Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2876673, 7 pages
https://doi.org/10.1155/2022/2876673

Research Article

Certain Geometric Properties of the Canonical Weierstrass Product of an Entire Function Associated with Conic Domains

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Received 24 July 2022; Accepted 20 August 2022; Published 20 September 2022

Academic Editor: Teodor Bulboaca

In this paper, we determine the radius of λ-uniform convexity, λ-starlikeness, and α-convexity of order δ for the Weierstrass canonical product of an entire function as a root having smallest modulus and argument ϕ of a functional equation. As special cases, we also determine the radius of λ-uniform convexity, λ-starlikeness, and α-convexity of order δ for the entire function $1/\Gamma$.

1. Introduction

Let $r>0$ be a real number and $\mathscr{A}$ be the class of analytic functions defined in the disk $\mathbb{U}(r)=\{w\in\mathbb{C}:|w|<r\}$ and satisfy the normalization conditions $f(0)=f'(0)-1=0$. Let (a_n), where $a_n\in\mathbb{C}$, $\forall n\geq 2$ be a sequence with

$$\frac{1}{\lim_{n\longrightarrow+\infty}\sup|a_n|^{1/n}}=r_f\geq 0, \tag{1}$$

where r_f means the radius of convergence of the series $w+\sum_{n=2}^{\infty}a_n w^n=f(w)\in\mathscr{A}$. If $\lim_{n\longrightarrow+\infty}\sup|a_n|^{1/n}=0$, then $r_f=+\infty$.

In 1999, Kanas and Wisniowska [9] (also refer Goodman [7, 8], Rønning [15], and Ma and Minda [12]) proposed the idea of λ-uniform convexity denoted by $\lambda-\mathscr{UCV}$.

A function $f\in\mathscr{A}$ is said to be in $\lambda-\mathscr{UCV}(\delta)$, the class of λ-uniformly Convex of order δ [3], iff

$$\operatorname{Re}\left(1+\frac{wf''(w)}{f'(w)}\right)>\lambda\left|\frac{wf''(w)}{f'(w)}\right|+\delta, \lambda\geq 0, \delta\in[0,1)\forall w\in\mathbb{U}(r). \tag{2}$$

A function $f\in\mathscr{A}$ is said to be in $\lambda-\mathscr{ST}(\delta)$, the class of λ-starlike function of order δ [10], iff

$$\operatorname{Re}\left(\frac{wf'(w)}{f(w)}\right)>\lambda\left|\frac{wf'(w)}{f(w)}-1\right|+\delta, \lambda\geq 0, \delta\in[0,1)\forall w\in\mathbb{U}(r). \tag{3}$$

Geometrically, the conditions (2) and (3) mean that for $f\in\lambda-\mathscr{UCV}(\delta)$ and $f\in\lambda-\mathscr{ST}(\delta)$, the images of $\mathbb{U}(r)$ under the functions $1+wf''(w)/f'(w)$ and $wf'(w)/f(w)$ are in the conic domain Ω_λ^δ contained in the right half plane for which $1\in\Omega_\lambda^\delta$ and $\partial\Omega_\lambda^\delta$ is the curve defined by the equation

$$\partial\Omega_\lambda^\delta=\left\{\omega=u+iv:(u-\delta)^2=\lambda^2\left[(u-1)^2+v^2\right]\right\}, \lambda\geq 0. \tag{4}$$

Moreover, Ω_λ^δ is an elliptic region for $\lambda>1$, parabolic for $\lambda=1$, and hyperbolic for $0<\lambda<1$, and finally, Ω_0^0 is the whole right half plane.

The radius of λ-uniform convexity of order δ denoted by $r_{uc(f)}^{\lambda}(\delta)$ and radius of λ-starlikeness of order δ denoted by $r_{st(f)}^{\lambda}(\delta)$ are defined by

$$r^{\lambda}_{uc(f)}(\delta)=\sup\left\{r\in(0,r_f):\ \operatorname{Re}\left(1+\frac{wf''(w)}{f'(w)}\right)>\lambda\left|\frac{wf''(w)}{f'(w)}\right|+\delta,\forall w\in\mathbb{U}(r)\right\},\tag{5}$$

$$r^{\lambda}_{st(f)}(\delta)=\sup\left\{r\in(0,r_f):\ \operatorname{Re}\left(\frac{wf'(w)}{f(w)}\right)>\lambda\left|\frac{wf'(w)}{f(w)}-1\right|+\delta,\forall w\in\mathbb{U}(r)\right\},\tag{6}$$

where $\lambda\geq 0, \delta\in[0,1)$.

By specializing the parameters, we observe $r^0_{uc(f)}(0)=r^c_f$, radius of convexity, $r^0_{uc(f)}(\delta)=r^c_f(\delta)$, radius of convexity of order δ, $r^0_{st(f)}(\delta)=r^*_f(\delta)$, radius of starlikeness of order δ and $r^0_{st(f)}(0)=r^*_f$, radius of starlikeness.

Let $\alpha\in\mathbb{R}$ and $\alpha\in[0,1)$. A function $f\in\mathscr{A}$ is said to be in $\mathscr{M}_\alpha(\delta)$, the class of α-convex functions (Mocanu functions) of order δ [14, 16] iff

$$\operatorname{Re}\left((1-\alpha)\frac{wf'(w)}{f(w)}+\alpha\left(1+\frac{wf''(w)}{f'(w)}\right)\right)>\delta,\ w\in\mathbb{U}(r),\ \delta\in[0,1).\tag{7}$$

The radius of α-convexity (Mocanu functions) of order δ denoted by $r^{\alpha}_{c(f)}(\delta)$ is defined by, for $0\leq\delta<1$,

$$r^{\alpha}_{c(f)}(\delta)=\sup\left\{r\in(0,r_f):\ \operatorname{Re}\left((1-\alpha)\frac{wf'(w)}{f(w)}+\alpha\left(1+\frac{wf''(w)}{f'(w)}\right)\right)>\delta,\ w\in\mathbb{U}(r)\right\}.\tag{8}$$

Addressing radius problems for some special functions is a new direction in the geometric function theory. For recent studies on radius problems, we refer to [2, 4, 6, 11].

By the Weierstrass factorization theorem [18], the function

$$\mathscr{B}(w)=we^{h(w)}\prod_{n=1}^{\infty}\left(1-\frac{w}{c_n}\right)\exp\left[\sum_{k=1}^{q_n}\frac{1}{k}\left(\frac{w}{c_n}\right)^k\right],\tag{9}$$

is an entire function for a proper choice of $q_n\leq n$ with zeros c_n and no other zeros, where $h(w)$ is an entire function with $h(0)=0$, $c_n\neq 0\forall n$, $q_n\geq 0$ are certain nonnegative integers, and for each n in which $q_n=0$, the value of exponential factor becomes 1.

The product (9) is called the canonical Weierstrass product [1]. In Theorem 3 of [13] Merkes et al. determined the radius of starlikeness of the canonical Weierstrass product $\mathscr{B}(w)$, and as a special case, the authors determined the radius of starlikeness of

$$\frac{1}{\Gamma(w)}=we^{w\gamma}\prod_{n=1}^{\infty}\left(1+\frac{w}{n}\right)e^{-w/n}.\tag{10}$$

Later in [17], Szasz obtained the radius of convexity for $\mathscr{B}(w)$.

Motivated by the results of Szász [17] and Merkes et al. [13], we determine the radius of λ-uniformly convexity, λ-starlikeness, and α-convexity of order δ for the function $\mathscr{B}(w)$ given by (9). Consequently, we also determine the radius of λ-uniform convexity, λ-starlikeness, and α-convexity of order δ for the function $1/\Gamma$ in this paper. In order to prove the main result, we require the following lemma.

Lemma 1 (see [17]). *If $a,b\in\mathbb{R}$ and $a>b>0$, then*

$$\left|\frac{a+w}{(b+w)^2}\right|\leq\frac{a-|w|}{(b-|w|)^2},\ \text{for } |w|<b, w\in\mathbb{U}=\mathbb{U}(1).\tag{11}$$

2. Main Results

Theorem 2. *Let $\{c_n\}_{n\in\mathbb{N}/\{0\}}$ be a sequence with $c_n=|c_n|e^{i\phi}\in\mathbb{C}$, $|c_n|\geq 1$ for $n\in\mathbb{N}/\{0\}$, $r_0=\inf\{|c_n|: n\in\mathbb{N}/\{0\}\}$, and let $h(w)$ be an analytic function in $\mathbb{U}(r_0)$ with $|w|e^{i\phi}h'(|w|e^{i\phi})\in\mathbb{R}$ and $|w|e^{i\phi}h'(|w|e^{i\phi})\leq\Re\{wh'(w)\}$, for $w\in\mathbb{U}(r_0)$. If the function $\mu:(0,r_0)\longrightarrow\mathbb{R}$ defined by $\mu(r)=re^{i\phi}h'(re^{i\phi})$ is decreasing with respect to r and $\mathscr{B}(w)$ is of the form (9) with $q_n\in\mathbb{N}/\{0\}$ for $n\in\mathbb{N}/\{0\}$, then the radius of λ-starlikeness of order δ of the function $\mathscr{B}(w)$ is $r^{\lambda}_{st(\mathscr{B})}(\delta)$, the absolute value of the root of the equation $(1+\lambda)w\mathscr{B}'(w)-(\lambda+\delta)\mathscr{B}(w)=0$ having the smallest modulus and argument ϕ.*

Proof. By logarithmic differentiation, (9) becomes

$$\frac{w\mathscr{B}'(w)}{\mathscr{B}(w)}=1+wh'(w)-\sum_{n=1}^{\infty}\frac{\left(\frac{w}{c_n}\right)^{q_n+1}}{1-\frac{w}{c_n}}.\tag{12}$$

For $w\in\mathbb{U}$ and $k,n\in\mathbb{N}$,

$$\Re\left[\frac{w^n}{(1-w)^k}\right]\leq\left|\frac{w^n}{(1-w)^k}\right|=\frac{|w|^n}{|1-w|^k}\leq\frac{|w|^n}{(1-|w|)^k}.\tag{13}$$

Since $|w/c_n| \leq 1$, (12) along with (13) implies

$$\begin{aligned}\Re\left\{\frac{w\mathscr{B}'(w)}{\mathscr{B}(w)}\right\} &\geq 1+\Re\left\{wh'(w)\right\}-\sum_{n=1}^{\infty}\left(\frac{|w/c_n|^{q_n+1}}{1-|w/c_n|}\right)\\ &\geq 1+|w|e^{i\phi}h'\left(|w|e^{i\phi}\right)-\sum_{n=1}^{\infty}\left(\frac{|w/c_n|^{q_n+1}}{1-|w/c_n|}\right)\\ &=\frac{|w|e^{i\phi}\mathscr{B}'\left(|w|e^{i\phi}\right)}{\mathscr{B}(|w|e^{i\phi})}.\end{aligned} \tag{14}$$

□

Also,

$$\begin{aligned}\left|\frac{w\mathscr{B}'(w)}{\mathscr{B}(w)}-1\right| &\leq \left|wh'(w)\right|+\sum_{n=1}^{\infty}\left(\frac{|w/c_n|^{q_n+1}}{1-|w/c_n|}\right)\\ &\leq -|w|e^{i\phi}h'\left(|w|e^{i\phi}\right)+\sum_{n=1}^{\infty}\left(\frac{|w/c_n|^{q_n+1}}{1-|w/c_n|}\right)\\ &=1-\frac{|w|e^{i\phi}\mathscr{B}'\left(|w|e^{i\phi}\right)}{\mathscr{B}(|w|e^{i\phi})}.\end{aligned} \tag{15}$$

From (14) and (15), we have

$$\begin{aligned}&\Re\left\{\frac{w\mathscr{B}'(w)}{\mathscr{B}(w)}\right\}-\lambda\left|\frac{w\mathscr{B}'(w)}{\mathscr{B}(w)}-1\right|-\delta\\ &\geq(1+\lambda)\frac{|w|e^{i\phi}\mathscr{B}'\left(|w|e^{i\phi}\right)}{\mathscr{B}(|w|e^{i\phi})}-(\lambda+\delta), \delta\in[0,1), \lambda\geq 0.\end{aligned} \tag{16}$$

By the virtue of minimum principle for harmonic functions,

$$\begin{aligned}\inf_{|w|<r}&\left\{\Re\left\{\frac{w\mathscr{B}'(w)}{\mathscr{B}(w)}\right\}-\lambda\left|\frac{w\mathscr{B}'(w)}{\mathscr{B}(w)}-1\right|-\delta\right\}\\ &=(1+\lambda)\frac{re^{i\phi}\mathscr{B}'\left(re^{i\phi}\right)}{\mathscr{B}(re^{i\phi})}-(\lambda+\delta), r\in(0,r_o).\end{aligned} \tag{17}$$

We observe that the function $\varphi:(0,r_0)\longrightarrow\mathbb{R}$ defined by

$$\varphi(r)=(1+\lambda)\frac{re^{i\phi}\mathscr{B}'\left(re^{i\phi}\right)}{\mathscr{B}(re^{i\phi})}-(\lambda+\delta) \tag{18}$$

is strictly decreasing; also, $\lim_{r\longrightarrow 0}\varphi(r)=(1-\delta)>0$ and $\lim_{r\longrightarrow r_0}\varphi(r)=-\infty$.

Hence, the equation $(1+\lambda)e^{i\phi}r\mathscr{B}'(re^{i\phi})-(\lambda+\delta)\mathscr{B}(e^{i\phi}r)=0$ has a unique root in $(0,r_0)$, and this root is $r^{\lambda}_{st(\mathscr{B})}(\delta)$.

Remark 3. $\lambda\geq 0$ in Theorem 2 means that, if $\mathscr{B}\in\lambda-\mathscr{ST}(\delta)$, then the image of $\mathbb{U}(r)$ under the function $w\mathscr{B}'(w)/\mathscr{B}(w)$ is in conic domain $\Omega^{\delta}_{\lambda}$ contained in the right half plane for which $1\in\Omega^{\delta}_{\lambda}$ and $\partial\Omega^{\delta}_{\lambda}$ is the curve defined by equation (4).

In the following remarks, we deduce the radius of some special classes by specializing the parameters in Theorem 2.

Remark 4. Taking $\lambda\geq 0$, $\delta=0$ in Theorem 2, we get $r^{\lambda}_{st(\mathscr{B})}$, the radius of λ -starlikeness of the function $\mathscr{B}(w)$. $r^{\lambda}_{st(\mathscr{B})}$ is the absolute value of the root of the equation $(1+\lambda)w\mathscr{B}'(w)-\lambda\mathscr{B}(w)=0$ having the smallest modulus and argument ϕ.

Remark 5. Letting $\lambda=0$, $0\leq\delta<1$ in Theorem 2, we get $r^{*}_{\mathscr{B}}(\delta)$, the radius of starlikeness of order δ of the function $\mathscr{B}(w)$. $r^{*}_{\mathscr{B}}(\delta)$ is the absolute value of the root of the equation $w\mathscr{B}'(w)-\delta\mathscr{B}(w)=0$ having the smallest modulus and argument ϕ.

In the following, we obtain the radius $r^{\lambda}_{uc(\mathscr{B})}(\delta)$ of λ -uniform convexity of order δ for $\mathscr{B}(w)$.

Theorem 6. *Let $\{c_n\}_{n\in\mathbb{N}/\{0\}}$ be a sequence with $c_n=|c_n|e^{i\phi}\in\mathbb{C}$, $|c_n|\geq 1$ for $n\in\mathbb{N}/\{0\}$, $r_0=\inf\{|c_n|: n\in\mathbb{N}/\{0\}\}$, and let $h(w)$ be an analytic function in $\mathbb{U}(r_0)$ with $|w|e^{i\phi}h'(|w|e^{i\phi})\in\mathbb{R}$, $|w|e^{i\phi}h'(|w|e^{i\phi})\leq\Re\{wh'(w)\}$, and $|w^2h''(w)|\leq-|w|^2e^{i2\phi}h''(|w|e^{i\phi})$ for $w\in\mathbb{U}(r_0)$. If the function $\mu:(0,r_0)\longrightarrow\mathbb{R}$ defined by $\mu(r)=re^{i\phi}h'(re^{i\phi})$ is decreasing, the function $\vartheta(r)$: $(0,r_0)\longrightarrow\mathbb{R}$ defined by $\vartheta(r)=-r^2e^{2i\phi}h''(re^{i\phi})$ is increasing with respect to r and $\mathscr{B}(w)$ is of the form (9) with $q_n\in\mathbb{N}/\{0\}$ for $n\in\mathbb{N}/\{0\}$; then, λ-uniform convexity of order δ of the function $\mathscr{B}(w)$ is $r^{\lambda}_{uc(\mathscr{B})}(\delta)$, the absolute value of the root of the equation $(1+\lambda)w\mathscr{B}''(w)+(1-2\lambda-\delta)\mathscr{B}'(w)=0$ having the smallest modulus and argument ϕ.*

Proof. From (9),

$$\begin{aligned}1+\frac{w\mathscr{B}''(w)}{\mathscr{B}'(w)}&=2+wh'(w)-\sum_{n=1}^{\infty}\frac{(w/c_n)^{q_n+1}}{1-w/c_n}\\ &-\frac{1-w^2h''(w)+\sum_{n=1}^{\infty}\left(\left(\left((q_n)(w/c_n)^{q_n+1}\right)-\left((q_n-1)(w/c_n)^{q_n+2}\right)\right)/(1-w/c_n)^2\right)}{1+wh'(w)-\sum_{n=1}^{\infty}\left(\left((w/c_n)^{q_n+1}\right)/(1-(w/c_n))\right)}\end{aligned} \tag{19}$$

$$=1+\frac{w\mathscr{B}'(w)}{\mathscr{B}(w)}-\frac{1-w^2h''(w)+\sum_{n=1}^{\infty}\left(\left((q_n)(w/c_n)^{q_n+1}\right)-\left((q_n-1)(w/c_n)^{q_n+2}\right)/(1-w/c_n)^2\right)}{w\mathscr{B}'(w)/\mathscr{B}(w)}. \tag{20}$$

Using (14) and the inequality of Lemma 1, we have

$$\begin{aligned}
&\Re\left(\frac{1-w^2h''(w)+\sum_{n=1}^{\infty}\left(\left((q_n)(w/c_n)^{q_n+1}-(q_n-1)(w/c_n)^{q_n+2}\right)/\left((1-(w/c_n))^2\right)\right)}{w\mathscr{B}'(w)/\mathscr{B}(w)}\right)\\
&\le \frac{\left|1-w^2h''(w)+\sum_{n=1}^{\infty}\left(\left((q_n)(w/c_n)^{q_n+1}-(q_n-1)(w/c_n)^{q_n+2}\right)/\left((1-(w/c_n))^2\right)\right)\right|}{\left|\left(w\mathscr{B}'(w)\right)/\mathscr{B}(w)\right|}\\
&\le \frac{1+\left|w^2h''(w)\right|+\sum_{n=1}^{\infty}\left(\left(|(w/c_n)|^{q_n+1}|(q_n)-(q_n-1)(w/c_n)|\right)/\left(|1-(w/c_n)|^2\right)\right)}{\Re\left(\left(w\mathscr{B}'(w)\right)/(\mathscr{B}(w))\right)}\\
&\le \frac{1-|w|^2e^{i2\phi}h''\left(|w|e^{i\phi}\right)+\sum_{n=1}^{\infty}\left(\left((q_n)|(w/c_n)|^{q_n+1}-(q_n-1)|(w/c_n)|^{q_n+2}\right)/\left(|1-(w/c_n)|^2\right)\right)}{\left(|w|e^{i\phi}\mathscr{B}'(|w|e^{i\phi})\right)/(\mathscr{B}(|w|e^{i\phi}))}.
\end{aligned} \tag{21}$$

From (14), (20), and (21),

$$\begin{aligned}
&\Re\left(1+\left(\left(w\mathscr{B}''(w)\right)/\mathscr{B}'(w)\right)\right)\\
&\ge 1+\Re\left(\left(w\mathscr{B}'(w)\right)/(\mathscr{B}(w))\right)-\Re\left(\frac{1-w^2h''(w)+\sum_{n=1}^{\infty}\left(\left((q_n)(w/c_n)^{q_n+1}-(q_n-1)(w/c_n)^{q_n+2}\right)/\left((1-(w/c_n))^2\right)\right)}{\left(w\mathscr{B}'(w)\right)/(\mathscr{B}(w))}\right)\\
&\ge 1+\frac{|w|e^{i\phi}\mathscr{B}'\left(|w|e^{i\phi}\right)}{\mathscr{B}(|w|e^{i\phi})}-\frac{1-|w|^2e^{i2\phi}h''\left(|w|e^{i\phi}\right)+\sum_{n=1}^{\infty}\left(\left((q_n)|w/c_n|^{q_n+1}-(q_n-1)|w/c_n|^{q_n+2}\right)/\left((1-|w/c_n|)^2\right)\right)}{\left(|w|e^{i\phi}\mathscr{B}'(|w|e^{i\phi})\right)/(\mathscr{B}(|w|e^{i\phi}))}\\
&=1+\frac{|w|e^{i\phi}\mathscr{B}''\left(|w|e^{i\phi}\right)}{\mathscr{B}'(|w|e^{i\phi})}.
\end{aligned} \tag{22}$$

Also, we have

$$\begin{aligned}
\left|\frac{w\mathscr{B}''(w)}{\mathscr{B}'(w)}\right| &\le \left|\frac{w\mathscr{B}'(w)}{\mathscr{B}(w)}-\frac{1-w^2h''(w)+\sum_{n=1}^{\infty}\left(\left((q_n)(w/c_n)^{q_n+1}-(q_n-1)(w/c_n)^{q_n+2}\right)/\left((1-(w/c_n))^2\right)\right)}{1+wh'(w)-\sum_{n=1}^{\infty}\left(\left((w/c_n)^{q_n+1}\right)/(1-(w/c_n))\right)}\right|\\
&\le \left|\frac{w\mathscr{B}'(w)}{\mathscr{B}(w)}\right|+\left|\frac{1-w^2h''(w)+\sum_{n=1}^{\infty}\left(\left((q_n)(w/c_n)^{q_n+1}-(q_n-1)(w/c_n)^{q_n+2}\right)/\left((1-(w/c_n))^2\right)\right)}{1+wh'(w)-\sum_{n=1}^{\infty}\left(\left((w/c_n)^{q_n+1}\right)/(1-(w/c_n))\right)}\right|\\
&\le 2-\frac{|w|e^{i\phi}\mathscr{B}'\left(|w|e^{i\phi}\right)}{\mathscr{B}(|w|e^{i\phi})}+\frac{1-|w|^2e^{i2\phi}h''\left(|w|e^{i\phi}\right)+\sum_{n=1}^{\infty}\left(\left((q_n)|w/c_n|^{q_n+1}-(q_n-1)|w/c_n|^{q_n+2}\right)/\left((1-|w/c_n|)^2\right)\right)}{1+|w|e^{i\phi}h'(|w|e^{i\phi})-\sum_{n=1}^{\infty}\left(\left(|w/c_n|^{q_n+1}\right)/(1-|w/c_n|)\right)}\\
&\le 2-\frac{|w|e^{i\phi}\mathscr{B}'\left(|w|e^{i\phi}\right)}{\mathscr{B}(|w|e^{i\phi})}+\frac{1-|w|^2e^{i2\phi}h''\left(|w|e^{i\phi}\right)+\sum_{n=1}^{\infty}\left(\left((q_n)|w/c_n|^{q_n+1}-(q_n-1)|w/c_n|^{q_n+2}\right)/\left((1-|w/c_n|)^2\right)\right)}{\left(|w|e^{i\phi}\mathscr{B}'(|w|e^{i\phi})\right)/(\mathscr{B}(|w|e^{i\phi}))}\\
&=2-\frac{|w|e^{i\phi}\mathscr{B}''\left(|w|e^{i\phi}\right)}{\mathscr{B}'(|w|e^{i\phi})}.
\end{aligned} \tag{23}$$

From (22) and (23),

$$\begin{aligned}&\Re\left(1+\frac{w\mathscr{B}''(w)}{\mathscr{B}'(w)}\right)-\lambda\left|\frac{w\mathscr{B}''(w)}{\mathscr{B}'(w)}\right|-\delta\\&\geq 1+\frac{|w|e^{i\phi}\mathscr{B}''\left(|w|e^{i\phi}\right)}{\mathscr{B}'\left(|w|e^{i\phi}\right)}-\lambda\left(2-\frac{|w|e^{i\phi}\mathscr{B}''\left(|w|e^{i\phi}\right)}{\mathscr{B}'\left(|w|e^{i\phi}\right)}\right)-\delta\\&=(1+\lambda)\left(\frac{|w|e^{i\phi}\mathscr{B}''\left(|w|e^{i\phi}\right)}{\mathscr{B}'\left(|w|e^{i\phi}\right)}\right)+(1-\delta-2\lambda),\delta\in[0,1),0\\&\leq\lambda<\frac{(1-\delta)}{2}.\end{aligned}\tag{24}$$

By the virtue of minimum principle for harmonic functions,

$$\begin{aligned}&\inf_{|w|<r}\left\{\Re\left(1+\frac{w\mathscr{B}''(w)}{\mathscr{B}'(w)}\right)-\lambda\left|\frac{w\mathscr{B}''(w)}{\mathscr{B}'(w)}\right|-\delta\right\}\\&=(1+\lambda)\left(\frac{re^{i\phi}\mathscr{B}''\left(re^{i\phi}\right)}{\mathscr{B}'\left(re^{i\phi}\right)}\right)+(1-\delta-2\lambda),\end{aligned}\tag{25}$$

where $r\in(0,r_0)$. The function $\psi:(0,r_0)\longrightarrow\mathbb{R}$, defined by $\psi(r)=(1+\lambda)(re^{i\phi}\mathscr{B}''(re^{i\phi})/\mathscr{B}'(re^{i\phi}))+(1-\delta-2\lambda)$ is strictly decreasing; also, observe that $\lim_{r\longrightarrow 0}\psi(r)=1-\delta-2\lambda>0$, $\lim_{r\longrightarrow r_0}\psi(r)=-\infty$. Thus, it follows that the equation $(1+\lambda)e^{i\phi}r\mathscr{B}''(e^{i\phi}r)+(1-2\lambda-\delta)\mathscr{B}'(e^{i\phi}r)=0$ has a unique root situated in $(0,r_0)$, and this root is $r^{\lambda}_{uc(\mathscr{B})}(\delta)$.

Remark 7. As $\delta\in[0,1)$ and $0\leq\lambda<((1-\delta)/2)\leq(1/2)$,we have $\lambda\in[0,1/2)$, which means that if $\mathscr{B}\in\lambda-\mathscr{UCV}(\delta)$, then the image of $\mathbb{U}(r)$ under the function contained in the right half plane for which $1+((w\mathscr{B}''(w))/(\mathscr{B}'(w)))$ is in hyperbolic domain $\Omega^{\delta}_{\lambda}$ contained in the right half plane for which $1\in\Omega^{\delta}_{\lambda}$ and $\partial\Omega^{\delta}_{\lambda}$ is the curve defined by equation (4).

By specializing the parameters in Theorem 6, we have

Remark 8. Substituting $\delta=0$ and $\lambda\in[0,1/2)$ in Theorem 6, we get the radius $r^{\lambda}_{uc(\mathscr{B})}$ of λ -uniform convexity given by the absolute value of the root of the equation $(1+\lambda)w\mathscr{B}''(w)+(1-2\lambda)\mathscr{B}'(w)=0$ having the smallest modulus and argument ϕ.

Remark 9 (see [17]). Taking $\lambda=0$, $0\leq\delta<1$ in Theorem 6, we get the radius $r^{c}_{\mathscr{B}}(\delta)$ of convexity of order δ given by the absolute value of the root of the equation $w\mathscr{B}''(w)+(1-\delta)\mathscr{B}'(w)=0$, having the smallest modulus and argument ϕ.

Theorem 10. *Let $\{c_n\}_{n\in\mathbb{N}/\{0\}}$ be a sequence with $c_n=|c_n|e^{i\phi}\in\mathbb{C}$, $|c_n|\geq 1$ for $n\in\mathbb{N}/\{0\}$, $r_0=\inf\{|c_n|: n\in\mathbb{N}/\{0\}\}$, and let $h(w)$ be an analytic function in $\mathbb{U}(r_0)$ with $|w|e^{i\phi}h'(|w|e^{i\phi})\in\mathbb{R}$, $|w|e^{i\phi}h'(|w|e^{i\phi})\leq\Re\{wh'(w)\}$, and $|w^2h''(w)|\leq-|w|^2e^{i2\phi}h''(|w|e^{i\phi})$ for $w\in\mathbb{U}(r_0)$. If the function $\mu:(0,r_0)\longrightarrow\mathbb{R}$ defined by $\mu(r)=re^{i\phi}h'(re^{i\phi})$ is decreasing, the function $\vartheta(r)$: $(0,r_0)\longrightarrow\mathbb{R}$ defined by $\vartheta(r)=-r^2e^{2i\phi}h''(re^{i\phi})$ is increasing with respect to r, and $\mathscr{B}(w)$ is of the form (9) with $q_n\in\mathbb{N}/\{0\}$ for $n\in\mathbb{N}/\{0\}$ and $\alpha\in[0,1)$; then, the radius of α -convexity of order δ of the function $\mathscr{B}(w)$ is the smallest positive root of the equation $(1-\alpha)(w\mathscr{B}'(w)/\mathscr{B}(w))+\alpha(1+w\mathscr{B}''(w)/\mathscr{B}'(w))=\delta$ having the smallest modulus and argument ϕ.*

Proof. Consider

$$\begin{aligned}&\Re\{\mathscr{M}(\alpha,\mathscr{B}(w))\}\\&=\Re\left\{(1-\alpha)\frac{w\mathscr{B}'(w)}{\mathscr{B}(w)}+\alpha\left(1+\frac{w\mathscr{B}''(w)}{\mathscr{B}'(w)}\right)\right\}\\&=1+wh'(w)-\sum_{n=1}^{\infty}\frac{(w/c_n)^{q_n+1}}{1-(w/c_n)}\\&\quad+\alpha\left(1-\frac{1-w^2h''(w)+\sum_{n=1}^{\infty}\left(\left((q_n)(w/c_n)^{q_n+1}-(q_n-1)(w/c_n)^{q_n+2}\right)/\left((1-(w/c_n))^2\right)\right)}{1+wh'(w)-\sum_{n=1}^{\infty}\left(\left((w/c_n)^{q_n+1}\right)/(1-(w/c_n))\right)}\right)\\&\geq 1+|w|e^{i\phi}h'\left(|w|e^{i\phi}\right)-\sum_{n=1}^{\infty}\left(\frac{|w/c_n|^{q_n+1}}{1-|w/c_n|}\right)\\&\quad+\alpha\left(1-\frac{1-|w|^2e^{i2\phi}h''\left(|w|e^{i\phi}\right)+\sum_{n=1}^{\infty}\left(\left((q_n)|w/c_n|^{q_n+1}-(q_n-1)|w/c_n|^{q_n+2}\right)/\left((1-|w/c_n|)^2\right)\right)}{1+|w|e^{i\phi}h'(|w|e^{i\phi})-\sum_{n=1}^{\infty}\left(\left(|w/c_n|^{q_n+1}\right)/(1-|w/c_n|)\right)}\right)\\&\geq(1-\alpha)\left(\frac{|w|e^{i\phi}\mathscr{B}'\left(|w|e^{i\phi}\right)}{\mathscr{B}(|w|e^{i\phi})}\right)+\alpha\left(1+\frac{|w|e^{i\phi}\mathscr{B}''\left(|w|e^{i\phi}\right)}{\mathscr{B}'\left(|w|e^{i\phi}\right)}\right)\\&=\mathscr{M}\left(\alpha,e^{i\phi}|\mathscr{B}(w)|\right),\end{aligned}\tag{26}$$

for every $|w| < r_0$, and the equality holds for $w = |w|e^{i\phi}$. By the virtue of minimum principle for harmonic functions,

$$\inf_{|w|<r} \Re\{\mathscr{M}(\alpha, \mathscr{B}(w))\} = \mathscr{M}\left(\alpha, re^{i\phi}\right), r \in (0, r_0), \tag{27}$$

Also, $\mathscr{M}(\alpha, re^{i\phi})$ is strictly decreasing; also, $\lim_{r\longrightarrow 0} \mathscr{M}(\alpha, re^{i\phi}) = 1 > 0$ and $\lim_{r\longrightarrow r_0} \mathscr{M}(\alpha, re^{i\phi}) = -\infty$. Hence, the equation $(1-\alpha)(re^{i\phi}\mathscr{B}'(re^{i\phi})/\mathscr{B}(re^{i\phi})) + \alpha(1 + re^{i\phi}\mathscr{B}''(re^{i\phi})/\mathscr{B}'(re^{i\phi})) = \delta$ has a unique root in $(0, r_0)$, and this root is $r^{\alpha}_{c(\mathscr{B})}(\delta)$.

Remark 11 (see [17]). Taking $\alpha = 0$ in Theorem 10, we get the radius $r^*_{\mathscr{B}}(\delta)$ of starlikeness of order δ, given by the absolute value of the root of the equation $w\mathscr{B}'(w) - \delta\mathscr{B}(w) = 0$, having the smallest modulus and argument ϕ.

Remark 12 (see [17]). Taking $\alpha = 1$, in Theorem 6, we get the radius $r^c_{\mathscr{B}}(\delta)$ of convexity of order δ, given by the absolute value of the root of the equation $w\mathscr{B}''(w) + (1-\delta)\mathscr{B}'(w) = 0$, having the smallest modulus and argument ϕ.

In the following remark, we discuss the radius of λ-starlikeness, λ-uniform convexity, and α-convexity of order δ for the function $1/\Gamma$.

Remark 13. Let $h(w) = \gamma w$ where γ is the Euler-Mascheroni constant [5], and let $q_n = 1, c_n = -n, n \in \mathbb{N}$, and $\phi = 0$. Then,

$$\mathscr{B}(w) = \frac{1}{\Gamma(w)} = we^{\gamma w}\prod_{n=1}^{\infty}\left(1 + \frac{w}{n}\right)e^{-w/n}. \tag{28}$$

We now have $wh'(w) = \gamma w$, $w^2h''(w) = \gamma w^2$, and it is easy to verify $\Re(wh'(w)) \geq \gamma|w|, w \in \mathbb{U}$ with equality iff $w \in \mathbb{R}$ and $|w^2h''(w)| = \gamma|w|^2$, $w \in \mathbb{U}$. The conditions of Theorem 2, Theorem 6, and Theorem 10 are satisfied.

By Theorem 2, the radius $r^{\lambda}_{st(1/\Gamma)}(\delta)$ of λ-starlikeness of order δ of the function $1/(\Gamma(w))$ is the modulus of the biggest negative root of the equation $((w\Gamma'(w))/\Gamma(w)) + ((\lambda+\delta)/(1+\lambda)) = 0$. Numerical approach gives $r^0_{st(1/\Gamma)}(0) = 0.504083$, $r^{1/2}_{st(1/\Gamma)}(0) = 0.416321$, $r^0_{st(1/\Gamma)}(1/2) = 0.358071$, $r^1_{st(1/\Gamma)}(1/4) = 0.30431$, and $r^2_{st(1/\Gamma)}(1/2) = 0.180823$.

By Theorem 6, the radius $r^{\lambda}_{uc(1/\Gamma)}(\delta)$ of λ-uniform convexity of order δ of the function $1/\Gamma(w)$ is the modulus of the biggest negative root of the equation

$$\frac{w\Gamma''(w)}{\Gamma'(w)} - \frac{2w\Gamma'(w)}{\Gamma(w)} + \frac{1-2\lambda-\delta}{1+\lambda} = 0. \tag{29}$$

Numerical approach gives $r^0_{uc(1/\Gamma)}(0) = 0.266701$, $r^0_{uc(1/\Gamma)}(1/2) = 0.190771$, $r^{1/3}_{uc(1/\Gamma)}(0) = 0.125966$, $r^{1/4}_{uc(1/\Gamma)}(1/4) = 0.108467$, and $r^{1/5}_{uc(1/\Gamma)}(1/3) = 0.116513$.

By Theorem 10, the radius $r^{\alpha}_{c(1/\Gamma)}(\delta)$ of α-convexity of order δ for the function $1/\Gamma(w)$ is the modulus of the biggest negative root of the equation

$$\alpha\left(1 + \frac{w\Gamma''(w)}{\Gamma'(w)}\right) + (1+\alpha)\frac{w\Gamma'(w)}{\Gamma(w)} = \delta. \tag{30}$$

Numerical approach gives $r^0_{c(1/\Gamma)}(0) = 0.504083$, $r^1_{c(1/\Gamma)}(0) = 0.266701$, $r^{1/3}_{c(1/\Gamma)}(1/2) = 0.258289$, $r^{1/2}_{c(1/\Gamma)}(1/3) = 0.269676$, and $r^{1/2}_{c(1/\Gamma)}(1/2) = 0.234978$.

In the following remark, we give an example which shows that the Theorems 2, 6, and 10 work even if $we^{h(w)}$ is not starlike. That is, the example given in the following remark shows that the hypotheses of Theorems 2, 6, and 10 are free from the hypothesis of Theorem 3 in [13], proved by Merkes et al.

Remark 14. Let $h(w) = w^2/(w^2-1)$ with $\phi = 0$, and let $q_n = 1, c_n = n$. Clearly, $we^{h(w)}$ is not starlike. Then, we have $wh'(w) = (-2w^2/((w^2-1)^2))$ and $w^2h''(w) = ((2w^2+6w^4)/((w^2-1)^3))$. Also, $\Re(wh'(w)) \geq (-2|w|^2/((|w|^2-1)^2))$ and $|w^2h''(w)| \geq ((2|w|^2+6|w|^4)/((1-|w|^2)^3))$, $w \in \mathbb{U}$ with equality iff $w \in \mathbb{R}$.

By Theorem 2, the radius $r^{\lambda}_{st(\mathscr{B})}(\delta)$ of λ-starlikeness of order δ of the function

$$\mathscr{B}_1(w) = we^{w^2/(w^2-1)}\prod_{n=1}^{\infty}\left(1 - \frac{w}{n}\right)e^{w/n} \tag{31}$$

is the smallest positive root of the equation $1 - (2r^2/((1-r^2)^2)) - \sum_{n=1}^{\infty}(r^2/(n(n-r))) - ((\lambda+\delta)/(1+\lambda)) = 0$. Numerical approach gives $r^0_{st(\mathscr{B}_1)}(0) = r^*_{\mathscr{B}_1} = 0.426948$, $r^0_{st(\mathscr{B}_1)}(1/2) = r^*_{\mathscr{B}_1}(1/2) = 0.325887$, $r^1_{st(\mathscr{B}_1)}(1/2) = 0.241843$, $r^2_{st(\mathscr{B}_1)}(1/2) = 0.201282$, and $r^{1/2}_{st(\mathscr{B}_1)}(1/2) = 0.274465$.

By Theorem 6, the radius $r^{\lambda}_{uc(\mathscr{B}_1)}(\delta)$ of λ-uniform convexity of order δ is the smallest positive root of the equation

$$\begin{aligned} 1 &- \frac{2r^2}{(1-r^2)^2} - \sum_{n=1}^{\infty}\frac{r^2}{n(n-r)} \\ &- \frac{1 + \left((2r^2+6r^4)/\left((1-r^2)^3\right)\right) + \sum_{n=1}^{\infty}\left(r^2/((n-r)^2)\right)}{1 - \left(2r^2/\left((1-r^2)^2\right)\right) - \sum_{n=1}^{\infty}(r^2/(n(n-r)))} \\ &+ \frac{(1-\delta-2\lambda)}{1+\lambda} = 0. \end{aligned} \tag{32}$$

Numerical approach gives $r^0_{uc(\mathscr{B}_1)}(0) = r^c_{\mathscr{B}_1} = 0.242015$, $r^0_{uc(\mathscr{B}_1)}(1/2) = r^c_{\mathscr{B}_1}(1/2) = 0.187093$, $r^{1/6}_{uc(\mathscr{B}_1)}(1/2) = 0.108455$, and $r^{1/4}_{uc(\mathscr{B}_1)}(1/4) = 0.126439$.

By Theorem 10, the radius $r^{\alpha}_{c(\mathscr{B}_1)}(\delta)$ of α-convexity of order δ is the smallest positive root of the equation

$$1-\frac{2r^2}{(1-r^2)^2}-\sum_{n=1}^{\infty}\frac{r^2}{n(n-r)}+\alpha\left(1-\frac{1+\left((2r^2+6r^4)/\left((1-r^2)^3\right)\right)+\sum_{n=1}^{\infty}\left(r^2/\left((n-r)^2\right)\right)}{1-\left(2r^2/\left((1-r^2)^2\right)\right)-\sum_{n=1}^{\infty}\left(r^2/(n(n-r))\right)}\right)=\delta. \tag{33}$$

Numerical approach gives $r^{0}_{c(\mathscr{B}_1)}(0)=r^{*}_{\mathscr{B}_1}=0.426948$, $r^{1}_{c(\mathscr{B}_1)}(0)=r^{c}_{\mathscr{B}_1}=0.242015$, $r^{0}_{c(\mathscr{B}_1)}(1/2)=r^{*}_{\mathscr{B}_1}(1/2)=0.325887$, $r^{1}_{c(\mathscr{B})}(1/2)=r^{c}_{\mathscr{B}_1}(1/2)=0.187093$, $r^{1/2}_{c(\mathscr{B}_1)}(1/2)=0.222952$, $r^{1/4}_{c(\mathscr{B}_1)}(1/4)=0.139653$, and $r^{1/4}_{c(\mathscr{B}_1)}(1/2)=0.254242$.

Data Availability

No data were used to support this study.

Conflicts of Interest

There is no conflict of interest regarding the publication of this article.

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