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Research Article

Examination of Generalized Statistical Convergence of Order α on Time Scales

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In this paper, we introduce the concepts of deferred statistical convergence of order α and strongly deferred Cesàro summable functions (real valued) of order α on time scales and give some relationships between deferred statistical convergence of order α and strongly deferred Cesàro summable functions (real valued) of order α on time scales.

1. Introduction

In mathematics, the concept of convergence has been of great importance for many years. This concept has been studied theoretically by many mathematicians in many different fields. Many types of convergence have been defined so far, and then, very valuable results and concepts have been presented to the mathematical community. One of these ideas is the converging statistics. In 1935, Zygmund [1] introduced the concept of statistical convergence to the mathematical community, Steinhaus [2] and Fast [3] independently introduced the concept of statistical convergence, and Schoenberg [4] reintroduced it in the year 1959. Then, it has been addressed under various titles including Fourier analysis, Ergodic theory, Number theory, Turnpike theory, Measure theory, Trigonometric series, and Banach spaces. The concept was later applied to summability theory by various authors such as Çinar et al. ([5, 6]), Çolak [7], Connor [8], Fridy [9], Altay et al. [10], Garcia and Kama [11], Isik et al. ([12–15]), Kucukaslan and Yılmazturk ([16, 17]), Šalát [18], Ercan et al. ([19–21]), and Parida et al. [22], and this concept has been extended to sequence spaces, accordingly, to the notions such as summability theory.

The natural density of a subset A of $\mathbb{N}$ is defined as

$$\delta(A) = \lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k), \tag{1}$$

provided that limit exists, where χ_A is the characteristic function of A. If

$$\delta(k \in \mathbb{N} : |x_k - L| \geq \varepsilon) = 0, \tag{2}$$

for each $\varepsilon > 0$, then $x = (x_k)$ is said to be statistically convergent to ℓ writing $S - \lim_{k \longrightarrow \infty} x_k = \ell$. Over the years, that notion has been presented in a variety of ways, and its relationship to aggregation has been investigated in several domains. In recent years, researchers have attempted to apply the relationship between statistical convergence and summability theory in applicable disciplines.

Now is the time to recall the key notions of our study deferred Cesàro mean and deferred statistical convergence.

Deferred Cesàro mean, defined by Agnew [23] in 1932, is a generalization of Cesàro mean, and its definition can be given as follows:

$$\left(D_{p,q}x\right)_m = \frac{1}{q(m)-p(m)} \sum_{p(m)+1}^{q(m)} x_k, \tag{3}$$

where $p = (p(m))$ and $q = (q(m))$ are the sequences of nonnegative integers satisfying

$$p(m) < q(m) \text{ and } \lim_{m\longrightarrow\infty} q(m) = \infty. \tag{4}$$

Küçükaslan and Yılmaztürk [16, 17] defined the concepts of derferred density and deferred statistical convergence by using the deferred Cesàro mean.

The deferred density of a subset A of the natural numbers $\mathbb{N}$ is defined by

$$\delta_{p,q}(A) = \lim_{m\longrightarrow\infty} \frac{1}{q(m)-p(m)} \left|A_{p,q}(m)\right|, \tag{5}$$

provided the limit exists, where $A_{p,q}(m) = \{p(m) < k \le q(m): k \in A\}$.

A real valued sequence $x = (x_k)$ is said to be deferred statistical convergent to L, if

$$\lim_{m\longrightarrow\infty} \frac{1}{q(m)-p(m)} \left|\{p(m) < k \le q(m): |x_k - L| \ge \varepsilon\}\right| = 0, \tag{6}$$

for each $\varepsilon > 0$ [16, 17].

Throughout the paper, we assume that the sequences $(p(m))$ and $(q(m))$ satisfy the following conditions

$$p(m) < q(m) \text{ and } \lim_{m\longrightarrow\infty} q(m) = \infty, \tag{7}$$

and additionally, $\lim_{m\longrightarrow\infty} (q(m)-p(m)) = \infty$.

In 1988, Hilger [24] proposed the time scale hypothesis. In 2001, Bohner and Peterson [25] published the first detailed explanation of the time scale theory. In 2003, Guseinov [26] developed a Measure theory on time scales. Cabada and Vivero [27] presented the Lebesque integral on time scales in 2006. These findings provide the foundation for time scale summability theory research. Many mathematicians in various domains have investigated the time scale calculus over the years [28]. As a result, it seems natural to generalize convergence on time scales in light of recent applications of time scales to real-world situations. Numerous writers in the literature have used statistical convergence to apply to time scales for various purposes (see [29–33]).

A nonempty closed subset of real numbers is called a time scale. Two basic concepts on the time scale are the forward jump operator and the backward jump operator. These can be given as follows:

(i) $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$, $\sigma(s) = \inf \{t \in \mathbb{T} : t > s\}$

(ii) $\rho(s) = \sup \{t \in \mathbb{T} : t < s\}$ for $s \in \mathbb{T}$

A Lebesque Δ-measure is defined on the family of intervals $[x, y)_{\mathbb{T}} = \{s \in \mathbb{T} : x \le s \le y\}$ on an arbitrary time scale $\mathbb{T}$ with the help of forward and backward jump operators. This defined measure is denoted by ν_Δ and provides the following properties:

(i) If $x \in \mathbb{T} \setminus \max\{\mathbb{T}\}$, then the set $\{x\}$ is Δ-measurable and $\nu_\Delta(x) = \sigma(x) - x$

(ii) If $x, y \in \mathbb{T}$ and $x \le y$, then $\nu_\Delta([x, y)_{\mathbb{T}}) = y - x$ and $\nu_\Delta((x, y)_{\mathbb{T}}) = y - \sigma(x)$

(iii) If $x, y \in \mathbb{T} \setminus \max\{\mathbb{T}\}$ and $x \le y$, then $\nu_\Delta((x, y]_{\mathbb{T}}) = \sigma(y) - \sigma(x)$ and $\nu_\Delta([x, y]_{\mathbb{T}}) = \sigma(y) - x$

Let Ω be a Δ-measurable subset of $\mathbb{T}$, and for $t \in \mathbb{T}$, write

$$\Omega(t) = \{s \in [t_0, t]: s \in \Omega\}. \tag{8}$$

Turan and Duman [32, 34] were defined the density and statistical convergence on time scales as follows:

$$\delta_{\mathbb{T}}(\Omega) = \lim_{t\longrightarrow\infty} \frac{\nu_\Delta(\Omega(t))}{\nu_\Delta([t_0, t])}, \tag{9}$$

provided that limit exists.

Let $f : \mathbb{T} \longrightarrow R$ be a Δ-measurable function. On $\mathbb{T}$, f is said to be statistically convergent to L if

$$\delta_{\mathbb{T}}(t \in \mathbb{T} : |f(t) - L| \ge \varepsilon) = 0, \tag{10}$$

for every $\varepsilon > 0$. In this case, we write $st_{\mathbb{T}} - \lim_{t\longrightarrow\infty} f(t) = L$.

2. Main Results

The aim of this study is to define the concepts of deferred Cesàro summability of order α and deferred statistical convergence of order α on the time scale and examine the relationships between them.

Definition 1. Let f be a Δ-delta measurable function on the time scale $\mathbb{T}$ and $\alpha \in (0, 1]$, and we say that f is deferred statistically convergent of order α on $\mathbb{T}$ to the number L if for every $\varepsilon > 0$,

$$\lim_{m\longrightarrow\infty} \frac{\nu_\Delta\{k \in (p(m), q(m)]_{\mathbb{T}} : |f(k) - L| \geqslant \varepsilon\}}{\nu_\Delta^\alpha((p(m), q(m)]_{\mathbb{T}})} = 0. \tag{11}$$

We will show this convergence with $D(st_{\mathbb{T}}^\alpha) - \lim f(t) = L$. We will denote by $DS_{\mathbb{T}}^\alpha[p, q]$ the set of all functions that deferred statistically convergent of order α on the time scale $\mathbb{T}$.

Obviously,

(i) If we get $q(s)=s$ and $p(s)=s_0$ and $\alpha=1$, then we get the definition of statistical convergence in time scale [32]

(ii) If we take $q(m)=k(m)$, $p(m)=k(m-1)$, and $\alpha=1$, then we get lacunary statistical convergence in time scale [34]

(iii) If we take $q(t)=t$, $p(t)=t-\lambda_t+t_0$, and $\alpha=1$, then we get λ statistical convergence in time scale [35]

(iv) If we get $\alpha=1$, then we get the definition of deferred statistical convergence in time scale [29]

Example 1. Let f be defined as in [36]. It is seen from the following inequality that the function f is deferred statistical convergence in $\mathbb{T}$ for $\alpha>1/2$

$$\begin{aligned}&\frac{\nu_\Delta\{k\in(p(m),q(m)]_\mathbb{T}:|f(k)|\geq\varepsilon\}}{\nu_\Delta^\alpha((p(m),q(m)]_\mathbb{T})}\\&\quad\leq\frac{\nu_\Delta\{(p(m),\sigma p(m)+[\sqrt{\nu_m}])\}}{\nu(m)^\alpha}\\&\quad=\frac{[\sqrt{\nu_m}]}{\nu(m)^\alpha}\longrightarrow0(m\longrightarrow\infty).\end{aligned}\tag{12}$$

Theorem 2. *If $f,g:\mathbb{T}\longrightarrow\mathbb{R}$ with $D(st_\mathbb{T}^\alpha)-\lim f(t)=L_1$ and $D(st_\mathbb{T}^\alpha)-\lim g(t)=L_2$, then the following statements hold*

(i) $D(st_\mathbb{T}^\alpha)-\lim(f(t)+g(t))=L_1+L_2$

(ii) $D(st_\mathbb{T}^\alpha)-\lim(cf(t))=cL_1$

Proof. (i) Let $D(st_\mathbb{T}^\alpha)-\lim f(t)=L_1$ and $D(st_\mathbb{T}^\alpha)-\lim g(t)=L_2$. We write

$$\begin{aligned}&\frac{\nu_\Delta\{k\in(p(m),q(m)]_\mathbb{T}:|(f(k)+g(k))-(L_1+L_2)|\geq\varepsilon\}}{\nu_\Delta^\alpha((p(m),q(m)]_\mathbb{T})}\\&\quad=\frac{\nu_\Delta\{k\in(p(m),q(m)]_\mathbb{T}:|f(k)-L_1|\geq\varepsilon\}}{\nu_\Delta^\alpha((p(m),q(m)]_\mathbb{T})}\\&\qquad+\frac{\nu_\Delta\{k\in(p(m),q(m)]_\mathbb{T}:|g(k)-L_2|\geq\varepsilon\}}{\nu_\Delta^\alpha((p(m),q(m)]_\mathbb{T})},\end{aligned}\tag{13}$$

for every $\varepsilon>0$. Taking limit as $m\longrightarrow\infty$, (i) will be proved.

(ii) Let $D(st_\mathbb{T}^\alpha)-\lim f(t)=L$. Assume that $c\neq0$; then, the proof of (ii) follows from

$$\begin{aligned}&\frac{\nu_\Delta\{k\in(p(m),q(m)]_\mathbb{T}:|cf(k)-cL|\geq\varepsilon\}}{\nu_\Delta^\alpha((p(m),q(m)]_\mathbb{T})}\\&\quad=\frac{\nu_\Delta\{k\in(p(m),q(m)]_\mathbb{T}:|f(k)-L|\geq(\varepsilon/|c|)\}}{\nu_\Delta^\alpha((p(m),q(m)]_\mathbb{T})}.\end{aligned}\tag{14}$$

□

Definition 3. Let f be a Δ-delta measurable function on the time scale $\mathbb{T}$. Then, f is strongly deferred Cesàro summable of order α to L if

$$\lim_{m\longrightarrow\infty}\frac{1}{\nu_\Delta^\alpha((p(m),q(m)]_\mathbb{T})}\int_{(p(m),q(m)]_\mathbb{T}}|f(k)-L|\Delta k=0.\tag{15}$$

By $D_\mathbb{T}^\alpha[p,q]$, we denote all strongly deferred Cesàro summable functions of order α on $\mathbb{T}$.

If we take the function f as follows, for $\alpha>1/2$,

$$f(t)=\begin{cases}1,\text{if}\,s\in(p(m),\sigma(p(m))+1),_\mathbb{T}\\1,\text{if}\,s\in[\sigma(p(m))+1,\sigma(p(m))+2)_\mathbb{T},\\\cdots\\1,\,if\,s\in[\sigma(p(m))+[\sqrt{\nu_m}]-1,\sigma(p(m))+[\sqrt{\nu_m}])_\mathbb{T},\\0,\text{otherwise},\end{cases}\tag{16}$$

which is Cesàro summable.

Theorem 4. *Let f be a Δ-delta measurable function on the time scale $\mathbb{T}$. If $f\in D_\mathbb{T}^\alpha[p,q]$, then $f\in DS_\mathbb{T}^\alpha[p,q]$.*

Proof. Let $\varepsilon>0$ and $D(\varepsilon)=\{s\in(p(m),q(m)]_\mathbb{T}:|f(s)-L|\geq\varepsilon\}$. The proof is obtained from the following inequality:

$$\int_{(p(m),q(m)]_\mathbb{T}}|f(s)-L|\Delta s\geq\int_{D(\varepsilon)}|f(s)-L|\Delta s\geq\varepsilon\mu_\Delta\{D(\varepsilon)\}.\tag{17}$$

□

Corollary 5. *Let f be a Δ-delta measurable function on the time scale $\mathbb{T}$ and $\alpha,\beta\in(0,1]$ such that $\alpha<\beta$. If $f\in D_\mathbb{T}^\alpha[p,q]$, then $f\in DS_\mathbb{T}^\beta[p,q]$.*

To show that the inverse of Theorem 4 and Corollary 5 is not true, we can consider the example on page 3 of Çolak's article [7].

The converse of Theorem 4 and Corollary 5 is usually not satisfied, but provided that f is bounded, by taking $\alpha=1$, we can give the following result.

Theorem 6. *Let f be a Δ-delta measurable function on the time scale $\mathbb{T}$. If $f \in DS_{\mathbb{T}}[p,q]$ and f is bounded, then $f \in D_{\mathbb{T}}[p,q]$.*

Proof. Suppose that $f \in DS_{\mathbb{T}}[p,q]$ and f is bounded. In this case, there is $K > 0$ such that $|f(k)| \leqslant K$ and also

$$\lim_{m\longrightarrow\infty} \frac{1}{\nu_\Delta((p(m),q(m)])} \nu_\Delta(\{k \in (p(m),q(m)]: |f(k)-L| \geqslant \varepsilon\}) = 0. \tag{18}$$

Therefore, we have

$$\begin{aligned} &\frac{1}{\nu_\Delta((p(m),q(m)])} \int_{(p(m),q(m)]_{\mathbb{T}}} |f(k)-L|\Delta k \\ &\quad = \frac{1}{\nu_\Delta(D(p(m),q(m)])} \int_{D(\varepsilon)} |f(k)-L|\Delta k \\ &\qquad + \frac{1}{\nu_\Delta((p(m),q(m)])} \int_{(p(m),q(m)]_{\mathbb{T}}\setminus D(\varepsilon)} |f(k)-L|\Delta k \\ &\quad \leqslant \frac{K}{\nu_\Delta((p(m),q(m)])} \int_{D(\varepsilon)} \Delta k \\ &\qquad + \frac{\varepsilon}{\nu_\Delta((p(m),q(m)])} \int_{(p(m),q(m)]_{\mathbb{T}}} \Delta k \\ &\quad = \frac{K\nu_\Delta(D(\varepsilon))}{\nu_\Delta((p(m),q(m)])} + \varepsilon. \end{aligned} \tag{19}$$

It is obvious that for $m \longrightarrow \infty$, the theorem is proved. □

Theorem 7. *Let f be a Δ-delta measurable function on the time scale $\mathbb{T}$ and $\nu_\Delta^\alpha((p(m),q(m)])/(\sigma(q(m)))^\alpha$ is bounded. If $st_{\mathbb{T}}^\alpha - \lim f(t) = L$, then $D(st_{\mathbb{T}}^\alpha) - \lim f(t) = L$.*

Proof. Since $st_{\mathbb{T}}^\alpha - \lim f(t) = L$, we write

$$\begin{aligned} &\frac{1}{\nu_\Delta^\alpha((t_0,q(m)])} \mu_\Delta\left(\left\{k \in (t_0,q(m)]_{\mathbb{T}} : |f(k)-L| \geqslant \varepsilon\right\}\right) \\ &\quad \geqslant \frac{\nu_\Delta(D(\varepsilon))}{(\sigma(q(m))-t_0)^\alpha} \\ &\quad \geqslant \frac{(\sigma(q(m))-\sigma(p(m)))^\alpha}{(\sigma(q(m)))^\alpha} \frac{\nu_\Delta(D(\varepsilon))}{(\sigma(q(m))-\sigma(p(m)))^\alpha}. \end{aligned} \tag{20}$$

Clearly, for $m \longrightarrow \infty$, the theorem is proved. □

Corollary 8. *Let $(q(m))$ be an arbitrary sequence with $q(m) \in t_0, t]$, and $\nu_\Delta^\alpha(t_0,t])/\nu_\Delta^\alpha((p(m),q(m)])$ is bounded. Then, f is statistical convergence of order α to L on $\mathbb{T}$ implies f is deferred statistical convergence of order α to L on $\mathbb{T}$.*

Let the four sequences $(p(m))$, $(q(m))$, $(p'(m))$, and $(q'(m))$ are nonnegative real numbers such that

$$p(m) \leqslant p'(m) < q'(m) \leqslant q(m), \tag{21}$$

for all $m \in \mathbb{N}$.

Theorem 9. *Let $(p(m))$, $(q(m))$, $(p'(m))$, and $(q'(m))$ be given as in (21). If*

$$\lim_{m\longrightarrow\infty} \frac{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)}{\nu_\Delta^\alpha((p(m),q(m)])} > 0, \tag{22}$$

then $f \in DS_{\mathbb{T}}^\alpha[p,q]$ implies $f \in DS_{\mathbb{T}}^\alpha[p',q']$.

Proof. Let $D'(\varepsilon) = \{k \in (p'(m),q'(m)]: |f(k)-L| \geqslant \varepsilon\}$. The proof is obtained from the following inequality:

$$\begin{aligned} &\frac{1}{\nu_\Delta^\alpha((p(m),q(m)])} \nu_\Delta(D(\varepsilon)) \\ &\quad \geqslant \frac{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)}{\nu_\Delta^\alpha((p(m),q(m)])} \frac{1}{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)} \nu_\Delta\left(D'(\varepsilon).\right). \end{aligned} \tag{23}$$

□

Corollary 10. *Let $(p(m))$, $(q(m))$, $(p'(m))$, and $(q'(m))$ be given as in (21) and $\alpha, \beta \in (0,1]$ such that $\alpha \leq \beta$. If (22) holds, then $f \in DS_{\mathbb{T}}^\alpha[p,q]$ implies $f \in DS_{\mathbb{T}}^\beta[p',q']$.*

Theorem 11. *Let $(p(m))$, $(q(m))$, $(p'(m))$, and $(q'(m))$ be given as in (21). If*

$$\lim_{m\longrightarrow\infty} \frac{\nu_\Delta^\alpha((p(m),q(m)])}{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)} > 0, \tag{24}$$

then f is deferred Cesàro summable to L of order α on $[p(m),q(m)]$ implies f is deferred Cesàro summable to L of order α on $[p'(m),q'(m)]$.

Proof. Proof follows from the following inequality:

$$\begin{aligned} &\frac{1}{\nu_\Delta^\alpha((p(m),q(m)])} \int_{(p(m),q(m)]_{\mathbb{T}}} |f(k)-L|\Delta k \\ &\quad \geqslant \frac{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)}{\nu_\Delta^\alpha((p(m),q(m)])} \frac{1}{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)} \int_{(p'(m),q'(m)]_{\mathbb{T}}} |f(k)-L|\Delta k. \end{aligned} \tag{25}$$

□

Corollary 12. *Let $(p(m))$, $(q(m))$, $(p'(m))$, and $(q'(m))$ be given as (21) and $\alpha, \beta \in (0,1]$, $\alpha \leq \beta$. If (24) holds, then f is deferred Cesàro summable to L of order α on $[p(m),q(m)]$ implies f is deferred Cesàro summable to L of order α on $[p'(m),q'(m)]$.*

Theorem 13. *Let* $(p(m))$, $(q(m))$, $(p'(m))$, *and* $(q'(m))$ *be given as (21). If*

$$\lim_{m\longrightarrow\infty}\frac{\nu_\Delta\left(\left(p(m),p'(m)\right]\right)}{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)}>0,\ \lim_{m\longrightarrow\infty}\frac{\nu_\Delta\left(\left(q(m),q'(m)\right]\right)}{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)}>0. \tag{26}$$

If $f\in DS_{\mathbb{T}}^\alpha[p',q']$ *and* f *is bounded, then* $f\in D_{\mathbb{T}}^\alpha[p,q]$.

Proof. Suppose that $f\longrightarrow LDS_{\mathbb{T}}^\alpha[p',q']$. Since f is bounded, there exists a positive number K such that $|f(s)|\leqslant K$. Then, we may write

$$\begin{aligned}
&\frac{1}{\nu_\Delta^\alpha((p(m),q(m)])}\int_{[p(m),q(m)]_{\mathbb{T}}}|f(k)-L|\Delta k\\
&\quad=\frac{1}{\nu_\Delta^\alpha((p(m),q(m)])}\\
&\qquad\cdot\left[\int_{\left[p(m),p'(m)\right]_{\mathbb{T}}}|f(k)-L|\Delta k+\int_{\left[p'(m),q'(m)\right]_{\mathbb{T}}}|f(k)-L|\Delta k\right.\\
&\qquad\left.+\int_{\left[q'(m),q(m)\right]_{\mathbb{T}}}|f(k)-L|\Delta s\right]\\
&\quad\leqslant\frac{1}{\nu_\Delta^\alpha((p(m),q(m)])}\left[\int_{\left[p(m),p'(m)\right]_{\mathbb{T}}}K\Delta k\right.\\
&\qquad\left.+\int_{\left[p'(m),q'(m)\right]_{\mathbb{T}}}|f(k)-L|\Delta k+K\int_{\left[q'(m),q(m)\right]_{\mathbb{T}}}\Delta k\right]\\
&\quad\leqslant\frac{1}{\nu_\Delta^\alpha((p(m),q(m)])}\left(\sigma\left(p'(m)-\sigma(p(m))\right)\right.\\
&\qquad+\left(\sigma\left(q(m)-\sigma\left(q'(m)\right)\right)+\frac{1}{\nu_\Delta^\alpha((p(m),q(m)])}\right.\\
&\qquad\cdot\left[\int_{\left\{[p(m),q(m)]_{\mathbb{T}}:|f(k)-L|\geqslant\varepsilon\right\}}|f(k)-L|\Delta k\right.\\
&\qquad+\int_{\left\{[p(m),q(m)]_{\mathbb{T}}:|f(k)-L|<\varepsilon\right\}}|f(k)-L|\Delta k\\
&\quad\leqslant\frac{\mu_\Delta\left((p(m),p'(m)\right])+\nu_\Delta\left((q(m),q'(m)\right])}{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)}.K\\
&\qquad+\frac{K}{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)}\nu_\Delta\left(\left\{p'(m)<k_1\leqslant q'(m):|f(k)\right.\right.\\
&\qquad\left.\left.-L|\geqslant\varepsilon\right\}\right)+\frac{\varepsilon}{\nu_\Delta^\alpha\left(\left(p'(m),q'(m)\right]\right)}.
\end{aligned}\tag{27}$$

This completes the proof. □

3. Conclusion

Various variations of statistical convergence have been studied throughout the years, yielding some extremely important conclusions. Deferred statistical convergence of order α is one of these versions. This variant of statistical convergence is investigated on arbitrary time scales in this paper, and a significant generalization is made. As a result, the current results constitute a particular case of our findings. Then, on temporal scales, strongly postponed Cesàro summability of order α is built. Finally, various inclusion relations for the newly obtained spaces are investigated. The concepts and theorems mentioned will vary as the time scale changes. This will have a significant impact on applications employing the notion of summability in numerous ways.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally, and they have read and approved the final manuscript for publication.

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