Hindawi
Journal of Function Spaces
Volume 2024, Article ID 9895201, 1 page
https://doi.org/10.1155/2024/9895201

Retraction

Retracted: Statistical Analysis of COVID-19 Data: Using A New Univariate and Bivariate Statistical Model

Journal of Function Spaces

Received 23 January 2024; Accepted 23 January 2024; Published 24 January 2024

Copyright © 2024 Journal of Function Spaces. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] R. A. R. Bantan, S. Shafiq, M. H. Tahir et al., "Statistical Analysis of COVID-19 Data: Using A New Univariate and Bivariate Statistical Model," *Journal of Function Spaces*, vol. 2022, Article ID 2851352, 26 pages, 2022.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2851352, 26 pages
https://doi.org/10.1155/2022/2851352

Research Article

Statistical Analysis of COVID-19 Data: Using A New Univariate and Bivariate Statistical Model

Rashad A. R. Bantan[1] **Shakaiba Shafiq**[2] **M. H. Tahir**[2] **Ahmed Elhassanein**[3,4] **Farrukh Jamal**[2] **Waleed Almutiry**[5] **and Mohammed Elgarhy**[6]

[1]*Department of Marine Geology, Faculty of Marine Science, King Abdulaziz University, Jeddah 21551, Saudi Arabia*
[2]*Department of Statistics, Faculty of Computing, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan*
[3]*Department of Mathematics, College of Science, University of Bisha, Bisha, Saudi Arabia*
[4]*Department of Mathematics, Faculty of Science, Damanhour University, Damanhour, Egypt*
[5]*Department of Mathematics, College of Science and Arts in Ar Rass, Qassim University, Buryadah 52571, Saudi Arabia*
[6]*The Higher Institute of Commercial Sciences, Al Mahalla Al Kubra, Algarbia 31951, Egypt*

Correspondence should be addressed to Farrukh Jamal; farrukh.jamal@iub.edu.pk
and Mohammed Elgarhy; m_elgarhy85@sva.edu.eg

Received 1 April 2022; Revised 8 May 2022; Accepted 17 May 2022; Published 23 June 2022

Academic Editor: Muhammad Gulzar

Copyright © 2022 Rashad A. R. Bantan et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, a new distribution named as unit-power Weibull distribution (UPWD) defined on interval (0,1) is introduced using an appropriate transformation to the positive random variable of the Weibull distribution. This work offers quantile function, linear representation of the density, ordinary and incomplete moments, moment-generating function, probability-weighted moments, L-moments, TL-moments, Rényi entropy, and MLE estimation. Additionally, several actuarial measures are computed. The real data applications are carried out to underline the practical usefulness of the model. In addition, a bivariate extension for the univariate power Weibull distribution named as bivariate unit-power Weibull distribution (BIUPWD) is also configured. To elucidate the bivariate extension, simulation analysis and application using COVID-19-associated fatality rate data from Italy and Belgium to conform a BIUPW distribution with visual depictions are also presented.

RETRACTED

1. Introduction

Many disciplines of applied science deal with the constraints of bounded variables measuring specific features of phenomena. Variables like proportions of a certain attribute, comparing prices of a grocery item, profit or loss in a business, checking an ability for a job, likes or dislikes about the product of a company, and rates set on the interval (0,1) are frequently encountered in metrology, biological studies, economics, and other sciences. For adequate modeling of these variables, continuous probability distributions with support of [0,1] also known as unit distributions are essential. Although the Beta distribution [1] and Kumaraswamy distribution [2] are most widely used models for modeling data sets on the interval [0,1], neither the beta distribution holds closed form expressions of cumulative distribution function nor Kumaraswamy distribution holds closed form expressions of moments. Many unit distributions as alternatives to these distributions are presented in the literature to meet this prerequisite. The most valuable unit distributions with a given set of parameters are Johnson S_B [3], Topp-Leone distribution [4], unit-Weibull distribution [5], unit-Gamma distribution [6], unit-Gompertz distribution [7], unit-inverse Gaussian distribution [8], unit-Lindley distribution [9, 10], unit-generalized half normal distribution [11], unit-modified Burr-III distribution [12], unit-Chen distribution [13], unit-Rayleigh distribution [14], unit power-logarithmic distribution [15] and unit Nadarajah and Haghighi [16].

The fundamental goal of the article under consideration is to introduce a new unit-power Weibull distribution

(UPWD *for short*) as well as to investigate its statistical characteristics. The following points provide sufficient incentive to study the proposed model. We specify it as follows: (i) we employed a unique transformation to develop UPWD instead of employing traditional transformation found in literature to propose unit distributions which include $Y = e^{-x}$, $(1+x)^{-1}$, or $Y = (x)(1+x)^{-1}$, depending upon the functional identifiability of the baseline model; (ii) recent developments in distribution theory have shown a significant rise in the analysis of bivariate extensions of univariate models; for further information, we may refer the readers to see in [17–20]. So, we introduced and thoroughly explored a bivariate extension of a unit distribution, known as the bivariate unit-power Weibull distribution (BIUPWD *for short*) as far as no bivariate extension has been explored for the unit distributions in the literature. This is accomplished through a simulation analysis and application based on risks associated with COVID-19 data; (iii) it is remarkable to observe the flexibility of the proposed model with the diverse graphical shapes of probability density functions (pdfs) and hazard rate functions (hrfs). So, the form analysis of the corresponding pdf and hrf has shown new characteristics, revealing the unseen fitting potential of UPWD; (iv) because of the enhanced flexibility of the postulated distribution in terms of tail features, it can now be applied to risk evaluation theory with substantially better outcomes; (v) not just limited to flexibility in terms of tails, a unique feature to capture the entire information available is also illustrated using Min–Max approach. Hence, the proposed model with three parameters can be implemented to fit data in diverse scientific entities. This ability of the model is explored using three real-life data sets proving the practical utility of the model being featured.

1.1. Paper Organization. The paper is structured as follows: In Section 2, the development of the proposed model UPWD after reparameterizing the Weibull distribution using an appropriate transformation is expressed. The distribution function (cdf), pdf, survival function (sf), and hrf along with asymptotes and graphical shapes for pdf and hrf are presented in this section. In Section 3, explicit expressions of some basic properties of the proposed UPWD such as quantile function, linear representation of the density, rth ordinary and sth incomplete moments, moment-generating function, probability-weighted moments, order statistics, entropy measure, L-moments, and Trimmed L- (TL-) moments are established. In Section 3.5, we carried out estimation using maximum likelihood estimation (MLE) to estimate the unknown parameters of the UPWD. In Section 4, a Monte Carlo simulation analysis is performed to examine the accuracy of the MLE parameters of UPWD. This simulation is replicated for $N = 2000$ times, each with different sample sizes as 25, 50, 100, 300, 500, and 750 for the random parametric combinations. In Section 5, we evaluated risk evaluation measures by studying value at risk, expected shortfall, tail value at risk, tail variance, and tail variance premium. Numerical illustration and plots of value at risk and expected shortfall are also presented in this section. In Section 6, we carried out application for the UPWD using three real data sets. We also presented the descriptive summary and total time on test (TTT) plots for the UPWD in this section. In addition, the proposed model is compared with five comparative models, namely, exponentiated Weibull (EW), Kumaraswamy exponential (KE), gamma Kumaraswamy (GK), and beta exponential (BE). In Section 7, we introduced a bivariate extension for the univariate unit-power Weibull distribution, namely, bivariate unit-power Weibull (BIUPW) for a bivariate continuous random vector (X, Y). The estimation, simulation, and application to real data set of COVID-19 along with graphical presentation for marginal densities are illustrated in this section. Finally, in Section 8, some concluding remarks of our findings for all sections of this paper are presented.

2. Unit-Power Weibull Distribution

Weibull distribution [21] initially proposed in 1951, is well-established model to assess the time to event phenomenon for bounded interval. The cdf of well-known Weibull model is as follows:

$$F(x) = 1 - e^{-\alpha x^{\beta}} \quad x > 0, \alpha, \beta > 0. \tag{1}$$

Restricting our focus on extensions of Weibull distribution in unit interval context, several extensions/modifications have been employed. For instance, in [5], the authors employed $Y = e^{-x}$ to propose unit-Weibull distribution.

$$F(y) = e^{-\alpha(-\log y)^{\beta}} \quad \alpha, \beta > 0. \tag{2}$$

For $\beta = 1$, the authors studied the unit-Rayleigh model in [14] and explored some of its interesting properties. Given the significance of Weibull distribution in lifetime analysis, for cdf defined in Equation (1), we use a new transformation $y = 1 - (1+x)^{-1/\lambda}$ to propose a novel UPWD with support on the unit interval.

Proposition 1. *Let $Y \sim UPWD(\alpha, \beta, \lambda)$ for $y \in (0, 1)$ and $\alpha, \beta, \lambda > 0$; then, its pdf and cdf, respectively, are given by the following equations:*

$$f(y) = \alpha\lambda\beta(1-y)^{-\lambda-1}\left[(1-y)^{-\lambda} - 1\right]^{\beta-1} e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}, \tag{3}$$

$$F(y) = 1 - e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}. \tag{4}$$

By using standard asymptotic arguments, we have

$$\lim_{y\longrightarrow 0} F(y) = 1 - e^{0} \longrightarrow 0, \qquad \lim_{y\longrightarrow 1} F(y) = 1 - e^{-\infty} \longrightarrow 1. \tag{5}$$

RETRACTED

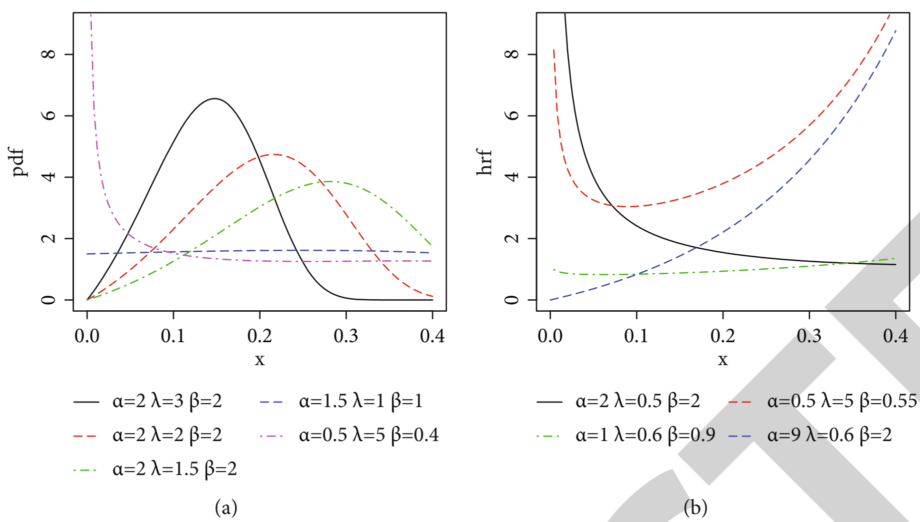


FIGURE 1: Plots of (a) UPWD density and (b) hazard function for random parametric values.

Proposition 2. *For $0 < y < 1$ and $\alpha, \beta, \lambda > 0$, the following results hold for UPWD density at the boundaries*

$$\lim_{y \longrightarrow 0} f(y) = 0,$$
$$\lim_{y \longrightarrow 1} f(y) = 0. \tag{6}$$

Proposition 3. *Let $Y \sim UPWD(\alpha, \beta, \lambda)$ for $y \in (0, 1)$ and $\alpha, \beta, \lambda > 0$; then, its survival function (sf) is given in the following equation:*

$$S(x) = e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}. \tag{7}$$

Proposition 4. *For $0 < y < 1$ and $\alpha, \beta, \lambda > 0$, at boundaries $h(y) = 0$ for $y = 0$ and $h(y) = \infty$ for $y = 1$, the hazard rate function (hrf) is given by the following expression:*

$$h(x) = \alpha\lambda\beta(1-y)^{-\lambda-1}\left[(1-y)^{-\lambda} - 1\right]^{\beta-1}. \tag{8}$$

In Figure 1, some shapes of pdf and hrf are displayed. In Figure 1(a), the possible shapes of UPWD density are featured while in Figure 1(b), the shapes of hrf are depicted. In addition to monotone (increasing, decreasing, and constant), nonmonotone shapes (bathtub) are also yielded which are suggestive of the added flexibility due to the resulting transformation. Additional graphical illustrations are presented in Figures 2 and 3.

3. Properties

This section provides the structural properties of the UPWD, defined in Equation (4), including explicit expressions for quantile function (qf), linear representation of the density, rth ordinary and sth incomplete moment, moment-generating function, probability-weighted moments, the expression of order statistics, uncertainty evaluating measure, L-moments, and TL-moments. Some graphical illustrations in relation to these characteristics are also featured.

3.1. Quantile Function. The qf is an accurate statistical metric which can be used to build artificial survival time data sets in biological case studies, determine percentiles in time to failure distributions, and examine particular risk indicators in actuarial context. The qf is also important to generate random variates. For $u \sim \text{uniform}(0, 1)$, the qf of the UPWD is given in Equation (9) as follows.

Proposition 5. *Let $Y \sim UPWD(\alpha, \beta, \lambda)$ for $y \in (0, 1)$ and $\alpha, \beta, \lambda > 0$; then, its quantile function is given in the following equation:*

$$Q_u = \left[1 - \left\{\left(\frac{1}{-\alpha}\log\left[1-u\right]\right)^{1/\beta} + 1\right\}^{-1/\lambda}\right]. \tag{9}$$

By replacing $u = 0.5$ in Equation (9), the median of the UPWD is readily available.

3.2. Useful Expansion. Here we showed the useful expansion of the UPWD density which can be used to drive several important properties of the UPWD. Here we use the following two series to obtain the expansion for UPWD.

Proposition 6. *The generalized binomial expansion is given in the following equation which holds for any real noninteger b and $|t| < 1$.*

$$(1-t)^b = \sum_{c=0}^{\infty}(-1)^c\binom{b}{c}t^c. \tag{10}$$

RETRACTED

RETRACTED

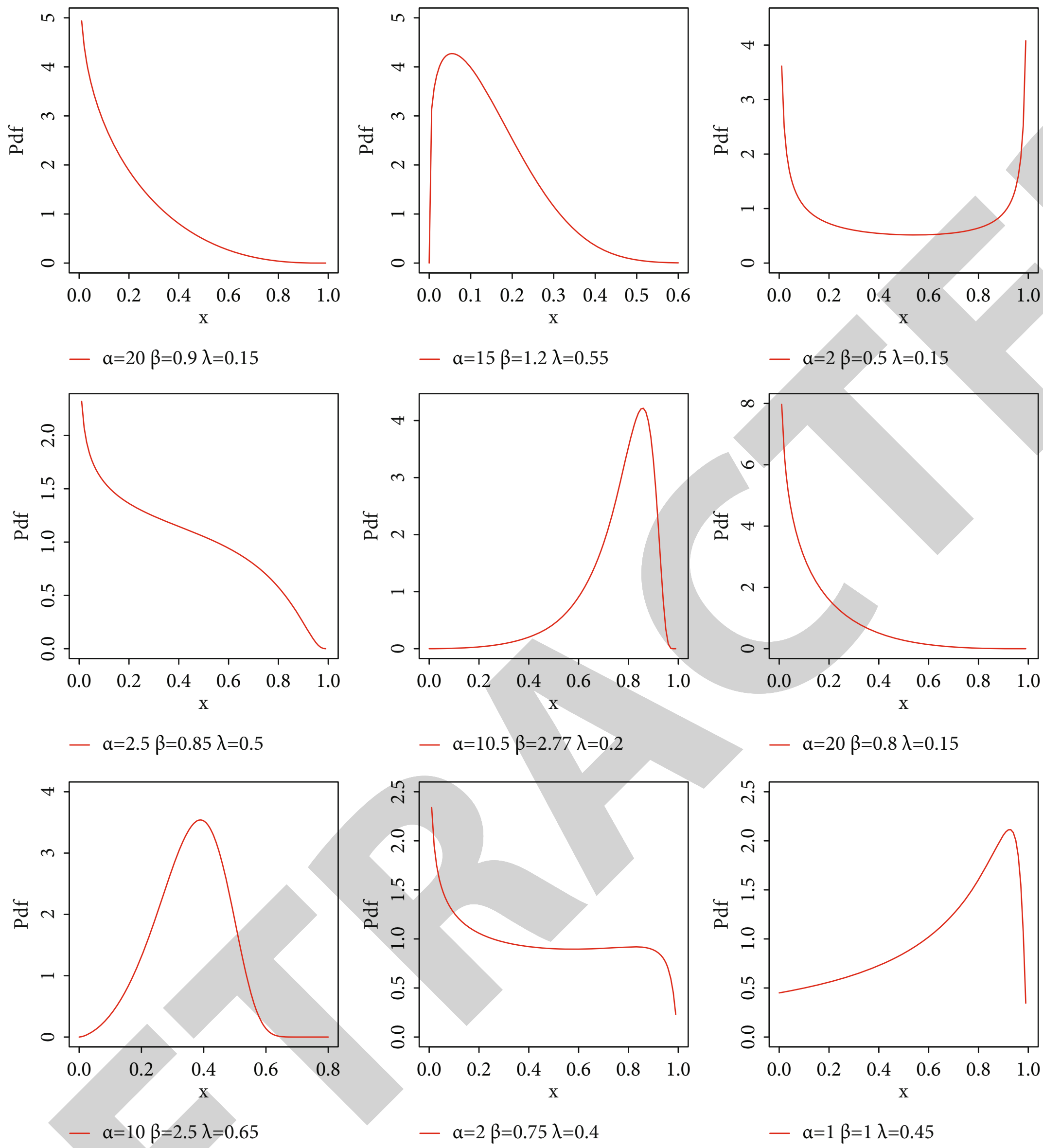


Figure 2: Plots of UPWD density for some parametric values.

Power series for exponential function, the series is also used by Bourguignon et al. [22].

$$\exp\left\{-\alpha[x]^{b}\right\} = \sum_{k=0}^{\infty}(-1)^{k}\alpha^{k}\frac{x^{kb}}{k!}. \tag{11}$$

By using Equation (3) and applying generalized binomial expansion (10)

$$f(y) = \alpha\lambda\beta(1-y)^{-\lambda-1}\underbrace{\left[(1-y)^{-\lambda}-1\right]^{\beta-1}}_{I}e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}. \tag{12}$$

For simplification, consider the term in I in above equation as

$$\left[(1-y)^{-\lambda}-1\right]^{\beta-1} = (-1)^{\beta-1}\sum_{z=0}^{\infty}(-1)^{z}\binom{\beta-1}{z}(1-y)^{-\lambda z}. \tag{13}$$

Now the term I reduced to

$$I = (-1)^{\beta-1}\sum_{z=0}^{\infty}(-1)^{z}\binom{\beta-1}{z}(1-y)^{-\lambda-\lambda z-1}. \tag{14}$$

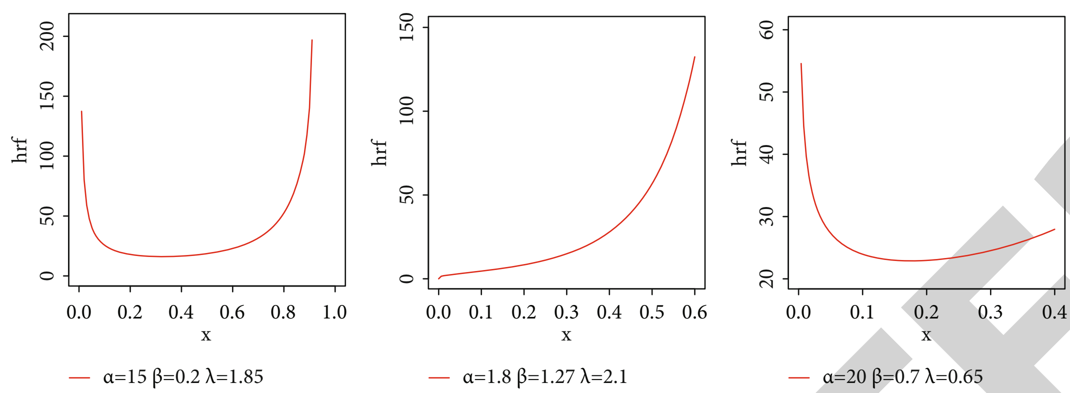


FIGURE 3: Plots of UPWD hrf for some parametric values.

Substituting the result of term I in Equation (3) reduced to

$$f(y)=\alpha\lambda\beta(-1)^{\beta-1}\sum_{z=0}^{\infty}(-1)^{z}\begin{pmatrix}\beta-1\\ z\end{pmatrix}(1-y)^{-\lambda-\lambda z-1}e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}. \tag{15}$$

Now, applying power series Equation (11) for exponential function and after some algebra, Equation (15) reduced to

$$f(y)=\alpha\lambda\beta\sum_{z,p=0}^{\infty}\sum_{t=0}^{\infty}(-1)^{z+p+t+\beta-1+p\beta}\frac{\alpha^{p}}{p!}\begin{pmatrix}\beta-1\\ z\end{pmatrix}\begin{pmatrix}p\beta\\ t\end{pmatrix}(1-y)^{-\lambda-\lambda z-1-\lambda t}, \tag{16}$$

where

$$w_t=\alpha\lambda\beta\sum_{z,p=0}^{\infty}(-1)^{z+p+t+\beta-1+p\beta}\frac{\alpha^{p}}{p!}\begin{pmatrix}\beta-1\\ z\end{pmatrix}\begin{pmatrix}p\beta\\ t\end{pmatrix}, \tag{17}$$

$$f(y)=\sum_{t=0}^{\infty}w_t(1-y)^{-\lambda-\lambda z-1-\lambda t}. \tag{18}$$

The above expansion in Equation (18) of UPWD can be used for driving several properties of the proposed UPWD by taking into account the beta function of first kind as $x\in(0,1)$.

3.3. rth Moment. The rth ordinary or raw moments is an important measure to find measures of dispersion of the distribution. The following relationship is used to obtain the central or actual moments, the first moment about mean is always equal to zero, and second moment about mean is equal to variance as $\mu_2=\mu_2'-(\mu_1')^2$, $\mu_3=\mu_3'-3\mu_1'\mu_2'+2(\mu_1')^3$, and $\mu_4=\mu_4'-4\mu_3'\mu_1'+6\mu_2'(\mu_1')^2-3(\mu_1')^4$. The moment-based measure of skewness and kurtosis is obtained by using $\beta_1=\mu_3^2/\mu_2^3$ and $\beta_2=\mu_4/\mu_2^2$, respectively. Pearson's coefficient of skewness is simply square root of β_1, and coefficient of kurtosis is computed as β_2-3.

Proposition 7. *Let $Y\sim UPWD(\alpha,\beta,\lambda)$ for $y\in(0,1)$ and $\alpha,\beta,\lambda>0$; then, its rth ordinary or raw moments by using Equation (18) and beta function of first kind $\beta(a,b)={}_0^1x^{a-1}(1-x)^{b-1}dx$ are given by*

$$\mu_r'=\sum_{t=0}^{\infty}w_t{}_0^1y^r(1-y)^{-\lambda-\lambda z-1-\lambda t}dx,$$
$$\mu_r'=\sum_{t=0}^{\infty}w_t\beta[(r+1),(1-(\lambda+\lambda z+1+\lambda t))]. \tag{19}$$

For r=1, the mean of UPWD is yielded as $\mu_1'=\sum_{t=0}^{\infty}w_t\beta[2,(1-(\lambda+\lambda z+1+\lambda t))]$ and $[\lambda(1+z+t)+1]<1$. 3D graphical illustrations of mean (a) and variance (b) in Figure 4 with skewness (a) and kurtosis (b) presented in Figure 5.

3.4. sth Incomplete Moment. The sth incomplete moment is an important measure and has wide applications in order to compute mean deviation from mean and median, mean waiting time, conditional moments, and income inequality measures.

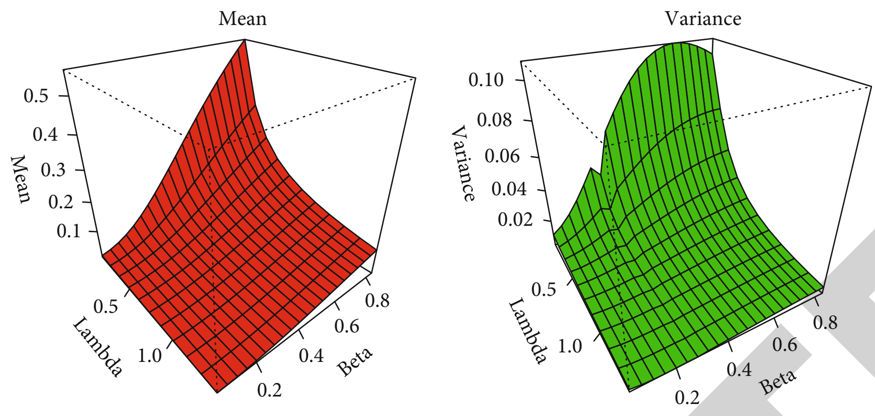
Proposition 8. *Let $Y\sim UPWD(\alpha,\beta,\lambda)$ for $y\in(0,1)$ and $\alpha,\beta,\lambda>0$; then, its sth incomplete moments by using (18) and incomplete beta function $\beta_l(a,b)={}_0^lx^{a-1}(1-x)^{b-1}dx$ are given by*

$$\varphi_s(l)=\sum_{t=0}^{\infty}w_t\beta_l[(s+1),(1-(\lambda+\lambda z+1+\lambda t))]. \tag{20}$$

Theoretically, Equation (20) is very useful by using the relationship between incomplete beta function and Gauss hypergeometric function as $\beta_x(a,b)=x^a/a^2F_1(a,1-b;a+1;x)$ to compute Bonferroni and Lorenz curve. The graphical representation of these measures is depicted in Figure 6. The readers are referred to Nadarajah and Kotz [23] for detailed discussion and various beta functions and its relationships.

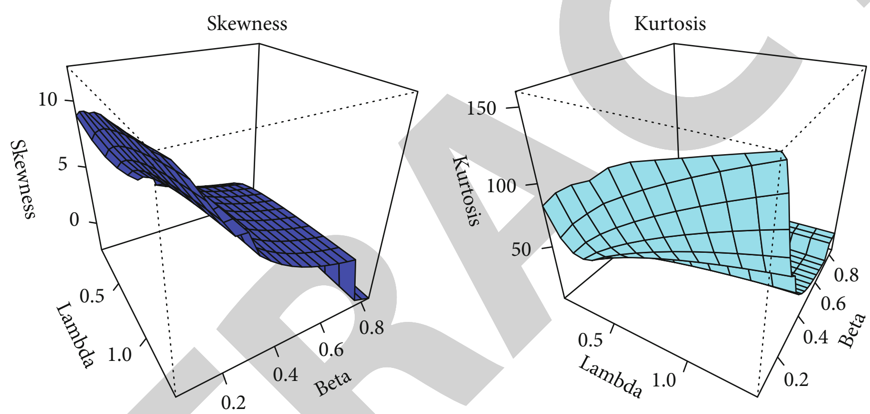
$$\varphi_s(l)=\sum_{t=0}^{\infty}w_t\frac{l^{s+1}}{(s+1)_2}F_1[s+1,(\lambda+\lambda z+1+\lambda t);s+2;l]. \tag{21}$$

RETRACTED



(a) (b)

Figure 4: Graphical illustration of (a) mean and (b) variance of UPWD model.



(a) (b)

Figure 5: Graphical illustration of (a) skewness and (b) kurtosis of UPWD model.

3.5. Moment-Generating Function. By definition, moment-generating function $M(t) = E[e^{ty}] = \int e^{ty} f(y)dy$ can be yielded as follows:

Proposition 9. *Let $Y \sim UPWD(\alpha, \beta, \lambda)$ for $y \in (0, 1)$ and α, $\beta, \lambda > 0$; then, its moment-generating function can be obtained by using (18) and replacing $e^{ty} = \sum_{m=0}^{\infty}(t^m/m!)y^m$ is given by*

$$E\left[e^{ty}\right] = \sum_{m=0}^{\infty}\sum_{t=0}^{\infty} w_t \frac{t^m}{m!}\beta[m+1, 1-(\lambda(1+z+t)+1)], \quad (22)$$

where $[\lambda(1+z+t)+1] < 1$.

3.6. Probability-Weighted Moments. The probability-weighted moments (PWMs) are the expectation of the certain functions of a random variable and can be defined for any random variable whose ordinary moments exist. In general, the PWM approach can be used to estimate distribution parameters whose inverted form cannot be specified directly. The (s, r) of the PWM of Y following the UPWD family, say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r} = E[Y^s F(y)^r] = {}_{-\infty}^{+\infty} Y^s F(y)^r f(y)dy. \quad (23)$$

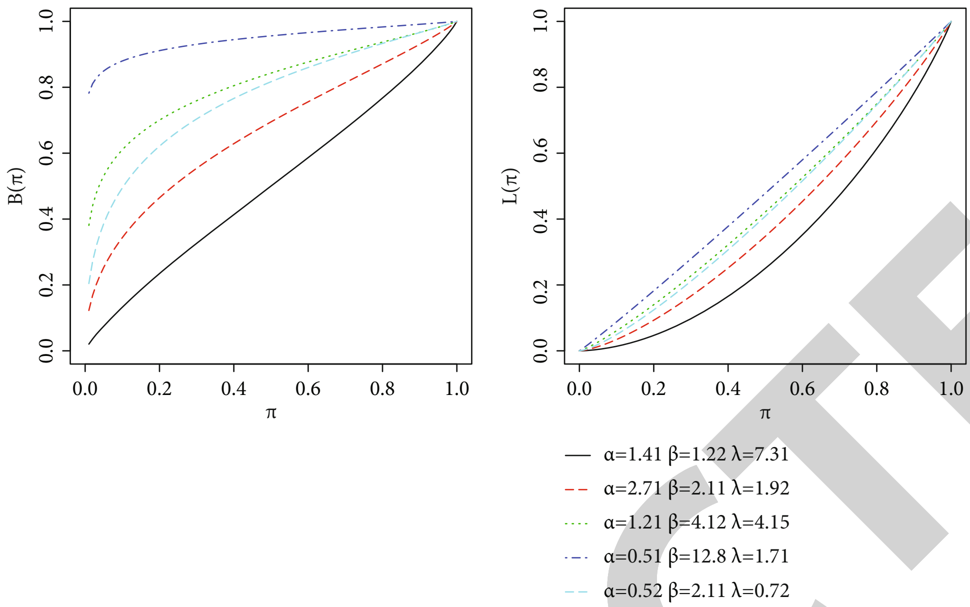
The expression in (23) is expanded in the same manner as Equation (18) using binomial expansion as follows:

$$\mathrm{II} = \sum_{t=0}^{\infty} K_t (1-y)^{-\lambda t-\lambda z-\lambda-1}, \quad (24)$$

where

$$K_t = \alpha\lambda\beta \sum_{p=0}^{\infty}\sum_{z,j=0}^{\infty} (-1)^{p+\beta-1+z+j+\beta j+t} \binom{r}{p}\binom{\beta-1}{z}\binom{\beta j}{t}\frac{[\alpha(1+p)]^j}{j!}. \quad (25)$$

RETRACTED



(a) (b)

Figure 6: Plot of (a) Bonferroni curve and (b) Lorenz curve of UPWD for some parametric values.

RETRACTED

By replacing Equations (23) and (24) and after some algebraic manipulation, we arrive at

$$\rho_{s,r} = E[Y^s F(y)^r] = \sum_{t=0}^{\infty} K_t \beta[s+1, -\lambda(t+z+1)]. \tag{26}$$

3.7. Order Statistics. The density function $f_{i:n}(y)$ of the ith-order statistic for $i = 1, \cdots, n$ from the values $y_1, \cdots, y_n$ can be expressed as

$$f_{i:n}(y) = \frac{1}{B(i, n-i+1)} f(y) \sum_{l=0}^{n-i} (-1)^l \binom{n-i}{l} [F(y)]^{i+l-1}. \tag{27}$$

Following the methodology to derive Equation (18), we arrive at

$$f_{i:n}(y) = \sum_{t=0}^{\infty} V_{i:n}^{(t)} (1-y)^{-\lambda z-\lambda-\lambda t-1}, \tag{28}$$

where

$$V_{i:n}^{(t)} = \frac{\alpha\lambda\beta}{B(i, n-i+1)} \sum_{l=0}^{n-i} \sum_{p=0}^{(i+l-1)} \sum_{z,j=0}^{\infty} (-1)^{p+\beta-1+z+j+\beta j+t+l} \binom{n-i}{l} \cdot \binom{i+l-1}{p} \times \binom{\beta-1}{z} \binom{\beta j}{t} \frac{[\alpha(1+p)]^j}{j!}. \tag{29}$$

The sth moment of order statistics can be yielded as

$$E(Y_{i:n}^s) = \sum_{t=0}^{\infty} V_{i:n}^{(j)} \beta[s+1, -\lambda(z+t+1)]. \tag{30}$$

To study the distributional behavior of the set of observation, we can use minimum and maximum (Min–Max) plot of the order statistics. Min–Max plot depends on extreme order statistics, and it is introduced to capture all information not only about the tails of the distribution but also about the whole distribution of the data. Figure 7 shows the Min- and the Max-order statistics for some parametric values and depends on $E(Y_{1:n})$ and $E(Y_{n:n})$, respectively

L-moments based on order statistics can be yielded by using the linear combinations of order statistics, and the following explicit expression of L-moments can be obtained by using (30).

$$L_r = \frac{1}{r} \sum_{d=0}^{r-1} (-1)^d \binom{r-1}{d} E(Y_{r-d:r}), \quad r \geq 1. \tag{31}$$

The first four L-moments are as under

$$\begin{aligned} L_1 &= E(Y_{1:1}), \\ L_2 &= \frac{1}{2} E(Y_{2:2} - Y_{1:2}), \\ L_3 &= \frac{1}{3} E(Y_{3:3} - 2Y_{2:3} + Y_{1:3}), \\ L_4 &= \frac{1}{4} E(Y_{4:4} - 3Y_{3:4} + 3Y_{2:4} - Y_{1:4}). \end{aligned} \tag{32}$$

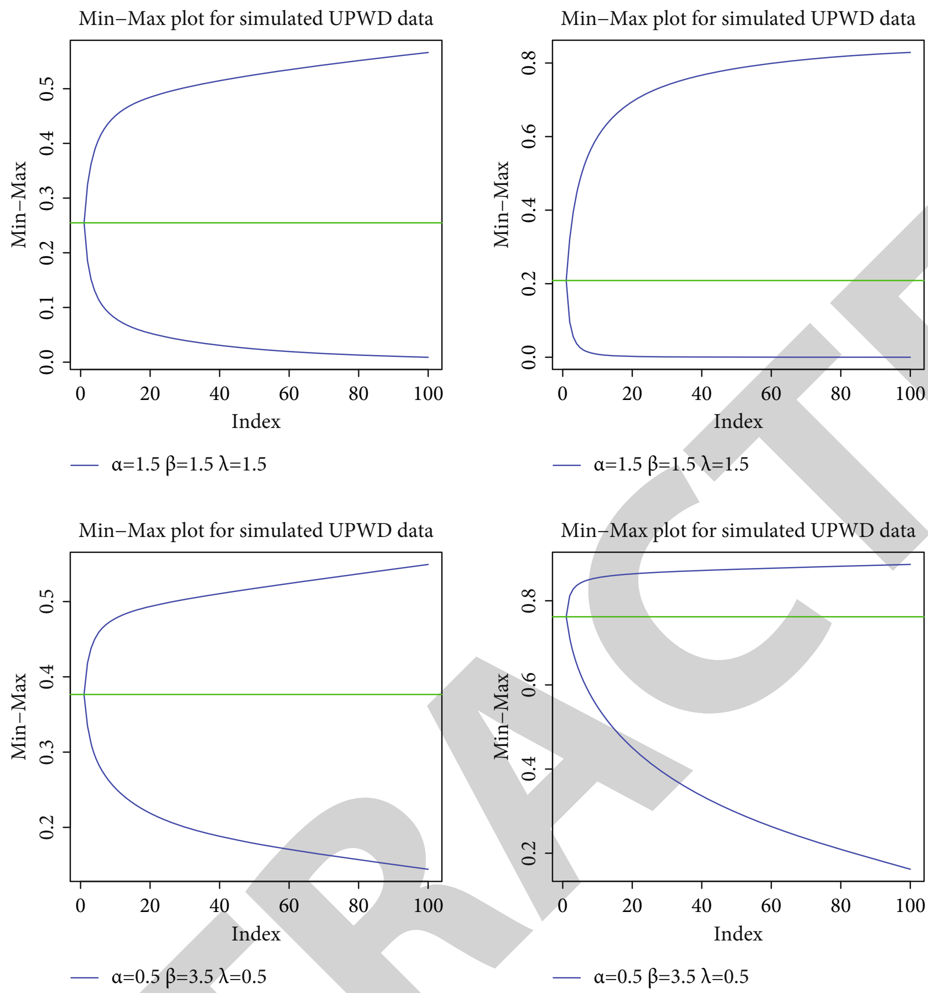


Figure 7: Min–Max plot of order statistics of UPWD model for some parametric values.

RETRACTED

By setting $s = 1$ in (30), we can simply get the L–moments L_r for Y.

3.8. TL-Moments. Trimmed or TL-moments are more robust than L-moments. If the distribution mean does not exist, one cannot yield the L-moments. On the other hand, TL-moments exist if the distribution does not have mean. The following expression yielded the rth TL-moments

$$\lambda_r^{(t_1 t_2)} = r^{-1} \sum_{k=0}^{r-1} (-)^k \binom{r-1}{k} E\left(Y_{r+t_1-k;r+t_1+t_2}\right), r = 1, 2, \cdots, \tag{33}$$

where t_1 and t_2 are the amount of lower and upper trimming. Here, we study a special case when $t_1 = t_2 = t$, and Equation (33) reduces to

$$\mathrm{TL}_r^{(t)} = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E\left(Y_{r+t-k;r+2t}\right), \quad r = 1, 2, \cdots. \tag{34}$$

The expectation of order statistics may be written as

$$\mathrm{TL}_r^{(t)} = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{(r+2t)!}{(r+t-k-1)!(t+k)!} \times {}_0^1 Q(u)\, u^{r+t-k-1} (1-u)^{t+k} du, r = 1, 2, \cdots. \tag{35}$$

When $t = 0$ in Equation (35), it reduces to ordinary L -moments and when $t = 1$, the first four TL-moments are given.

$$\mathrm{TL}_1^{(1)} = E(Y_{2:3}) = 6{}_0^1 Q(u)\, u\,(1-u)\, du,$$

$$\mathrm{TL}_2^{(1)} = \frac{1}{2} E(Y_{3:4} - Y_{2:4}) = 6{}_0^1 Q(u)\, u\,(1-u)\,(1-2u)\, du,$$

$$\mathrm{TL}_3^{(1)} = \frac{1}{3} E(Y_{4:5} - 2Y_{3:5} + Y_{2:5}) = \frac{20^1}{3\ {}_0} Q(u)\, u\,(1-u)\left(5u^2 - 5u - 1\right) du,$$

$$\begin{aligned} \mathrm{TL}_4^{(1)} &= \frac{1}{4} E(Y_{5:6} - 3Y_{4:6} + 3Y_{3:6} - Y_{2:6}) \\ &= \frac{15}{2}\int_0^1 Q(u)\, u\,(1-u) \times \left(14u^3 - 21u^2 + 9u - 1\right) du. \end{aligned} \tag{36}$$

One can get TL-moments by using mathematical software Mathematica or Maple to solve the complex integral by using (9).

3.9. Entropy Measures. Entropies are a measure of a system's variation, instability, or unpredictability. The Rényi entropy is important in ecology and statistics as index of diversity. For $\delta > 0$ and $\delta \neq 1$, it is defined by the following expression:

$$I_\delta(Y) = (1-\delta)^{-1} \log \int_{-\infty}^{+\infty} f(y)^\delta dy. \tag{37}$$

Again, we use the series expansions and mathematical maneuvering as we did to derive Equation (18), to arrive at

$$I_\delta(Y) = (1-\delta)^{-1} \log \left[\sum_{t=0}^{\infty} h_t \beta[1, 1 - (\lambda z + \delta(\lambda+1) + \lambda t)]\right]. \tag{38}$$

4. Estimation

In this section, we perform an estimation of unknown parameters of the UPWD model by taking into account the popular estimation framework known as maximum likelihood estimation (MLE). The MLE has an edge over other estimation methods, as it enjoys the required properties of normality conditions that can be used in constructing confidence intervals as well as in delivering simple approximation which is very handy while working for a finite sample case. The well-known R package called AdequacyModel is implemented to estimate the unknown parameters in the application section. The likelihood function L for the vector of parameters $\Phi = (\alpha, \beta, \lambda)^T$ for a UPWD is given in (3) is given by

$$\begin{aligned} & n \log(\alpha) + n \log(\beta) + n \log(\lambda) - (\lambda+1) \\ & \cdot \sum_{i=1}^{n} \log(1-y_i) + (\beta-1) \times \sum_{i=1}^{n} \log\left[(1-y_i)^{-\lambda} - 1\right] \\ & - \alpha \sum_{i=1}^{n} \left[(1-y_i)^{-\lambda} - 1\right]^{\beta}. \end{aligned} \tag{39}$$

Proposition 10. *Let $y_1, y_2, \cdots \cdots, y_n$ be a random sample from UPWD; then, the computed score vector $(\Phi_\alpha, \Phi_\beta, \Phi_\lambda)$ is given by*

$$\begin{aligned} \Phi_\alpha &= \frac{n}{\alpha} - \sum_{i=1}^{n} \left[(1-y_i)^{-\lambda} - 1\right]^{\beta}, \\ \Phi_\beta &= \frac{n}{\beta} + \sum_{i=1}^{n} \log\left[(1-y_i)^{-\lambda} - 1\right] \\ &\quad - \alpha \sum_{i=1}^{n} \left[(1-y_i)^{-\lambda} - 1\right]^{\beta} \log\left[(1-y_i)^{-\lambda} - 1\right], \\ \Phi_\lambda &= \frac{n}{\lambda} - \sum_{i=1}^{n} \log(1-y_i) - (\beta-1) \sum_{i=1}^{n} \frac{(1-y_i)^{-\lambda} \log(1-y_i)}{\left[(1-y_i)^{-\lambda} - 1\right]} \\ &\quad + \alpha\beta \sum_{i=1}^{n} \left[(1-y_i)^{-\lambda} - 1\right]^{\beta-1} (1-y_i)^{-\lambda} \log(1-y_i). \end{aligned} \tag{40}$$

By replacing $\Phi_\alpha = 0, \Phi_\beta = 0$ and $\Phi_\lambda = 0$, the maximum likelihood estimates can be attained by solving the above nonlinear equations simultaneously.

5. Simulation Analysis Univariate Case

In this section, Monte Carlo numerical study is carried out in order to assess the accuracy of the MLE parameters of UPWD distribution. The simulation study is replicated for $N = 2000$ times at varying sample sizes 25, 50,$\cdots$, 750 for the following scenario: I $= [\alpha = 1, \beta = 2.5, \lambda = 0.5]$, II $= [\alpha = 1, \beta = 0.85, \lambda = 1]$, and III $= [\alpha = 1, \beta = 1, \lambda = 1.5]$. The detailed summary of simulation analysis is shown in Table 1. The results reveal that MLEs perform well for estimating the parameters of UPWD with reduced mean square error (MSE) and bias as sample size increases. Therefore, the MLEs and their asymptotic results can be used for estimating and constructing confidence intervals for the model parameters. Readers are referred to Sigal and Chalmers [24] for designing simulation algorithm using R programming language. The plots of MLE estimates, MSE, bias, and absolute bias of simulation study at varying sample sizes are given in Figure 8.

$$\begin{aligned} \mathrm{Bias}\left(\hat{\theta}\right) &= \sum_{i=1}^{750} \frac{\hat{\theta}_i}{750} - \theta, \\ \mathrm{MSE}\left(\hat{\theta}\right) &= \sum_{i=1}^{750} \frac{\left(\hat{\theta}_i - \theta\right)^2}{750}. \end{aligned} \tag{41}$$

6. Actuarial Measures

The current hostile environment of the world has made the financial markets vulnerable to fatal risks associated with uncertainties. The primary risk assessment tools in this regard include value at risk (VaR), expected shortfall (ES), tail value at risk (TVaR), tail variance (TV), and tail variance premium (TVP). In this part, we shall obtain major expressions to obtain these measure using Equation (9). Some graphical representations are also illustrated.

6.1. Value at Risk. VaR is extensively used as a standard volatile measure in financial markets. It plays an important role

RETRACTED

Table 1: Detailed summary of simulation analysis of UPWD.

MLE estimates				MSE				Bias			
n	α	β	λ	n	α	β	λ	n	α	β	λ
Scenario-I											
25	1.8876	2.0252	1.1593	25	6.1626	0.7005	1.2541	25	0.8876	-0.4748	0.6593
50	1.9927	2.1372	0.9631	50	6.2471	0.5123	0.7997	50	0.9927	-0.3628	0.4631
100	1.9036	2.2393	0.7897	100	5.7165	0.3310	0.3688	100	0.9036	-0.2607	0.2897
300	1.5198	2.3428	0.6398	300	3.7357	0.1359	0.1121	300	0.5198	-0.1572	0.1398
500	1.2870	2.3867	0.6007	500	2.5413	0.0821	0.0565	500	0.2870	-0.1133	0.1007
750	1.0805	2.3896	0.5883	750	1.5961	0.0585	0.0400	750	0.0805	-0.1104	0.0883
Scenario-II											
25	1.7326	0.8591	1.3876	25	3.5884	0.0621	1.2088	25	0.7326	0.0091	0.3876
50	1.5576	0.8631	1.1797	50	2.4523	0.0323	0.5502	50	0.5576	0.0131	0.1797
100	1.3405	0.8536	1.0931	100	1.3348	0.0176	0.2544	100	0.3405	0.0036	0.0931
300	1.1233	0.8531	1.0165	300	0.2701	0.0060	0.0741	300	0.1233	0.0031	0.0165
500	1.0525	0.8490	1.0165	500	0.0958	0.0033	0.0413	500	0.0525	-0.0010	0.0165
750	1.0414	0.8496	1.0088	750	0.0681	0.0025	0.0290	750	0.0414	-0.0004	0.0088
Scenario-III											
25	1.8783	1.0088	2.0576	25	4.3963	0.0830	2.3403	25	0.8783	0.0088	0.5576
50	1.7145	1.0118	1.7917	50	3.2473	0.0456	1.3098	50	0.7145	0.0118	0.2917
100	1.4699	1.0028	1.6540	100	2.0009	0.0257	0.6335	100	0.4699	0.0028	0.1540
300	1.1961	1.0055	1.5318	300	0.5326	0.0097	0.2077	300	0.1961	0.0055	0.0318
500	1.0947	1.0006	1.5287	500	0.2082	0.0058	0.1257	500	0.0947	0.0006	0.0287
750	1.0693	1.0006	1.5199	750	0.1366	0.0043	0.0911	750	0.0693	0.0006	0.0199

in many business decisions, the uncertainty regarding foreign market, commodity price, and government policies can affect significantly firm earnings. The loss portfolio value is specified by the certain degree of confidence say q (90%, 95%, or 99%). VaR of random variable Y is simply the qth quantile of its cdf. If X follows the UPWD model, then its VaR is defined by the following expression:

$$\mathrm{VaR}_q = \left[1 - \left\{\left(\frac{1}{-\alpha}\log\left[1-q\right]\right)^{1/\beta} + 1\right\}^{-1/\lambda}\right]. \tag{42}$$

6.2. Expected Shortfall. The other important financial risk measure is expected shortfall (ES), introduced by [25], and generally considered a better measure than value at risk. It is defined by the following expression:

$$\mathrm{ES}_q(y) = \frac{1}{q}\int_0^q \mathrm{VaR}_y dy, \tag{43}$$

for $0 < q < 1$, using Equation (42) in Equation (43), yielded ES for UPWD.

6.3. Tail Value at Risk. One of the most pressing issues in portfolio management is the issue of risk measurement. From finance and insurance perspective, TVaR or tail conditional expectation or conditional tail expectation is an important measure and is defined as the expected value of the loss, given the loss is greater than the VaR measure.

$$\mathrm{TVaR}_q(y) = \frac{1}{1-q}\int_{\mathrm{VaR}_q}^{\infty} y f(y) dy. \tag{44}$$

By using (18) in (44), the yielded TVaR is as under

$$\mathrm{TV\alpha R}_q(y) = \frac{1}{1-q}\sum_{t=0}^{\infty}\omega_t \frac{(\mathrm{V\alpha R}q)^2}{2}\,{}_2F_1[2, (\lambda + \lambda z + 1 + \lambda t)\,;3\,;\mathrm{V\alpha R}q]. \tag{45}$$

6.4. Tail Variance. Tail variance (TV) is yet another important risk measure because it considers the variability of the risk along the tail of distribution and is defined by the following expression:

$$\mathrm{TV}_q(y) = E\left[Y^2 \mid Y > y_q\right] - \left[\mathrm{TVaR}_q\right]^2. \tag{46}$$

Consider $I = E[Y^2 \mid Y > y_q]$.

$$I = \mathrm{TV\alpha R}_q(y) = \frac{1}{1-q}\int_{\mathrm{V\alpha R}_q}^{\infty} y^2 f(y) dy, \tag{47}$$

RETRACTED

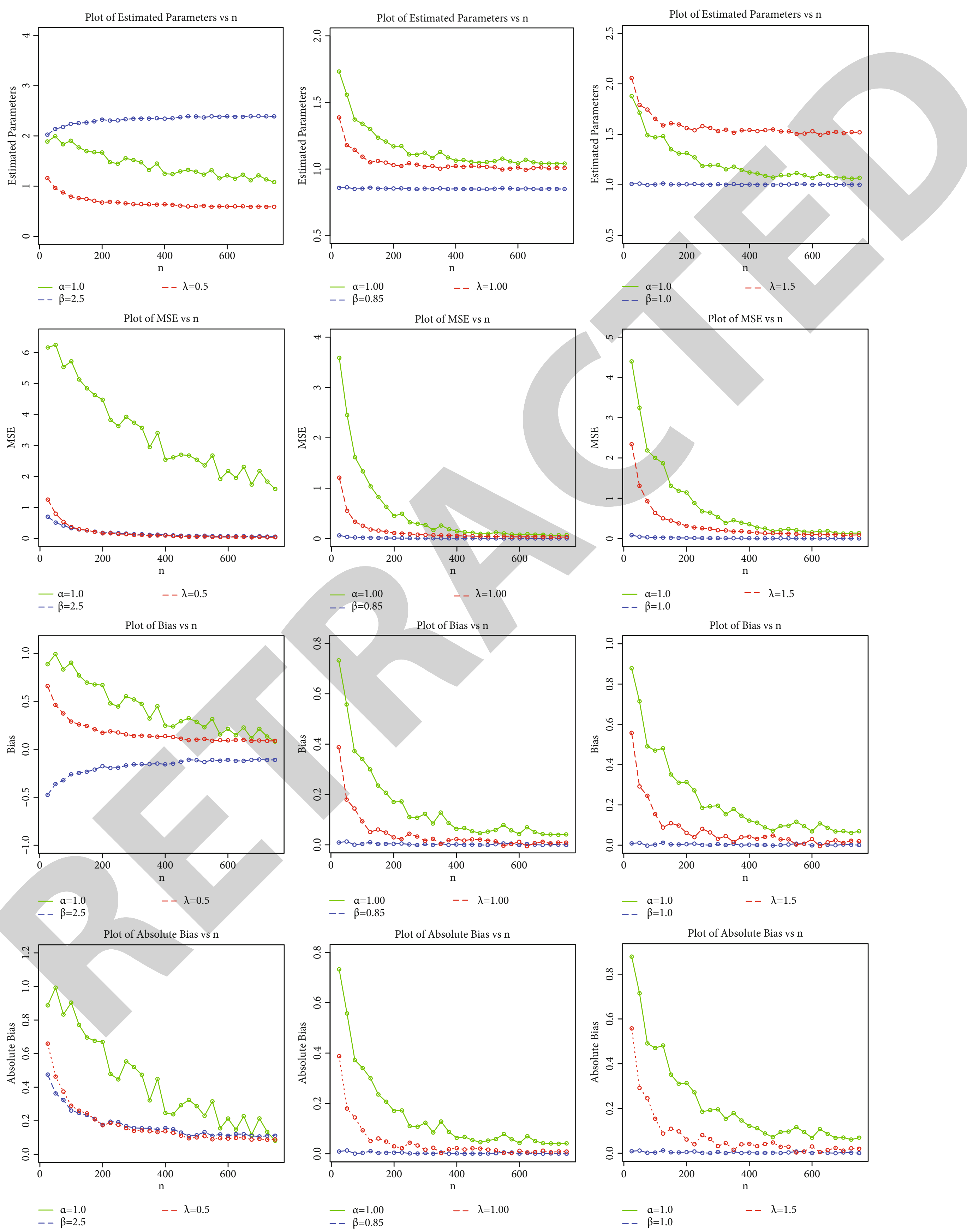
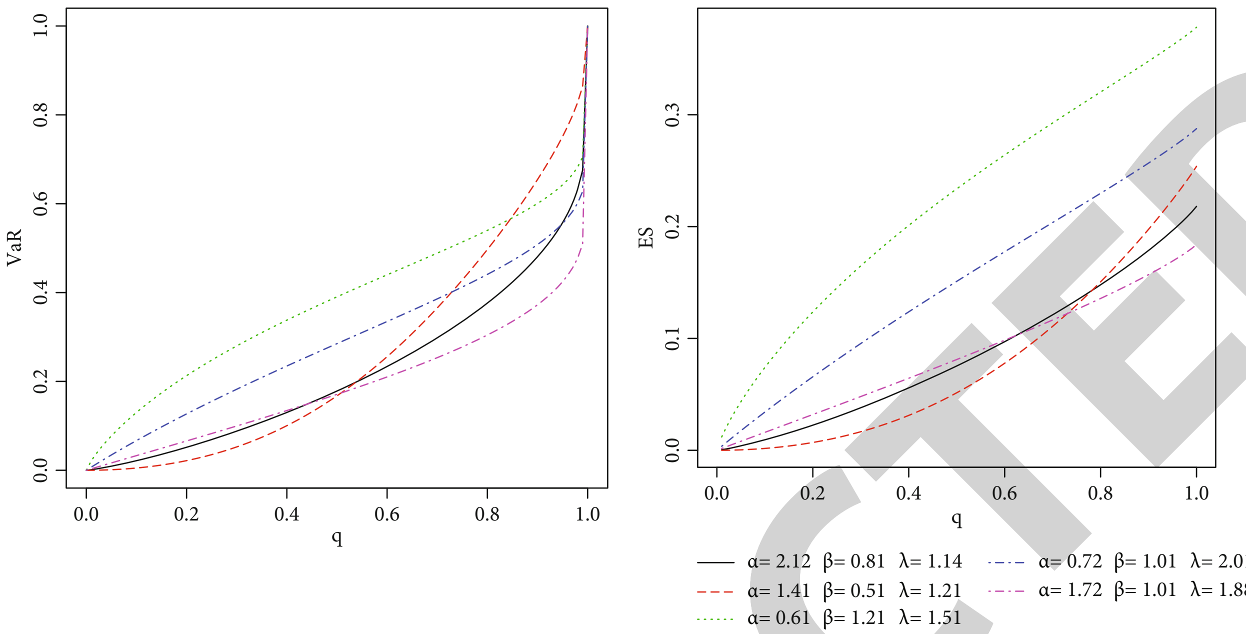


FIGURE 8: Plots of MLE estimates, MSE, bias, and absolute bias of simulation study at varying sample sizes.

RETRACTED

RETRACTED



(a) (b)

FIGURE 9: Plot of (a) VaR and (b) ES of UPWD for some parametric values.

TABLE 2: Numerical illustration of ES and VaR of UPWD based on MLE values of insurance claim.

q	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	0.99
ES	0.3749	0.3817	0.3881	0.3942	0.4000	0.4055	0.4110	0.4163	0.4217	0.4263
VaR	0.4530	0.4612	0.4691	0.4770	0.4850	0.4933	0.5023	0.5128	0.5266	0.5484

$$I = \mathrm{TV\alpha R}_q(y) = \frac{1}{1-q}\sum_{t=0}^{\infty}\omega_t \frac{\left(\mathrm{V\alpha R}_q\right)^3}{3} {}_2F_1\left[3, \left(\lambda + \lambda z + 1 + \lambda t\, ; 4\, ; V\alpha R_q\right)\right]. \quad (48)$$

using (45) and (48) in (46), we obtain the expression for TV for UPWD model.

6.5. Tail Variance Premium. Tail variance premium (TVP) yet is another crucial risk measure. It is the combination of both central tendency and dispersion statistics, so it can measure variability of loss along the right tail better. TVP could be alternative risk measure, especially when risk that is bigger than a certain threshold is concerned.

$$\mathrm{TVP}_q(Y) = \mathrm{TVaR}_q + \delta \mathrm{TV}_q, \quad (49)$$

where $0 < \delta < 1$. Using the expressions (46) and (45) in (49), we obtain the tail variance premium for UPWD model.

A sample of 100 is randomly drawn, and the effect of shape and scale parameters of the proposed models are underlined for both risk measures. Various combinations of the scale and shape parameters are executed $\mathrm{I} = [\alpha = 2.12, \beta = 0.81, \lambda = 1.14]$, $\mathrm{II} = [\alpha = 1.41, \beta = 0.51, \lambda = 1.21]$, $\mathrm{III} = [\alpha = 0.61, \beta = 1.21, \lambda = 1.51]$, $\mathrm{IV} = [\alpha = 0.72, \beta = 1.01, \lambda = 2.01]$, and $\mathrm{V} = [\alpha = 1.72, \beta = 1.01, \lambda = 1.88]$, and changes in the curve of VaR and ES are illustrated in Figure 9.

6.6. Numerical Illustration of VaR and ES. Here we demonstrate the numerical as well as graphical presentation of the two important risk measures ES and VaR for UPWD. It is worth emphasis that a model with higher values of the risk measures is said to have a heavier tail. Table 2 provides the numerical illustration of the ES and VaR for UPWD of both the risk measures. The graphical demonstration of the UPWD is presented in Figure 10. The readers are referred to Chan et al. [26] for detail discussion of VaR and ES and their computation by using an R programming language.

7. Application

The real data application of the UPWD distribution is carried out in this section by using the unemployment claims form July 2008 to April 2013, reported by the Department of Labour, Licencing and Regulation, USA. The data set consists of 21 variables, and we used the variable 5, i.e., new claims filed with total observation for each variable is 58. Recently, the data has been studied by [27]. The second

RETRACTED

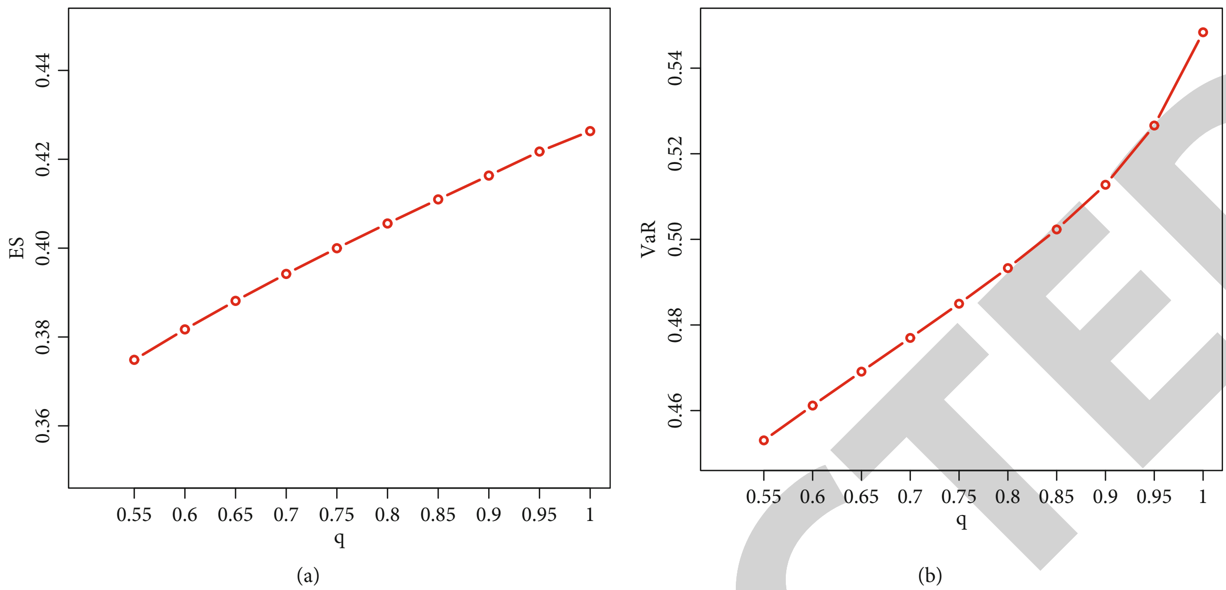


Figure 10: Plot of (a) ES and (b) VaR of UPWD based on MLEs.

Table 3: Real data sets along with descriptive summary.

Data 1									
.188	0.202	0.195	0.385	0.489	0.545	0.541	0.535	0.521	0.508
.512	0.507	0.519	0.493	0.487	0.460	0.490	0.460	0.490	0.500
.400	0.350	0.370	0.410	0.400	0.400	0.410	0.400	0.420	0.450
.450	0.420	0.390	0.340	0.360	0.400	0.440	0.390	0.410	0.450
.460	0.470	0.490	0.460	0.410	0.390	0.400	0.440	0.420	0.420
.450	0.470	0.530	0.420	0.490	0.440	0.420	0.400	–	–
Descriptive summary									
n	Min	Max	Q1	Q3	Mean	Median	SD	S	K
58	0.188	0.545	0.400	0.4898	0.4322	0.440	0.0754	-1.376	2.826
Data 2									
.853	0.759	0.866	0.809	0.717	0.544	0.492	0.403	0.344	0.213
.116	0.116	0.092	0.07	0.059	0.048	0.036	0.029	0.021	0.014
.011	0.008	0.006	–	–	–	–	–	–	–
Descriptive summary									
n	Min	Max	Q1	Q3	Mean	Median	SD	S	K
23	0.006	0.866	0.0325	0.518	0.2881	0.116	0.3181	0.768	-1.026
Data 3									
.853	0.759	0.874	0.8000	0.716	0.557	0.503	0.399	0.334	0.207
.118	0.118	0.097	0.078	0.067	0.056	0.044	0.036	0.026	0.019
.014	0.010	–	–	–	–	–	–	–	–
Descriptive summary									
n	Min	Max	Q1	Q3	Mean	Median	SD	S	K
22	0.01	0.874	0.047	0.5435	0.3039	0.118	0.3178	0.711	-1.116

and third data sets are based on computer algorithm computation timing of SC16 and P3. This data set is also used by [28]. Three real data sets along with descriptive summary are illustrated in Table 3. The total time on test (TTT) plots are presented in Figure 11 which show that the first data set has increasing hazard rate, whereas the second and third data sets have decreasing-increasing hazard rates, which means these data sets can better be fitted under the proposed UPWD. The comparative studies of the proposed UPWD with some commonly used well-known models, namely,

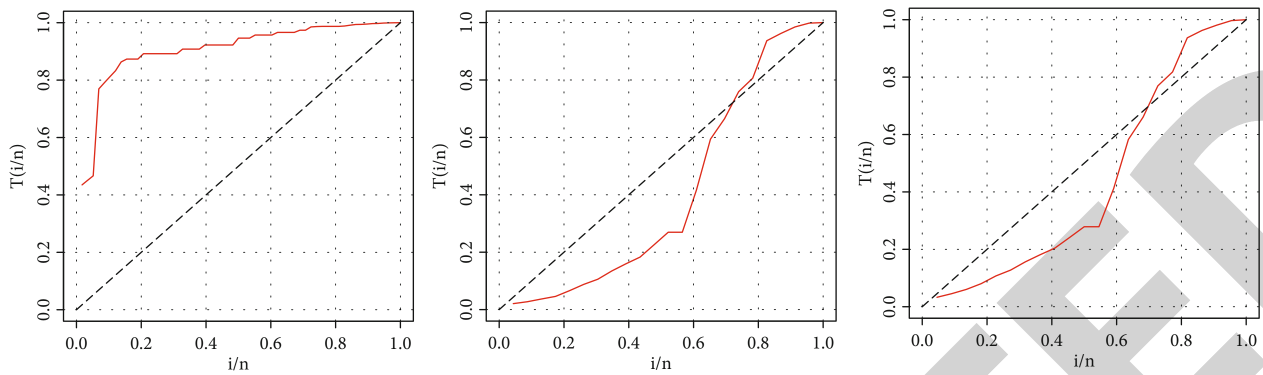


Figure 11: TTT plots of data sets.

Table 4: ML estimates along with SEs of the fitted models.

		Data 1		Data 2		Data 3	
Dist.	Para.	Estimates	SEs	Estimates	SEs	Estimates	SEs
	$\hat{\alpha}$	0.0033	0.0009	81.548	77.154	75.932	66.223
UPWD	$\hat{\beta}$	1.1965	1.0788	0.6319	0.1256	0.6844	0.1145
	$\hat{\lambda}$	7.6044	6.8914	0.0028	0.0015	0.0044	0.0036
	$\hat{\alpha}$	2600.02	1113.18	2.6566	0.0345	2.1645	0.0362
EW	$\hat{\beta}$	10.9715	0.7798	7.0785	0.0041	6.9040	0.0391
	$\hat{a}$	0.5438	0.1095	0.0682	0.0142	0.0771	0.0164
	$\hat{\alpha}$	2.7196	0.6347	32.2780	0.2934	30.9026	0.1577
KE	$\hat{a}$	13.2509	2.9500	0.5132	0.2235	0.7668	0.2217
	$\hat{b}$	90.3433	71.6273	0.1011	0.0215	0.1038	0.0223
	$\hat{a}$	0.0296	0.0130	0.5287	0.9954	0.5414	0.8432
	$\hat{b}$	0.4896	0.0805	0.0353	0.0964	0.0400	0.1144
GK	$\hat{\alpha}$	0.0699	0.0336	0.0317	0.0933	0.0339	0.1048
	$\hat{\beta}$	74.1031	18.4738	0.9366	1.7654	1.0301	1.6026
Beta	$\hat{a}$	16.8273	3.0994	0.4869	0.1208	0.5540	0.1423
	$\hat{b}$	22.2035	4.1044	1.1679	0.3578	1.2198	0.3758
	$\hat{\alpha}$	1.2130	1.2726	34.9869	0.0601	30.6664	0.3138
BE	$\hat{a}$	26.0074	4.9457	0.5828	0.2442	0.7679	0.3430
	$\hat{b}$	38.2259	49.6452	0.0914	0.0202	0.1030	0.0233

exponentiated Weibull (EW), Kumaraswamy exponential (KE) [29], gamma Kumaraswamy (GK) [30], and beta exponential (BE) [31] are considered to establish the practical versatility of the UPWD. The ML estimates along with standard errors (SEs) of the all fitted models are presented in Table 4 and goodness of fit test in Table 5. The analysis of data revealed that UPWD is outperforming its competitive models based on goodness of fit criterion, namely, Akaike information criterion (AIC), Bayesian information criterion (BIC), corrected Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC). The Anderson Darling (A^*), Cramer-Von-Mises (W^*), and Kolmogorov-Smirnov (K-S) test also used for model selection. The graphical illustration of all three data set of estimated pdf, cdf, failure rate, and probability-probability (P-P) plot is presented from Figures 12–14 which show a good agreement between actual and predicted.

8. Bivariate Extension

Here we introduce a bivariate extension for the univariate unit-power Weibull distribution Equation (4), namely, bivariate unite-power Weibull distribution (BIUPW). A bivariate continuous random vector (X, Y) will be called BIUPW distribution with parameters $(\alpha, \lambda, \beta, \theta_1, \theta_2, \theta_3)$, where $\alpha, \lambda, \beta > 0$, $-1 < \theta_1 + \theta_3 < 1$, $-1 < \theta_2 + \theta_3 < 1$, $0 < x < 1$, and $0 < y < 1$ if its cdf is given by

RETRACTED

Table 5: Detailed summary of accuracy measures of the fitted models.

Dist.	$2\hat{\ell}$	AIC	CAIC	BIC	HQIC	A^*	W^*	K.S	P value
Data 1									
UPWD	-74.71	-143.42	-142.98	-137.24	-141.01	0.827	0.143	0.116	0.417
EW	-74.04	-142.07	141.63	-135.89	-139.66	0.853	0.120	0.117	0.400
KE	-69.58	-133.16	-132.71	-126.98	-130.75	1.406	0.167	0.139	0.212
GK	-69.07	-130.14	-129.39	-121.90	-126.93	1.493	0.179	0.140	0.203
Beta	-65.53	-127.05	-126.84	-122.93	-125.45	2.107	0.268	0.169	0.074
BE	-61.10	-116.20	-115.75	-110.02	-113.79	2.897	0.394	0.191	0.029
Data 2									
UPWD	9.833	-13.67	-12.40	-10.26	-12.81	0.598	0.093	0.154	0.647
EW	-9.348	-12.70	-11.43	-9.289	-11.84	0.744	0.119	0.187	0.396
KE	-6.309	-6.618	-5.355	-3.211	-5.761	0.786	0.123	0.206	0.284
GK	-9.667	-11.33	-9.111	-6.791	-10.19	0.691	0.110	0.182	0.430
Beta	-9.607	-15.21	-14.61	-12.94	-14.64	0.690	0.110	0.184	0.420
BE	6.542	-7.083	-5.820	-3.677	-6.227	0.790	0.124	0.199	0.323
Data 3									
UPWD	7.034	-8.067	-6.734	-4.794	-7.296	0.633	0.103	0.185	0.440
EW	6.378	-6.757	-5.423	-3.483	-5.986	0.751	0.123	0.205	0.313
KE	4.299	-2.599	-1.266	0.674	-1.828	0.721	0.114	0.212	0.277
GK	-6.837	-5.674	-3.321	-1.309	-4.646	0.715	0.118	0.199	0.351
Beta	-6.782	-9.564	-8.932	-7.382	-9.050	0.712	0.117	0.341	0.200
BE	4.391	-2.783	-1.450	0.490	-2.012	0.731	0.116	0.205	0.313

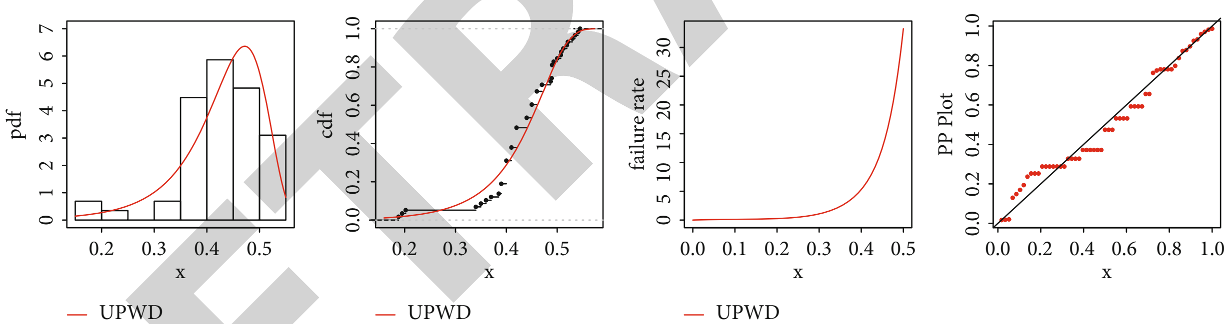


Figure 12: Graphical illustration of estimated pdf, cdf, failure rate, and P-P for data 1.

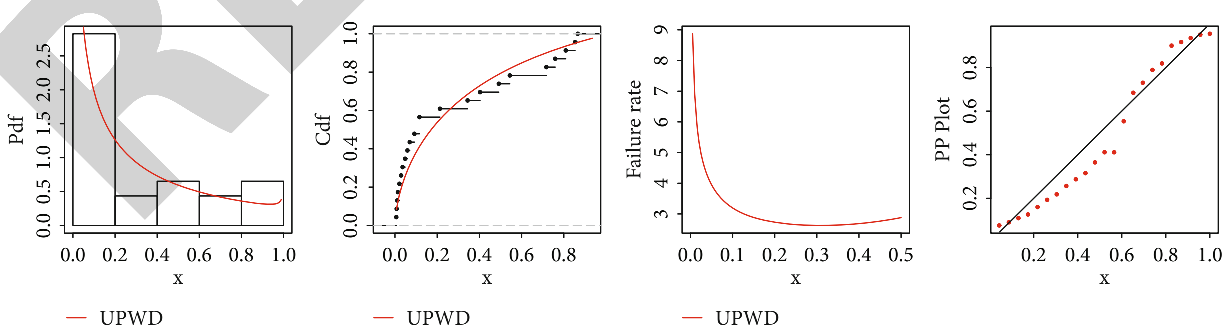


Figure 13: Graphical illustration of estimated pdf, cdf, failure rate, and P-P for data 2.

RETRACTED

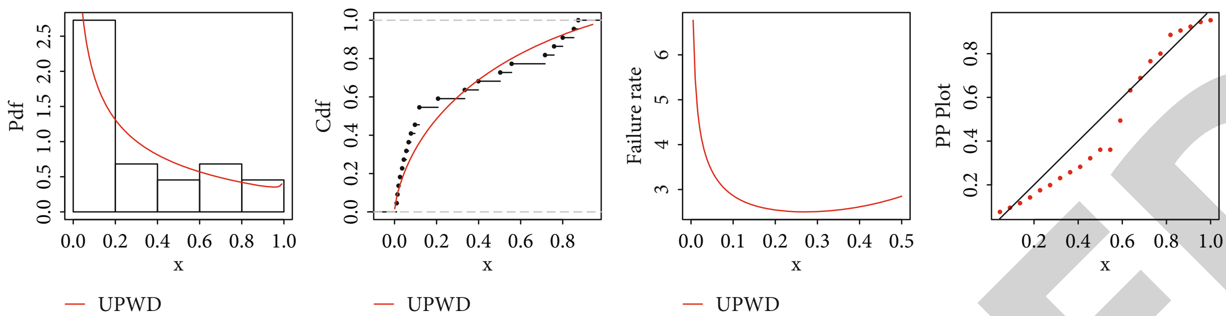


Figure 14: Graphical illustration of estimated pdf, cdf, failure rate, and P-P for data 3.

$$F_{X,Y}(x,y,\alpha,\lambda,\beta) = \left(1-e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}\right)\left(1-e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}\right) \times \left\{1+(\theta_1+\theta_3)e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}+(\theta_2+\theta_3)e^{-\left[\alpha(1-y)^{-\lambda}-1\right]^{\beta}}\right\}. \tag{50}$$

It will be denoted by $(X, Y) \sim \mathrm{BIUPW}(\Theta)$, where $\Theta =: (\lambda, \beta, \theta_1, \theta_2, \theta_3)$. The readers are referred to [32–34].

Proposition 11. *Let $(X, Y) \sim BIUPW(\alpha, \lambda, \beta, \theta_1, \theta_2, \theta_3)$. Then, its pdf is given by*

$$f_{X,Y}(x,y,\Theta) = (\alpha\lambda\beta)^2(1-x-y+xy)^{-\lambda-1}\left[(1-x-y+xy)^{-\lambda}-(1-x)^{-\lambda}-(1-y)^{-\lambda}+1\right]^{\beta-1}e^{-\alpha\left\{\left[(1-x)^{-\lambda}-1\right]^{\beta}+\left[(1-y)^{-\lambda}-1\right]^{\beta}\right\}} \times \left\{1+(\theta_1+\theta_3)\left(2e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}-1\right)+(\theta_2+\theta_3)\left(2e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}-1\right)\right\}. \tag{51}$$

Proposition 12. *Let $(X, Y) \sim BIUPW(\Theta)$. Then, its marginals are given by*

$$\begin{aligned}
F_X(x,\Theta) &= \left(1-e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}\right)\times\left\{1+(\theta_1+\theta_3)e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}\right\},\\
F_Y(y,\Theta) &= \left(1-e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}\right)\times\left\{1+(\theta_2+\theta_3)e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}\right\},\\
f_X(x,\Theta) &= \alpha\lambda\beta(1-x)^{-\lambda-1}\left[(1-x)^{-\lambda}-1\right]^{\beta-1}e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}\left\{1+(\theta_1+\theta_3)\left(2e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}-1\right)\right\},\\
f_Y(y,\Theta) &= \alpha\lambda\beta(1-y)^{-\lambda-1}\left[(1-y)^{-\lambda}-1\right]^{\beta-1}e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}\left\{1+(\theta_2+\theta_3)\left(2e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}-1\right)\right\}.
\end{aligned} \tag{52}$$

Proposition 13. *Proposition. 3. Let $(X, Y) \sim BIUPW(\Theta)$. Then,*

$$\begin{aligned}
f_{Y/X}\left(\frac{y}{x}\right) &= \alpha\lambda\beta(1-y)^{-\lambda-1}\left[(1-y)^{-\lambda}-1\right]^{\beta-1}e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}\left(1+\frac{\phi(\theta_1,y)}{1+\phi(\theta_2,x)}\right),\\
f_{X/Y}\left(\frac{x}{y}\right) &= \alpha\lambda\beta(1-x)^{-\lambda-1}\left[(1-x)^{-\lambda}-1\right]^{\beta-1}e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}\left(1+\frac{\phi(\theta_2,x)}{1+\phi(\theta_1,y)}\right),
\end{aligned} \tag{53}$$

RETRACTED

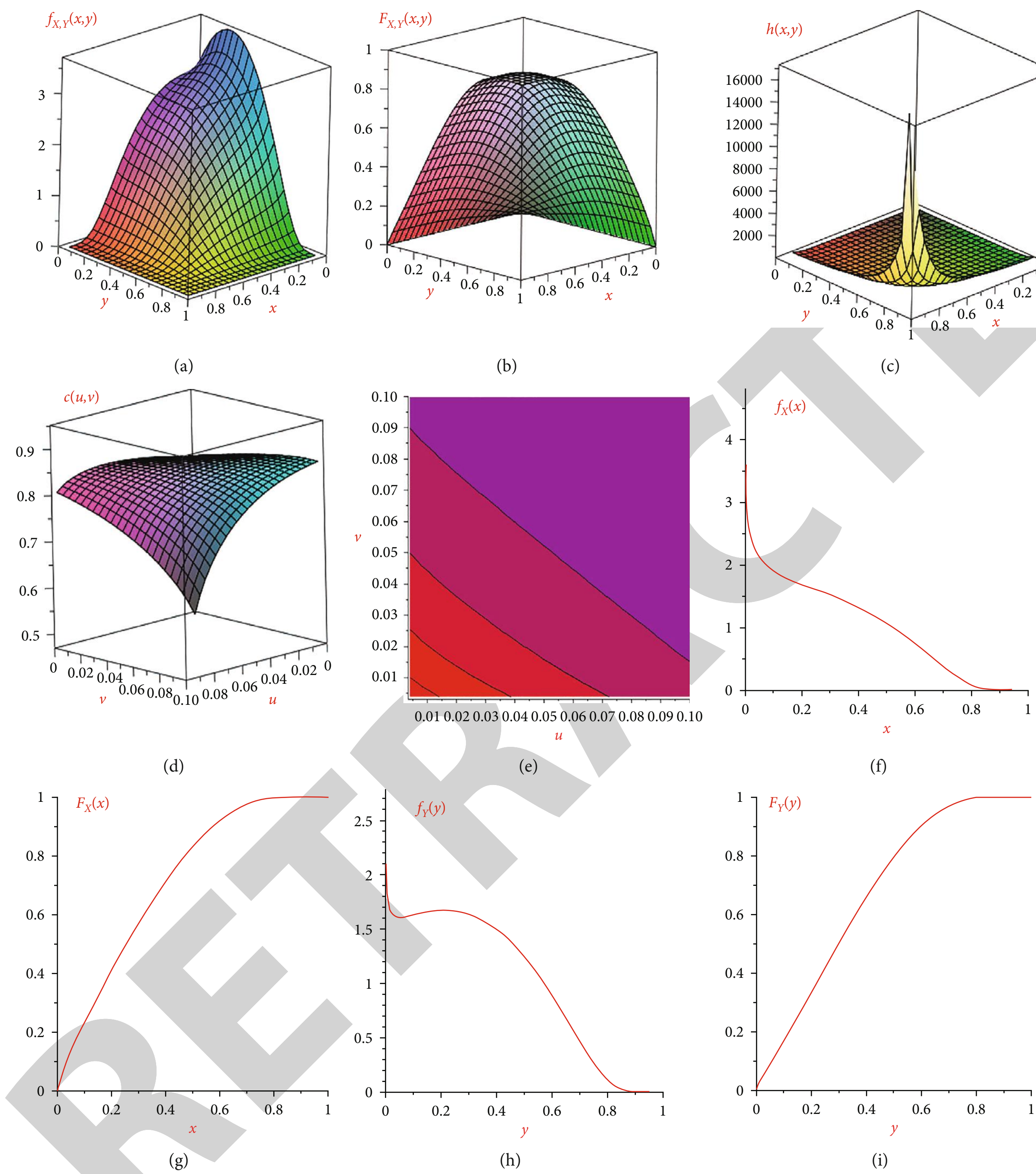


FIGURE 15: Statistical quantities for $(X, Y) \sim \mathrm{BIUPW}(2.0,1.0,0.8,0.2,-0.1,-0.5)$.

RETRACTED

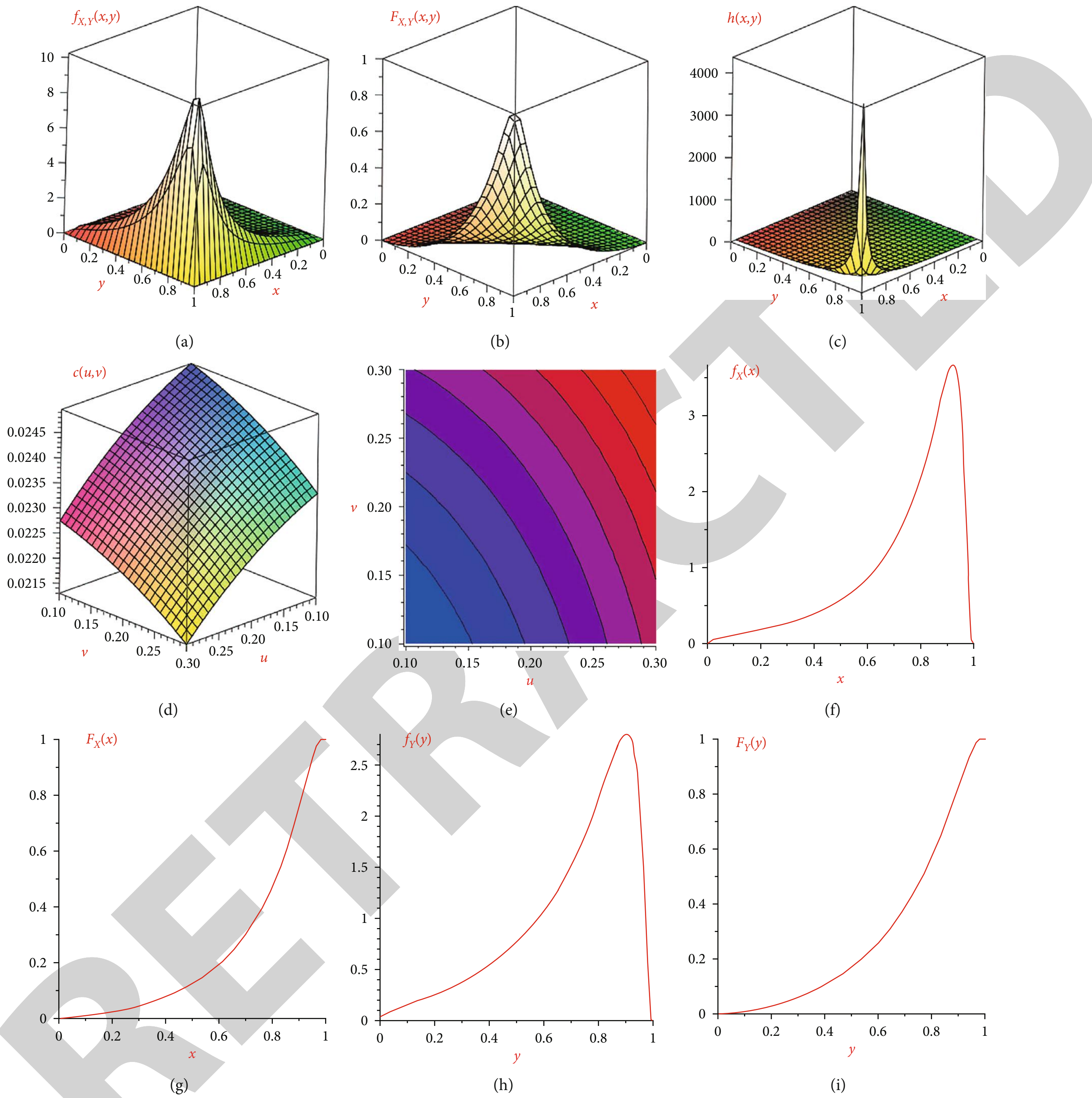


Figure 16: Statistical quantities for $(X, Y) \sim \text{BIUPW}(0.8, 0.4, 1.5, -0.8, -0.4, 0.7)$.

RETRACTED

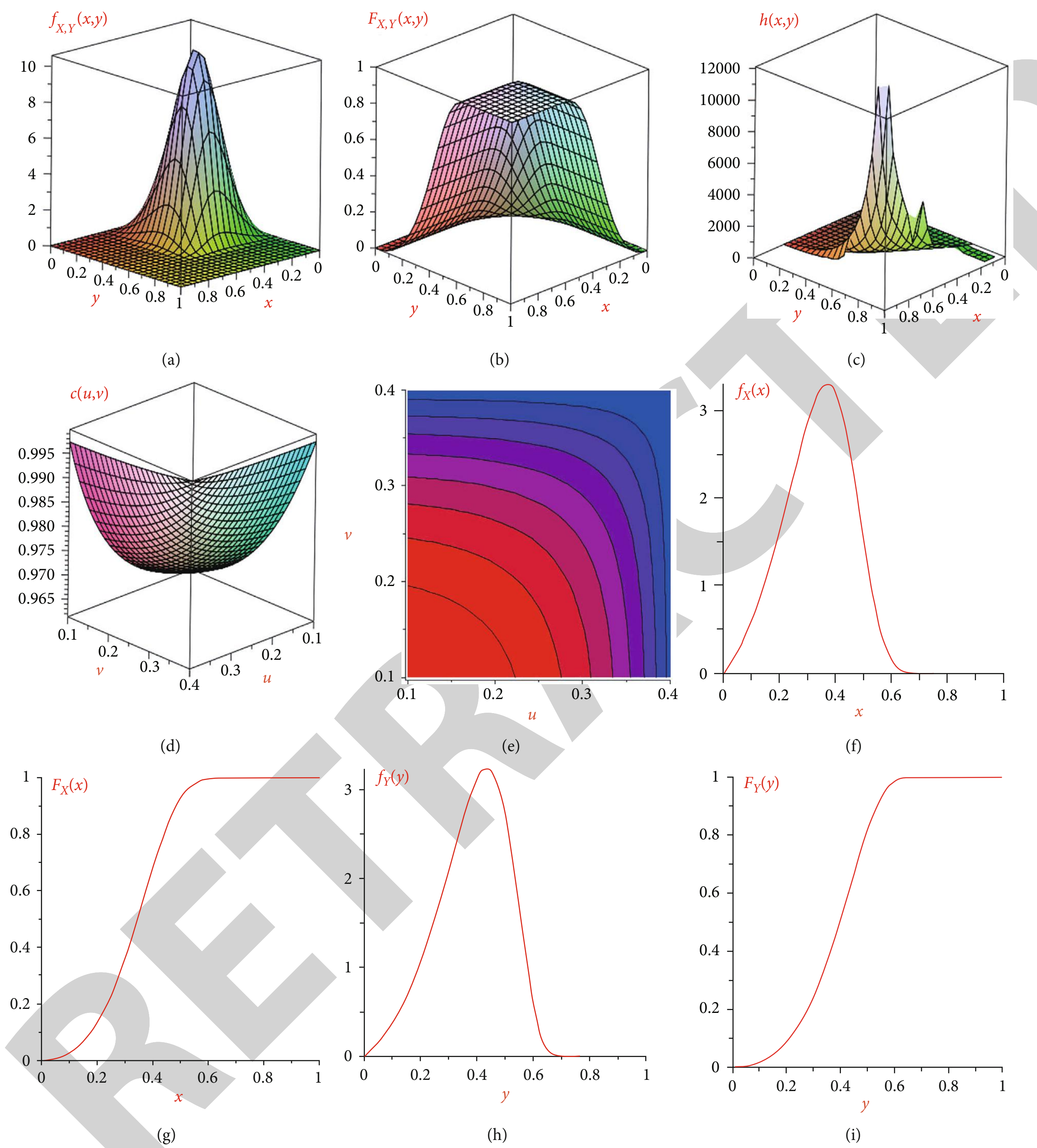


FIGURE 17: Statistical quantities for $(X, Y) \sim \text{BIUPW}(0.6,1.4,2.0,0.3,-0.4,0.5)$.

RETRACTED

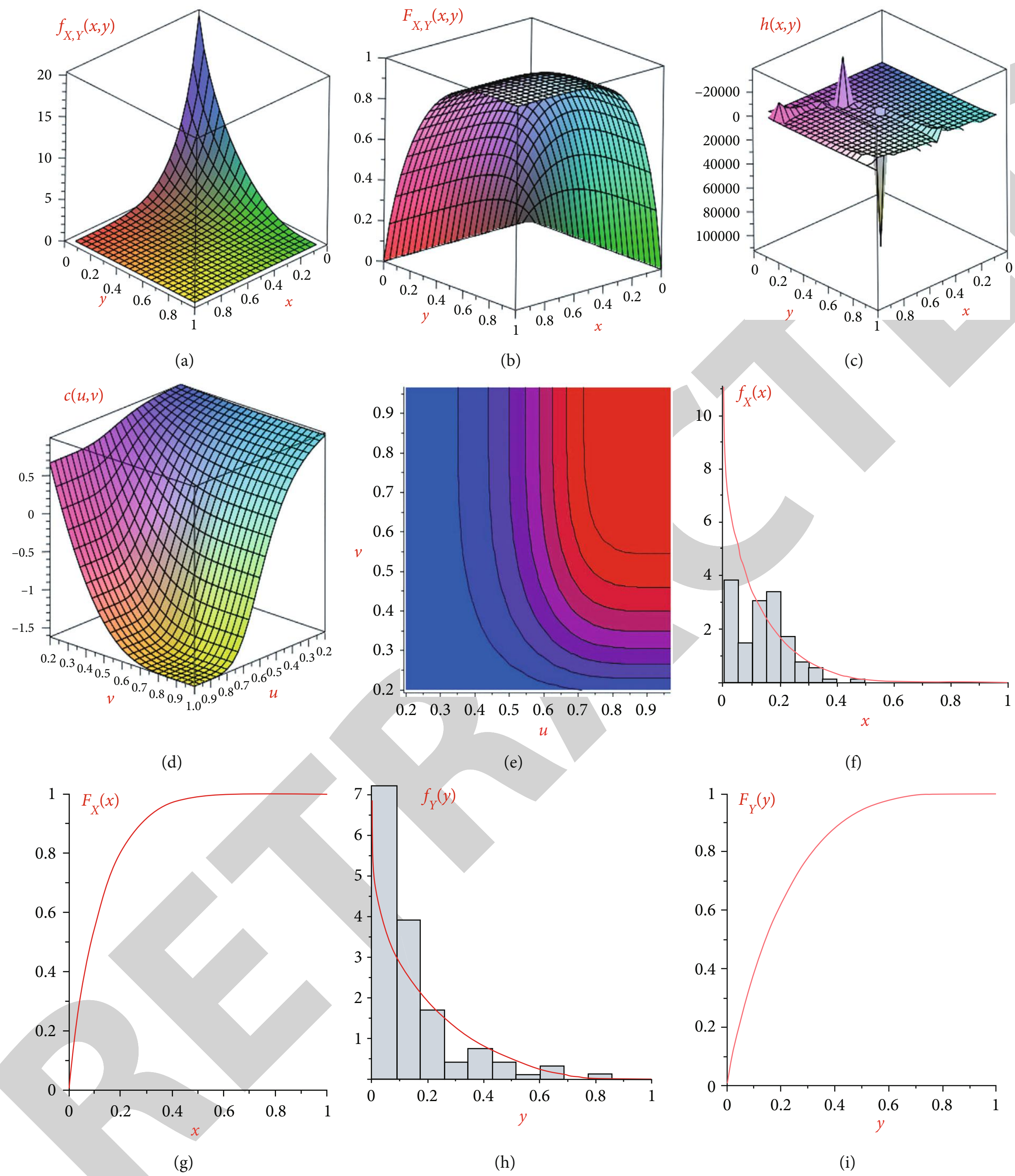


FIGURE 18: Statistical quantities of the random variable $(X, Y) \sim$ BIUPW at MLEs.

RETRACTED

where

$$\phi(\theta,\xi)=(\theta+\theta_3)\left(2e^{-\alpha\left[(1-\xi)^{-\lambda}-1\right]^{\beta}}-1\right). \tag{54}$$

Proposition 14. *Let $(X,Y)\sim BIUPW(\Theta)$. Then,*

$$\begin{aligned}\mu^{r}_{X/Y}(y)=E\left(\frac{X^r}{Y}=y\right)&=\left(1+\frac{\theta_1+\theta_3}{1+\phi(\theta_2,y)}\right)r\sum_{i=0}^{\infty}\sum_{j=0}^{\beta i}(-1)^{(\beta+1)i-j}\frac{\alpha^i}{i!}\binom{\beta i}{j}B(r,1-\lambda j)\\&-\frac{2r(\theta_1+\theta_3)}{1+\phi(\theta_2,y)}\sum_{i=0}^{\infty}\sum_{j=0}^{\beta i}(-1)^{(\beta+1)i-j}\left(1-2^{i-1}\right)\frac{\alpha^i}{i!}\binom{\beta i}{j}B(r,1-\lambda j),\end{aligned} \tag{55}$$

$$\begin{aligned}\mu^{r}_{Y/X}(x)=E\left(\frac{Y^r}{X}=x\right)&=\left(1+\frac{\theta_2+\theta_3}{1+\phi(\theta_1,x)}\right)r\sum_{i=0}^{\infty}\sum_{j=0}^{\beta i}(-1)^{(\beta+1)i-j}\frac{\alpha^i}{i!}\binom{\beta i}{j}B(r,1-\lambda j)\\&-\frac{2r(\theta_2+\theta_3)}{1+\phi(\theta_1,x)}\sum_{i=0}^{\infty}\sum_{j=0}^{\beta i}(-1)^{(\beta+1)i-j}\left(1-2^{i-1}\right)\frac{\alpha^i}{i!}\binom{\beta i}{j}B(r,1-\lambda j),\end{aligned} \tag{56}$$

where $\lambda j<1$ and $\phi(\theta,\xi)$ is given by (54).

Proposition 15. *Let $(X,Y)\sim BIUPW(\Theta)$. Then,*

$$\begin{aligned}\mu_{r,s}=E(X^rY^s)&=(1+\theta_1+\theta_2+2\theta_3)\times rs\sum_{i=0}^{\infty}\sum_{j=0}^{\beta i}(-1)^{(\beta+1)i-j}\frac{\alpha^i}{i!}\binom{\beta i}{j}B(r,1-\lambda j)\\&\times\sum_{i=0}^{\infty}\sum_{j=0}^{\beta i}(-1)^{(\beta+1)i-j}\frac{\alpha^i}{i!}\binom{\beta i}{j}B(s,1-\lambda j)-2\left\{(\theta_1+\theta_3)rs\sum_{i=0}^{\infty}\sum_{j=0}^{\beta i}(-1)^{(\beta+1)i-j}\frac{\alpha^i}{i!}\binom{\beta i}{j}B(s,1-\lambda j)\right.\\&\times\sum_{i=0}^{\infty}\sum_{j=0}^{\beta i}(-1)^{(\beta+1)i-j}\left(1-2^{i-1}\right)\frac{\alpha^i}{i!}\binom{\beta i}{j}B(r,1-\lambda j)+(\theta_2+\theta_3)rs\sum_{i=0}^{\infty}\sum_{j=0}^{\beta i}(-1)^{(\beta+1)i-j}\frac{\alpha^i}{i!}\binom{\beta i}{j}B(r,1-\lambda j)\\&\left.\times\sum_{i=0}^{\infty}\sum_{j=0}^{\beta i}(-1)^{(\beta+1)i-j}\left(1-2^{i-1}\right)\frac{\alpha^i}{i!}\binom{\beta i}{j}B(s,1-\lambda j)\right\},\end{aligned} \tag{57}$$

where $\lambda j<1$.

Proposition 16. *Let $(X,Y)\sim BIUPW(\Theta)$. Then, the bivariate reliability function is given by*

$$\begin{aligned}R(x,y)&=1-\left(1-e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}\right)\left\{1+(\theta_1+\theta_3)e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}\right\}-\left(1-e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}\right)\left\{1+(\theta_2+\theta_3)e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}\right\}\\&+\left(1-e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}\right)\left(1-e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}\right)\times\left\{1+(\theta_1+\theta_3)e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}+(\theta_2+\theta_3)e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}\right\}.\end{aligned} \tag{58}$$

RETRACTED

Proposition 17. *Let $(X, Y) \sim BIUPW(\Theta)$. Then, the bivariate hazard rate function [35] is given by*

$$\begin{aligned} h(x, y) ={}& (\alpha\lambda\beta)^2(1-x-y+xy)^{-\lambda-1}\left[(1-x-y+xy)^{-\lambda}-(1-x)^{-\lambda}-(1-y)^{-\lambda}+1\right]^{\beta-1} e^{-\alpha\left\{\left[(1-x)^{-\lambda}-1\right]^{\beta}+\left[(1-y)^{-\lambda}-1\right]^{\beta}\right\}} \\ &\times\left\{1+(\theta_1+\theta_3)\left(2e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}-1\right)+(\theta_2+\theta_3)\left(2e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}-1\right)\right\} \\ &\times\left[1-\left(1-e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}\right)\left\{1+(\theta_1+\theta_3)e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}\right\}-\left(1-e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}\right)\left\{1+(\theta_2+\theta_3)e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}\right\}\right. \\ &\left.+\left(1-e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}\right)\left(1-e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}\right)\times\left\{1+(\theta_1+\theta_3)e^{-\alpha\left[(1-x)^{-\lambda}-1\right]^{\beta}}+(\theta_2+\theta_3)e^{-\alpha\left[(1-y)^{-\lambda}-1\right]^{\beta}}\right\}\right]^{-1}. \end{aligned} \tag{59}$$

Proposition 18. *Let $(X, Y) \sim BIUPW(\Theta)$. Then, its copula function [36] is given by*

$$c(u, v) = \frac{1+\phi(\theta_1, x)+\phi(\theta_2, y)}{(1+\phi(\theta_1, x))(1+\phi(\theta_2, y))}, \tag{60}$$

where $\phi(\theta, \xi)$ is given by in Equation (54).

8.1. Estimation

Proposition 19. *Let $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ be a random sample from a random variable $BIUPW(\Theta)$. Then, the maximum log-likelihood function is given by*

$$\begin{aligned} L(\Phi) ={}& 2n\ln(\alpha\lambda\beta)-(\lambda+1)\sum_{i=1}^{n}\{\ln(1-x_i)+\ln(1-y_i)\}+(\beta-1)\sum_{i=1}^{n}\left\{\ln\left[(1-x_i)^{-\lambda}-1\right]+\ln\left[(1-y_i)^{-\lambda}-1\right]\right\}-\alpha\sum_{i=1}^{n}\left\{\ln\left[(1-x_i)^{-\lambda}-1\right]^{\beta}\right. \\ &\left.+\ln\left[(1-y_i)^{-\lambda}-1\right]^{\beta}\right\}+\sum_{i=1}^{n}\left\{\ln\left[1+(\theta_1+\theta_3)\left(2e^{-\alpha\left[(1-x_i)^{-\lambda}-1\right]^{\beta}}-1\right)+(\theta_2+\theta_3)\left(2e^{-\alpha\left[(1-y_i)^{-\lambda}-1\right]^{\beta}}-1\right)\right]\right\}, \end{aligned} \tag{61}$$

where $\Phi = (\alpha, \lambda, \beta, \theta_1, \theta_2, \theta_3)'$.

Proposition 20. *Proposition 6. Let $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ be a random sample from a random variable $BIUPW(\Theta)$. Then, the score vector $(\Phi_\alpha, \Phi_\lambda, \Phi_\beta, \Phi_{\theta_1}, \Phi_{\theta_2}, \Phi_{\theta_3})'$ is given by*

$$\begin{aligned} \Phi_\alpha ={}& \frac{2n}{\alpha}-\sum_{i=1}^{n}\left\{\ln\left[(1-x_i)^{-\lambda}-1\right]^{\beta}+\ln\left[(1-y_i)^{-\lambda}-1\right]^{\beta}\right\} \\ &-2(\theta_1+\theta_3)\sum_{i=1}^{n}\frac{\left[(1-x_i)^{-\lambda}-1\right]^{\beta}e^{-\alpha\left[(1-x_i)^{-\lambda}-1\right]^{\beta}}}{1+\phi(\theta_1, x_i)+\phi(\theta_2, y_i)} \\ &-2(\theta_2+\theta_3)\sum_{i=1}^{n}\frac{\left[(1-y_i)^{-\lambda}-1\right]^{\beta}e^{-\alpha\left[(1-y_i)^{-\lambda}-1\right]^{\beta}}}{1+\phi(\theta_1, x_i)+\phi(\theta_2, y_i)}, \end{aligned}$$

$$\begin{aligned} \Phi_\lambda ={}& \frac{2n}{\lambda}-\sum_{i=1}^{n}\{\ln(1-x_i)+\ln(1-y_i)\}+(\beta-1) \\ &\times\sum_{i=1}^{n}\left\{\frac{(1-x_i)^{-\lambda}\ln(1-x_i)}{\left[(1-x_i)^{-\lambda}-1\right]}+\frac{(1-y_i)^{-\lambda}\ln(1-y_i)}{\left[(1-y_i)^{-\lambda}-1\right]}\right\} \\ &-\alpha\beta\sum_{i=1}^{n}\left\{\frac{(1-x_i)^{-\lambda}\left[(1-x_i)^{-\lambda}-1\right]^{\beta-1}\ln(1-x_i)}{\left[(1-x_i)^{-\lambda}-1\right]^{\beta}}\right. \\ &\left.+\frac{(1-y_i)^{-\lambda}\left[(1-y_i)^{-\lambda}-1\right]^{\beta-1}\ln(1-y_i)}{\left[(1-y_i)^{-\lambda}-1\right]^{\beta}}\right\} \\ &-2(\theta_1+\theta_3)\alpha\beta \\ &\times\sum_{i=1}^{n}\frac{(1-x_i)^{-\lambda}\left[(1-x_i)^{-\lambda}-1\right]^{\beta-1}\ln(1-x_i)e^{-\alpha\left[(1-x_i)^{-\lambda}-1\right]^{\beta}}}{1+\phi(\theta_1, x_i)+\phi(\theta_2, y_i)} \end{aligned}$$

RETRACTED

Table 6: The estimates of $(\alpha, \lambda, \beta, \theta_1, \theta_2, \theta_3)$ with AIC and BIC.

Parameter	Estimates	AIC	BIC
α	5.501707		
λ	0.536157		
β	0.900586	545.9356	563.0485
θ_1	0.264058		
θ_2	-0.471280		
θ_3	0.656245		

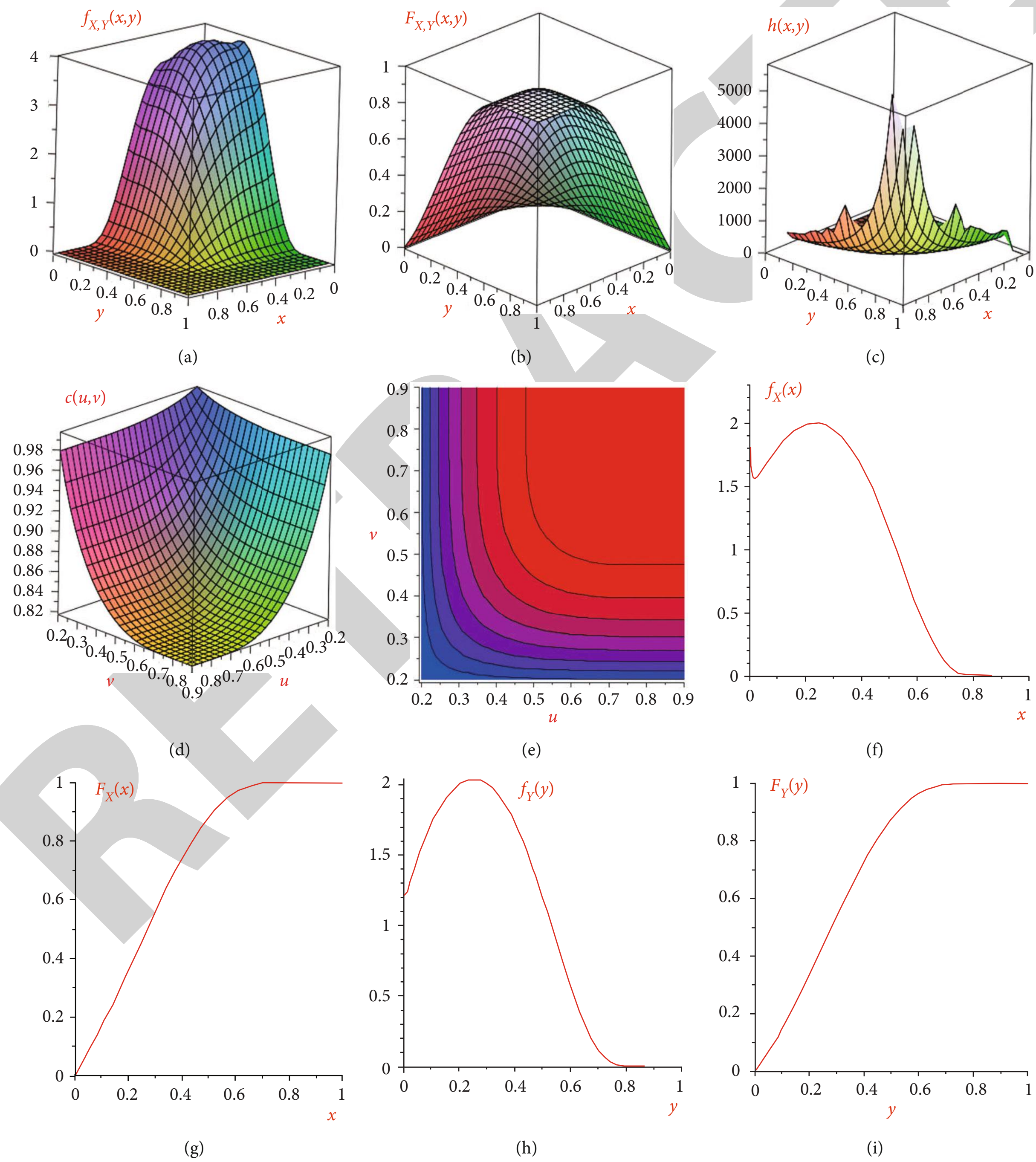


Figure 19: Statistical quantities for $(X, Y) \sim \text{BIUPW}(1.5,1.6,0.8,-0.2,-0.3,-0.5)$.

RETRACTED

$$
\begin{aligned}
&- 2(\theta_2+\theta_3)\alpha\beta \\
&\times \sum_{i=1}^{n} \frac{(1-y_i)^{-\lambda}\left[(1-y_i)^{-\lambda}-1\right]^{\beta-1} \ln{(1-y_i)} e^{-\alpha\left[(1-y_i)^{-\lambda}-1\right]^{\beta}}}{1+\phi(\theta_1,x_i)+\phi(\theta_2,y_i)},
\end{aligned}
$$

$$
\begin{aligned}
\Phi_\beta = {} & \frac{2n}{\beta} + \sum_{i=1}^{n}\left\{\ln\left[(1-x_i)^{-\lambda}-1\right] + \ln\left[(1-y_i)^{-\lambda}-1\right]\right\} \\
& - \alpha \sum_{i=1}^{n}\left\{\frac{\left[(1-x_i)^{-\lambda}-1\right]^{\beta}\ln\left[(1-x_i)^{-\lambda}-1\right]}{\left[(1-x_i)^{-\lambda}-1\right]^{\beta}}\right. \\
& \left. + \frac{\left[(1-y_i)^{-\lambda}-1\right]^{\beta}\ln\left[(1-y_i)^{-\lambda}-1\right]}{\left[(1-y_i)^{-\lambda}-1\right]^{\beta}}\right\} \\
& - 2(\theta_1+\theta_3)\alpha \\
& \times \sum_{i=1}^{n}\frac{\left[(1-x_i)^{-\lambda}-1\right]^{\beta}\ln\left[(1-x_i)^{-\lambda}-1\right] e^{-\alpha(1-x_i)^{-\lambda}-1^{\beta}}}{1+\phi(\theta_1,x_i)+\phi(\theta_2,y_i)} \\
& - 2(\theta_2+\theta_3)\alpha \\
& \times \sum_{i=1}^{n}\frac{\left[(1-y_i)^{-\lambda}-1\right]^{\beta}\ln\left[(1-y_i)^{-\lambda}-1\right] e^{-\alpha\left[(1-y_i)^{-\lambda}-1\right]^{\beta}}}{1+\phi(\theta_1,x_i)+\phi(\theta_2,y_i)},
\end{aligned}
$$

$$
\Phi_{\theta_1} = \sum_{i=1}^{n}\frac{2e^{-\alpha\left[(1-x_i)^{-\lambda}-1\right]^{\beta}}-1}{1+\phi(\theta_1,x_i)+\phi(\theta_2,y_i)},
$$

$$
\Phi_{\theta_2} = \sum_{i=1}^{n}\frac{2e^{-\alpha\left[(1-y_i)^{-\lambda}-1\right]^{\beta}}-1}{1+\phi(\theta_1,x_i)+\phi(\theta_2,y_i)},
$$

$$
\Phi_{\theta_3} = \sum_{i=1}^{n}\frac{2e^{-\alpha\left[(1-x_i)^{-\lambda}-1\right]^{\beta}}-1}{1+\phi(\theta_1,x_i)+\phi(\theta_2,y_i)} + \sum_{i=1}^{n}\frac{2e^{-\alpha\left[(1-y_i)^{-\lambda}-1\right]^{\beta}}-1}{1+\phi(\theta_1,x_i)+\phi(\theta_2,y_i)}, \tag{62}
$$

where $\phi(\theta,\xi)$ is given by (54).

8.2. Simulation Analysis Bivariate Case. This section discusses numerically the properties of statistical quantities related to the BIUPW(Θ) for different values of Θ. Let $(X, Y) \sim$ BIUPW (2.0,1.0,0.8,0.2,–0.1,–0.5). The joint density function with joint cumulative function is given in Figures 15(a) and 15(b). Figure 15(a) may be considered as a closed surface with unimodal and light right tail. The hazard function has a zero value for $(x, y) \in (0,0.6)^2$. It begins to increase approximately for $x, y \geq 0.6$, until approaching the maximum value at $(x, y) = (1, 1)$, Figure 15(c). The stochastic independence is displayed in Figures 15(d) and 15(e). It is observed that the marginal density function for the random variable X is a decreasing function, Figure 15(f) but it changes its behavior from decreasing to increasing and again to decreasing for the random variable Y, Figure 15(h). For $(X, Y) \sim$ BIUPW(0.8,0.4,1.5,–0.8,–0.4,0.7), the joint density function is a closed surface with heavy left tail, Figure 16(a), with cumulative function in Figure 16(b). Although the density function has different behavior in comparison with Figure 15(a), the hazard function has the approximately the same characteristics, Figure 16(c). The two random variables X and Y have low independence structure, Figures 16(d) and 16(e). Marginal densities are characterized by heavy left tail, Figures 16(f) and 16(h), with marginal cumulatives in Figures 16(g) and 16(i). For $(X, Y) \sim$ BIUPW(0.6,1.4,2.0,0.3,–0.4,0.5), the density function is approximately symmetric, Figure 17(a). The cumulative function and hazard function are given in Figures 17(b) and 17(c), respectively. The high independence structure can be noted in Figures 17(d) and 17(e). Approximately symmetry of marginal densities can be observed, Figures 17(f) and 17(h) with cumulatives in Figures 17(g) and 17(i). Statistical quantities for the random variable (1.5,1.6,0.8,–0.2,–0.3,–0.5)($X, Y) \sim$ BIUPW are given in Figure 18.

8.3. Application. Here, we used a COVID-19-related mortality rate data of Italy and Belgium to fit a BIUPW distribution. The data is available [37]. It covers the interval from 1 April to 20 August 2020. Table 6 shows the MLEs with AIC an BIC, whereas the graphical demonstration is depicted in Figure 19. The fitted model yielded good agreement to uncover the trend of COVID-19-related mortality rates.

9. Concluding Remarks

In this article, we presented a unit-power Weibull distribution after reparameterizing the Weibull distribution using an appropriate transformation. The proposed model shows greater flexibility. Some basic properties of the UPWD include quantile function, linear representation of the density, rth ordinary and sth incomplete moments, moment-generating function, probability-weighted moments, the expression of order statistics entropy measure, L-moments, and TL-moments. We performed an estimation of unknown parameters of UPWD using maximum likelihood estimation (MLE). A Monte Carlo simulation study is carried out to check the accuracy of the MLE parameters of UPWD. The actuarial measures are also computed, namely, value at risk, value at risk, expected shortfall, tail value at risk, tail variance, and tail variance premium are expressed. We performed the application using three real data sets which shows a good agreement between actual and predicted. The proposed model is compared with some well-known models such as exponentiated Weibull (EW), Kumaraswamy exponential (KE), gamma Kumaraswamy (GK), and beta exponential (BE). Bivariate extension of the model is presented and called bivariate unit-power Weibull. The estimation, simulation, and application to the real data set of COVID-19 along with the graphical presentation for marginal densities are illustrated. The findings depict that the fitted model uncovers the COVID-19 trend by effective means. The statistical quantities for $(X, Y) \sim$ BIUPW (1.5,1.6,0.8,–0.2,–0.3 ,–0.5) are given in Figure 19.

Data Availability

The data used in the findings of the study are included within the article.

RETRACTED

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the funds of the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant no. IFPIP-1479-150-1442.

References

[1] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous univariate distributions*, John wiley and sons.1, 1995.

[2] P. A. Mitnik, “New properties of the Kumaraswamy distribution,” *Communications in Statistics-Theory and Methods*, vol. 42, no. 5, pp. 741–755, 2013.

[3] N. L. Johnson, “Systems of frequency curves generated by methods of translation,” *Biometrika*, vol. 36, no. 1-2, pp. 149–176, 1949.

[4] C. W. Topp and F. C. Leone, “A family of J-shaped frequency functions,” *Journal of the American Statistical Association*, vol. 50, no. 269, pp. 209–219, 1955.

[5] J. Mazucheli, A. F. B. Menezes, and M. E. Ghitany, “The unit-Weibull distribution and associated inference,” *Journal of Applied Probability and Statistics*, vol. 13, no. 2, pp. 1–22, 2018.

[6] J. Mazucheli, A. F. B. Menezes, and S. Dey, “Improved maximum-likelihood estimators for the parameters of the unit-gamma distribution,” *Communications in Statistics-Theory and Methods*, vol. 47, no. 15, pp. 3767–3778, 2018.

[7] J. Mazucheli, A. F. Menezes, and S. Dey, “Unit-Gompertz distribution with applications,” *Statistica*, vol. 79, no. 1, pp. 25–43, 2019.

[8] M. E. Ghitany, J. Mazucheli, A. F. B. Menezes, and F. Alqallaf, “The unit-inverse Gaussian distribution: a new alternative to two-parameter distributions on the unit interval,” *Communications in Statistics-Theory and Methods*, vol. 48, no. 14, pp. 3423–3438, 2019.

[9] J. Mazucheli, A. F. B. Menezes, and S. Chakraborty, “On the one parameter unit-Lindley distribution and its associated regression model for proportion data,” *Journal of Applied Statistics*, vol. 46, no. 4, pp. 700–714, 2019.

[10] J. Mazucheli, S. R. Bapat, and A. F. B. Menezes, “A new one-parameter unit-Lindley distribution,” *Chilean Journal of Statistics (ChJS)*, vol. 11, no. 1, 2020.

[11] M. Ç. Korkmaz, “The unit generalized half normal distribution: a new bounded distribution with inference and application,” *UPB Scientific Bulletin, Series A*, vol. 82, no. 2, 2020.

[12] S. Hashmi, K. Aidi, P. L. Ramos, and F. Louzada, “Unit modified Burr-III distribution: estimation, characterizations and validation test,” *Annals of Data Science*, vol. 7, pp. 1–26, 2020.

[13] M. C. Korkmaz, A. Emrah, C. Chesneau, and H. M. Yousof, “On the unit-Chen distribution with associated quantile regression and applications,” *Math Solovaca*, vol. 72, 2021.

[14] R. A. Bantan, C. Chesneau, F. Jamal et al., “Some new facts about the unit-Rayleigh distribution with applications,” *Mathematics*, vol. 8, no. 11, p. 1954, 2020.

[15] C. Chesneau, “Study of a unit power-logarithmic distribution,” *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 218–235, 2021.

[16] I. Shah, B. Iqbal, M. Farhan Akram, S. Ali, and S. Dey, “Unit Nadarajah and Haghighi distribution: properties and applications in quality control,” *Scientia Iranica*, vol. 28, 2021.

[17] Q. Ramzan, M. Amin, A. Elhassanein et al., “The extended generalized inverted Kumaraswamy Weibull distribution: properties and applications,” *AIMS Mathematics*, vol. 6, no. 9, pp. 9955–9980, 2021.

[18] Q. Ramzan, S. Qamar, M. Amin, H. Alshanbari, A. Nazeer, and A. Elhassanein, “On the extended generalized inverted Kumaraswamy distribution,” *Computational Intelligence and Neuroscience*, vol. 2022, 18 pages, 2022.

[19] A. Algarni, A. M. Almarashi, F. Jamal, C. Chesneau, and M. Elgarhy, “Truncated inverse Lomax generated family of distributions with applications to biomedical data,” *Journal of Medical Imaging and Health Informatics*, vol. 11, pp. 2425–2439, 2021.

[20] R. A. ZeinEldin, C. Chesneau, F. Jamal, M. Elgarhy, A. M. Almarashi, and S. Al-Marzouki, “Generalized truncated Fréchet generated family distributions and their applications,” *Journal of Computer Modeling in Engineering and Sciences*, vol. 126, no. 2, pp. 791–819, 2021.

[21] W. Weibull, “A statistical distribution function of wide applicability,” *Journal of Applied Mechanics*, vol. 18, no. 3, pp. 293–297, 1951.

[22] M. Bourguignon, R. B. Silva, and G. M. Cordeiro, “The Weibull–G family of probability distributions,” *Journal of Data Science*, vol. 12, no. 1, pp. 53–68, 2014.

[23] S. Nadarajah and S. Kotz, “Some beta distributions,” *Bulletin of the Brazilian Mathematical Society*, vol. 37, no. 1, pp. 103–125, 2006.

[24] M. J. Sigal and R. P. Chalmers, “Play it again: teaching statistics with Monte Carlo simulation,” *Journal of Statistics Education*, vol. 24, no. 3, pp. 136–156, 2016.

[25] P. Artzner, F. Delbaen, J. M. Eber, and D. Heath, “Coherent measures of risk,” *Mathematical Finance*, vol. 9, no. 3, pp. 203–228, 1999.

[26] S. Chan, S. Nadarajah, and E. Afuecheta, “An r package for value at risk and expected shortfall,” *Communications in Statistics–Simulation and Computation*, vol. 45, no. 9, pp. 3416–3434, 2016.

[27] A. Z. Afify, A. M. Gemeay, and N. A. Ibrahim, “The heavy–tailed exponential distribution: risk measures, estimation, and application to actuarial data,” *Mathematics*, vol. 8, no. 8, p. 1276, 2020.

[28] A. Biswas and S. Chakraborty, “A new method for constructing continuous distributions on the unit interval,” 2021, http://arxiv.org/abs/2101.04661.

[29] S. Nadarajah, G. M. Cordeiro, and E. M. Ortega, “General results for the kumaraswamy g distribution,” *Journal of Statistical Computation and Simulation*, vol. 82, no. 7, pp. 951–979, 2012.

[30] I. Ghosh and G. G. Hamedani, “The Gamma–Kumaraswamy distribution: an alternative to Gamma distribution,” *Communications in Statistics–Theory and Methods*, vol. 47, no. 9, pp. 2056–2072, 2018.

[31] M. C. Jones, “Families of distributions arising from distributions of order statistics,” *Test*, vol. 13, no. 1, pp. 1–43, 2004.

[32] J. A. Darwish and M. Q. Shahbaz, “Bivariate transmuted Burr distribution: properties and applcations,” *Pakistan Journal of Statistics & Operation Research*, vol. 17, no. 1, pp. 15–24, 2021.

RETRACTED

[33] M. Ganji, H. Bevraninand, and H. Golzar, "A new method for generating continuous bivariate families," *Journal of The Iranian Statistical Society*, vol. 17, no. 1, pp. 109–129, 2018.

[34] M. Ragab and A. Elhassanein, "A new bivariate extended generalized inverted Kumaraswamy Weibull distribution," *Advances in Mathematical Physics*, vol. 2022, 13 pages, 2022.

[35] J. Navarro, "Characterizations using the bivariate failure rate function," *Statistics and Probability Letters*, vol. 78, no. 12, pp. 1349–1354, 2008.

[36] M. V. O. Peres, J. A. Achcar, and E. Z. Martinez, "Bivariate lifetime models in presence of cure fraction: a comparative study with many different copula functions," *Heliyon*, vol. 6, no. 6, p. e03961, 2020.

[37] A. Algarni, A. M. Almarashi, I. Elbatal et al., "Type I half logistic Burr X-G family: properties, Bayesian, and non-Bayesian estimation under censored samples and applications to COVID-19 data," *Mathematical Problems in Engineering*, vol. 2021, 21 pages, 2021.

RETRACTED