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Retraction

Retracted: Approach to Multiattribute Decision-Making Problems Based on Neutrality Aggregation Operators of Picture Fuzzy Information

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] M. Javed, S. Javeed, J. Ahmad, K. Ullah, and L. Zedam, "Approach to Multiattribute Decision-Making Problems Based on Neutrality Aggregation Operators of Picture Fuzzy Information," *Journal of Function Spaces*, vol. 2022, Article ID 2762067, 16 pages, 2022.

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Research Article

Approach to Multiattribute Decision-Making Problems Based on Neutrality Aggregation Operators of Picture Fuzzy Information

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This manuscript is aimed at developing some novel operational laws named scalar neutrality operation and neutrality addition on picture fuzzy numbers (PFNs). The main focus of this work is to involve the neutral behaviour of the experts towards the priorities of entities where it presents equal degrees to independent membership functions. Moreover, based on these operations, some novel aggregation operators are established to aggregate the different priorities of experts. Some useful relations and characteristics are examined thoroughly. Lastly, the multiattribute group decision-making algorithm in accordance with the suggested operation is illustrated and examined a case study in order to choose a suitable mining company for a mining project along with several numerical examples. The advantages, as well as the superiority of the suggested approach, are exhibited by comparing the results with a few existing methods.

1. Introduction

Zadeh's idea of fuzzy set (FS) [1] opened a new horizon for the researchers to deal with real-life scenarios which includes uncertainty. Atanassov [2] generalized the idea of FSs and suggested the framework of intuitionistic fuzzy set (IFS) which provided us a significant tool to handle the imprecision. Atanassov and Gargov [3] and later on Atanassov [4] introduced interval valued intuitionistic fuzzy sets (IVIFSs) which used real numbers in the form of intervals to represent a membership degree (MD), a nonmembership degree (NMD), and hesitancy level. Researchers have frequently applied the concept of FSs to many real-life situations where decision-making is one of the prominent areas among them. Some very interesting fields including Xu et al. [5, 6] developed clustering algorithms, De et al. and Xiao and Ding [7, 8] presented their theories on medical diagnosis, and studies on TOPSIS and TODIM methods were conducted by several authors [9–12]. Pamučar et al. [13] introduced the MABAC method for interval valued fuzzy rough numbers. Mu et al. [14] extended the study by working on Maclaurin symmetric mean based on interval valued Pythagorean fuzzy set (PyFS). As (3,2) FS has larger domain in terms of membership degree as compared to PyFS, so Ibrahim et al. [15] recently investigated the relation of (3,2) FS with other fuzzy sets. Zeng [16] made decision-making more effective by utilizing uncertain intelligence. Muhiuddin et al. [17, 18] recorded their extension in the field of interval valued m polar fuzzy structure. Pan and Deng [19] established a new similarity measure to discuss the clustering problems under IFS. Similarly, Yager [20, 21] adopted the fusion process to aggregate the information during the multiattribute decision-making (MADM) problems. Xu and Yager [22] introduced some innovative aggregation operators (AOs) under IFS environment. Wang and Liu [23] made extension to these AOs by using Einstein operations. Ghani and Isa [24] proposed some interactive AOs based on IFS. Xu et al. [25] established innovative operators by fusing Einstein operator under IFS. Wan et al. [26] have established a decision-making scheme by using IVIFS with the help of incomplete attribute weight. Recently, few authors utilized linear Diophantine fuzzy sets and presented their work [27, 28]. Some useful approaches [29–31] have been adopted to use the fuzzy information more smoothly Furthermore, many authors dealt with MADM problems with more powerful aggregating tools [32].

It is clear from above studies that PFSs have the ability to handle the vagueness of data efficiently. Many researchers have taken keen interest in IVIFS and contributed a lot (see [33, 34] for reference). Although IFSs have been utilized to solve many issues, there are so many real-life situations which cannot be represented by using IFSs. One of them is voting which needs three independent functions membership, nonmembership, and neutral membership function to represent the human opinion. To handle such situations, Cuong [35, 36] proposed the concept of picture fuzzy set (PFS) and examined some basic characteristics of PFS. Singh [37] used correlation coefficients to conduct the clustering analysis for PFS. Son [38] presented a new clustering scheme and weather forecasting technique on the basis of PFS. Thong [39] applied PFNs on medical diagnosis to study and support the health care system. Wei and Jana et al. [40–42] presented AOs based on PFSs and solved MADMPs. Due to immense utility of the PFS and SFS, the authors really paid their attention and produced valuable researches [43–46]. Recently, Liu et al. [47], Ullah et al. [48, 49], and Akram et al. [50] made valuable addition in the field of pattern recognition, decision-making, and performance evaluation of solar energy cells by using similarity measures and aggregation operators under picture fuzzy and interval valued T-spherical fuzzy environment.

The above-given studies show that different authors have made their contributions to establish different AOs under IFS, IVIFS, PFS, etc. to manage the ambiguities in the data. Yet, it has been observed that all are not unbiased solely. For instance, if a decision-maker has assigned equal degree to MD, NMD, and AD, then the comprehensive values are not equal by utilizing existing tools for aggregation. It shows that the obtained value is not neutral. In order to enhance the utility of the AOs, it is necessary to add the neutrality character to the operational laws and their respective AOs. To achieve this, we construct probability sum (PS) function and the interaction between the MD, NMD, and AD to establish some novel operational laws on PFS. Furthermore, we suggest the picture fuzzy weighted neutral averaging (PFWNA) operator and picture fuzzy ordered neutral averaging (PFOWNA) operator to aggregate the different values of the experts. Finally, we develop a scheme to treat the MAGDMPs and give a numerical example to explore the validity of the scheme.

The rest of the paper is set out as follows. In the next section, we review some basic definitions regarding PFS. In Section 3, we define some neutral operational laws and its salient features for PFS. In Section 4, we define AOs on the basis of newly proposed operational laws. An innovative algorithm is presented in Section 5 to manage the MAGDMPs. A feasibility study of the scheme and comparison study are conducted in Section 6. Finally, Section 7 gives us final notes to conclude the manuscript.

2. Preliminaries

In this section, some fundamental definitions associated with PFS are revised. In this manuscript, X denote a nonempty set and $\dot{s}, i, d$, and r denote membership degree (MD), abstinence degree (AD), nonmembership degree (NMD), and degree of refusal (RD), respectively. The concept of PFS was proposed by Cuong [36] by narrating the fuzzy data by using a MD, NMD, AD, and r.

Definition 1 (see [36]). A PFS $\tilde{N}$ on X is defined as

$$\tilde{\mathrm{N}} = \left\{ \begin{array}{c} \left(x, \left(\dot{s}, i, d\right)\right) : \dot{s} : X \longrightarrow [0,1], i : X \longrightarrow [0,1] \text{ and } d : X \longrightarrow [0,1] \forall x \in X, \\ 0 \le \dot{s}(x) + i(x) + d(x) \le 1, \\ r(x) = 1 - \left(\dot{s}(x) + i(x) + d(x)\right). \end{array} \right\} \tag{1}$$

Definition 2. For two PFNs $\tilde{N}_1 = (\dot{s}_1, i_1, d_1)$ and $\tilde{N}_2 = (\dot{s}_2, i_2, d_2)$ and a real $\lambda > 0$, we have some dcharacteristic axioms defined as

(1) $\tilde{\mathrm{N}}_1^C = (d_1, i_1, \dot{s}_1)$

(2) $\tilde{N}_1 \subseteq \tilde{N}_2$ if $\dot{s}_1 \le \dot{s}_2$, $i_1 \le i_2$ and $d_1 \ge d_2$

(3) $\tilde{N}_1 = \tilde{N}_2$ if $\tilde{N}_1 \subseteq \tilde{N}_2$ and $\tilde{N}_2 \subseteq \tilde{N}_1$

(4) $\tilde{\mathrm{N}}_1 \bigoplus \tilde{\mathrm{N}}_2 = (\dot{s}_1 + \dot{s}_{2,} - \dot{s}_1\dot{s}_{2,}, i_1 . i_2, d_1 . d_2)$

(5) $\tilde{N}_1 \otimes \tilde{N}_2 = (\dot{s}_1\dot{s}_{2,}, i_1 + i_2 - i_1 i_2, d_1^q + d_2^q - d_1^q d_2^q)$

(6) $\lambda\tilde{\mathrm{N}}_1 = (1 - (1 - \dot{s}_1^q)^\lambda, i_1^\lambda, d_1^\lambda)$

(7) $\tilde{\mathrm{N}}_1{}^\lambda = (\dot{s}_1^\lambda, 1 - (1 - i_1)^\lambda, 1 - (1 - d_1)^\lambda)$

Definition 3 (see [51]). For a PFN $\tilde{N} = (\dot{s}(x), i(x), d(x))$, the score function and accuracy function for PFNs are defined as under

$$\mathrm{SC}(\tilde{\mathrm{N}}) = \dot{s}(x) - d(x), \text{and } \mathrm{SC}(\tilde{\mathrm{N}}) \in [-1, 1] \tag{2}$$

$$\mathrm{AC}(\tilde{\mathrm{N}}) = \dot{s}(x) + i(x) + d(x), \text{and } \mathrm{AC}(\tilde{\mathrm{N}}) \in [0, 1] \tag{3}$$

Definition 4. Let $\tilde{N}_1$ and $\tilde{N}_2$ be two PFNs; $\mathrm{SC}(\tilde{N})$ is the "score function"; and $AC(\tilde{N})$ is "accuracy function"; then, $\tilde{N}_1 > \tilde{N}_2$, where $>$ refers to "preferred to" if either $\mathrm{SC}(\tilde{N}_1) > \mathrm{SC}(\tilde{N}_2)$ or $\mathrm{SC}(\tilde{N}_1) = \mathrm{SC}(\tilde{N}_2)$ and $\mathrm{AC}(\tilde{N}_1) > \mathrm{AC}(\tilde{N}_2)$ hold.

Table 1: Comparison of score functions.

Examples	Score values by using Equation (2)	Comparison. By using Equation (5)	Ordering
$\tilde{N}_1 = (0.3, 0.2, 0.25)$ $\tilde{N}_2 = (0.4.0.2, 0.35)$	$SC(\tilde{N}_1) = SC(\tilde{N}_2)$	$\mathcal{S}(\tilde{N}_1) < \mathcal{S}(\tilde{N}_2)$	$\tilde{N}_1 < \tilde{N}_2$
$\tilde{N}_1 = (0.7, 0.1, 0)$ $\tilde{N}_2 = (0.8.0.05, 0.1)$	$SC(\tilde{N}_1) = SC(\tilde{N}_2)$	$\mathcal{S}(\tilde{N}_1) < \mathcal{S}(\tilde{N}_2)$	$\tilde{N}_1 < \tilde{N}_2$
$\tilde{N}_1 = (0.45, 0.15, 0.35)$ $\tilde{N}_2 = (0.4.0.2, 0.3)$	$SC(\tilde{N}_1) = SC(\tilde{N}_2)$	$\mathcal{S}(\tilde{N}_1) > \mathcal{S}(\tilde{N}_2)$	$\tilde{N}_1 > \tilde{N}_2$

For a collection of PFNs $\tilde{N}_j = (\dot{s}_j, i_j, d_j)$, $j = 1, 2, \cdots, n$, the weighted averaging AOs are defined as follows.

Definition 5 (see [52]). Let $\tilde{N}_j = (\dot{s}_j, i_j, d_j)$, $j = 1, 2, \cdots, n$, be a collection of PFNs and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ be the weight vector of $\tilde{N}_j$ with $\sum_{j=1}^n \omega_j = 1$; the weighted averaging and ordered weighted averaging AOs are defined as

$$\mathrm{PFWA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_n) = \left(1 - \prod_{j=1}^n (1 - \dot{s}_j)^{\omega_j}, \prod_{j=1}^n (i_j)^{\omega_j}, \prod_{j=1}^n \left(d_j\right)^{\omega_j}\right),$$

$$\mathrm{PFOWA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_n) = \left(1 - \prod_{j=1}^n \left(1 - \dot{s}_{\sigma(j)}\right)^{\omega_j}, \prod_{j=1}^n \left(i_{\sigma(j)}\right)^{\omega_j}, \prod_{j=1}^n \left(d_{\sigma(j)}\right)^{\omega_j}\right), \tag{4}$$

where σ is a transformation map of $(1, 2, \cdots, n)$ such that $\tilde{N}_{\sigma(j-1)} \geq \tilde{N}_{\sigma(j)}$ for all $j = 2, 3, \cdots, n$.

3. New Operational Laws on PFNs

It is not possible with the help of score function given in Definition 3 to obtain the accurate ranking for all PFNs. For instance, $\tilde{N}_1 = (0.3, 0.2, 0.25)$ and $\tilde{N}_2 = (0.4.0.2, 0.35)$ are two PFNs and by using score function according to Definition 3, we get $SC(\tilde{N}_1) = SC(\tilde{N}_2)$. But we are well aware that $\tilde{N}_1 \neq \tilde{N}_2$. To manage such cases under PFSs, we define a new score function for PFS.

3.1. A New Score Function

Definition 6. For a PFN $\tilde{N} = (\dot{s}, i, d)$, an innovative score function is defined as

$$\mathcal{S}(\tilde{N}) = \frac{e^{\dot{s}(x) - i(x) - d(x)}}{1 + r}, \tag{5}$$

where degree of refusal is denoted by r and calculated as $r = 1 - \dot{s}(x) - i(x) - d(x)$.

Theorem 7. *Let $\tilde{N} = (\dot{s}, i, d)$ be a PFN, and $\mathcal{S}$ is the score function; then, $\mathcal{S}$ monotonically increases for $\dot{s}$ and decreases for i and d.*

Proof. To prove the required result, firstly, we partially differentiate $\mathcal{S}$ with respect to $\dot{s}$ which gives $\partial\mathcal{S}/\partial\dot{s} = ((2 + r) e^{\dot{s}-i-d}/(2 - \dot{s} - i - d)^2) \geq 0$ and then with respect to i and d gives $\partial\mathcal{S}/\partial i = (-(r)e^{\dot{s}-i-d}/(2 - \dot{s} - i - d)^2) \leq 0$ and $\partial\mathcal{S}/\partial d = (-(r)e^{\dot{s}-i-d}/(2 - \dot{s} - i - d)^2) \leq 0$. Hence, the result is proved. □

Theorem 8. *For PFN $\tilde{N}$, the score function $\mathcal{S}$ holds the following:*

(1) $e^{-1} \leq \mathcal{S} \leq e(6)$

(2) $\mathcal{S}(\tilde{N}) = e$ iff $\tilde{N} = (1, 0, 0)$

(3) $\mathcal{S}(\tilde{N}) = e^{-1}$ iff $\tilde{N} = (0, 1, 0)$ and $\tilde{N} = (0, 0, 1)$

This result has trivial proof.

Proposed score function $\mathcal{S}$ is validated with the help of comparison analysis given in Table 1. Clearly, we observe advantages of newly proposed score functions over the existing function. We justify our claim in Theorem 9.

Theorem 9. *For two PFNs $\tilde{N}_1$ and $\tilde{N}_2$, if $SC(\tilde{N}_1) = SC(\tilde{N}_2)$ and $AC(\tilde{N}_1) > AC(\tilde{N}_2)$, then $\mathcal{S}(\tilde{N}_1) > \mathcal{S}(\tilde{N}_2)$. Further, if $SC(\tilde{N}_1) = SC(\tilde{N}_2)$ and $AC(\tilde{N}_1) = AC(\tilde{N}_2)$, then $\mathcal{S}(\tilde{N}_1) = \mathcal{S}(\tilde{N}_2)$.*

Proof. Let $\tilde{N}_1 = (\dot{s}_1, i_1, d_1)$ and $\tilde{N}_2 = (\dot{s}_2 . i_2, d_2)$ be two PFNs; then, by using Definition 4, if $SC(\tilde{N}_1) = SC(\tilde{N}_2)$ and $AC(\tilde{N}_1) > AC(\tilde{N}_2)$, we have $\dot{s}_1 - d_1 = \dot{s}_{2,} - d_2$ and $\dot{s}_1 + i_1 + d_1 > \dot{s}_{2,} + i_2 + d_2$. Therefore, we can have $\mathcal{S}(\tilde{N}_1) = (e^{\dot{s}_1 - i_1 - d_1}/(2 - (\dot{s}_1 + i_1 + d_1))2 - (\dot{s}_1 + i_1 + d_1)) \geq (e^{\dot{s}_{2,} - i_2 - d_2}/(2 - (\dot{s}_{2,} + i_2 + d_2))2 - (\dot{s}_{2,} + i_2 + d_2)) = \mathcal{S}(\tilde{N}_2)$. □

In the next subsection, we intend to establish some novel operations on the basis of probability sum (PS) and interaction coefficients. Major advantage of such newly proposed operation for PFS environment can be obtained with a fair decision in the case when all the membership degrees are equal for a PFN.

3.2. Meaning of PS and Rules of PFNs. For PFNs $\tilde{N}_1 = (\dot{s}_1, i_1, d_1)$ and $\tilde{N}_2 = (\dot{s}_2 . i_2, d_2)$, the geometric characteristic of the (PS) is explained in Figure 1. In this figure, $\dot{s}_1$,$\dot{s}_2$ represents the MDs of $\tilde{N}_1$ and $\tilde{N}_2$ while i_1, i_2 and d_1, d_2 represent

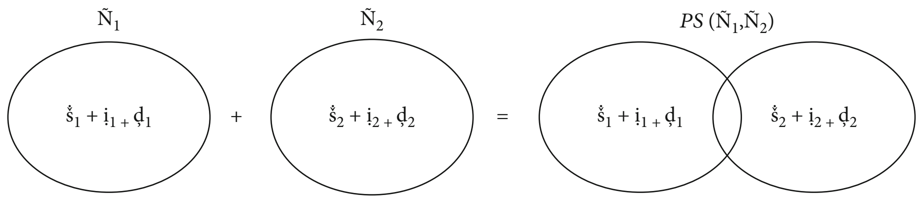

Figure 1: The geometric interpretation of PS function.

the ADs and NMDs, respectively. Obviously, $\dot{s}_1 + i_1 + d_1$ and $\dot{s}_{2,} + i_2 + d_2$ are two such events which do not depend on each other for PSF. Express$\dot{s}_\varepsilon + i_\varepsilon + d_\varepsilon$ to be the PS of occurring at least independent events of$\dot{s}_1 + i_1 + d_1$ and $\dot{s}_{2,} + i_2 + d_2$; thus,

$$\dot{s}_\varepsilon + i_\varepsilon + d_\varepsilon = \mathrm{PS}\left(\dot{s}_1 + i_1 + d_1, \dot{s}_{2,} + i_2 + d_2\right). \tag{6}$$

From it, we observe that the aggregated values of the PFNs $\tilde{N}_1$and $\tilde{N}_2$ can be accomplished by the following:

(1) Aggregate the MD, NMD, AD, and RD of the PFNs $\tilde{N}_1$and $\tilde{N}_2$ by utilizing the algebraic sum operations $T(x, y) = x + y$; as a result, we obtain $\dot{s}_1 + \dot{s}_2$, $i_1 + i_2$, $d_1 + d_2$ and $r_1 + r_2$ as it is shown with the help of different colours in Figure 2. It is absolutely apparent that their sum exceeds from 1 which is no more a PFN

(2) In order to calculate the overall aggregated values as PFN, we assign a mutual interaction coefficient $(1 - r_1 r_{2.})/(\dot{s}_1 + \dot{s}_2 + i_1 + i_2 + d_1 + d_2)$ to $\dot{s}_1 + \dot{s}_2$, $i_1 + i_2$, and $d_1 + d_2$ which can be considered as MD, AD, and NMD. Also, their sum is not greater than 1 and corresponding RD is $r_1 r_{2.}$. So

$$\begin{aligned}
\dot{s}_\varepsilon &= \frac{\dot{s}_1 + \dot{s}_2}{\dot{s}_1 + \dot{s}_2 + i_1 + i_2 + d_1 + d_2}(1 - r_1 r_2),\\
i_\varepsilon &= \frac{i_1 + i_2}{\dot{s}_1 + \dot{s}_2 + i_1 + i_2 + d_1 + d_2}(1 - r_1 r_2),\\
d_\varepsilon &= \frac{d_1 + d_2}{\dot{s}_1 + \dot{s}_2 + i_1 + i_2 + d_1 + d_2}(1 - r_1 r_2)
\end{aligned} \tag{7}$$

Moreover, PS function used in Equation (6) is$t$ -conorm for PFS and properties like commutativity, associativity, monotonicity, and boundedness which are satisfied. Hence, on the basis of this erudition, we intend to establish some novel operation laws under PFS environment.

3.3. Neutral Operational Laws. This section presents novel operational rules for PFS by fusing the neutral attitude of the experts into preferences.

Definition 10. Let $\tilde{N}_1 = (\dot{s}_1, i_1, d_1)$ and $\tilde{N}_2 = (\dot{s}_2 . i_2, d_2)$ be two PFNs. The neutrality operation of $\tilde{N}_1$and $\tilde{N}_2$ is given as

$$\tilde{N}_1 \ominus \tilde{N}_2 = \begin{pmatrix}
\dfrac{\mathrm{MCS}(\tilde{N}_1, \tilde{N}_2)}{\mathrm{MCS}(\tilde{N}_1, \tilde{N}_2) + \mathrm{NCS}(\tilde{N}_1, \tilde{N}_2) + \mathrm{ACS}(\tilde{N}_1, \tilde{N}_2)} . \mathrm{PS}\left(\dot{s}_1 + i_1 + d_1, \dot{s}_2 + i_2 + d_2\right),\\
\dfrac{\mathrm{NCS}(\tilde{N}_1, \tilde{N}_2)}{\mathrm{MCS}(\tilde{N}_1, \tilde{N}_2) + \mathrm{NCS}(\tilde{N}_1, \tilde{N}_2) + \mathrm{ACS}(\tilde{N}_1, \tilde{N}_2)} . \mathrm{PS}\left(\dot{s}_1 + i_1 + d_1, \dot{s}_2 + i_2 + d_2\right),\\
\dfrac{\mathrm{ACS}(\tilde{N}_1, \tilde{N}_2)}{\mathrm{MCS}(\tilde{N}_1, \tilde{N}_2) + \mathrm{NCS}(\tilde{N}_1, \tilde{N}_2) + \mathrm{ACS}(\tilde{N}_1, \tilde{N}_2)} . \mathrm{PS}\left(\dot{s}_1 + i_1 + d_1, \dot{s}_2 + i_2 + d_2\right),
\end{pmatrix} \tag{8}$$

where $\mathrm{MCS}(\tilde{N}_1, \tilde{N}_2) = \dot{s}_1 + \dot{s}_2$, $\mathrm{NCS}(\tilde{N}_1, \tilde{N}_2) = d_1 + d_2$, and $\mathrm{ACS}(\tilde{N}_1, \tilde{N}_2) = i_1 + i_2$ represent the MD, NMD, and AD coefficient sum of $\tilde{N}_1$and $\tilde{N}_2$, respectively. Also, $\mathrm{PS}(x, y) = 1 - (1 - x)(1 - y)$ is used to represent the PS of the FN $\tilde{N}_1$ and $\tilde{N}_2$.

For $\tilde{N}_1 = (\dot{s}_1, i_1, d_1)$ and real $\tau > 1$, we get

$$\mathrm{PS}\left(\tau\left(\dot{s}_1 + i_1 + d_1\right)\right) = \mathrm{PS}\left(\dot{s}_1 + i_1 + d_1, \mathrm{PS}\left((\tau - 1)\left(\dot{s}_1 + i_1 + d_1\right)\right)\right). \tag{9}$$

Hence, we asserted following propositions.

Proposition 11. *For PFN $\tilde{N}_1 = (\dot{s}_1, i_1, d_1)$ and a real $\tau > 0$, we get*

$$PS(\tau(\dot{s}_1 + i_1 + d_1)) = 1 - {r_1}^{\tau}. \tag{10}$$

Proof. We apply mathematical induction on τ.

Step 1. For $\tau = 2$ and by using Equation (9), we have

$$\begin{aligned} PS(\tau(\dot{s}_1 + i_1 + d_1)) &= PS(\dot{s}_1 + i_1 + d_1, PS((\tau-1)(\dot{s}_1 + i_1 + d_1))) \\ &= PS(\dot{s}_1 + i_1 + d_1, 1 - r_1) \\ &= 1 - (1 - \dot{s}_1 - i_1 - d_1)(1 - 1 + r_1) = 1 - (r_1)^2. \end{aligned} \tag{11}$$

The given statement is satisfied for $\tau = 2$.

Step 2. Assume Equation (10) is true for $\tau = n$, then, for $\tau = n + 1$,

$$\begin{aligned} PS((n+1)(\dot{s}_1 + i_1 + d_1)) &= PS((\dot{s}_1 + i_1 + d_1), PS(n(\dot{s}_1 + i_1 + d_1))) \\ &= PS((\dot{s}_1 + i_1 + d_1), 1 - (r_1)^n) \\ &= 1 - (1 - \dot{s}_1 - i_1 - d_1)(1 - 1 + (r_1)^n) \\ &= 1 - (r_1)^{n+1}. \end{aligned} \tag{12}$$

Hence, by using induction, Equation (10) is satisfied for all τ.

For a group of "n," PFNs $\tilde{N}_j = (\dot{s}_j, i_j, d_j)$, $j = 1, 2, \cdots, n$, such that $MCS(\tilde{N}_j) = \dot{s}_j$ and $MCS(\tilde{N}_1, \cdots \tilde{N}_n) = MCS(\tilde{N}_1, \cdots \tilde{N}_{n-1}) + \dot{s}_n$. Thus, we have $MCS(\tilde{N}_1, \cdots \tilde{N}_n) = \sum_{j=1}^{n} \dot{s}_j$. Likewise, we have $ACS(\tilde{N}_1, \cdots \tilde{N}_n) = \sum_{j=1}^{n} i_j$ and $NCS(\tilde{N}_1, \cdots \tilde{N}_n) = \sum_{j=1}^{n} d_j$. By using this, we have the below given proposition. □

Proposition 12. *For PFN $\tilde{N}_1 = (\dot{s}_1, i_1, d_1)$ and a real number $\tau > 0$, we have $MCS(\tau\tilde{N}_1) = \tau.MCS(\tilde{N}_1)$, $NCS(\tau\tilde{N}_1) = \tau.NCS(\tilde{N}_1)$, and $ACS(\tau\tilde{N}_1) = \tau.ACS(\tilde{N}_1)$.*

Proof. If we take $\tilde{N}_1 = \tilde{N}_2$ in MCS, then $MCS(2\tilde{N}_1) = MCS(\tilde{N}_1, \tilde{N}_1) = (\dot{s}_1 + \dot{s}_1) = 2MCS(\tilde{N}_1)$. Applying induction on τ, we can easily have that $MCS(\tau\tilde{N}_1) = \tau.MCS(\tilde{N}_1)$, $NCS(\tau\tilde{N}_1) = \tau.NCS(\tilde{N}_1)$ and $ACS(\tau\tilde{N}_1) = \tau.ACS(\tilde{N}_1)$. □

Definition 13. For PFN $\tilde{N}_1 = (\dot{s}_1, i_1, d_1)$ and a real $\tau \geq 0$, the "scalar neutrality operation" is specified as

$$\tau.\tilde{N}_1 = \begin{pmatrix} \dfrac{MCS(\tau.\tilde{N}_1)}{MCS(\tau.\tilde{N}_1) + NCS(\tau.\tilde{N}_1) + ACS(\tau.\tilde{N}_1)}.PS(\tau.(\dot{s}_1 + i_1 + d_1)), \\ \dfrac{NCS(\tau.\tilde{N}_1)}{MCS(\tau.\tilde{N}_1) + NCS(\tau.\tilde{N}_1) + ACS(\tau.\tilde{N}_1)}.PS(\tau.(\dot{s}_1 + i_1 + d_1)), \\ \dfrac{ACS(\tilde{N}_1, \tilde{N}_2)}{MCS(\tau.\tilde{N}_1) + NCS(\tau.\tilde{N}_1) + ACS(\tau.\tilde{N}_1)}.PS(\tau.(\dot{s}_1 + i_1 + d_1)), \end{pmatrix} \tag{13}$$

where $(\tau\tilde{N}_1) = \tau\dot{s}_1$, $NCS(\tau\tilde{N}_1) = \tau d_1$, and $ACS(\tau\tilde{N}_1) = \tau i_1$.

From Equation (8) and Equation (13), we note that decision-makers have neutral behaviour to MD, NMD, and AD which is shown in following preposition.

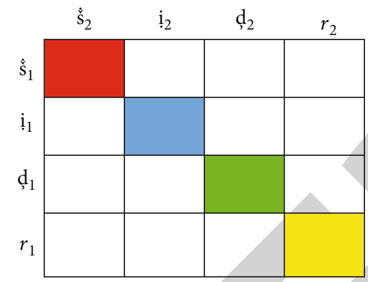

Figure 2: Geometrical description of the suggested operation.

Proposition 14. *For PFNs $\tilde{N}_1 = (\dot{s}_1, i_1, d_1)$ and $\tilde{N}_2 = (\dot{s}_2.i_2, d_2)$, if $\dot{s}_1 = i_1 = d_1$ and $\dot{s}_2 = i_2 = d_2$, then $\dot{s}_{\tilde{N}_1 \ominus \tilde{N}_2} = d_{\tilde{N}_1 \ominus \tilde{N}_2} = i_{\tilde{N}_1 \ominus \tilde{N}_2}$.*

Proof. If $\dot{s}_1 = i_1 = d_1$ and $\dot{s}_2 = i_2 = d_2$, then by using the neutrality operation, we have $\dot{s}_{\tilde{N}_1 \ominus \tilde{N}_2} / i_{\tilde{N}_1 \ominus \tilde{N}_2} = (MCS(\tilde{N}_1, \tilde{N}_2)/ACS(\tilde{N}_1, \tilde{N}_2)) = (\dot{s}_1 + \dot{s}_2 / i_1 + i_2) = 1$. Also, $\dot{s}_{\tilde{N}_1 \ominus \tilde{N}_2} / d_{\tilde{N}_1 \ominus \tilde{N}_2} = (MCS(\tilde{N}_1, \tilde{N}_2)/NCS(\tilde{N}_1, \tilde{N}_2)) = (\dot{s}_1 + \dot{s}_2 / d_1 + d_2) = 1$. Thus, $\dot{s}_{\tilde{N}_1 \ominus \tilde{N}_2} = d_{\tilde{N}_1 \ominus \tilde{N}_2} = i_{\tilde{N}_1 \ominus \tilde{N}_2}$. Hence, it makes clear that suggested operations provide us fair decision even if the PFN has equal degrees. □

Remark 15. This fact is observed from Definition 2 that when $\dot{s}_1 = i_1 = d_1$ and $\dot{s}_2 = i_2 = d_2$, we can obtain $\dot{s}_{\tilde{N}_1 \ominus \tilde{N}_2} \neq d_{\tilde{N}_1 \ominus \tilde{N}_2} \neq i_{\tilde{N}_1 \ominus \tilde{N}_2}$. Therefore, it shows the neutrality of newly proposed operations.

Now, by utilizing the definition of MCS, NCS, ACS, and PS function, we can write Equation (8) as

$$\tilde{N}_1 \ominus \tilde{N}_2 = \begin{pmatrix} \dfrac{\dot{s}_1 + \dot{s}_2}{\dot{s}_1 + \dot{s}_2 + i_1 + i_2 + d_1 + d_2}(1 - r_1 r_2), \\ \dfrac{i_1 + i_2}{\dot{s}_1 + \dot{s}_2 + i_1 + i_2 + d_1 + d_2}(1 - r_1 r_2), \\ \dfrac{d_1 + d_2}{\dot{s}_1 + \dot{s}_2 + i_1 + i_2 + d_1 + d_2}(1 - r_1 r_2). \end{pmatrix} \tag{14}$$

Subsequently, scalar neutrality operation is introduced by using Definition 13. Equation (8) gives

$$\begin{aligned} \tilde{N}_1 \ominus \tilde{N}_1 &= 2\tilde{N}_1 \\ &= \begin{pmatrix} \dfrac{MCS(\tilde{N}_1, \tilde{N}_1)}{MCS(\tilde{N}_1, \tilde{N}_1) + NCS(\tilde{N}_1, \tilde{N}_1) + ACS(\tilde{N}_1, \tilde{N}_1)}.PS(\dot{s}_1 + i_1 + d_1, \dot{s}_1 + i_1 + d_1), \\ \dfrac{MCS(\tilde{N}_1, \tilde{N}_1)}{MCS(\tilde{N}_1, \tilde{N}_1) + NCS(\tilde{N}_1, \tilde{N}_1) + ACS(\tilde{N}_1, \tilde{N}_1)}.PS(\dot{s}_1 + i_1 + d_1, \dot{s}_1 + i_1 + d_1), \\ \dfrac{MCS(\tilde{N}_1, \tilde{N}_1)}{MCS(\tilde{N}_1, \tilde{N}_1) + NCS(\tilde{N}_1, \tilde{N}_1) + ACS(\tilde{N}_1, \tilde{N}_1)}.PS(\dot{s}_1 + i_1 + d_1, \dot{s}_1 + i_1 + d_1) \end{pmatrix} \\ &= \left(\frac{\dot{s}_1}{\dot{s}_1 + i_1 + d_1}(1 - (r_1)^2), \frac{i_1}{\dot{s}_1 + i_1 + d_1}(1 - (r_1)^2), \frac{d_1}{\dot{s}_1 + i_1 + d_1}(1 - (r_1)^2) \right). \end{aligned} \tag{15}$$

Similarly,

$$3\tilde{N}_1 = \left(\frac{\dot{s}_1}{\dot{s}_1 + i_1 + d_1}(1 - (r_1)^3), \frac{i_1}{\dot{s}_1 + i_1 + d_1}(1 - (r_1)^3), \frac{d_1}{\dot{s}_1 + i_1 + d_1}(1 - (r_1)^3) \right). \tag{16}$$

In the same way, for any real positive τ, we have

$$\tau\tilde{N}_1 = \left(\frac{\dot{s}_1}{\dot{s}_1 + i_1 + d_1}\left(1-(r_1)^{\tau}\right), \frac{i_1}{\dot{s}_1 + i_1 + d_1}\left(1-(r_1)^{\tau}\right), \frac{d_1}{\dot{s}_1 + i_1 + d_1}\left(1-(r_1)^{\tau}\right)\right). \tag{17}$$

Proposition 16. *The MCS, NCS, and ACS for two PFNs $\tilde{N}_1$ and $\tilde{N}_2$ satisfy the following properties and τ, τ_1, τ_2 are positive real numbers; then,*

(1) $MCS(\tilde{N}_1, \tilde{N}_2) = MCS(\tilde{N}_2, \tilde{N}_1)$

(2) $MCS(\tau(\tilde{N}_1, \tilde{N}_2)) = MCS(\tau\tilde{N}_1, \tau\tilde{N}_2)$

(3) $MCS(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1) = MCS((\tau_1 + \tau_{21})\tilde{N}_1)$

(4) $NCS(\tilde{N}_1, \tilde{N}_2) = NCS(\tilde{N}_2, \tilde{N}_1)$

(5) $NCS(\tau(\tilde{N}_1, \tilde{N}_2)) = NCS(\tau\tilde{N}_1, \tau\tilde{N}_2)$

(6) $NCS(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1) = NCS((\tau_1 + \tau_2)\tilde{N}_1)$

(7) $ACS(\tilde{N}_1, \tilde{N}_2) = ACS(\tilde{N}_2, \tilde{N}_1)$

(8) $ACS(\tau(\tilde{N}_1, \tilde{N}_2)) = ACS(\tau\tilde{N}_1, \tau\tilde{N}_2)$

(9) $ACS(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1) = ACS((\tau_1 + \tau_{21})\tilde{N}_1)$

Proof. For PFNs $\tilde{N}_1 = (\dot{s}_1, i_1, d_1)$and $\tilde{N}_2 = (\dot{s}_2.i_2, d_2)$, we have $\mathrm{MCS}(\tilde{N}_1, \tilde{N}_2) = \dot{s}_1 + \dot{s}_2$, $\mathrm{NCS}(\tilde{N}_1, \tilde{N}_2) = d_1 + d_2$, and $\mathrm{ACS}(\tilde{N}_1, \tilde{N}_2) = i_1 + i_2$ and $\mathrm{MCS}(\tau\tilde{N}_1) = \tau\dot{s}_1$, $\mathrm{NCS}(\tau\tilde{N}_1) = \tau d_1$, and $\mathrm{ACS}(\tau\tilde{N}_1) = \tau i_1$. Thus, $\mathrm{MCS}(\tau\tilde{N}_1, \tau\tilde{N}_2) = \tau\dot{s}_1 + \tau\dot{s}_2 = \tau(\dot{s}_1 + \dot{s}_2) = \tau\,\mathrm{MCS}(\tilde{N}_1, \tilde{N}_2) = \mathrm{MCS}(\tau(\tilde{N}_1, \tilde{N}_2))$. Similarly, the remaining parts can be proven □

3.4. Characteristics of the Proposed Operations. The next subsection discusses some remarkable properties of the suggested operational laws as under.

Theorem 17. *If $\tilde{N}_1 = (\dot{s}_1, i_1, d_1)$ and $\tilde{N}_2 = (\dot{s}_2.i_2, d_2)$ are two distinct PFNs, then operations of $\tilde{N}_1$ and $\tilde{N}_2$, $\tilde{N}_1 \ominus \tilde{N}_2$, and $\tau\tilde{N}_1$ are also PFNs for $\tau > 0$.*

Proof. For PFNs $\tilde{N}_1 = (\dot{s}_1, i_1, d_1)$ and $\tilde{N}_2 = (\dot{s}_2.i_2, d_2)$, we have $\dot{s}_1, i_1, d_1\dot{s}_2, i_2, d_2 \in [0, 1]$, $r_1 = 1 - \dot{s}_1 - i_1 - d_1 \in [0, 1]$ and $r_2 = 1 - \dot{s}_2 - i_2 - d_2 \in [0, 1]$. Let $\tilde{N}_1 \ominus \tilde{N}_2 = (\dot{s}_{\tilde{N}_1\ominus\tilde{N}_2}, i_{\tilde{N}_1\ominus\tilde{N}_2}, d_{\tilde{N}_1\ominus\tilde{N}_2})$. To obtain the required result $\tilde{N}_1 \ominus \tilde{N}_2 \in$ PFN, we will just show that $\dot{s}_{\tilde{N}_1\ominus\tilde{N}_2}, i_{\tilde{N}_1\ominus\tilde{N}_2}, d_{\tilde{N}_1\ominus\tilde{N}_2} \in [0, 1]$ and $\dot{s}_{\tilde{N}_1\ominus\tilde{N}_2} + i_{\tilde{N}_1\ominus\tilde{N}_2} + d_{\tilde{N}_1\ominus\tilde{N}_2} \leq 1$. Since $\dot{s}_1 + i_1 + d_1 \leq 1$, $\dot{s}_2 + i_2 + d_2 \leq 1$ and hence, $((\dot{s}_1 + \dot{s}_2)/(\dot{s}_1 + \dot{s}_2 + i_1 + i_2 + d_1 + d_2)\dot{s}_1 + \dot{s}_2 + i_1 + i_2 + d_1 + d_2) \leq 1$, provided $\dot{s}_1 + \dot{s}_2 + i_1 + i_2 + d_1 + d_2 \neq 0$. Also, $r_1, r_2 \in [0, 1]$ which implies that $1 - r_1 r_2 \in [0, 1]$. Hence, it can be obtained that $\dot{s}_{\tilde{N}_1\ominus\tilde{N}_2}, i_{\tilde{N}_1\ominus\tilde{N}_2}, d_{\tilde{N}_1\ominus\tilde{N}_2} \in [0, 1]$. Further, $\dot{s}_{\tilde{N}_1\ominus\tilde{N}_2} + i_{\tilde{N}_1\ominus\tilde{N}_2} + d_{\tilde{N}_1\ominus\tilde{N}_2} = 1 - r_1 r_2 \in [0, 1]$. Hence, $\tilde{N}_1 \ominus \tilde{N}_2$ is a PFN. Similarly, $\tau\tilde{N}_1$ can also be proven as a PFN. □

Theorem 18. *Let $\tilde{N}_1 = (\dot{s}_1, i_1, d_1)$ and $\tilde{N}_2 = (\dot{s}_2.i_2, d_2)$ be two PFN and $\tau, \tau_1\tau_2 \geq 0$ be a real number; then,*

(1) $\tilde{N}_1 \ominus \tilde{N}_2 = \tilde{N}_2 \ominus \tilde{N}_1$

(2) $\tau(\tilde{N}_1 \ominus \tilde{N}_2) = (\tau\tilde{N}_1 \ominus \tau\beta_2)$

(3) $\tau_1\tilde{N}_1 \ominus \tau_2\tilde{N}_1 = (\tau_1 + \tau_2)\tilde{N}_1$

Proof.

(1) Easily can be followed from Equation (14)

(2) For PFNs $\tilde{N}_1$and $\tilde{N}_2$,we get

$$\begin{aligned}
\tau\tilde{N}_1 \ominus \tau\tilde{N}_2 &= \begin{pmatrix} \dfrac{\mathrm{MCS}(\tau\tilde{N}_1, \tau\tilde{N}_2)}{\mathrm{MCS}(\tau\tilde{N}_1, \tau\tilde{N}_2) + \mathrm{NCS}(\tau\tilde{N}_1, \tau\tilde{N}_2) + \mathrm{ACS}(\tau\tilde{N}_1, \tau\tilde{N}_2)}.\mathrm{PS}(\mathrm{PS}(\tau(\dot{s}_1 + i_1 + d_1)), \mathrm{PS}(\tau(\dot{s}_2 + i_2 + d_2))), \\ \dfrac{\mathrm{NCS}(\tau\tilde{N}_1, \tau\tilde{N}_2)}{\mathrm{MCS}(\tau\tilde{N}_1, \tau\tilde{N}_2) + \mathrm{NCS}(\tau\tilde{N}_1, \tau\tilde{N}_2) + \mathrm{ACS}(\tau\tilde{N}_1, \tau\tilde{N}_2)}.\mathrm{PS}(\mathrm{PS}(\tau(\dot{s}_1 + i_1 + d_1)), \mathrm{PS}(\tau(\dot{s}_2 + i_2 + d_2))), \\ \dfrac{\mathrm{ACS}(\tau\tilde{N}_1, \tau\tilde{N}_2)}{\mathrm{MCS}(\tau\tilde{N}_1, \tau\tilde{N}_2) + \mathrm{NCS}(\tau\tilde{N}_1, \tau\tilde{N}_2) + \mathrm{ACS}(\tau\tilde{N}_1, \tau\tilde{N}_2)}.\mathrm{PS}(\mathrm{PS}(\tau(\dot{s}_1 + i_1 + d_1)), \mathrm{PS}(\tau(\dot{s}_2 + i_2 + d_2))) \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{\tau(\dot{s}_1 + \dot{s}_2)}{\tau(\dot{s}_1 + \dot{s}_2) + \tau(i_1 + i_2) + \tau(d_1 + d_2)}.\mathrm{PS}(1 - {r_1}^{\tau}, 1 - {r_2}^{\tau}), \\ \dfrac{\tau(i_1 + i_2)}{\tau(\dot{s}_1 + \dot{s}_2) + \tau(i_1 + i_2) + \tau(d_1 + d_2)}.\mathrm{PS}(1 - {r_1}^{\tau}, 1 - {r_2}^{\tau}), \\ \dfrac{\tau(d_1 + d_2)}{\tau(\dot{s}_1 + \dot{s}_2) + \tau(i_1 + i_2) + \tau(d_1 + d_2)}.\mathrm{PS}(1 - {r_1}^{\tau}, 1 - {r_2}^{\tau}) \end{pmatrix} = \begin{pmatrix} \dfrac{(\dot{s}_1 + \dot{s}_2)}{(\dot{s}_1 + \dot{s}_2) + (i_1 + i_2) + (d_1 + d_2)}(1 - (r_1)^{\tau}(r_2)^{\tau}), \\ \dfrac{(i_1 + i_2)}{(\dot{s}_1 + \dot{s}_2) + (i_1 + i_2) + (d_1 + d_2)}(1 - (r_1)^{\tau}(r_2)^{\tau}), \\ \dfrac{(d_1 + d_2)}{(\dot{s}_1 + \dot{s}_2) + (i_1 + i_2) + (d_1 + d_2)}(1 - (r_1)^{\tau}(r_2)^{\tau}) \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{(\dot{s}_1 + \dot{s}_2)}{(\dot{s}_1 + \dot{s}_2) + (i_1 + i_2) + (d_1 + d_2)}\mathrm{PS}(\tau(1 - r_1 r_2)), \\ \dfrac{(i_1 + i_2)}{(\dot{s}_1 + \dot{s}_2) + (i_1 + i_2) + (d_1 + d_2)}\mathrm{PS}(\tau(1 - r_1 r_2)), \\ \dfrac{(d_1 + d_2)}{(\dot{s}_1 + \dot{s}_2) + (i_1 + i_2) + (d_1 + d_2)}\mathrm{PS}(\tau(1 - r_1 r_2)) \end{pmatrix} = \tau(\tilde{N}_1 \ominus \tilde{N}_2).
\end{aligned} \tag{18}$$

For $\tau_1, \tau_2 > 0$,

$$\tau_1\tilde{N}_1 = \left(\frac{\dot{s}_1}{\dot{s}_1 + i_1 + d_1}\left(1-(r_1)^{\tau_1}\right), \frac{i_1}{\dot{s}_1 + i_1 + d_1}\left(1-(r_1)^{\tau_1}\right), \frac{d_1}{\dot{s}_1 + i_1 + d_1}\left(1-(r_1)^{\tau_1}\right)\right),$$
$$\tau_2\tilde{N}_1 = \left(\frac{\dot{s}_1}{\dot{s}_1 + i_1 + d_1}\left(1-(r_1)^{\tau_2}\right), \frac{i_1}{\dot{s}_1 + i_1 + d_1}\left(1-(r_1)^{\tau_2}\right), \frac{d_1}{\dot{s}_1 + i_1 + d_1}\left(1-(r_1)^{\tau_2}\right)\right). \tag{19}$$

Thus, by using Equation (8), we get

$$\tau_1\tilde{N}_1 \ominus \tau_2\tilde{N}_1 = \begin{pmatrix} \dfrac{\mathrm{MCS}(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1)}{\mathrm{MCS}(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1) + \mathrm{NCS}(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1) + \mathrm{ACS}(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1)} . \mathrm{PS}\left(\mathrm{PS}\left(\tau_1\left(\dot{s}_1 + i_1 + d_1\right)\right), \mathrm{PS}\left(\tau_2\left(\dot{s}_2 + i_2 + d_2\right)\right)\right), \\ \dfrac{\mathrm{NCS}(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1)}{\mathrm{MCS}(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1) + \mathrm{NCS}(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1) + \mathrm{ACS}(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1)} . \mathrm{PS}\left(\mathrm{PS}\left(\tau_1\left(\dot{s}_1 + i_1 + d_1\right)\right), \mathrm{PS}\left(\tau_2\left(\dot{s}_2 + i_2 + d_2\right)\right)\right), \\ \dfrac{\mathrm{ACS}(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1)}{\mathrm{MCS}(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1) + \mathrm{NCS}(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1) + \mathrm{ACS}(\tau_1\tilde{N}_1, \tau_2\tilde{N}_1)} . \mathrm{PS}\left(\mathrm{PS}\left(\tau_1\left(\dot{s}_1 + i_1 + d_1\right)\right), \mathrm{PS}\left(\tau_2\left(\dot{s}_2 + i_2 + d_2\right)\right)\right) \end{pmatrix}$$
$$= \left(\frac{\dot{s}_1}{\dot{s}_1 + i_1 + d_1}\left(1-(r_1)^{\tau_1+\tau_2}\right), \frac{i_1}{\dot{s}_1 + i_1 + d_1}\left(1-(r_1)^{\tau_1+\tau_2}\right), \frac{d_2}{\dot{s}_1 + i_1 + d_1}\left(1-(r_1)^{\tau_1+\tau_2}\right)\right) = (\tau_1 + \tau_2)\tilde{N}_1. \tag{20}$$

□

4. Neutral Aggregation Operators for PFNs

In the next section, the PFWN and PFOWNA operators of PFNs are discussed, including their related characteristics. To get this, we assume Ω to be the group of PFNs.

Definition 19. Let $\tilde{N}_i$ be a collection of "n" PFNs. The PFWNA of $(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_n)$ is given by

$$\mathrm{PFWNA}\left(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_n\right) = \ominus_{i=1}^{n} \omega_i \tilde{N}_i, \tag{21}$$

where $\omega_i > 0$ is the weighting vector of $\tilde{N}_i$and $\sum_{i=1}^{n}\omega_i = 1$.

Theorem 20. *The aggregated result of PFNs with the help of Definition 19 is also a PFN and given by*

$$PFWNA\left(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_n\right) = \begin{pmatrix} \dfrac{\sum_{i=1}^{n}\omega_i \dot{s}_i}{\sum_{i=1}^{n}\omega_i\left(\dot{s}_i + i_i + d_i\right)} \cdot \left(1 - \prod_{i=1}^{n}(r_i)^{\omega_i}\right), \\ \dfrac{\sum_{i=1}^{n}\omega_i i_i}{\sum_{i=1}^{n}\omega_i\left(\dot{s}_i + i_i + d_i\right)} \cdot \left(1 - \prod_{i=1}^{n}(r_i)^{\omega_i}\right), \\ \dfrac{\sum_{i=1}^{n}\omega_i d_i}{\sum_{i=1}^{n}\omega_i\left(\dot{s}_i + i_i + d_i\right)} \cdot \left(1 - \prod_{i=1}^{n}(r_i)^{\omega_i}\right). \end{pmatrix} \tag{22}$$

Proof. For "n" PFNs $\tilde{N}_i$ and real $\omega_i > 0$, by utilizing Theorem 7, the first result is trivially satisfied. For the existence of Equation (22), we apply mathematical induction on "n" which is compiled as follows:

Step 1. For $n = 1$, we have $\tilde{N}_i = (\dot{s}_i, i_i, d_i)$ and $\omega_i = 1$. Therefore, it can be written that

$$\mathrm{PFWNA}\left(\tilde{N}_1\right) = \omega_1\tilde{N}_1 = \left(\dot{s}_1, i_1, d_1\right)$$
$$= \left(\frac{\omega_1\dot{s}_1}{\omega_1\left(\dot{s}_1 + i_1 + d_1\right)}\left(1-(r_1)^{\omega_1}\right), \cdot \frac{\omega_1 i_1}{\omega_1\left(\dot{s}_1 + i_1 + d_1\right)}\left(1-(r_1)^{\omega_1}\right), \cdot \frac{\omega_1 d_1}{\omega_1\left(\dot{s}_1 + i_1 + d_1\right)}\left(1-(r_1)^{\omega_1}\right)\right). \tag{23}$$

Thus, Equation (22) is satisfied.

Step 2. Assume Equation (22) holds for $n = k$, that is,

$$\mathrm{PFWNA}\left(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k\right) = \begin{pmatrix} \dfrac{\sum_{i=1}^{k}\omega_i \dot{s}_i}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_i + i_i + d_i\right)} \cdot \left(1 - \prod_{i=1}^{k}(r_i)^{\omega_i}\right), \\ \dfrac{\sum_{i=1}^{k}\omega_i i_i}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_i + i_i + d_i\right)} \cdot \left(1 - \prod_{i=1}^{k}(r_i)^{\omega_i}\right), \\ \dfrac{\sum_{i=1}^{k}\omega_i d_i}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_i + i_i + d_i\right)} \cdot \left(1 - \prod_{i=1}^{k}(r_i)^{\omega_i}\right). \end{pmatrix} \tag{24}$$

Now, for $n = k + 1$, we get

$$\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_{k+1}) = \mathrm{TSFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k) \ominus (\omega_{k+1}\tilde{N}_{k+1})$$
$$= \left(\begin{array}{l} \dfrac{\mathrm{MCS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1})}{\left(\begin{array}{c} \mathrm{MCS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1}) \\ +\mathrm{NCS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1}) + \mathrm{ACS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1}) \end{array} \right)} .\mathrm{PS}\left(1 - \prod_{i=1}^{k} (r_i)^{\omega_i}, 1 - (r_{k+1})^{\omega_{k+1}}\right), \\ \dfrac{\mathrm{ACS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1})}{\left(\begin{array}{c} \mathrm{MCS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1}) \\ +\mathrm{NCS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1}) + \mathrm{ACS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1}) \end{array} \right)} .\mathrm{PS}\left(1 - \prod_{i=1}^{k} (r_i)^{\omega_i}, 1 - (r_{k+1})^{\omega_{k+1}}\right), \\ \dfrac{\mathrm{NCS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1})}{\left(\begin{array}{c} \mathrm{MCS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1}) \\ +\mathrm{NCS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1}) + \mathrm{ACS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1}) \end{array} \right)} .\mathrm{PS}\left(1 - \prod_{i=1}^{k} (r_i)^{\omega_i}, 1 - (r_{k+1})^{\omega_{k+1}}\right). \end{array} \right) \tag{25}$$

After substituting the expressions of MCS, ACS, and NCS, we get

$$\begin{aligned} &\mathrm{MCS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1}) \\ &\quad = \mathrm{MCS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k)) + \mathrm{MCS}(\omega_{k+1}\tilde{N}_{k+1}) \\ &\quad = \sum_{i=1}^{k} \omega_i \dot{s}_i + \omega_{k+1}\dot{s}_{k+1} = \sum_{i=1}^{k+1} \omega_i \dot{s}_i. \end{aligned} \tag{26}$$

Similarly, we have

$$\begin{aligned} &\mathrm{ACS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1}) \\ &\quad = \mathrm{ACS}(\mathrm{TSFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k)) + \mathrm{ACS}(\omega_{k+1}\tilde{N}_{k+1}) \\ &\quad = \sum_{i=1}^{k} \omega_i i_i + \omega_{k+1} i_{k+1} = \sum_{i=1}^{k+1} \omega_i i_i, \\ &\mathrm{NCS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k), \omega_{k+1}\tilde{N}_{k+1}) \\ &\quad = \mathrm{NCS}(\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_k)) + \mathrm{NCS}(\omega_{k+1}\tilde{N}_{k+1}) \\ &\quad = \sum_{i=1}^{k} \omega_i d_i + \omega_{k+1} d_{k+1} = \sum_{i=1}^{k+1} \omega_i d_i. \end{aligned} \tag{27}$$

By using the definition of PS, we get

$$\begin{aligned} &\mathrm{PS}\left(1 - \prod_{i=1}^{k} (r_i)^{\omega_i}, 1 - (r_{k+1})^{\omega_{k+1}}\right) \\ &\quad = 1 - \left(1 - 1 + \prod_{i=1}^{k} (r_i)^{\omega_i}\right)(1 - 1 + (r_{k+1})^{\omega_{k+1}}) = 1 - \prod_{i=1}^{k+1} (r_i)^{\omega_i}. \end{aligned} \tag{28}$$

Thus,

$$\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_{k+1}) = \left(\begin{array}{l} \dfrac{\sum_{i=1}^{k+1} \omega_i \dot{s}_i}{\sum_{i=1}^{k+1} \omega_i (\dot{s}_i + i_i + d_i)} . \left(1 - \prod_{i=1}^{k+1} (r_i)^{\omega_i}\right), \\ \dfrac{\sum_{i=1}^{k+1} \omega_i i_i}{\sum_{i=1}^{k+1} \omega_i (\dot{s}_i + i_i + d_i)} . \left(1 - \prod_{i=1}^{k+1} (r_i)^{\omega_i}\right), \\ \dfrac{\sum_{i=1}^{k+1} \omega_i d_i}{\sum_{i=1}^{k+1} \omega_i (\dot{s}_i + i_i + d_i)} . \left(1 - \prod_{i=1}^{k+1} (r_i)^{\omega_i}\right). \end{array} \right) \tag{29}$$

Equation (22) holds for $n = k + 1$. Therefore, by applying induction, Equation (22) is true for all n. □

Example 1. Let us consider four PFNs $\tilde{N}_1 = (0.4, 0.3, 0.1)$, $\tilde{N}_2 = (0.5, 0.2, 0.2)$, $\tilde{N}_3 = (0.1, 0.2, 0.4)$, and $\tilde{N}_4 = (0.3, 0.3, 0.2)$; the weigh vector associated with them is $\omega = (0.2, 0.4, 0.3, 0.1)$; we use here PFWNA to calculate the four PFNs, since

$$\mathrm{PFWNA}(\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{N}_4) = \left(\begin{array}{l} \dfrac{\sum_{i=1}^{4} \omega_i \dot{s}_i}{\sum_{i=1}^{4} \omega_i (\dot{s}_i + i_i + d_i)} . \left(1 - \prod_{i=1}^{k+1} (r_i)^{\omega_i}\right), \\ \dfrac{\sum_{i=1}^{4} \omega_i i_i}{\sum_{i=1}^{4} \omega_i (\dot{s}_i + i_i + d_i)} . \left(1 - \prod_{i=1}^{k+1} (r_i)^{\omega_i}\right), \\ \dfrac{\sum_{i=1}^{4} \omega_i d_i}{\sum_{i=1}^{4} \omega_i (\dot{s}_i + i_i + d_i)} . \left(1 - \prod_{i=1}^{k+1} (r_i)^{\omega_i}\right). \end{array} \right) \tag{30}$$

Further,

$$\frac{\sum_{i=1}^{4}\omega_i\dot{s}_i}{\sum_{i=1}^{4}\omega_i\left(\dot{s}_i+i_i+d_i\right)}\cdot\left(1-\prod_{i=1}^{k+1}\left(r_i\right)^{\omega_i}\right)=0.3479,$$
$$\frac{\sum_{i=1}^{4}\omega_i i_i}{\sum_{i=1}^{4}\omega_i\left(\dot{s}_i+i_i+d_i\right)}\cdot\left(1-\prod_{i=1}^{k+1}\left(r_i\right)^{\omega_i}\right)=0.2353, \tag{31}$$
$$\frac{\sum_{i=1}^{4}\omega_i d_i}{\sum_{i=1}^{4}\omega_i\left(\dot{s}_i+i_i+d_i\right)}\cdot\left(1-\prod_{i=1}^{k+1}\left(r_i\right)^{\omega_i}\right)=0.2456.$$

Finally, we get

$$\mathrm{PFWNA}\left(\tilde{N}_1,\tilde{N}_2,\tilde{N}_3,\tilde{N}_4\right)=(0.3479, 0.2553, 0.2456), \tag{32}$$

which is also a PFN.

Theorem 21 (Idempotency). *Let* $\tilde{N}_i=(\dot{s}_i, i_i, d_i)(i=1,2,3\ldots,n)$ *be a set of PFNs. If* $\tilde{N}_i=(\dot{s}_0, i_0, d_0)$ *for all* i, *then* $PFWNA(\tilde{N}_1,\tilde{N}_2,\cdots,\tilde{N}_n)=(\dot{s}_0, i_0, d_0)$.

Proof. For "n" PFNs $\tilde{N}_i=(\dot{s}_i, i_i, d_i)$ and $\tilde{N}_0=(\dot{s}_0, i_0, d_0)$ such that $\tilde{N}_i=\tilde{N}_0$, we have $\dot{s}_i=\dot{s}_0$, $i_i=i_0$ and $d_i=d_0$ for all i. Then, by Equation (22) and $\omega_i>0$ with $\sum_{i=1}^{n}\omega_i=1$, we get

$$\mathrm{TSFWNA}\left(\tilde{\mathrm{N}}_1,\tilde{\mathrm{N}}_2,\cdots,\tilde{\mathrm{N}}_n\right)=\left(\begin{array}{c}\dfrac{\dot{s}_0}{\dot{s}_0+i_0+d_0}\cdot\left(1-(r_0)^{\sum_{i=1}^{n}\omega_i}\right),\\ \dfrac{i_0}{\dot{s}_0+i_0+d_0}\cdot\left(1-(r_0)^{\sum_{i=1}^{n}\omega_i}\right),\\ \dfrac{d_0}{\dot{s}_0+i_0+d_0}\cdot\left(1-(r_0)^{\sum_{i=1}^{n}\omega_i}\right)\end{array}\right)=\left(\begin{array}{c}\dfrac{\dot{s}_0}{\dot{s}_0+i_0+d_0}\cdot(\dot{s}_0+i_0+d_0),\\ \dfrac{i_0}{\dot{s}_0+i_0+d_0}\cdot(\dot{s}_0+i_0+d_0),\\ \dfrac{d_0}{\dot{s}_0+i_0+d_0}\cdot(\dot{s}_0+i_0+d_0)\end{array}\right)=\tilde{\mathrm{N}}_0. \tag{33}$$

□

Theorem 22. *Consider a collection of "n" PFNs* $\tilde{N}_i$; *we have*

(1) $\min\{\dot{s}_i+i_i+d_i\}\le\dot{s}_p+i_p+d_p\le\max\{\dot{s}_i+i_i+d_i\}$

(2) $\min\{\dot{s}_i+i_i+d_i\}.\min\{\dot{s}_i\}/\max\{\dot{s}_i+i_i+d_i\}\le\dot{s}_p\le\max\{\dot{s}_i+i_i+d_i\}.\max\{\dot{s}_i\}/\min\{\dot{s}_i+i_i+d_i\}$

(3) $\min\{\dot{s}_i+i_i+d_i\}.\min\{i_i\}/\max\{\dot{s}_i+i_i+d_i\}\le i_p\le\max\{\dot{s}_i+i_i+d_i\}.\max\{i_i\}/\min\{\dot{s}_i+i_i+d_i\}$

(4) $\min\{\dot{s}_i+i_i+d_i\}.\min\{d_i\}/\max\{\dot{s}_i+i_i+d_i\}\le d_p\le\max\{\dot{s}_i+i_i+d_i\}.\max\{d_i\}/\min\{\dot{s}_i+i_i+d_i\}$

where $PFWNA(\tilde{N}_1,\tilde{N}_2,\cdots,\tilde{N}_n)=(\dot{s}_p, i_p, d_p)$. *This result is known as boundedness.*

Proof. For a collection of "n" PFNs $\tilde{N}_i=(\dot{s}_i, i_i, d_i)$, we have

$$\begin{aligned}\min\left\{\dot{s}_i+i_i+d_i\right\}&=1-\left(1-\min\left\{\dot{s}_i+i_i+d_i\right\}\right)^{\sum_{i=1}^{n}\omega_i}\\&=1-\prod_{i=1}^{n}\left(1-\min\left\{\dot{s}_i+i_i+d_i\right\}\right)^{\omega_i}\\&\le 1-\prod_{i=1}^{n}\left(1-\dot{s}_i-i_i-d_i\right)^{\omega_i}\\&\le 1-\prod_{i=1}^{n}\left(1-\max\left\{\dot{s}_i+i_i+d_i\right\}\right)^{\omega_i}\\&=1-\left(1-\max\left\{\dot{s}_i+i_i+d_i\right\}\right)^{\sum_{i=1}^{n}\omega_i}\\&=\max\left\{\dot{s}_i+i_i+d_i\right\}\end{aligned} \tag{34}$$

Thus, we have

$$\min\left\{\dot{s}_i+i_i+d_i\right\}\le 1-\prod_{i=1}^{n}\left(1-\dot{s}_i-i_i-d_i\right)^{\omega_i}\le\max\left\{\dot{s}_i+i_i+d_i\right\}. \tag{35}$$

By using Theorem 20,

$$\dot{s}_p=\frac{\sum_{i=1}^{k}\omega_i\dot{s}_i}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_i+i_i+d_i\right)}\cdot\left(1-\prod_{i=1}^{k}\left(r_i\right)^{\omega_i}\right),$$
$$i_p=\frac{\sum_{i=1}^{k}\omega_i i_i}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_i+i_i+d_i\right)}\cdot\left(1-\prod_{i=1}^{k}\left(r_i\right)^{\omega_i}\right), \tag{36}$$
$$d_p=\frac{\sum_{i=1}^{k}\omega_i d_i}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_i+i_i+d_i\right)}\cdot\left(1-\prod_{i=1}^{k}\left(r_i\right)^{\omega_i}\right).$$

Therefore,

$$\dot{s}_p+i_p+d_p=1-\prod_{i=1}^{n}\left(1-\dot{s}_i-i_i-d_i\right)^{\omega_i}. \tag{37}$$

Hence, we get $\min\{\dot{s}_i+i_i+d_i\}\le\dot{s}_p+i_p+d_p\le\max\{\dot{s}_i+i_i+d_i\}$.

(1) Since, $\dot{s}_i\ge\min\{\dot{s}_i\}$,so by expression $\dot{s}_p$, we have

$$\begin{aligned}\dot{s}_p &\geq \frac{\sum_{i=1}^{n}\omega_i\left(\min\left\{\dot{s}_i\right\}\right)}{\sum_{i=1}^{n}\omega_i\left(\max\left\{\dot{s}_i+i_i+d_i\right\}\right)}\left[1-\prod_{i=1}^{n}\left(1-\min\left\{\dot{s}_i+i_i+d_i\right\}\right)^{\omega_i}\right]\\ &= \frac{\min\left\{\dot{s}_i\right\}}{\max\left\{\dot{s}_i+i_i+d_i\right\}}\left[1-\left(1-\min\left\{\dot{s}_i+i_i+d_i\right\}\right)^{\sum_{i=1}^{n}\omega_i}\right]\\ &= \frac{\min\left\{\dot{s}_i+i_i+d_i\right\}\min\left\{\dot{s}_i\right\}}{\max\left\{\dot{s}_i+i_i+d_i\right\}}\end{aligned} \tag{38}$$

Further,

$$\begin{aligned}\dot{s}_p &\leq \frac{\sum_{i=1}^{n}\omega_i\left(\max\left\{\dot{s}_i\right\}\right)}{\sum_{i=1}^{n}\omega_i\left(\min\left\{\dot{s}_i+i_i+d_i\right\}\right)}\left[1-\prod_{i=1}^{n}\left(1-\max\left\{\dot{s}_i+i_i+d_i\right\}\right)^{\omega_i}\right]\\ &= \frac{\max\left\{\dot{s}_i\right\}}{\min\left\{\dot{s}_i+i_i+d_i\right\}}\left[1-\left(1-\max\left\{\dot{s}_{\mathrm{i}}+i_{\mathrm{i}}+d_i\right\}\right)^{\sum_{i=1}^{\mathrm{n}}\omega_{\mathrm{i}}}\right]\\ &= \frac{\max\left\{\dot{s}_i+i_i+d_i\right\}\max\left\{\dot{s}_i\right\}}{\min\left\{\dot{s}_i+i_i+d_i\right\}} = \frac{\min\left\{\dot{s}_i+i_i+d_i\right\}\min\left\{\dot{s}_i\right\}}{\max\left\{\dot{s}_i+i_i+d_i\right\}}\\ &\leq \dot{s}_p \leq \frac{\max\left\{\dot{s}_i+i_i+d_i\right\}.\max\left\{\dot{s}_i\right\}}{\min\left\{\dot{s}_i+i_i+d_i\right\}}.\end{aligned} \tag{39}$$

(2) Similar to part (2)

(3) Similar to part (2)

□

Theorem 23 (Monotonicity). *Let* $\tilde{N}_i=(\dot{s}_{\tilde{N}_i}, i_{\tilde{N}_i}, d_{\tilde{N}_i})$ *and* $M_i=(\dot{s}_{M_i}, i_{M_i}, d_{M_i})$ *be collection of "n" PFNs. Then,*

(1) $\dot{s}_{p_{\tilde{N}}}+i_{p_{\tilde{N}}}+d_{p_{\tilde{N}}} \leq \dot{s}_{p_M}+i_{p_M}+d_{p_M}$ *if* $\dot{s}_{\tilde{N}_i}+i_{\tilde{N}_i}+d_{\tilde{N}_i} \leq \dot{s}_{M_i}+i_{M_i}+d_{M_i}$

(2) $\dot{s}_{p_{\tilde{N}}} \leq \dot{s}_{p_M}$, $i_{p_{\tilde{N}}} \geq i_{p_M}$, $d_{p_{\tilde{N}}} \geq d_{p_M}$ *if* $\dot{s}_{\tilde{N}_i}+i_{\tilde{N}_i}+d_{\tilde{N}_i}=\dot{s}_{M_i}+i_{M_i}+d_{M_i}$ *and* $\dot{s}_{\tilde{N}_i} \leq \dot{s}_{M_i}$

(3) $PFWNA(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_n) \leq PFWNA(M_1, M_2, \cdots, M_n)$ *if* $\dot{s}_{\tilde{N}_i}+i_{\tilde{N}_i}+d_{\tilde{N}_i}=\dot{s}_{M_i}+i_{M_i}+d_{M_i}$ *and* $\dot{s}_{\tilde{N}_i} \leq$

Proof. For "n" PFNs $\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_n$ and $M_1, M_2, \cdots, M_n$ and by using Theorem 20, we get PFWNA $(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_n)=(\dot{s}_{p_{\tilde{N}}}, i_{p_{\tilde{N}}}, d_{p_{\tilde{N}}})$ and PFWNA$(M_1, M_2, \cdots, M_n)=(\dot{s}_{p_M}, i_{p_M}, d_{p_M})$, where

$$\begin{aligned}\dot{s}_{p_{\mathrm{N}}} &= \frac{\sum_{i=1}^{k}\omega_i\dot{s}_{\beta i}}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{p_{\mathrm{N}}}+i_{p_{\mathrm{N}}}+d_{p_{\mathrm{N}}}\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{p_{\mathrm{N}}}-i_{p_{\mathrm{N}}}-d_{p_{\mathrm{N}}}\right)^{\omega_i}\right],\\ i_{p_{\tilde{N}}} &= \frac{\sum_{i=1}^{k}\omega_i i_{\tilde{N}_i}}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{p_{\tilde{N}}}+i_{p_{\tilde{N}}}+d_{p_{\tilde{N}}}\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{p_{\tilde{N}}}-i_{p_{\tilde{N}}}-d_{p_{\tilde{N}}}\right)^{\omega_i}\right],\\ d_{p_{\tilde{N}}} &= \frac{\sum_{i=1}^{k}\omega_i d_{\beta i}}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{p_{\tilde{N}}}+i_{p_{\tilde{N}}}+d_{p_{\tilde{N}}}\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{p_{\tilde{N}}}-i_{p_{\tilde{N}}}-d_{p_{\tilde{N}}}\right)^{\omega_i}\right],\\ \dot{s}_{p_M} &= \frac{\sum_{i=1}^{k}\omega_i\dot{s}_{M_i}}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{p_M}+i_{p_M}+d_{p_M}\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{p_M}-i_{p_M}-d_{p_M}\right)^{\omega_i}\right],\\ i_{p_M} &= \frac{\sum_{i=1}^{k}\omega_i i_{M_i}}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{p_M}+i_{p_M}+d_{p_M}\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{p_M}-i_{p_M}-d_{p_M}\right)^{\omega_i}\right],\\ d_{p_M} &= \frac{\sum_{i=1}^{k}\omega_i d_{M_i}}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{p_M}+i_{p_M}+d_{p_M}\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{p_M}-i_{p_M}-d_{p_M}\right)^{\omega_i}\right].\end{aligned} \tag{40}$$

On the basis of above information, we have

(1) If $\dot{s}_{\tilde{N}_i}+i_{\tilde{N}_i}+d_{\tilde{N}_i}=\dot{s}_{M_i}+i_{M_i}+d_{M_i}$, then we get

$$\begin{aligned}\dot{s}_{p_{\tilde{\mathrm{N}}}}+i_{p_{\tilde{\mathrm{N}}}}+d_{p_{\tilde{\mathrm{N}}}} &\leq 1-\prod_{i=1}^{n}\left(1-\dot{s}_{\tilde{\mathrm{N}}_i}-i_{\tilde{\mathrm{N}}_i}-d_{\tilde{\mathrm{N}}_i}\right)^{\omega_i}\\ &\leq 1-\prod_{i=1}^{n}\left(1-\dot{s}_{\mathrm{M}_i}-i_{\mathrm{M}_i}-d_{\mathrm{M}_i}\right)^{\omega_i}\\ &= \dot{s}_{p_M}+i_{p_M}+d_{p_M}\end{aligned} \tag{41}$$

(2) If $\dot{s}_{\tilde{N}_i}+i_{\tilde{N}_i}+d_{\tilde{N}_i}=\dot{s}_{M_i}+i_{M_i}+d_{M_i}$, and $\dot{s}_{\tilde{N}_i} \leq \dot{s}_{M_i}$, then we get

$$\begin{aligned}\dot{s}_{p_{\tilde{\mathrm{N}}}} &= \frac{\sum_{i=1}^{k}\omega_i\dot{s}_{\tilde{\mathrm{N}}_i}}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{\tilde{\mathrm{N}}_i}+i_{\tilde{\mathrm{N}}_i}+d_{\tilde{\mathrm{N}}_i}\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{\tilde{\mathrm{N}}_i}-i_{\tilde{\mathrm{N}}_i}-d_{\tilde{\mathrm{N}}_i}\right)^{\omega_i}\right]\\ &\leq \frac{\sum_{i=1}^{k}\omega_i\dot{s}_{\mathrm{M}_i}}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{\mathrm{M}_i}+i_{\mathrm{M}_i}+d_{\mathrm{M}_i}\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{\mathrm{M}_i}-i_{\mathrm{M}_i}-d_{\mathrm{M}_i}\right)^{\omega_i}\right]\\ &= \dot{s}_{p_{\mathrm{M}}},\\ i_{p\tilde{N}} &= \frac{\sum_{i=1}^{k}\omega_i i_{\tilde{N}_i}}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{\tilde{N}_i}+i_{\tilde{N}_i}+d_{\tilde{N}_i}\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{\tilde{N}_i}-i_{\tilde{N}_i}-d_{\tilde{N}_i}\right)^{\omega_i}\right]\\ &\leq \frac{\sum_{i=1}^{k}\omega_i i_{M_i}}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{M_i}+i_{M_i}+d_{M_i}\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{M_i}-i_{M_i}-d_{M_i}\right)^{\omega_i}\right]=i_{pM},\\ d_{p\tilde{N}} &= \frac{\sum_{i=1}^{k}\omega_i d_{\tilde{N}_i}}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{\tilde{N}_i}+i_{\tilde{N}_i}+d_{\tilde{N}_i}\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{\tilde{N}_i}-i_{\tilde{N}_i}-d_{\tilde{N}_i}\right)^{\omega_i}\right]\\ &\leq \frac{\sum_{i=1}^{k}\omega_i d_{M_i}}{\sum_{i=1}^{k}\omega_i\left(\dot{s}_{M_i}+i_{M_i}+d_{M_i}\right)}\left[1-\prod_{i=1}^{n}\left(1-\dot{s}_{M_i}-i_{M_i}-d_{M_i}\right)^{\omega_i}\right]=d_{pM}\end{aligned} \tag{42}$$

(3) From part (2), we obtain $\dot{s}_{p_{\tilde{N}}} \leq \dot{s}_{p_M}$, $\underset{.}{i}_{p_{\tilde{N}}} \geq \underset{.}{i}_{p_M}$, $\underset{.}{d}_{p_{\tilde{N}}} \geq \underset{.}{d}_{p_M}$. Hence, by utilizing the score function of Equation (2), we get $\dot{s}_{p_{\tilde{N}}} - \underset{.}{i}_{p_{\tilde{N}}} - \underset{.}{d}_{p_{\tilde{N}}} \leq \dot{s}_{p_M} - \underset{.}{i}_{p_M} - \underset{.}{d}_{p_M}$. So, by an order relation, we get $\mathrm{PFWNA}(\tilde{\mathrm{N}}_1, \tilde{\mathrm{N}}_2, \cdots, \tilde{\mathrm{N}}_n) \leq \mathrm{PFWNA}(\mathrm{M}_1, \mathrm{M}_2, \cdots, \mathrm{M}_n)$

□

Definition 24. Let $\tilde{N}_i = (\dot{s}_i, \underset{.}{i}_i, \underset{.}{d}_i) (i = 1, 2, 3 \ldots, n)$ be a set of PFNs. The PFOWNA operator of $\tilde{N}_i$ is defined by

$$\mathrm{PFOWNA}\left(\tilde{\mathrm{N}}_1, \tilde{\mathrm{N}}_2, \cdots, \tilde{\mathrm{N}}_n\right) = \ominus_{i=1}^{n} \omega_i \tilde{\mathrm{N}}_{\sigma(i)}, \tag{43}$$

where σ is the permutation map of $1, 2, 3 \ldots, n$ such that $\tilde{N}_{\sigma(i-1)} \geq \tilde{N}_{\sigma(i)}$ and $\tilde{N}_i \in \Omega$ and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector associated with PFOWNA operator, satisfying $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$.

Theorem 25. *The aggregated value by using Definition 24 for "n" PFNs $\tilde{N}_i = (\dot{s}_i, \underset{.}{i}_i, \underset{.}{d}_i)$ is also a PFN, where*

$$\begin{aligned} & PFOWNA\left(\tilde{N}_1, \tilde{N}_2, \cdots, \tilde{N}_n\right) \\ & = \left(\begin{array}{l} \dfrac{\sum_{i=1}^{n} \omega_i \dot{s}_{\sigma(i)}}{\sum_{i=1}^{n} \omega_i \left(\dot{s}_{\sigma(i)} + \underset{.}{i}_{\sigma(i)} + \underset{.}{d}_{\sigma(i)} \right)} \cdot \left(1 - \prod_{i=1}^{n} \left(r_{\sigma(i)} \right)^{\omega_i} \right), \\ \dfrac{\sum_{i=1}^{n} \omega_i \underset{.}{i}_{\sigma(i)}}{\sum_{i=1}^{n} \omega_i \left(\dot{s}_{\sigma(i)} + \underset{.}{i}_{\sigma(i)} + \underset{.}{d}_{\sigma(i)} \right)} \cdot \left(1 - \prod_{i=1}^{n} \left(r_{\sigma(i)} \right)^{\omega_i} \right), \\ \dfrac{\sum_{i=1}^{n} \omega_i \underset{.}{d}_{\sigma(i)}}{\sum_{i=1}^{n} \omega_i \left(\dot{s}_{\sigma(i)} + \underset{.}{i}_{\sigma(i)} + \underset{.}{d}_{\sigma(i)} \right)} \cdot \left(1 - \prod_{i=1}^{n} \left(r_{\sigma(i)} \right)^{\omega_i} \right). \end{array} \right) \end{aligned} \tag{44}$$

The proof of this Theorem 25 is omitted because it is similar to Theorem 20.

Remark 26. The PFOWNA operator satisfies all the properties which are satisfied by the PFWNA operator mentioned in Theorems 20, 21, 22, and 23. Also, their proofs are omitted due to similarity.

5. Proposed MAGDM Approach Based on Aggregation Operators

This section introduces an inventive scheme to work out the MAGDM problem under the PFS conditions. Consider that a collection of alternatives $\mathfrak{F} = \{\mathfrak{F}_1, \mathfrak{F}_2, \cdots \mathfrak{F}_m\}$ and objective is to choose the best possible one under the evaluation of a set of attributes $\mathscr{H} = \{\mathscr{H}_1, \mathscr{H}_2, \cdots \mathscr{H}_n\}$ with weight vector $\omega > 0$such that $\sum_{j=1}^{n} \omega_j = 1$. The set $\mathscr{H}$ containing attributes is split into two mutually exclusive sets, that is, cost type and benefit type attributes, respectively (F_1) and (F_2). To rank the given alternatives, in this event, a team of experts $\widehat{E} = \{\widehat{E}^{(1)}, \widehat{E}^{(2)}, \cdots \widehat{E}^{(l)}\}$ is selected with weight vector $w_k > 0$ such that $\sum_{k=1}^{l} w_k = 1$, with a task to propose the best possible solution. For the sake of smooth assessment, decision-makers will utilize PFS environment as $\delta_{ij}^{(k)} = (\dot{s}_{ij}^{(k)}, \underset{.}{i}_{ij}^{(k)}, \underset{.}{d}_{ij}^{(k)})$ with $0 \leq \dot{s}_{ij}^{(k)}, \underset{.}{i}_{ij}^{(k)}, \underset{.}{d}_{ij}^{(k)} \leq 1$ s.t. $0 \leq \dot{s}_{ij}^{(k)} + \underset{.}{i}_{ij}^{(k)} + \underset{.}{d}_{ij}^{(k)} \leq 1$ for $i = 1, 2 \cdots, m, j = 1, 2 \cdots n$, and $k = 1, 2, \cdots l$. Henceforth, a PF decision matrix can be considered as

$$\mathfrak{N}^k = \begin{pmatrix} \delta_{11}^{(k)} & \delta_{12}^{(k)} & \cdots & \delta_{1n}^{(k)} \\ \delta_{21}^{(k)} & \delta_{22}^{(k)} & \cdots & \delta_{2n}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{m1}^{(k)} & \delta_{m2}^{(k)} & \cdots & \delta_{mn}^{(k)} \end{pmatrix}. \tag{45}$$

Now, the detailed strategy of new technique is presented below:

Step 1. A decision matrix $\mathfrak{N}$ is formulated under PFS environment corresponding to each expert $\widehat{E}$.

Step 2. By using a weight vector and suitable AOs, that is, PFWNA or PFOWNA aggregate all the decision-makers' preferences $\delta_{ij}^{(k)}, k = 1, 2, \cdots, l$ into $\delta_{ij} = (\dot{s}_{ij}, \underset{.}{i}_{ij}, \underset{.}{d}_{ij})$.

Step 3. As there are two types of attribute indicators, namely, the cost (F_1) and the benefit (F_2), so we normalize δ_{ij} into r_{ij} if it is essential.

$$r_{ij} = \begin{cases} \left(\underset{.}{d}_{ij}, \underset{.}{i}_{ij}, \dot{s}_{ij} \right), & \text{for } F_1 \text{ criteria}, \\ \left(\dot{s}_{ij}, \underset{.}{i}_{ij}, \underset{.}{d}_{ij} \right), & \text{for } F_2 \text{ criteria}. \end{cases} \tag{46}$$

Step 4. In the case if weights of the attributes are already given, then use them as they are. Contrarily, if the detail about the attributes is partially known, which is signified in set $\mathscr{G}$, then an optimization model is formulated to calculate an unknown weight vector for each attribute.

$$\max f = \sum_{i=1}^{m} \sum_{j=1}^{n} \omega_{ij} \mathscr{S}_{ij} \tag{47}$$

$$\text{s.t.} \sum_{j=1}^{n} \omega_j = 1, \quad \omega_j \geq 0, \omega \in \mathscr{G}, \tag{48}$$

where $\mathscr{S}_{ij}$ is the score of each alternative, $\mathfrak{F}_i$; the above model gives$\omega = (\omega_1, \omega_2, \cdots \omega_n)$.

Step 5. Using weights and utilizing suggested PF averaging AOs to compute the comprehensive values $r_i = (\dot{s}_i, \underset{.}{i}_i, \underset{.}{d}_i)$ of alternatives $\mathfrak{F}_i (i = 1, 2, \cdots, m)$.

Table 2: The decision matrix from experts.

Experts	Alternatives	$\mathscr{H}_1$	$\mathscr{H}_2$	$\mathscr{H}_3$	$\mathscr{H}_4$
	$\mathbf{F}_1$	$(0.5, 0.19, 0.3)$	$(0.4, 0.25, 0.3)$	$(0.2, 0.4, 0.3)$	$(0.3, 0.3, 0.3)$
$\widehat{E}^{(1)}$	$\mathbf{F}_2$	$(0.4, 0.39, 0.2)$	$(0.45, 0.24, 0.3)$	$(0.37, 0.5, 0.1)$	$(0.27, 0.32, 0.4)$
	$\mathbf{F}_3$	$(0.5, 0.19, 0.3)$	$(0.34, 0.49, 0.1)$	$(0.29, 0.4, 0.2)$	$(0.4, 0.35, 0.21)$
	$\mathbf{F}_4$	$(0.21, 0.31, 0.33)$	$(0.51, 0.27, 0.2)$	$(0.32, 0.27, 0.4)$	$(0.42, 0.25, 0.3)$
	$\mathbf{F}_1$	$(0.5, 0.19, 0.3)$	$(0.5, 0.22, 0.2)$	$(0.3, 0.23, 0.2)$	$(0.28, 0.28, 0.28)$
$\widehat{E}^{(2)}$	$\mathbf{F}_2$	$(0.34, 0.42, 0.2)$	$(024, 0.3, 0.0.39)$	$(0.25, 0.3, 0.39)$	$(0.41, 0.28, 0.3)$
	$\mathbf{F}_3$	$(0.35, 0.35, 0.25)$	$(0.44, 0.35, 0.18)$	$(0.35, 0.34, 0.3)$	$(0.5, 0.29, 0.2)$
	$\mathbf{F}_4$	$(0.34, 0.42, 0.2)$	$(0.39, 0.3, 0.3)$	$(0.3, 0.35, 0.29)$	$(0.4, 0.37, 0.2)$
	$\mathbf{F}_1$	$(0.5, 0.38, 0.1)$	$(0.4, 0.28, 0.3)$	$(0.2, 0.39, 0.4)$	$(0.33, 0.33, 0.33)$
$\widehat{E}^{(3)}$	$\mathbf{F}_2$	$(0.37, 0.31, 0.3)$	$(0.29, 0.33, 0.32)$	$(0.32, 0.25, 0.4)$	$(0.4, 0.55, 0.04)$
	$\mathbf{F}_3$	$(0.4, 0.3, 0.29)$	$(0.54, 0.22, 0.2)$	$(0.34, 0.34, 0.3)$	$(0.31, 0.38, 0.3)$
	$\mathbf{F}_4$	$(0.51, 0.23, 0.2)$	$(0.39, 0.32, 0.27)$	$(0.32, 0.42, 0.21)$	$(0.34, 0.31, 0.3)$

Table 3: Aggregated values of the experts by the PFWNA operator.

Alternatives	$\mathscr{H}_1$	$\mathscr{H}_2$	$\mathscr{H}_3$	$\mathscr{H}_4$
$\mathcal{F}_1$	$(0.5004, 0.2548, 0.2322)$	$(0.4350, 0.2376, 0.2781)$	$(0.2313, 0.3436, 0.3945)$	$(0.3160, 0.3160, 0.3160)$
$\mathcal{F}_2$	$(0.3748, 0.3721, 0.2347)$	$(0.3432, 0.2904, 0.3353)$	$(0.3212, 0.3617, 0.2809)$	$(0.3543, 0.3899, 0.2385)$
$\mathcal{F}_3$	$(0.4278, 0.2721, 0.2846)$	$(0.4363, 0.3615, 0.1561)$	$(0.3301, 0.13711, 0.2665)$	$(0.3982, 0.3456, 0.2390)$
$\mathcal{F}_4$	$(0.3520, 0.3169, 0.2542)$	$(0.44371, 0.2953, 0.2510)$	$(0.3176, 0.2458, 0.3086)$	$(0.3878, 0.3031, 0.2733)$

Step 6. Equation (49) is used to calculate the score values $r_i = (\dot{s}_i, \underset{\cdot}{i}_i, \underset{\cdot}{d}_i)$, $i = 1, 2, \cdots, m$.

$$\mathcal{S}(r_i) = \frac{e^{\dot{s}_i - \underset{\cdot}{i}_i - \underset{\cdot}{d}_i}}{2 - \dot{s}_i - \underset{\cdot}{i}_i - \underset{\cdot}{d}_i}. \tag{49}$$

Step 7. By using Definition 4, the alternatives $\mathbf{F}_i (i = 1, 2, \cdots, m)$ are ranked.

6. Illustrative Examples

To validate the suggested method, we assume a MAGDM problem and examine the performance of the proposed method further; for the sake of comparison, some existing methods are compared with the suggested method.

6.1. Case Study. Mineral resources are the most vital among the category of natural resources. Minerals provide raw material to all kind of industries which is considered as a backbone of economic growth. Mining is the process through which different minerals are extracted from the earth. Mining sector has made great contributions for human development. Especially in developing countries, it has really helped to raise the standards of health and education. Pakistan is an Asian country and full of natural resources. Balochistan is a resource rich province of Pakistan. A well-known geological survey team has reported that Balochistan has round about ninety mineral resources with sufficient layers. Most of them have not been discovered yet. Recently, there are three main projects working in Balochistan named as Reko-diq, Dudder, and Saindak. All of them are being managed by the Chinese companies. The management of Balochistan province wants to explore more mineral resources. In order to speed up the mining work, the government has invited four different companies with four required attributes for selection named as technical capability ($\mathscr{H}_1$), financial status ($\mathscr{H}_2$), company back ground ($\mathscr{H}_3$), and employability ($\mathscr{H}_4$). There are three panels $\widehat{E}^{(1)}, \widehat{E}^{(2)}$, and $\widehat{E}^{(3)}$ of decision-makers which are appraising the four companies: $\mathbf{F}_1$ ("China National Coal Group"), $\mathbf{F}_2$ ("China Northern Rare Earth Group"), $\mathbf{F}_3$ ("High-Tech Mining Associates"), and $\mathbf{F}_4$ ("HBIS Group") with the abovementioned four attributes. Suppose that $w = (0.39, 0.27, 0.34)$ is the weight vector of the experts and their assessment matrices $\mathfrak{N}^1, \mathfrak{N}^2$ and $\mathfrak{N}^3$ under PFNs are presented in Table 2. The purpose of the study is to choose the appropriate company for new mining project. The steps of the proposed MAGDM process are performed as follows:

Step 1. All the values evaluated by the experts are summarized in Table 2.

Step 2. We assume the corresponding weight vector $w = (0.39, 0.27, 0.34)$ and PFWNA operator to aggregate the information of each alternative. The values are listed in Table 3.

Table 4: Influence of alteration of AOs and ranking order.

Operators used Step 2	Step 5	Score values $\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	$\mathcal{F}_4$	Ranking
PFWNA	PFWNA	0.8221	0.7452	0.8407	0.8046	$\mathcal{F}_3 > \mathcal{F}_4 > \mathcal{F}_2 > \mathcal{F}_1$
PFOWNA	PFWNA	0.8135	0.7368	0.8144	0.7742	$\mathcal{F}_3 > \mathcal{F}_4 > \mathcal{F}_2 > \mathcal{F}_1$
PFWNA	PFOWNA	0.3961	0.3332	0.3978	0.3728	$\mathcal{F}_3 > \mathcal{F}_4 > \mathcal{F}_2 > \mathcal{F}_1$
PFOWNA	PFOWNA	0.8351	0.7436	0.8506	0.8107	$\mathcal{F}_3 > \mathcal{F}_4 > \mathcal{F}_2 > \mathcal{F}_1$

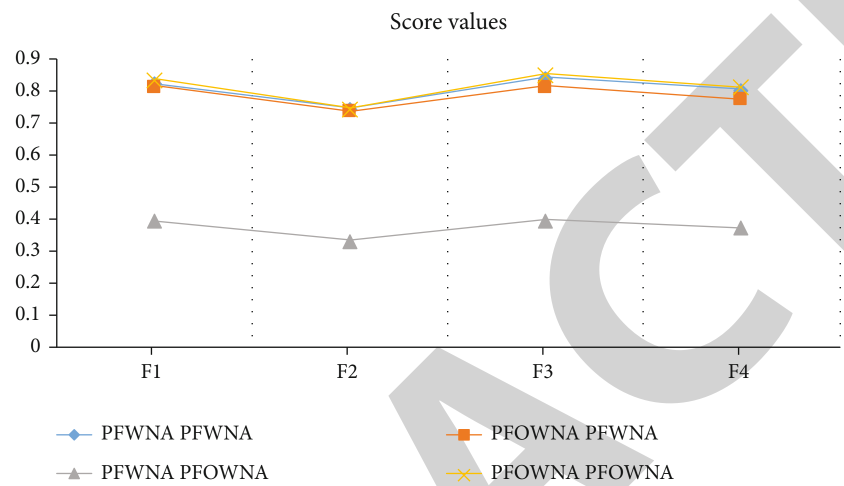

Figure 3: Geometrical interpretation of alteration of AOs.

Step 3. All of the given attributes are the same type, so it is not essential to normalize the information.

Step 4. The partial weight information regarding attribute importance as proposed by decision-makers is $\mathcal{G} = \{0.2 \leq \omega_1 \leq 0.3,\ 0.25 \leq \omega_2 \leq 0.4,\ 0.12 \leq \omega_3 \leq 0.4,\ 0.2 \leq \omega_4 \leq 0.35,\ \omega_1 + \omega_3 \leq \omega_2,\ \omega_1 - \omega_4 \leq \omega_3\}$; an optimization model has been formulated by using Equation (48), and after some calculations, we obtained $\omega = (0.26.0.4, 0.14, 0.2)$.

Step 5. In accordance with the PFWNA operator and weights ω, we obtain the overall value r_i of each alternative; values are given as $r_1 = (0.4018, 0.2740, 0.2917)$, $r_2 = (0.3509, 0.4021, 0.3335)$, $r_3 = (0.4130, 0.3376, 0.2223)$, and $r_4 = (0.3911, 0.3117, 0.2661)$.

Step 6. By using Equation (5), score values are calculated as $\mathcal{S}(r_1) = 0.8221$, $\mathcal{S}(r_2) = 0.7452$, $\mathcal{S}(r_3) = 0.8407$, and $\mathcal{S}(r_4) = 0.8046$.

Step 7. As $\mathcal{S}(r_3) > \mathcal{S}(r_1) > S(r_4) > S(r_2)$, therefore the preferences are ordered as $\mathbf{F}_3 > \mathbf{F}_1 > \mathbf{F}_4 > \mathbf{F}_2$. So $\mathbf{F}_3$ is the optimal choice.

6.2. Alteration of AOs. We can choose a different pair of proposed AOs except from the above-given analysis. Here, we examine the ranking pattern of the alternatives if experts select distinct AOs in Step 2 and Step 5, respectively. As it is quite clear that every AO has its own importance and specific features as per given situation. For instance, PFWNAO gives more weight to PFNs; on the other hand, PFWONAO gives more importance to the position of PFNs after ordering them. In this way, decision-makers can alter the AOs as per requirement. Table 4 shows different values under alteration of AOs. It shows the stable behaviour of ranking pattern, and $\mathbf{F}_3$ remains the best choice throughout the ranking calculations. Figure 3 clearly demonstrates the stability of newly proposed AOs. Use of PFOWNA in Step 2 and PFWNA in step 5 generates relatively low score values of alternatives. However, the overall ranking pattern and best possible alternatives remain unchanged.

6.3. Comparative Analysis

Example 2. Again, utilizing the data from Table 2, here, we will calculate the aggregation values and validate the proposed AOs with the help of different tools which already exist in the literature. It is necessary to mention here that the already available work by Garg et al. [52, 53] on neutrality operators is not sufficient enough to manage the information recorded in Table 2. Here, we apply the operators proposed by Garg [52], Wei [40], and Jana at el. [42] on the information given in Table 2 in order to conduct a comparative study among different aggregating tools. The ranking patterns obtained by using the aforementioned existing AOs are enlisted in Table 5. Results obtained using the proposed PFWNA and PFOWNA operators have consistency. Ranking patterns are quite similar with results obtained by

Table 5: Comparative analysis by using existing and proposed methods.

Operator	Reference	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	$\mathcal{F}_4$	Ranking
PFDWA	Jana at el. [42]	0.4657	0.4367	0.5045	0.4809	$\mathcal{F}_3 > \mathcal{F}_4 > \mathcal{F}_1 > \mathcal{F}_2$
PFDWG	Jana at el. [42]	0.2217	0.2346	0.2788	0.2477	$\mathcal{F}_3 > \mathcal{F}_4 > \mathcal{F}_2 > \mathcal{F}_1$
PFWA	Garg [52]	−0.1245	−0.2272	−0.1136	−0.1668	$\mathcal{F}_3 > \mathcal{F}_1 > \mathcal{F}_4 > \mathcal{F}_2$
PFWG	Wei [40]	−0.1824	−0.2946	−0.1621	−0.2003	$\mathcal{F}_3 > \mathcal{F}_1 > \mathcal{F}_4 > \mathcal{F}_2$
PFWNA	This paper	0.8220	0.7452	0.8407	0.8046	$\mathcal{F}_3 > \mathcal{F}_1 > \mathcal{F}_4 > \mathcal{F}_2$
PFOWNA	This paper	0.8135	0.7364	0.8144	0.7742	$\mathcal{F}_3 > \mathcal{F}_1 > \mathcal{F}_4 > \mathcal{F}_2$

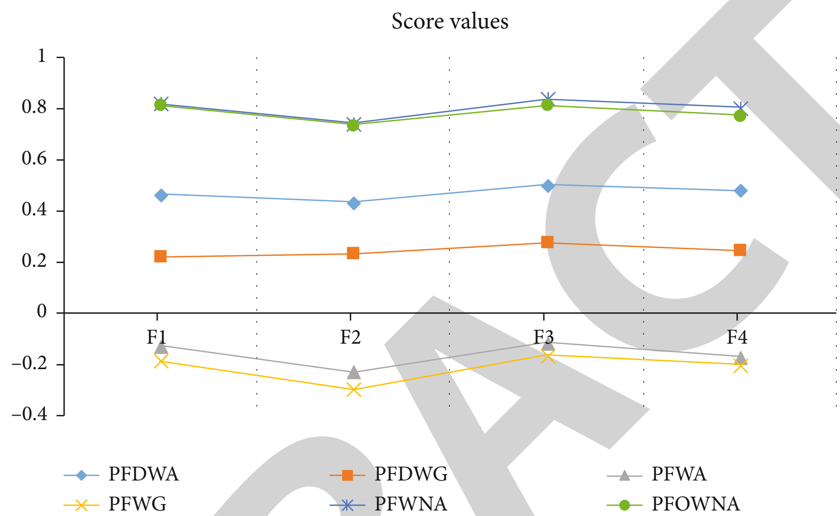

Figure 4: Pictorial analysis of validity and comparison.

Garg [52], Wei [40], and Jana at el. [42]. Moreover, neutrality towards the selection of alternatives plays an important part during the overall decision-making process so, in this way, the proposed operators provide better environment for decision-making. The validity of suggested AOs is evident from Figure 4. Throughout the analysis, there is no change in ranking patterns which shows that results obtained using novel operators agree with the results that exist in the literature. Furthermore, the stability of the results makes it clear from Figure 4 that the approach prescribed in this manuscript is much effective than the remaining methods recorded in Table 5.

6.4. Advantages of the Suggested Approach. Some salient features and advantages of the suggested approach are elaborated in the following:

(1) Suggested AOs can handle all the human aspects; in this way, decision-makers can handle various real-life situations more efficiently

(2) Degree of refusal plays an important role to choose the best alternative; during the information aggregation, by using AOs presented in this, work we can control refusal degree

(3) The main characteristic of this novel extension is the inclusion of decision-makers' attitude

(4) The ability to handle neutral behaviour of the experts makes this work more effective as compared to other studies presented in Table 5

7. Conclusion

Aggregation operators have a pivotal role during the MADM problems. Hence, we suggest some novel AOs in this paper named as PFWNA and PFOWNA on the basis of PFSs. Many researchers have made enormous contributions for IFSs which consider only MD and NMD. It has been noticed that some real-life scenarios cannot be represented clearly by using IFSs. During this work, we represented our information under PFS environment which extends the idea of IFS. Also, a novel MAGDM scheme is proposed on the basis of newly suggested AOs for PFSs. Some numerical examples have been demonstrated to prove the effectiveness of this approach. Finally, a comparative analysis is presented which shows the supremacy and advantages of this scheme. The concept can be further extended to develop neutrality AOs for spherical and T-spherical fuzzy sets [54, 55].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[3] K. Atanassov and G. Gargov, "Interval valued intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 31, no. 3, pp. 343–349, 1989.

[4] K. T. Atanassov, "Operators over interval valued intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 64, no. 2, pp. 159–174, 1994.

[5] Z. Xu, J. Chen, and J. Wu, "Clustering algorithm for intuitionistic fuzzy sets," *Information Sciences*, vol. 178, no. 19, pp. 3775–3790, 2008.

[6] Z. Xu and J. Wu, "Intuitionistic fuzzy C-means clustering algorithms," *Journal of Systems Engineering and Electronics*, vol. 21, no. 4, pp. 580–590, 2010.

[7] S. K. De, R. Biswas, and A. R. Roy, "An application of intuitionistic fuzzy sets in medical diagnosis," *Fuzzy Sets and Systems*, vol. 117, pp. 209–213, 2001.

[8] F. Xiao and W. Ding, "Divergence measure of Pythagorean fuzzy sets and its application in medical diagnosis," *Applied Soft Computing*, vol. 79, pp. 254–267, 2019.

[9] K. Kumar and H. Garg, "TOPSIS method based on the connection number of set pair analysis under interval-valued intuitionistic fuzzy set environment," *Computational and Applied Mathematics*, vol. 37, no. 2, pp. 1319–1329, 2018.

[10] P. Liu, "Multi-attribute decision-making method research based on interval vague set and TOPSIS method," *Technological and Economic Development of Economy*, vol. 15, no. 3, pp. 453–463, 2009.

[11] R. Lourenzutti and R. A. Krohling, "A study of TODIM in a intuitionistic fuzzy and random environment," *Expert Systems with Applications*, vol. 40, no. 16, pp. 6459–6468, 2013.

[12] G. Wei, "TODIM method for picture fuzzy multiple attribute decision making," *Informatica*, vol. 29, no. 3, pp. 555–566, 2018.

[13] D. Pamučar, I. Petrović, and G. Ćirović, "Modification of the Best-Worst and MABAC methods: A novel approach based on interval-valued fuzzy-rough numbers," *Expert Systems with Applications*, vol. 91, pp. 89–106, 2018.

[14] Z. Mu, S. Zeng, and P. Wang, "Novel approach to multi-attribute group decision-making based on interval- valued Pythagorean fuzzy power Maclaurin symmetric mean operator," *Computers & Industrial Engineering*, vol. 155, article 107049, 2021.

[15] H. Z. Ibrahim, T. M. Al-Shami, and O. Elbarbary, "(3, 2)-Fuzzy sets and their applications to topology and optimal choices," *Computational Intelligence and Neuroscience*, vol. 2021, Article ID 1272266, 14 pages, 2021.

[16] S. Zeng, "Uncertain intelligent computational decision-making methods," *Recent Advances in Computer Science and Communications*, vol. 14, pp. 2465-2466, 2021.

[17] G. Muhiuddin, D. Al-Kadi, A. Mahboob, and A. Albjedi, "Interval-valued m-polar fuzzy positive implicative ideals in BCK-algebras," *Mathematical Problems in Engineering*, vol. 2021, Article ID 1042091, 9 pages, 2021.

[18] G. Muhiuddin, D. Al-Kadi, A. Mahboob, and A. Aljohani, "Generalized fuzzy ideals of BCI-algebras based on interval valued m-polar fuzzy structures," *International Journal of Computational Intelligence Systems*, vol. 14, no. 1, pp. 1–9, 2021.

[19] L. Pan and Y. Deng, "A novel similarity measure in intuitionistic fuzzy sets and its applications," *Engineering Applications of Artificial Intelligence*, vol. 107, article 104512, 2022.

[20] R. R. Yager, "On ordered weighted averaging aggregation operators in multicriteria decisionmaking," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 18, no. 1, pp. 183–190, 1988.

[21] R. R. Yager and J. Kacprzyk, *The Ordered Weighted Averaging Operators: Theory and Applications*, Springer Science & Business Media, 2012.

[22] Z. Xu and R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *International Journal of General Systems*, vol. 35, no. 4, pp. 417–433, 2006.

[23] W. Wang and X. Liu, "Intuitionistic fuzzy information aggregation using Einstein operations," *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 5, pp. 923–938, 2012.

[24] A. S. A. Ghani and N. A. M. Isa, "Enhancement of low quality underwater image through integrated global and local contrast correction," *Applied Soft Computing*, vol. 37, pp. 332–344, 2015.

[25] Y. Xu, H. Wang, and J. M. Merigó, "Intuitionistic fuzzy Einstein Choquet integral operators for multiple attribute decision making," *Technological and Economic Development of Economy*, vol. 20, no. 2, pp. 227–253, 2014.

[26] S.-P. Wan, G.-L. Xu, F. Wang, and J.-Y. Dong, "A new method for Atanassov's interval-valued intuitionistic fuzzy MAGDM with incomplete attribute weight information," *Information Sciences*, vol. 316, pp. 329–347, 2015.

[27] T. Mahmood, Z. Ali, M. Aslam, and R. Chinram, "Generalized Hamacher aggregation operators based on linear Diophantine uncertain linguistic setting and their applications in decision-making problems," *IEEE Access*, vol. 9, pp. 126748–126764, 2021.

[28] N. Khan, N. Yaqoob, M. Shams, Y. U. Gaba, and M. Riaz, "Solution of linear and quadratic equations based on triangular linear Diophantine fuzzy numbers," *Journal of Function Spaces*, vol. 2021, Article ID 8475863, 14 pages, 2021.

[29] M. Riaz, D. Pamucar, A. Habib, and M. Riaz, "A new TOPSIS approach using cosine similarity measures and cubic bipolar

fuzzy information for sustainable plastic recycling process," *Mathematical Problems in Engineering*, vol. 2021, Article ID 4309544, 18 pages, 2021.

[30] H. M. A. Farid and M. Riaz, "Some generalized q-rung orthopair fuzzy Einstein interactive geometric aggregation operators with improved operational laws," *International Journal of Intelligent Systems*, vol. 36, no. 12, pp. 7239–7273, 2021.

[31] C. Jana, G. Muhiuddin, and M. Pal, "Multi-criteria decision making approach based on SVTrN Dombi aggregation functions," *Artificial Intelligence Review*, vol. 54, no. 5, pp. 3685–3723, 2021.

[32] E. K. Zavadskas, J. Antucheviciene, S. H. R. Hajiagha, and S. S. Hashemi, "Extension of weighted aggregated sum product assessment with interval-valued intuitionistic fuzzy numbers (WASPAS-IVIF)," *Applied Soft Computing*, vol. 24, pp. 1013–1021, 2014.

[33] H. Bustince and P. Burillo, "Correlation of interval-valued intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 74, no. 2, pp. 237–244, 1995.

[34] G. Wei and X. Zhao, "Minimum deviation models for multiple attribute decision making in intuitionistic fuzzy setting," *International Journal of Computational Intelligence Systems*, vol. 4, no. 2, pp. 174–183, 2011.

[35] B. C. Cuong and V. Kreinovich, "Picture fuzzy sets-a new concept for computational intelligence problems," in *2013 third world congress on information and communication technologies (WICT 2013)*, pp. 1–6, Hanoi, Vietnam, 2013.

[36] B. Cuong, *Picture fuzzy sets-first results. Part 1, seminar neuro-fuzzy systems with applications*, Institute of Mathematics, Hanoi, 2013.

[37] P. Singh, "Correlation coefficients for picture fuzzy sets," *Journal of Intelligent & Fuzzy Systems*, vol. 28, no. 2, pp. 591–604, 2015.

[38] L. H. Son, "A novel kernel fuzzy clustering algorithm for geodemographic analysis," *Information Sciences—Informatics and Computer Science, Intelligent Systems, Applications: An International Journal*, vol. 317, pp. 202–223, 2015.

[39] N. T. Thong and L. H. Son, "HIFCF: an effective hybrid model between picture fuzzy clustering and intuitionistic fuzzy recommender systems for medical diagnosis," *Expert Systems with Applications*, vol. 42, no. 7, pp. 3682–3701, 2015.

[40] G. Wei, "Picture fuzzy aggregation operators and their application to multiple attribute decision making," *Journal of Intelligent & Fuzzy Systems*, vol. 33, no. 2, pp. 713–724, 2017.

[41] G. Wei, "Picture fuzzy Hamacher aggregation operators and their application to multiple attribute decision making," *Fundamenta Informaticae*, vol. 157, no. 3, pp. 271–320, 2018.

[42] C. Jana, T. Senapati, M. Pal, and R. R. Yager, "Picture fuzzy Dombi aggregation operators: application to MADM process," *Applied Soft Computing*, vol. 74, pp. 99–109, 2019.

[43] C. Zuo, A. Pal, and A. Dey, "New concepts of picture fuzzy graphs with application," *Mathematics*, vol. 7, no. 5, p. 470, 2019.

[44] L. H. Son, P. Van Viet, and P. Van Hai, "Picture inference system: a new fuzzy inference system on picture fuzzy set," *Applied Intelligence*, vol. 46, no. 3, pp. 652–669, 2017.

[45] O. Barukab, S. Abdullah, S. Ashraf, M. Arif, and S. A. Khan, "A new approach to fuzzy TOPSIS method based on entropy measure under spherical fuzzy information," *Entropy*, vol. 21, no. 12, p. 1231, 2019.

[46] M. Mathew, R. K. Chakrabortty, and M. J. Ryan, "A novel approach integrating AHP and TOPSIS under spherical fuzzy sets for advanced manufacturing system selection," *Engineering Applications of Artificial Intelligence*, vol. 96, article 103988, 2020.

[47] P. Liu, M. Munir, T. Mahmood, and K. Ullah, "Some similarity measures for interval-valued picture fuzzy sets and their applications in decision making," *Information*, vol. 10, no. 12, p. 369, 2019.

[48] K. Ullah, "Picture fuzzy Maclaurin symmetric mean operators and their applications in solving multiattribute decision-making problems," *Mathematical Problems in Engineering*, vol. 2021, Article ID 1098631, 13 pages, 2021.

[49] K. Ullah, T. Mahmood, N. Jan, and Z. Ahmad, "Policy decision making based on some averaging aggregation operators of t-spherical fuzzy sets; a multi-attribute decision making approach," *Annals of Optimization Theory and Practice*, vol. 3, pp. 69–92, 2020.

[50] M. Akram, K. Ullah, and D. Pamucar, "Performance evaluation of solar energy cells using the interval-valued T-spherical fuzzy Bonferroni mean operators," *Energies*, vol. 15, no. 1, p. 292, 2022.

[51] S.-M. Chen and J.-M. Tan, "Handling multicriteria fuzzy decision-making problems based on vague set theory," *Fuzzy Sets and Systems*, vol. 67, no. 2, pp. 163–172, 1994.

[52] H. Garg, "Some picture fuzzy aggregation operators and their applications to multicriteria decision-making," *Arabian Journal for Science and Engineering*, vol. 42, no. 12, pp. 5275–5290, 2017.

[53] Y. He, H. Chen, Z. He, and L. Zhou, "Multi-attribute decision making based on neutral averaging operators for intuitionistic fuzzy information," *Applied Soft Computing*, vol. 27, pp. 64–76, 2015.

[54] T. Mahmood, K. Ullah, Q. Khan, and N. Jan, "An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets," *Neural Computing and Applications*, vol. 31, no. 11, pp. 7041–7053, 2019.

[55] H. Garg, K. Ullah, T. Mahmood, N. Hassan, and N. Jan, "T-spherical fuzzy power aggregation operators and their applications in multi-attribute decision making," *Journal of Ambient Intelligence and Humanized Computing*, vol. 12, no. 10, pp. 9067–9080, 2021.