psi}_{(\ell^F_{\tau(.)})_h})^\lambda(\Delta, \Lambda)$; hence, $(\lambda_x(X))_{x=0}^\infty \in (\ell^F_{\tau(.)})_h$ and $\|X - \bar{\rho}(\lambda_x(X), \bar{0})I\| = 0$, for all $x \in \mathcal{N}$. We have $X = \bar{\rho}(\lambda_x(X), \bar{0})I$, for all $x \in \mathcal{N}$, so

$$\bar{\rho}(s_x(\bar{X}), \bar{0}) = \bar{\rho}\left(s_x(\bar{\rho}(\bar{\bar{\lambda}}_x(X), \bar{0})I), \bar{0}\right) = \bar{\rho}(\lambda_x(X), \bar{0}), \quad (94)$$

for every $x \in \mathcal{N}$. Therefore, $(s_x(\bar{X}))_{x=0}^\infty \in (\ell^F_{\tau(.)})_h$; hence, $X \in \bar{\Psi}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$. Next, suppose $X \in \bar{\Psi}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda)$. Hence, $(s_x(\bar{X}))_{x=0}^\infty \in (\ell^F_{\tau(.)})_h$. One gets $\sum_{x=0}^\infty (1/\tau_x)[\bar{\rho}(s_x(\bar{X}), \bar{0})]^{\tau_x} < \infty$. Then, $\lim_{x\longrightarrow\infty}\bar{\rho}(s_x(\bar{X}), \bar{0}) = 0$. Assume $\|X - \bar{\rho}(s_x(\bar{X}), \bar{0})I\|^{-1}$ exists, with $x \in \mathcal{N}$. Then, $\|X - \bar{\rho}(s_x(\bar{X}), \bar{0})I\|^{-1}$ exists and is bounded, for all $x \in \mathcal{N}$. So, $\lim_{x\longrightarrow\infty}\|X - \bar{\rho}(s_x(\bar{X}), \bar{0})I\|^{-1} = \|X\|^{-1}$ exists and is bounded. As $(\bar{\Psi}_{(\ell^F_{\tau(.)})_h}, H)$ is a pre-quasi-mappings' ideal, we have

$$\begin{aligned} I = XX^{-1} \in \bar{\Psi}_{(\ell^F_{\tau(.)})_h}(\Delta, \Lambda) &\Rightarrow \left(s_x(\bar{I})\right)_{x=0}^\infty \in \ell^F_{\tau(.)} \\ &\Rightarrow \lim_{x\longrightarrow\infty}\bar{\rho}\left(s_x(\bar{I}), \bar{0}\right) = 0. \end{aligned} \quad (95)$$

This gives a contradiction, as $\lim_{x\longrightarrow\infty}\bar{\rho}(s_x(\bar{I}), \bar{0}) = 1$. Therefore, $\|X - \bar{\rho}(s_x(\bar{X}), \bar{0})I\| = 0$, with $x \in \mathcal{N}$. This explains $X \in (\bar{\Psi}_{(\ell^F_{\tau(.)})_h})^\lambda(\Delta, \Lambda)$. □

7. Applications

Consider the summable equations which are presented by many authors [38–40]:

$$Y_q = R_q + \sum_{r=0}^\infty D(q, r)m(r, Y_r), \quad (96)$$

where $D : \mathcal{N}^2 \longrightarrow \mathfrak{R}$, $m : \mathcal{N} \times \mathfrak{R}[0, 1] \longrightarrow \mathfrak{R}[0, 1]$, $R : \mathcal{N} \longrightarrow \mathfrak{R}[0, 1]$, and assume $V : (\ell^F_{\tau(.)})_h \longrightarrow (\ell^F_{\tau(.)})_h$, where $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$ and $h(Y) = [\sum_{j=0}^\infty (1/\tau_j)(\bar{\rho}(Y_j, \bar{0}))^{\tau_j}]^{1/K}$, for every $Y \in \ell^F(\tau)$, defined by

$$V(Y_q)_{q\in\mathcal{N}} = \left(R_q + \sum_{r=0}^\infty D(q, r)m(r, Y_r)\right)_{q\in\mathcal{N}}. \quad (97)$$

Example 15. The summable equation (96) has a solution in $(\ell^F_{\tau(.)})_h$, if

$$\begin{aligned} &K\left[\sum_{q=0}^\infty \frac{1}{\tau_q}\left(\bar{\rho}\left(R_q - Y_q + \sum_{r=0}^\infty D(q, r)m(r, Y_r), \bar{0}\right)\right)^{\tau_q}\right]^{1/K} \\ &\quad \le \ln \frac{\sum_{q=0}^\infty (1/\tau_q)\left(\bar{\rho}\left(R_q + \sum_{r=0}^\infty D(q, r)m(r, Y_r), \bar{0}\right)\right)^{\tau_q}}{\sum_{q=0}^\infty (1/\tau_q)\left(\bar{\rho}\left(Y_q, \bar{0}\right)\right)^{\tau_q}}. \end{aligned} \quad (98)$$

Evidently, we have

$$\begin{aligned} h(VY - Y) &= \left[\sum_{q\in\mathcal{N}} \frac{1}{\tau_q}\left(\bar{\rho}\left(VY_q - Y_q, \bar{0}\right)\right)^{\tau_q}\right]^{1/K} \\ &= \left[\sum_{q=0}^\infty \frac{1}{\tau_q}\left(\bar{\rho}\left(R_q - Y_q + \sum_{r=0}^\infty D(q, r)m(r, Y_r), \bar{0}\right)\right)^{\tau_q}\right]^{1/K} \\ &\le \frac{1}{K}\ln \frac{\sum_{q=0}^\infty (1/\tau_q)\left(\bar{\rho}\left(R_q + \sum_{r=0}^\infty \mathrm{D}(q, r)m(r, Y_r), \bar{0}\right)\right)^{\tau_q}}{\sum_{q=0}^\infty (1/\tau_q)\left(\bar{\rho}\left(Y_q, \bar{0}\right)\right)^{\tau_q}} \\ &= \ln(h(VY)) - \ln(h(Y)). \end{aligned} \quad (99)$$

By Theorem 20, one gets a solution of equation (96) in $(\ell^F_{\tau(.)})_h$.

Example 16. The summable equation (96) has a solution in $(\ell^F_{\tau(.)})_h$, if

$$\begin{aligned} &\left[\sum_{q=0}^\infty \frac{1}{\tau_q}\left(\bar{\rho}\left(R_q - Y_q + \sum_{r=0}^\infty D(q, r)m(r, Y_r), \bar{0}\right)\right)^{\tau_q}\right]^{1/K} \\ &\le \left[\sum_{q=0}^\infty \frac{1}{\tau_q}\left(\bar{\rho}\left(Y_q, \bar{0}\right)\right)^{\tau_q}\right]^{1/K} - \left[\sum_{q=0}^\infty \frac{1}{\tau_q}\left(\bar{\rho}\left(R_q + \sum_{r=0}^\infty D(q, r)m(r, Y_r), \bar{0}\right)\right)^{\tau_q}\right]^{1/K}. \end{aligned} \quad (100)$$

Clearly, we have

$$h(VY-Y)=\left[\sum_{q\in\mathcal{N}}\frac{1}{\tau_q}\left(\bar{\rho}\left(VY_q-Y_q,\bar{0}\right)\right)^{\tau_q}\right]^{1/K}$$
$$=\left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left(\bar{\rho}\left(R_q-Y_q+\sum_{r=0}^{\infty}D(q,r)m(r,Y_r),\bar{0}\right)\right)^{\tau_q}\right]^{1/K}$$
$$\leq\left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left(\bar{\rho}\left(Y_q,\bar{0}\right)\right)^{\tau_q}\right]^{1/K}$$
$$-\left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left(\bar{\rho}\left(R_q+\sum_{r=0}^{\infty}D(q,r)m(r,Y_r),\bar{0}\right)\right)^{\tau_q}\right]^{1/K}$$
$$=h(Y)-h(VY). \tag{101}$$

By Theorem 20, one gets a solution of equation (96) in $(\ell^F_{\tau(.)})_h$.

We conclude the following two applications in view of Theorem 22.

Example 17. The summable equation (96) has a solution in $(\ell^F_{\tau(.)})_h$, where $h(Y)=\sum_{q=0}^{\infty}(1/\tau_q)[\bar{\rho}(Y_q,\bar{0})]^{\tau_q}$, for all $Y\in\ell^F(\tau)$, if

(1) $h(V^{l+1}G-V^lG)\leq\alpha^l\ln h(V(G))/h(G)$, with $G\in(\ell^F_{\tau(.)})_h$

(2) V is h-sequentially continuous at $Z\in(\ell^F_{\tau(.)})_h$

(3) there is $Y\in(\ell^F_{\tau(.)})_h$ with $\{V^lY\}$ which has $\{V^{l_j}Y\}$ converging to Z

Then, $Z\in(\ell^F_{\tau(.)})_h$ is a fixed point of V.

Example 18. The summable equation (96) has a solution in $(\ell^F_{\tau(.)})_h$, where $h(Y)=\sum_{q=0}^{\infty}(1/\tau_q)[\bar{\rho}(Y_q,\bar{0})]^{\tau_q}$, for all $Y\in\ell^F(\tau)$, if

(1) $h(V^{l+1}G-V^lG)\leq\alpha^l(h(G)-h(V(G)))$, with $G\in(\ell^F_{\tau(.)})_h$

(2) V is h-sequentially continuous at $Z\in(\ell^F_{\tau(.)})_h$

(3) there is $Y\in(\ell^F_{\tau(.)})_h$ with $\{V^lY\}$ which has $\{V^{l_j}Y\}$ converging to Z

Then, $Z\in(\ell^F_{\tau(.)})_h$ is a fixed point of V.

In this part, we search for a solution to nonlinear matrix equations (102) at $M\in\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta,\Lambda)$, where Δ and Λ are Banach spaces, $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ with $\tau_0>0$, and $H(G)=[\sum_{q=0}^{\infty}(1/\tau_q)(\bar{\rho}(s_q(\bar{G}),\bar{0}))^{\tau_q}]^{1/K}$, for all $G\in\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta,\Lambda)$. Consider the summable equations

$$s_q(\bar{G})=s_q(\bar{P})+\sum_{r=0}^{\infty}D(q,r)m\left(r,s_r(\bar{G})\right), \tag{102}$$

where $D:\mathcal{N}^2\longrightarrow\mathfrak{R}, m:\mathcal{N}\times\mathfrak{R}[0,1]\longrightarrow\mathfrak{R}[0,1]$, and suppose $V:\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta,\Lambda)\longrightarrow\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta,\Lambda)$ defined by

$$H(V(G))=\left[\sum_{q=0}^{\infty}\frac{1}{\tau_q}\left(\bar{\rho}\left(s_q(\bar{P})+\sum_{r=0}^{\infty}D(q,r)m\left(r,s_r(\bar{G})\right),\bar{0}\right)\right)^{\tau_q}\right]^{1/K}. \tag{103}$$

We conclude the following two applications in view of Theorem 45.

Example 19. If

(i) there is $\alpha\in[0,1)$ so that $H(V^{l+1}G-V^lG)\leq\alpha^l\ln(H(V(G))/H(G))$, with $G\in\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta,\Lambda)$

(ii) V is H-sequentially continuous at an element $M\in\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta,\Lambda)$

(iii) there are $G\in\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta,\Lambda)$ such that the sequence of iterates $\{V^rG\}$ has a $\{V^{r_m}G\}$ converging to M

Then, $M\in\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta,\Lambda)$ is a fixed point of V.

Example 20. Suppose

(i) there is $\alpha\in[0,1)$ so that $H(V^{l+1}G-V^lG)\leq\alpha^l(H(G)-H(V(G)))$, with $G\in\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta,\Lambda)$

(ii) V is H-sequentially continuous at an element $M\in\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta,\Lambda)$

(iii) there are $G\in\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta,\Lambda)$ such that the sequence of iterates $\{V^rG\}$ has a $\{V^{r_m}G\}$ converging to M

Then, M $\in\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}(\Delta,\Lambda)$ is a fixed point of V.

8. Conclusion

We proposed in this paper the notions of premodular spaces of fuzzy numbers and extended s-fuzzy numbers to construct large spaces of solutions to many nonlinear summable and matrix equations of fuzzy numbers. We discuss some topological and geometric structures of $(\ell^F_{\tau(.)})_h$, of the multiplication mappings defined on $(\ell^F_{\tau(.)})_h$, of the class $\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h}$, and of the class $(\bar{\mathfrak{F}}_{(\ell^F_{\tau(.)})_h})^\lambda$. Moreover, the existence of Caristi's fixed point in $(\ell^F_{\tau(.)})_h$ is investigated. We also presented some examples and illustrated the implication of the new results in the study of the existence of solutions for a class of nonlinear summable and matrix equations.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work was funded by the University of Jeddah, Saudi Arabia, under grant No. UJ-21-DR-76. The authors, therefore, acknowledge with thanks the University's technical and financial support.

References

[1] K. R. Rajagopal and M. Růžička, "On the modeling of electrorheological materials," *Mechanics research communications*, vol. 23, pp. 401–407, 1996.

[2] M. Růžička, "Electrorheological fluids. Modeling and mathematical theory," in *Lecture Notes in Mathematics*, p. 1748, Springer, Berlin. Germany, 2000.

[3] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[4] K. Javed, F. Uddin, H. Aydi, A. Mukheimer, and M. Arshad, "Ordered-theoretic fixed point results in fuzzy b-metric spaces with an application," *Journal of Mathematics*, vol. 2021, Article ID 6663707, 7 pages, 2021.

[5] S. U. Rehman and H. Aydi, "Rational fuzzy cone contractions on fuzzy cone metric spaces with an application to Fredholm integral equations," *Journal of Function Spaces*, vol. 2021, Article ID 5527864, 13 pages, 2021.

[6] K. Javed, H. Aydi, F. Uddin, and M. Arshad, "On orthogonal partial b-metric spaces with an application," *Journal of Mathematics*, vol. 2021, Article ID 6692063, 7 pages, 2021.

[7] H. A. Hammad, M. Sarwar, and M. De la Sen, "Existence theorem for a unique solution to a coupled system of impulsive fractional differential equations in complex-valued fuzzy metric spaces," *Advances in Difference Equations*, vol. 2021, 26 pages, 2021.

[8] M. Sarwar and R. Rodríguez-López, "Fixed point results via α -admissibility in extended fuzzy rectangular b-metric spaces with applications to integral equations," *Mathematics*, vol. 9, p. 2009, 2021.

[9] S. Nanda, "On sequences of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 33, no. 1, pp. 123–126, 1989.

[10] F. Nuray and E. Savas, "Statistical convergence of sequences of fuzzy numbers," *Mathematica Slovaca*, vol. 45, pp. 269–273, 1995.

[11] M. Matloka, "Sequences of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 28, pp. 28–37, 1986.

[12] H. Altinok, R. Colak, and M. Et, "λ-Difference sequence spaces of fuzzy numbers," *Fuzzy Sets Systems*, vol. 160, no. 21, pp. 3128–3139, 2009.

[13] R. Colak, H. Altinok, and M. Et, "Generalized difference sequences of fuzzy numbers," *Chaos Solitons Fractals*, vol. 40, no. 3, pp. 1106–1117, 2009.

[14] B. Hazarika and E. Savas, "Some I-convergent lambda-summable difference sequence spaces of fuzzy real numbers defined by a sequence of Orlicz functions," *Mathematical and Computer Modelling*, vol. 54, no. 11–12, pp. 2986–2998, 2011.

[15] A. Pietsch, "Einigie neu Klassen von Kompakten linearen Abbildungen. revue Roum," *Journal des Mathématiques Pures et Appliquées*, vol. 8, pp. 427–447, 1963.

[16] A. Pietsch, "s-Numbers of operators in Banach spaces," *Studia Mathematica*, vol. 51, no. 3, pp. 201–223, 1974.

[17] A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1980.

[18] A. Pietsch, "Small ideals of operators," *Studia Mathematica*, vol. 51, no. 3, pp. 265–267, 1974.

[19] G. Constantin, "Operators of ces – p type," *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti*, vol. 52, no. 8, pp. 875–878, 1972.

[20] B. M. Makarov and N. Faried, "Some properties of operator ideals constructed by s numbers (in Russian)," in *Theory of operators in functional spaces*, pp. 206–211, Academy of Science. Siberian section, Novosibirsk. Russia, 1977.

[21] N. Tita, "On Stolz mappings," *Math Japonica*, vol. 26, no. 4, pp. 495-496, 1981.

[22] N. Tita, *Ideale de operatori generate de s numere*, Editura Universității" Transilvania", 1998.

[23] A. Maji and P. D. Srivastava, "Some results of operator ideals on s-type $|A, p|$ operators," *Tamkang Journal of Mathematics*, vol. 45, no. 2, pp. 119–136, 2014.

[24] E. E. Kara and M. Ilkhan, "On a new class of s-type operators," *Konuralp Journal of Mathematics (KJM)*, vol. 3, no. 1, pp. 1–11, 2015.

[25] T. Yaying, B. Hazarika, and M. Mursaleen, "On sequence space derived by the domain of q-Cesàro matrix in ℓ_p space and the associated operator ideal," *Journal of Mathematical Analysis and Applications*, vol. 493, article 124453, 2021.

[26] B. S. Komal, S. Pandoh, and K. Raj, "Multiplication operators on Cesàro sequence spaces," *Demonstratio mathematica*, vol. 49, no. 4, pp. 430–436, 2016.

[27] M. Ilkhan, S. Demiriz, and E. E. Kara, "Multiplication operators on Cesàro second order function spaces," *Positivity*, vol. 24, pp. 605–614, 2020.

[28] V. Kumar, A. Sharma, K. Kumar, and N. Singh, "On limit points and cluster points of sequences of fuzzy numbers," *International Mathematical Forum*, vol. 2, no. 57, pp. 2815–2822, 2007.

[29] B. Altay and F. Basar, "Generalization of the sequence space $\ell(p)$ derived by weighted means," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 147–185, 2007.

[30] H. Nakano, "Modulared sequence spaces," *Proceedings of the Japan Academy*, vol. 27, pp. 508–512, 1951.

[31] C. Farkas, "A generalized form of Ekeland's variational principle," *Analele Universitatii" Ovidius" Constanta-Seria Matematica*, vol. 20, no. 1, pp. 101–112, 2012.

[32] T. Mrowka, “A brief introduction to linear analysis: Fredholm operators,” in *Geometry of Manifolds. Fall*, Massachusetts Institute of Technology: MIT Open Couse Ware, 2004.

[33] B. E. Rhoades, “Operators of A – p type,” *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti*, vol. 59, pp. 238–241, 1975.

[34] A. Pietsch, *Eigenvalues and* s-*Numbers*, Cambridge University Press, New York. NY, USA, 1986.

[35] N. Faried and A. A. Bakery, “Small operator ideals formed by s numbers on generalized Cesàro and Orlicz sequence spaces,” *Journal of Inequalities and Applications*, vol. 2018, 14 pages, 2018.

[36] A. A. Bakery, E. A. E. Mohamed, and O. S. K. Mohamed, “On the domain of Cesàro matrix defined by weighted means in $\ell_{t(.)}$, and its pre-quasi ideal with some applications,” *Journal of Mathematics and Computer Science*, vol. 26, no. 1, pp. 41–66, 2021.

[37] A. Pietsch, *Operator Ideals*, VEB Deutscher Verlag derWissenschaften, Berlin, 1978.

[38] P. Salimi, A. Latif, and N. Hussain, “Modified α-ψ-contractive mappings with applications,” *Fixed Point Theory and Applications*, vol. 2013, 19 pages, 2013.

[39] R. P. Agarwal, N. Hussain, and M.-A. Taoudi, “Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations,” *Abstract and Applied Analysis*, vol. 2012, Article ID 245872, 2012.

[40] N. Hussain, A. R. Khan, and R. P. Agarwal, “Krasnosel’skii and Ky Fan type fixed point theorems in ordered Banach spaces,” *Journal of Nonlinear and Convex Analysis*, vol. 11, no. 3, pp. 475–489, 2010.