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Research Article

A Note on Lacunary Sequence Spaces of Fractional Difference Operator of Order (α, β)

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In the present paper, we defined lacunary sequence spaces of fractional difference operator of order (α, β) over n-normed spaces via Musielak-Orlicz function $\mathscr{M} = (\mathfrak{I}_k)$. Our aim in this paper is to study some topological properties and inclusion relation between the spaces $\mathscr{I} - N_\alpha^\beta(A, \mathscr{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$, $\mathscr{I} - N_\alpha^\beta(A, \mathscr{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)$, and $\mathscr{I} - N_\alpha^\beta(A, \mathscr{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_\infty$.

1. Introduction and Preliminaries

The concept of statistical convergence was introduced by Fast [1] and Schoenberg [2] independently. Many authors studied the concept of statistical convergence from the past few years we may refer to ([3–19]) and references therein.

The sequence $\xi = (\xi_k)$ is statistically convergent of order α to ℓ (see Çolak) if there is a complex number ℓ such that

$$\lim_{n\longrightarrow\infty} \frac{1}{n^\alpha} |\{k \le n : |\xi_k - \ell| \ge \varepsilon\}| = 0. \tag{1}$$

Let $0 < \alpha \le \beta \le 1$. We define the (α, β)-density of the subset E of $\mathbb{N}$ by

$$\delta_\alpha^\beta(E) = \lim_n \frac{1}{n^\alpha} |\{k \le n : k \in E\}|^\beta, \tag{2}$$

provided the limit exists, where $|\{k \le n : k \in E\}|^\beta$ denotes the βth power of number of elements of E not exceeding n ([20–22]).

By a lacunary sequence $\theta = (\theta_r)$, we mean a sequence of positive integers such that $\theta_0 = 0$, $0 < \theta_r < \theta_{r+1}$, and $\phi_r = \theta_r - \theta_{r-1} \longrightarrow \infty$ as $r \longrightarrow \infty$. The intervals determined by θ will be denoted by $J_r = (\theta_{r-1}, \theta_r]$ and $t_r = \theta_r/\theta_{r-1}$. Freedman et al. [23] defined the space of lacunary strongly convergent sequences by

$$N_\theta = \left\{ \xi \in w : \lim_{r\longrightarrow\infty} \frac{1}{\phi_r} \sum_{k\in J_r} |\xi_k - l| = 0, \text{for some } l \right\}. \tag{3}$$

Definition 1. Let $\theta = (\theta_r)$ be a lacunary sequence. The sequence $\xi = (\xi_k)$ is $S_\alpha^\beta(\theta)$-statistically convergent (or lacunary statistically convergent of order (α, β)) (see [20]) if there is a real number L such that

$$\lim_{r\longrightarrow\infty} \frac{1}{\phi_r^\alpha} |\{k \in J_r : |\xi_k - L| \ge \varepsilon\}|^\beta = 0, \tag{4}$$

where $J_r = (\theta_{r-1}, \theta_r]$ and ϕ_r^α denotes the αth power $(\phi_r)^\alpha$ of ϕ_r, that is, $\phi^\alpha = (\phi_r^\alpha) = (\phi_1^\alpha, \phi_2^\alpha, \cdots, \phi_r^\alpha, \cdots)$. In this case, we write $S_\alpha^\beta(\theta) - \lim \xi_k = L$. The set of all $S_\alpha^\beta(\theta)$-statistically convergent sequences is denoted by $S_\alpha^\beta(\theta)$. If $\alpha=\beta=1$ and $\theta=(2^r)$, then, we will write S instead of $S_\alpha^\beta(\theta)$.

A family $\mathscr{I}\subset 2^X$ of subsets of a nonempty set X is said to be an ideal in X if

(1) $\phi\in\mathscr{I}$

(2) $A,B\in\mathscr{I}$ imply $A\cup B\in\mathscr{I}$

(3) $A\in\mathscr{I}$, $B\subset A$ imply $B\in\mathscr{I}$

while an admissible ideal $\mathscr{I}$ of X further satisfies $\{\xi\}\in\mathscr{I}$ for each $\xi\in X$ (see [24]).

A sequence $(\xi_n)_{n\in\mathbb{N}}$ in X is said to be $\mathscr{I}$-convergent to $\xi\in X$ (see [24]), if $A(\varepsilon)=\{n\in\mathbb{N}:\|\xi_n-\xi\|\geq\varepsilon\}\in\mathscr{I}$, for each $\varepsilon>0$.

A sequence $(\xi_n)_{n\in\mathbb{N}}$ in X is said to be $\mathscr{I}$-bounded to $\xi\in X$ if there exists an $K>0$ such that $\{n\in\mathbb{N}:|\xi_n|>K\}\in\mathscr{I}$. Many authors studied the topological properties and applications of ideal, we refer to ([25–37]) and references therein.

The concept of difference sequence spaces was introduced in [38] and further generalized in [39].

In [40], Baliarsingh defined the fractional difference operator as follows:

Let $\xi=(\xi_k)\in w$ and γ be a real number, then, the fractional difference operator $\Delta^{(\gamma)}$ is defined by

$$\Delta^{(\gamma)}\xi_k=\sum_{i=0}^{k}\frac{(-\gamma)_i}{i!}\xi_{k-i}, \tag{5}$$

where $(-\gamma)_i$ denotes the Pochhammer symbol defined as

$$(-\gamma)_i=\begin{cases}1, & \text{if } \gamma=0 \text{ or } i=0, \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+i-1), \text{otherwise},\\ \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+i-1), & \text{otherwise}.\end{cases} \tag{6}$$

The concept of difference sequences, Orlicz function, Musielak-Orlicz function, and n-normed spaces was used by many authors and proves some topological properties (see [41–50]) and references therein. For details about n-normed spaces, we refer to ([51–55]), difference sequence spaces ([38, 39]), Orlicz function ([56–58]). Ideal convergence and fractional difference operator Δ^α has been studied in [59, 60]. We continue in this connection and construct new sequence spaces as follows.

Let $M=(\mathfrak{J}_k)$ be a Musielak-Orlicz function, $u=(u_k)$ be a bounded sequence of positive real numbers, and $0<\alpha\leq\beta\leq 1$. We define the following sequence spaces in the present paper

$$\mathscr{I}-N_\alpha^\beta\left(A,\mathscr{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|\right)_0=\left\{\xi\in w:\mathscr{I}-\lim_r\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta=0, \text{for some } \rho>0\right\}, \tag{7}$$

$$\mathscr{I}-N_\alpha^\beta\left(A,\mathscr{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|\right)=\left\{\xi\in w:\mathscr{I}-\lim_r\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)-L}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta=0, \text{for some } L \text{ and } \rho>0\right\}, \tag{8}$$

$$\mathscr{I}-N_\alpha^\beta\left(A,\mathscr{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|\right)_\infty=\left\{\xi\in w:\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \text{is bounded, for some } \rho>0\right\}. \tag{9}$$

If we take $\mathscr{M}(\xi)=\xi$, the above spaces reduces to $\mathscr{I}-N_\alpha^\beta(A,u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_0$, $\mathscr{I}-N_\alpha^\beta(A,u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)$, and $\mathscr{I}-N_\alpha^\beta(A,u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_\infty$.

If we take $u=(u_k)=1$, the above spaces reduces to $\mathscr{I}-N_\alpha^\beta(A,\mathscr{M},\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_0$, $\mathscr{I}-N_\alpha^\beta(A,\mathscr{M},\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)$, and $\mathscr{I}-N_\alpha^\beta(A,\mathscr{M},\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_\infty$.

The following inequality will be used in the proceeding results. If $0 \le u_k \le \sup u_k = H$, $D = \max(1, 2^{H-1})$, then

$$|r_k + s_k|^{u_k} \le D\{|r_k|^{u_k} + |s_k|^{u_k}\}, \tag{10}$$

for all k and $r_k, s_k \in \mathbb{C}$. Also $|r|^{u_k} \le \max(1, |r|^H)$ for all $r \in \mathbb{C}$.

2. Main Results

In this section, we study topological properties and prove some inclusion relations. In what follows, we will take $M = (\mathfrak{I}_k)$ a Musielak-Orlicz function and $u = (u_k)$ a bounded sequence of positive real numbers.

Theorem 2. *The spaces* $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|$, $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)$, *and* $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_\infty$ *are linear spaces.*

Proof. Let $\xi_1, \xi_2 \in \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_0$ and let μ, ν be scalars. Then, there exist two positive numbers ρ_1 and ρ_2 for $\varepsilon > 0$

$$D_1 = \left\{ r \in \mathbb{N} : \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi_1\right)}{\rho_1}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \ge \frac{\varepsilon}{2D} \right\} \in \mathcal{I}, \tag{11}$$

$$D_2 = \left\{ r \in \mathbb{N} : \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi_2\right)}{\rho_2}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \ge \frac{\varepsilon}{2D} \right\} \in \mathcal{I}. \tag{12}$$

Let $\rho_3 = \max\{2|\mu|\rho_1, 2|\nu|\rho_2\}$ and by inequality (1), we have

$$\begin{aligned} &\frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}(\mu\xi_1 + \nu\xi_2)\right)}{\rho_3}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \\ &\le \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{\mu A_k\left(\Delta^{(\gamma)}\xi_1\right)}{\rho_3}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \\ &\quad + \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{\nu A_k\left(\Delta^{(\gamma)}\xi_2\right)}{\rho_3}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \\ &\le \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi_1\right)}{\rho_1}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \\ &\quad + \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi_2\right)}{\rho_2}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta. \end{aligned} \tag{13}$$

Now by (11) and (12), we get

$$\left\{ r \in \mathbb{N} : \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}(\mu\xi_1 + \nu\xi_2)\right)}{\rho_3}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta > \varepsilon \right\} \subset D_1 \cup D_2. \tag{14}$$

Therefore, $\mu\xi_1 + \nu\xi_2 \in \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_0$. Hence, $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_0$ is a linear space. On a similar way, we can prove that $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)$ and $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_\infty$ are linear spaces. □

Theorem 3. *The inclusions* $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_0 \subset \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|) \subset \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_\infty$ *hold.*

Proof. The inclusion $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_0 \subset \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)$ is obvious. We prove $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|) \subset \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_\infty$. For this, let $\xi \in \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)$. Then, there exists $\rho_1 > 0$ such that for every $\varepsilon > 0$

$$B_1 = \left\{ r \in \mathbb{N} : \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi\right) - L}{\rho_1}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \ge \varepsilon \right\} \in \mathcal{I}. \tag{15}$$

We put $\rho = 2\rho_1$ and $\mathcal{M} = (\mathfrak{I}_k)$ is a Musielak-Orlicz function, we have

$$\begin{aligned} &\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1, \cdots, x_{n-1} \right\| \right) \\ &\le \mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi\right) - L}{\rho_1}, x_1, \cdots, x_{n-1} \right\| \right) \\ &\quad + \mathfrak{I}_k \left(\left\| \frac{L}{\rho_1}, x_1, \cdots, x_{n-1} \right\| \right). \end{aligned} \tag{16}$$

Suppose that $r \notin B_1$. Hence by above inequality and (1), we have

$$\begin{aligned} &\frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \\ &\le D \left\{ \frac{1}{\phi_r^\alpha} \left[\sum_{k \in J_r} \left[\mathfrak{I}_k \left(\left\| \frac{A_k\left(\Delta^{(\gamma)}\xi\right) - L}{\rho_1}, x_1, \cdots, x_{n-1} \right\| \right) \right]^{u_k} \right]^\beta \right. \end{aligned}$$

$$+\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{L}{\rho_1},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\Bigg\}$$

$$< D\left\{\varepsilon+\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{L}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\right\}. \quad (17)$$

By using $[\mathfrak{I}_k(\|(L/\rho_1),x_1,\cdots,x_{n-1}\|)]^{u_k}\le \max\{1,[\mathfrak{I}_k(\|(L/\rho_1),x_1,\cdots,x_{n-1}\|)]^H\}$, we have

$$\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{L}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}<\infty. \quad (18)$$

Put $K=D\{\varepsilon+1/\phi_r^{\alpha}[\sum_{k\in J_r}[\mathfrak{I}_k(\|(L/\rho),x_1,\cdots,x_{n-1}\|)]^{u_k}]^{\beta}\}$. It follows that

$$\left\{r\in\mathbb{N}:\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}>K\right\}\in\mathcal{I}. \quad (19)$$

This shows that $\xi\in\mathcal{I}-N_{\alpha}^{\beta}(A,\mathcal{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_{\infty}$, which completes the proof. □

Theorem 4. *The space $\mathcal{I}-N_{\alpha}^{\beta}(A,\mathcal{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_{\infty}$ is a paranormed space with paranorm defined by*

$$g(x)=\inf\left\{\rho>0:\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\le 1\right\}. \quad (20)$$

Proof. Since $g(\xi)=g(-\xi)$ and $\mathfrak{I}_k(0)=0$, we have $g(0)=0$. Let $\xi_1,\xi_2\in\mathcal{I}-N_{\alpha}^{\beta}(A,\mathcal{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_{\infty}$. Let

$$B(\xi_1)=\left\{\rho>0:\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi_1\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\le 1\right\}, \quad (21)$$

$$B(\xi_2)=\left\{\rho>0:\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi_2\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\le 1\right\}. \quad (22)$$

Let $\rho_1\in B(\xi_1)$ and $\rho_2\in B(\xi_2)$ and $\rho=\rho_1+\rho_2$, we have

$$\begin{aligned}&\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}(\xi_1+\xi_2)\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^{\beta}\\&\le\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi_1\right)}{\rho_1},x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^{\beta}\\&\quad+\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi_2\right)}{\rho_2},x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^{\beta}.\end{aligned} \quad (23)$$

Thus, $1/\phi_r^{\alpha}[\sum_{k\in J_r}[\mathfrak{I}_k(\|(A_k(\Delta^{(\gamma)}(\xi_1+\xi_2))/\rho_1+\rho_2),x_1,\cdots,x_{n-1}\|)]^{u_k}]^{\beta}\le 1$ and

$$\begin{aligned}g(\xi_1+\xi_2)&\le\inf\{(\rho_1+\rho_2)>0:\rho_1\in B(\xi_1),\rho_2\in B(\xi_2)\}\\&\le\inf\{\rho_1>0:\rho_1\in B(\xi_1)\}\\&\quad+\inf\{\rho_2>0:\rho_2\in B(\xi_2)\}\\&=g(\xi_1)+g(\xi_2).\end{aligned} \quad (24)$$

Let $\sigma^s\longrightarrow\sigma$ where $\sigma,\sigma^s\in\mathbb{C}$ and let $g(\xi^s-\xi)\longrightarrow 0$ as $s\longrightarrow\infty$. We have to show that $g(\sigma^s\xi^s-\sigma\xi)\longrightarrow 0$ as $s\longrightarrow\infty$. Let

$$B(\xi^s)=\left\{\rho_s>0:\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi^s\right)}{\rho_s},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\le 1\right\}, \quad (25)$$

$$B(\xi^s-\xi)=\left\{\rho_s'>0:\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}(\xi^s-\xi)\right)}{\rho_s'},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^{\beta}\le 1\right\}. \quad (26)$$

If $\rho_s\in B(\xi^s)$ and $\rho_s'\in B(\xi^s-\xi)$; then, we have

$$\begin{aligned}&\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\left(\sigma^s\xi^s-\sigma\xi\right)\right)}{\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|},x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^{\beta}\\&\le\frac{1}{\phi_r^{\alpha}}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\left(\sigma^s\xi^s-\sigma\xi^s\right)\right)}{\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|}\right.\right.\right.\right.\\&\quad\left.\left.\left.\left.+\frac{\left|A_k\left(\Delta^{(\gamma)}\left(\sigma\xi^s-\sigma\xi\right)\right)\right|}{\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|},x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^{\beta}\end{aligned}$$

$$\leq \frac{|\sigma^s-\sigma|\rho_s}{\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|}\frac{1}{\phi_r^\alpha}\cdot\left[\sum_{k\in J_r}\left[\mathfrak{J}_k\left(\left\|\frac{\left(A_k\left(\Delta^{(\gamma)}\left(\xi^s\right)\right)\right)}{\rho_s},x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^\beta + \frac{|\sigma|\rho_s'}{\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|}\frac{1}{\phi_r^\alpha}\cdot\left[\sum_{k\in J_r}\left[\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\left(\xi^s-\xi\right)\right)}{\rho_s'},x_1,\cdots,x_{n-1}\right\|\right)\right]\right]^\beta. \quad (27)$$

From the above inequality, it follows that

$$\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\left(\sigma^s\xi^s-\sigma\xi\right)\right)}{\rho_s\mid\sigma^s-\sigma\mid+\rho_s'\mid\sigma\mid},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\leq 1, \quad (28)$$

and consequently,

$$\begin{aligned} g(\sigma^s\xi^s-\sigma\xi)&\leq\inf\left\{\left(\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|\right)>0:\rho_s\in B\left(\xi^s\right),\rho_s'\in B\left(\xi^s-\xi\right)\right\}\\ &\leq(|\sigma^s-\sigma|)>0\inf\left\{\rho>0:\rho_s\in B\left(\xi^s\right)\right\}+(|\sigma|)\\ &>0\inf\left\{\left(\rho_s'\right)^{u_n/H}:\rho_s'\in B\left(\xi^s-\xi\right)\right\}\longrightarrow 0\, as\, s\longrightarrow\infty, \end{aligned} \quad (29)$$

which completes the proof. □

Theorem 5. *Let $\mathscr{M}=(\mathfrak{J}_k)$ and $\mathscr{M}'=(\mathfrak{J}_k')$ be Musielak-Orlicz functions that satisfy the Δ_2-condition. Then*

$$\mathscr{I}-N_\alpha^\beta\left(A,\mathscr{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|\right)_0\subseteq\mathscr{I}-N_\alpha^\beta\left(A,\mathscr{M}'\circ\mathscr{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|\right)_0, \quad (30)$$

$$\mathscr{I}-N_\alpha^\beta\left(A,\mathscr{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|\right)\subseteq\mathscr{I}-N_\alpha^\beta\left(A,\mathscr{M}'\circ\mathscr{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|\right), \quad (31)$$

$$\mathscr{I}-N_\alpha^\beta\left(A,\mathscr{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|\right)_\infty\subseteq\mathscr{I}-N_\alpha^\beta\left(A,\mathscr{M}'\circ\mathscr{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|\right)_\infty. \quad (32)$$

Proof. (i) Let $\xi\in\mathscr{I}-N_\alpha^\beta(A,\mathscr{M},u,\theta,\Delta^{(\gamma)},\|\cdot,\cdots,\cdot\|)_0$. Then, there exists $K_1>0$ such that

$$B_1=\left\{r\in\mathbb{N}:\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta\geq K_1\right\}\in\mathscr{I}, \quad (33)$$

for $\rho>0$. Since $\mathscr{M}'$ is a Musielak-Orlicz function which satisfies Δ_2-condition, we have

$$\begin{aligned} &\frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ \mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)>\delta}}\left[\mathfrak{J}_k'\left(\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right)\right]^{u_k}\right]^\beta\\ &\leq\max\left\{1,\left(K\frac{1}{\delta}\mathfrak{J}_k'(2)\right)^H\right\}\frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ \mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)>\delta}}\left(\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right)^{u_k}\right]^\beta, \end{aligned} \quad (34)$$

for $K\geq 1$. By continuity of $\mathscr{M}'$, we have

$$\begin{aligned} &\frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ \mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\leq\delta}}\left[\mathfrak{J}_k'\left(\mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right)\right]^{u_k}\right]^\beta\\ &\leq\frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ \mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\leq\delta}}\varepsilon^{u_k}\right]^\beta\\ &\leq\frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ \mathfrak{J}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\leq\delta}}\max\left\{\varepsilon^h,\varepsilon^H\right\}\right]^\beta. \end{aligned} \quad (35)$$

Suppose $r \notin B_1$. Then, by using (34) and (35), we have

$$\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}'_k\left(\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right)\right]^{u_k}\right]^\beta = \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ \mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)>\delta}}\left(\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right)^{u_k}\right]^\beta + \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ \mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\le\delta}}\left(\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right)^{u_k}\right]^\beta \le \max\left\{1,\left(K\frac{1}{\delta}\mathfrak{I}'_k(2)\right)^H\right\}K_1 + \max\left\{\varepsilon^h,\varepsilon^H\right\} = K_2. \quad (36)$$

Hence, $r \notin B_2 = \{r\in\mathbb{N} : 1/\phi_r^\alpha[\sum_{k\in J_r}[\mathfrak{I}'_k(\mathfrak{I}_k(\|A_k(\Delta^{(\gamma)}\xi)/\rho, x_1,\cdots,x_{n-1}\|))]^{u_k}]^\beta > K_2\}$ and so $B_2 \subset B_1$ which implies $B_2 \in \mathcal{I}$. This shows that $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}' \circ \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Hence, $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0 \subseteq \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}' \circ \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Similarly, we can prove (ii) and (iii) part. □

Corollary 6. *Let $\mathcal{M} = (\mathfrak{I}_k)$ satisfy Δ_2-condition. Then,*

$$\mathcal{I} - N_\alpha^\beta\left(A, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0 \subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0, \quad (37)$$

$$\mathcal{I} - N_\alpha^\beta\left(A, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right) \subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right), \quad (38)$$

$$\mathcal{I} - N_\alpha^\beta\left(A, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_\infty \subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_\infty. \quad (39)$$

Proof. If we put $\mathfrak{I}_k(x) = x$ and $\mathfrak{I}'_k(x) = \mathfrak{I}_k(x) \forall x \in [0,\infty)$ in Theorem 5, the result follows. □

Theorem 7. *Let $\mathcal{M} = (\mathfrak{I}_k)$ and $\mathcal{M}' = (\mathfrak{I}'_k)$ be Musielak-Orlicz functions that satisfy the Δ_2-condition. Then,*

$$\mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0 \cap \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}', u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0 \subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}' + \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_0, \quad (40)$$

$$\mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right) \cap \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}', u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right) \subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}' + \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right), \quad (41)$$

$$\mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_\infty \cap \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_\infty \subseteq \mathcal{I} - N_\alpha^\beta\left(A, \mathcal{M}' + \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|\right)_\infty. \quad (42)$$

Proof. (i) Let $\xi \in \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0 \cap \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}', u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Then, there exists $K_1 > 0$ and $K_2 > 0$ such that

$$B_1 = \left\{r\in\mathbb{N} : \frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \ge K_1\right\} \in \mathcal{I}, \quad (43)$$

$$B_2 = \left\{r\in\mathbb{N} : \frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}'_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \ge K_2\right\} \in \mathcal{I}, \quad (44)$$

for some $\rho > 0$. Let $r \notin B_1 \cup B_2$. Then, we have

$$\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k + \mathfrak{I}'_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \le D\left\{\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left(\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right)^{u_k}\right]^\beta + \frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left(\mathfrak{I}'_k\left(\left\|\frac{A_k\left(\Delta^{(\alpha)}\xi\right)}{\rho},x_1,\cdots,x_{n-1}\right\|\right)\right)^{u_k}\right]^\beta\right\} < \{K_1 + K_2\}. \quad (45)$$

$r \notin B = \{r\in\mathbb{N} : 1/\phi_r^\alpha[\sum_{k\in J_r}[(\mathfrak{I}'_k + \mathfrak{I}_k)(A_k(\Delta^{(\gamma)}\xi)/\rho, x_1,\cdots, x_{n-1})]^{u_k}]^\beta > K\}$. We have $B_1 \cup B_2 \in \mathcal{I}$ and so $B \subset B_1 \cup B_2$ which implies $B \in \mathcal{I}$. This shows that $x \in \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}' + \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0$. Hence, $\mathcal{I} - N_\alpha^\beta(A, \mathcal{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0 \cap \mathcal{I} - N_\alpha^\beta(A, \mathcal{M}', u, \theta, \Delta^{(\gamma)}, \|\cdot,\cdots,\cdot\|)_0 \subseteq \mathcal{I} -$

$N_\alpha^\beta(A, \mathscr{M}' + \mathscr{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_0$. Similarly, we can prove (ii) and (iii) part of the theorem. □

Theorem 8. *Let $0 < u_k \leq v_k$ and (v_k/u_k) be bounded. Then, the following inclusions hold*

$$\mathscr{I} - N_\alpha^\beta\left(A, \mathscr{M}, v, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|\right)_0 \subseteq \mathscr{I} - N_\alpha^\beta\left(A, \mathscr{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|\right)_0, \tag{46}$$

$$\mathscr{I} - N_\alpha^\beta\left(A, \mathscr{M}, v, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|\right) \subseteq \mathscr{I} - N_\alpha^\beta\left(A, \mathscr{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|\right). \tag{47}$$

Proof. (i) Let $\xi \in \mathscr{I} - N_\alpha^\beta(A, \mathscr{M}, v, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_0$. Write $s_k = [\mathfrak{I}_k(\|A_k(\Delta^{(\gamma)}\xi)/\rho, x_1, \cdots, x_{n-1}\|)]^{v_k}$ and $\lambda_k = u_k/v_k$, so that $0 < \lambda < \lambda_k \leq 1$. By using Hölder inequality, we have

$$\begin{aligned}
\frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}(s_k)^{\lambda_k}\right]^\beta &= \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ s_k\geq 1}}(s_k)^{\lambda_k}\right]^\beta + \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ s_k<1}}(s_k)^{\lambda_k}\right]^\beta \\
&\leq \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ s_k\geq 1}}(s_k)\right]^\beta + \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ s_k<1}}(s_k)^{\lambda}\right]^\beta \\
&= \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ s_k\geq 1}}(s_k)\right]^\beta + \left[\sum_{\substack{k\in J_r\\ s_k<1}}\left(\frac{1}{\phi_r^\alpha}s_k\right)^{\lambda}\left(\frac{1}{\phi_r^\alpha}\right)^{1-\lambda}\right]^\beta \\
&\leq \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ s_k\geq 1}}(s_k)\right]^\beta + \left[\left(\sum_{\substack{k\in J_r\\ s_k<1}}\left[\left(\frac{1}{\phi_r^\alpha}s_k\right)^{\lambda}\right]^{1/\lambda}\right)^{\lambda}\right]^\beta \\
&\quad \cdot \left[\left(\sum_{\substack{k\in J_r\\ s_k<1}}\left[\left(\frac{1}{\phi_r^\alpha}\right)^{1-\lambda}\right]^{1/\lambda-1}\right)^{1-\lambda}\right]^\beta \\
&\leq \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ s_k\geq 1}}(s_k)\right]^\beta + \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ s_k<1}}(s_k)^{\lambda}\right]^\beta.
\end{aligned} \tag{48}$$

Hence, for every $\varepsilon > 0$, we have

$$\begin{aligned}
&\left\{r\in\mathbb{N} : \frac{1}{\phi_r^\alpha}\left[\sum_{k\in J_r}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1, \cdots, x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \geq \varepsilon\right\} \\
&\subset \left\{r\in\mathbb{N} : \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ s_k\geq 1}}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1, \cdots, x_{n-1}\right\|\right)\right]^{u_k}\right]^\beta \geq \frac{\varepsilon}{2}\right\} \\
&\cup \left\{r\in\mathbb{N} : \frac{1}{\phi_r^\alpha}\left[\sum_{\substack{k\in J_r\\ s_k<1}}\left[\mathfrak{I}_k\left(\left\|\frac{A_k\left(\Delta^{(\gamma)}\xi\right)}{\rho}, x_1, \cdots, x_{n-1}\right\|\right)\right]^{v_k}\right]^\beta \geq \left(\frac{\varepsilon}{2}\right)^{1/\lambda}\right\}.
\end{aligned} \tag{49}$$

This implies that $\{r \in \mathbb{N} : 1/\phi_r^\alpha[\sum_{k\in I_r}[\mathfrak{I}_k(\|A_k(\Delta^{(\gamma)}\xi)/\rho, x_1, \cdots, x_{n-1}\|)]^{u_k}]^\beta \geq \varepsilon\} \in \mathscr{I}$ and so $\xi \in \mathscr{I} - N_\alpha^\beta(A, \mathscr{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_0$. Hence, $\mathscr{I} - N_\alpha^\beta(A, \mathscr{M}, v, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_0 \subseteq \mathscr{I} - N_\alpha^\beta(A, \mathscr{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)_0$. Similarly, we can prove $\mathscr{I} - N_\alpha^\beta(A, \mathscr{M}, v, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|) \subseteq \mathscr{I} - N_\alpha^\beta(A, \mathscr{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|)$. □

Corollary 9. *If $0 < \inf u_k \leq 1$. Then, the following inclusions hold:*

$$\mathscr{I} - N_\alpha^\beta\left(A, \mathscr{M}, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|\right)_0 \subseteq \mathscr{I} - N_\alpha^\beta\left(A, \mathscr{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|\right)_0 \tag{50}$$

$$\mathscr{I} - N_\alpha^\beta\left(A, \mathscr{M}, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|\right) \subseteq \mathscr{I} - N_\alpha^\beta\left(A, \mathscr{M}, u, \theta, \Delta^{(\gamma)}, \|\cdot, \cdots, \cdot\|\right). \tag{51}$$

Proof. The proof follows from Theorem 8. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to writing this paper. All authors read and approved the manuscript.

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