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Research Article

Fractional Minkowski-Type Integral Inequalities via the Unified Generalized Fractional Integral Operator

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This paper is aimed at presenting the unified integral operator in its generalized form utilizing the unified Mittag-Leffler function in its kernel. We prove the boundedness of this newly defined operator. A fractional integral operator comprising a unified Mittag-Leffler function is used to establish further Minkowski-type integral inequalities. Several related fractional integral inequalities that have recently been published in various articles can be inferred.

1. Introduction

Integral operators are useful in the study of differential equations and in the formation of real-world problems in integral equations. They also behave like integral transformations in particular cases. In the past few decades, fractional integral operators have been defined extensively (see [1–4]). Recently, in [5] a unified integral operator is studied which has interesting consequences in the theory of fractional integral operators.

This paper is aimed at presenting a unified integral operator in the more generalized form via the unified Mittag-Leffler function introduced in [6]. The boundedness of the newly defined integral operator is studied. By taking the power function $\xi^{\beta}; \beta > 1$, a unified generalized extended fractional integral operator is deduced and analyzed to construct Minkowski-type integral inequalities. This is the extension of our previous work on Minkowski-type integral inequalities [7]. The connection of the results of this paper is established with many published results of references [7–9]. We begin by reviewing several key Minkowski-type inequalities as well as some definitions that will be useful in our subsequent work.

The well-known Minkowski inequality is given as follows:

Theorem 1. Let ϕ**,** $\psi \in L_m[u, v]$**.** *Then for* $m \geq 1$*, we have*

$$\left(\int_u^v (\phi(\xi) + \psi(\xi))^m d\xi\right)^{1/m} \leq \left(\int_u^v \phi^m(\xi) d\xi\right)^{1/m} + \left(\int_u^v \psi^m(\xi) d\xi\right)^{1/m}. \quad (1)$$

Some more Minkowski-type inequalities are stated in the next results.

Theorem 2. ([10]). *Let* ϕ, $\psi \in L_m[u, v]$. *Also* ϕ, $\psi \in \Re^+$ *such that* $0 < k_1 \leq (\phi(\xi))/(\psi(\xi)) \leq k_2 \forall \xi \in [u, v]$. *Then for* $m \geq 1$, *the following inequality holds true*

$$\left(\int_u^v \phi^m(\xi) d\xi\right)^{1/m} + \left(\int_u^v \psi^m(\xi) d\xi\right)^{1/m} \leq \left(1 + \frac{k_2 - k_1}{(k_1 + 1)(k_2 + 1)}\right)\left(\int_u^v (\phi(\xi) + \psi(\xi))^m d\xi\right)^{1/m}. \quad (2)$$

Theorem 3. ([11]). *Under the assumptions of Theorem 2, we* *have the following inequality:*

$$\left(\int_u^v \phi^m(\xi)d\xi\right)^{2/m} + \left(\int_u^v \psi^m(\xi)d\xi\right)^{2/m} \geq \left(\frac{2+(k_1-1)(k_2+1)}{k_2}\right)\left(\int_u^v \phi^m(\xi)d\xi\right)^{2/m}\left(\int_u^v \psi^m(\xi)d\xi\right)^{2/m}. \quad (3)$$

Theorem 4. ([9]). *Let $\omega \in \mathbb{R}, \alpha, \beta, \gamma > 0, \theta > \lambda > 0$ with $s \geq 0$, $r > 0$ and $0 < k \leq r + \alpha$. Let $m \geq 1$ and $\phi, \psi \in L_m[u, v]$ be positive functions satisfying*

$$0 < k_1 \leq \frac{\phi(\xi)}{\psi(\xi)} \leq k_2, \xi \in [u, v]. \quad (4)$$

Then the following inequality holds:

$$[(\varepsilon\phi^m)(\xi;s)]^{1/m} + [(\varepsilon\psi^m)(\xi;s)]^{1/m} \leq \left(1 + \frac{k_2 - k_1}{(k_1+1)(k_2+1)}\right)[(\varepsilon(\phi+\psi)^m)(\xi;s)]^{1/m}. \quad (5)$$

Theorem 5. [9]. *Let $m, n > 1$ such that $1/m + 1/n = 1$. Then under the assumptions of Theorem 4, we have*

$$[(\varepsilon\phi)(\xi;s)]^{1/m}[(\varepsilon\psi)(\xi;s)]^{1/n} \leq \left(\frac{k_2}{k_1}\right)^{1/mn}\left[\left(\varepsilon\phi^{1/m}\psi^{1/n}\right)(\xi;s)\right]. \quad (6)$$

A special function known as the Mittag-Leffler function was introduced by a Swedish mathematician Gosta Mittag-Leffler [12] by the following series:

$$E_\alpha(t) = \sum_{l=0}^{\infty} \frac{t^l}{\Gamma(\alpha l + 1)}, \quad (7)$$

where $t, \alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$.

This function is a direct extension of the exponential function that can be used to construct solutions of fractional differential equations. Due to its wide range of applications, this function has received considerable attention in recent decades. Many researchers provided its numerous generalized forms due to its intriguing results. We refer the readers to [2, 4, 13–16] for the study of generalized versions of the Mittag-Leffler function. In [17], Bhatnagar et al. introduced the generalization of Mittag-Leffler function in the form of generalized Q function as follows:

Definition 6. The generalized Q function denoted by ($Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}$)(.;.,.) is defined by the following series:

$$Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t;\underline{a},\underline{b}) = \sum_{l=0}^{\infty} \frac{\Pi_{i=1}^n B(b_i,l)(\lambda)_{\rho l}(\theta)_{kl}t^l}{\Pi_{i=1}^n B(a_i,l)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)}, \quad (8)$$

where $\underline{a} = (a_1, a_2, \cdots, a_n)$, $\underline{b} = (b_1, b_2, \cdots, b_n)$, $\alpha, \beta, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta, a_i, b_i \in \mathbb{C}$, $k \in (0,1) \cup \mathbb{N}$ and $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\theta), \Re(\lambda), \Re(\delta), \Re(\rho)\} > 0$.

Recently, in [7], we introduced the fractional integral operator associated with generalized Q function as follows:

$${}_Q I^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu} f(\xi;\underline{a},\underline{b}) = \int_u^\xi (\xi-t)^{\beta-1} Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(\xi-t)^\alpha;\underline{a},\underline{b}\right)f(t)dt, \quad (9)$$

$${}_Q I^{\omega,\lambda,\rho,\theta,k,n}_{v^-,\alpha,\beta,\gamma,\delta,\mu,\nu} f(\xi;\underline{a},\underline{b}) = \int_\xi^v (t-\xi)^{\beta-1} Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(\omega(t-\xi)^\alpha;\underline{a},\underline{b}\right)f(t)dt. \quad (10)$$

Andrić et al. in [2] introduced an extended and generalized Mittag-Leffler function along with the corresponding fractional integral operator as follows:

Definition 7. The extended and generalized Mittag-Leffler function ($E^{\delta,\mu,k,\nu}_{\alpha,\beta,\gamma}$) is defined by the following series:

$$E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma}(t;s) = \sum_{l=0}^{\infty} \frac{B_s(\lambda+lk,\theta-\lambda)(\theta)_{lk}t^l}{B(\lambda,\theta-\lambda)(\gamma)_{lr}\Gamma(\alpha l+\beta)}, \quad (11)$$

where $t, \alpha, \beta, \gamma, \theta, \lambda \in \mathbb{C}$, $\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\theta), \Re(\lambda) > 0$, $\Re(\theta) > \Re(\lambda)$ with $s \geq 0$, $r > 0$, $0 < k \leq r + \Re(\alpha)$, and $(\theta)_{lk} = (\Gamma(\theta+lk))/(\Gamma(\theta))$.

Definition 8. Let $f \in L_1[u, v]$. Then for $\xi \in [u, v]$, the fractional integral operator corresponding to (11) is defined by the following integrals:

$$\varepsilon^{\omega,\lambda,\theta,k,r}_{u^+,\alpha,\beta,\gamma} f(\xi;s) = \int_u^\xi (\xi-t)^{\beta-1} E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma,}\left(\omega(\xi-t)^\alpha;s\right)f(t)dt,$$

$$\varepsilon^{\omega,\lambda,\theta,k,r}_{v^-,\alpha,\beta,\gamma} f(\xi;s) = \int_\xi^v (t-\xi)^{\beta-1} E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma,}\left(\omega(\xi-t)^\alpha;s\right)f(t)dt. \quad (12)$$

In [5], Farid defined the unified integral operator based on the extended and generalized Mittag-Leffler function (11) as follows:

Definition 9. Let $\omega, \beta, \gamma, \lambda, \theta \in \mathbb{C}$, $\Re(\beta), \Re(\gamma) > 0$, $\Re(\theta) > \Re(\lambda) > 0$ with $s \geq 0$, $\alpha, r > 0$ and $0 < k \leq r + \alpha$. Let $\phi \in L_1[u, v]$, $0 < u < v < \infty$, be a positive function. Let $g : [u, v] \longrightarrow \mathbb{R}$ be a differentiable function, strictly increasing. Also let $\zeta(\xi)/\xi$ be an increasing function on $[u, \infty)$ and $\xi \in [u, v]$. Then the left-sided integral is defined by

$$\left({}^\zeta_g F^{\omega,\lambda,\theta,k,r}_{u^+,\alpha,\beta,\gamma}\phi\right)(\xi;s) = \int_u^\xi \frac{\zeta(g(\xi)-g(t))}{g(\xi)-g(t)} E^{\lambda,\theta,k,r}_{\alpha,\beta,\gamma}\left(\omega(g(\xi)-g(t))^\alpha;s\right)g'(t)\phi(t)dt. \quad (13)$$

In [6], we have presented a further generalized unified Mittag-Leffler function and the associated fractional integral operator as follows:

Definition 10. For $\underline{a} = (a_1, a_2, \cdots, a_n)$, $\underline{b} = (b_1, b_2, \cdots, b_n)$, $\underline{c} = (c_1, c_2, \cdots, c_n)$, where $a_i, b_i, c_i \in \mathbb{C}; i = 1, \cdots, n$ such that $\mathfrak{R}(a_i), \mathfrak{R}(b_i), \mathfrak{R}(c_i) > 0$. Also let $, \beta, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta, t \in \mathbb{C}$, $\min\{\mathfrak{R}(\alpha), \mathfrak{R}(\beta), \mathfrak{R}(\gamma), \mathfrak{R}(\delta), \mathfrak{R}(\lambda), \mathfrak{R}(\theta)\} > 0$ and $k \in (0, 1) \cup \mathbb{N}$ with $s \geq 0$. *Let* $k + \mathfrak{R}(\rho) < \mathfrak{R}(\delta + \nu + \alpha)$ with $\operatorname{Im}(\rho) = \operatorname{Im}(\delta + \nu + \alpha)$, then the unified Mittag-Leffler function is defined as follows

$$M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(t; \underline{a}, \underline{b}, \underline{c}, s) = \sum_{l=0}^{\infty} \frac{\Pi_{i=1}^{n} B_s(b_i, a_i)(\lambda)_{\rho l}(\theta)_{kl}}{\Pi_{i=1}^{n} B(c_i, a_i)(\gamma)_{\delta l}(\mu)_{\nu l}} \frac{t^l}{\Gamma(\alpha l + \beta)}. \tag{14}$$

Definition 11. Let $\phi \in L_1[u, v]$. Then for $\xi \in [u, v]$, the fractional integral operators corresponding to (14) are defined by

$$I^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\phi(\xi; \underline{a}, \underline{b}, \underline{c}, s) = \int_u^{\xi} (\xi - t)^{\beta-1} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\xi - t)^{\alpha}; \underline{a}, \underline{b}, \underline{c}, s)\phi(t)dt, \tag{15}$$

$$I^{\omega,\lambda,\rho,\theta,k,n}_{v^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\phi(\xi; \underline{a}, \underline{b}, \underline{c}, s) = \int_{\xi}^{v} (t - \xi)^{\beta-1} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(t - \xi)^{\alpha}; \underline{a}, \underline{b}, \underline{c}, s)\phi(t)dt. \tag{16}$$

Fractional integral operators are used to extend different types of integral inequalities such as Opial-type inequalities [2, 18–22], Hadamard- and Fejér-Hadamard-type inequalities [23–32], Pólya-Szegö-, Chebyshev-, and Grüss-type inequalities [33–36] (see references therein), and Minkowski-type fractional inequalities [7–9]. In this paper we study Minkowski-type fractional inequalities via the unified Mittag-Leffler function.

In Section 2, we give the definition of further generalized integral operator containing the unified Mittag-Leffler function. The boundedness of this integral operator is proved under the conditions stated in the definition. In Section 3, by applying a particular fractional integral operator for the power function, Minkowski-type fractional integral inequalities are established. In Section 4, reverse Minkowski-type fractional integral inequalities are presented. The connection of these inequalities with previous work is stated in the form of remarks and corollaries.

2. Generalized Version of a Unified Integral Operator

In this section, we introduce a generalized version of a unified integral operator containing a unified Mittag-Leffler function in its kernel and also discuss its boundedness.

Definition 12. Let $\omega, \underline{a} = (a_1, a_2, \cdots, a_n)$, $\underline{b} = (b_1, b_2, \cdots, b_n)$, $\underline{c} = (c_1, c_2, \cdots, c_n)$, where $a_i, b_i, c_i \in \mathbb{C}; i = 1, \cdots, n$ such that $\mathfrak{R}(a_i), \mathfrak{R}(b_i), \mathfrak{R}(c_i) > 0 \forall i$. Also let $, \gamma, \mu, \lambda, \theta, t \in \mathbb{C}$, $\min\{\mathfrak{R}(\beta), \mathfrak{R}(\gamma), \mathfrak{R}(\mu), \mathfrak{R}(\lambda), \mathfrak{R}(\theta)\} > 0$, $\rho, \delta, \nu, \alpha > 0$ and $k \in (0, 1) \cup \mathbb{N}$. Let $k + \rho < \delta + \nu + \alpha$ with $s \geq 0$. Let $\phi \in L_1[u, v]$, $0 < u < v < \infty$ be a positive function, and let $g : [u, v] \longrightarrow \mathbb{R}$ be a differentiable and strictly increasing function. Also let $\zeta(\xi)/\xi$ be an increasing function on $[u, \infty]$ for $\xi \in [u, v]$. Then the unified integral operator in its generalized form is defined by the following integral:

$$\left({}_g^{\zeta}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\phi\right)(\xi; s) = \int_u^{\xi} \frac{\zeta(g(\xi) - g(t))}{g(\xi) - g(t)} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(g(\xi) - g(t))^{\alpha}; s)g'(t)\phi(t)dt. \tag{17}$$

On a particular case, by taking $\zeta(\xi) = \xi^{\beta}; \beta > 1$ and replacing $\mathbb{C}$ by $\mathbb{R}$, the above operator takes the following form:

$$\left({}_g\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\phi\right)(\xi; s) = \int_u^{\xi} (g(\xi) - g(t))^{\beta-1} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(g(\xi) - g(t))^{\alpha}; s)g'(t)\phi(t)dt, \tag{18}$$

where $\omega, \underline{a} = (a_1, a_2, \cdots, a_n)$, $\underline{b} = (b_1, b_2, \cdots, b_n)$, $\underline{c} = (c_1, c_2, \cdots, c_n)$, $a_i, b_i, c_i \in \mathbb{R}; i = 1, \cdots, n$ such that $a_i, b_i, c_i > 0 \forall i$. Also $\alpha, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta > 0$, $\beta > 1$ and $k \in (0, 1) \cup \mathbb{N}$ with $k + \rho < \delta + \nu + \alpha$ and $s \geq 0$.

Definition 13. By setting $a_i = l$, $s = 0$ and $\rho > 0$ in (18), we will get the following integral operator:

$$\left({}_Q^g\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\phi\right)(\xi; s) = \int_u^{\xi} (g(\xi) - g(t))^{\beta-1} Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(g(\xi) - g(t))^{\alpha}; s)g'(t)\phi(t)dt. \tag{19}$$

Remark 14.

(i) By considering $n = 1$, $b_1 = \lambda + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, $\rho = \nu = 0$, $\delta > 0$ in (17), the unified integral operator given in (13) is deduced

(ii) By considering the function g to be an identity function in (19), the fractional integral operator given in (9) is deduced

(iii) By considering $n = 1$, $b_1 = \lambda + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, $\rho = \nu = 0$, and $\delta > 0$ in (18), the generalized fractional integral operator ([8], Definition 1.4) is deduced

(iv) By considering $s = 0 = \omega$ in (18), then the left-sided Riemann-Liouville fractional integral operator of a function ϕ with respect to another function g of order β given in [1, 3] is deduced

(v) By considering g to be an identity function, (18) is deduced to (15)

(vi) By considering g as identity function and setting $n = 1$, $b_1 = \lambda + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, $\rho = \nu = 0$, and $\delta > 0$, in (18), the generalized fractional integral operator (21) is deduced

For simplicity, we will use the following notations throughout this paper: $M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu} := \mathbf{M}$, $I^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu} = \mathbf{I}$, ${}_Q I^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu} = {}_Q\mathbf{I}$, ${}_g^{\zeta}\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu} := {}^{\zeta}\mathbf{\Omega}$, ${}_g\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu} := \mathbf{\Omega}$, ${}_Q^g\Omega^{\omega,\lambda,\rho,\theta,k,n}_{u^+,\alpha,\beta,\gamma,\delta,\mu,\nu} := {}_Q\mathbf{\Omega}$.

Next, we discuss the boundedness of the newly defined generalized form of unified fractional integral operator.

Theorem 15. *Let $\omega \in \mathbb{R}$, $\underline{a} = (a_1, a_2, \cdots, a_n)$, $\underline{b} = (b_1, b_2, \cdots, b_n)$, $\underline{c} = (c_1, c_2, \cdots, c_n)$, $a_i, b_i, c_i \in \mathbb{R}$; $i = 1, \cdots, n$ such that $a_i, b_i, c_i > 0$. Also $\alpha, \beta, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta, t \in \mathbb{R}$, $k \in (0, 1) \cup \mathbb{N}$ and $\min\{\alpha, \beta, \gamma, \delta, \lambda, \theta\} > 0$ with $k + \rho < \delta + \nu + \alpha$ with $s \geq 0$. Let $\phi \in L_1[u, v]$, $0 < u < v < \infty$ be a positive function, and let $g : [u, v] \longrightarrow \mathbb{R}$ be a differentiable and strictly increasing function. Also let ζ/ξ be an increasing function on $[u, \infty]$. Then for $\xi \in [u, v]$, we get*

$$\left({}^{\zeta}\mathbf{\Omega}\phi\right)(\xi ; s) \leq \zeta(g(\xi) - g(u))\mathbf{M}\left(\omega(g(\xi) - g(u))^{\alpha} ; s\right)\phi_{[u,\xi]}, \tag{20}$$

$$\left({}^{\zeta}\mathbf{\Omega}\phi\right)(\xi ; s) \leq \zeta(g(v) - g(\xi))\mathbf{M}\left(\omega(g(\xi) - g(u))^{\alpha} ; s\right)\phi_{[\xi,v]}, \tag{21}$$

where $\|\phi\|_{[u,\xi]} = sup_{t\in[u,\xi]}|\phi(t)|$ and $\|\phi\|_{[\xi,v]} = sup_{t\in[\xi,v]}|\phi(t)|$.

Proof. According to the statement, ζ/ξ is an increasing function; therefore, the following inequality prevails:

$$\frac{\zeta(g(\xi) - g(t))}{g(\xi) - g(t)} \leq \frac{\zeta(g(\xi) - g(u))}{g(\xi) - g(u)}. \tag{22}$$

Since g is differentiable and increasing and ϕ is a positive function, so the above inequality remains preserved by multiplying it with $g'(t)\phi(t)$. Therefore, we obtain the following inequality:

$$\frac{\zeta(g(\xi) - g(t))}{g(\xi) - g(t)} g'(t)\phi(t) \leq \frac{\zeta(g(\xi) - g(u))}{g(\xi) - g(u)} g'(t)\phi(t). \tag{23}$$

Multiplying (23) by $\mathbf{M}(\omega(g(\xi) - g(t))^{\alpha} ; s)$ and integrating over $[u, \xi]$ one can get

$$\left({}^{\zeta}\mathbf{\Omega}\phi\right)(\xi ; s) \leq \frac{\zeta(g(\xi) - g(u))}{g(\xi) - g(u)} \|\phi\|_{[u,\xi]} \int_u^{\xi} \mathbf{M}\left(\omega(g(\xi) - g(t))^{\alpha} ; s\right) g'(t)dt. \tag{24}$$

Solving the above definite integral, we get

$$\left({}^{\zeta}\mathbf{\Omega}\phi\right)(\xi ; s) \leq \zeta(g(\xi) - g(u))\mathbf{M}\left(\omega(g(\xi) - g(u))^{\alpha} ; s\right)\|\phi\|_{[u,\xi]}. \tag{25}$$

Similarly, one can easily prove (21). □

3. Unified Versions of Minkowski-Type Fractional Integral Inequalities

In this section, we give proof of unified versions of generalized Minkowski-type integral inequalities.

Theorem 16. *Let $\omega \in \mathbb{R}$, $\underline{a} = (a_1, a_2, \cdots, a_n)$, $\underline{b} = (b_1, b_2, \cdots, b_n)$, $\underline{c} = (c_1, c_2, \cdots, c_n)$, $a_i, b_i, c_i \in \mathbb{R}$; $i = 1, \cdots, n$ such that $a_i, b_i, c_i > 0$. Also $\alpha, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta > 0$, $\beta > 1$ and $k \in (0, 1) \cup \mathbb{N}$ with $k + \rho < \delta + \nu + \alpha$ with $s \geq 0$. Let $g : [u, v] \longrightarrow \mathbb{R}$ be a differentiable and strictly increasing function, and let $\phi, \psi, \zeta_1, \zeta_2$ be $m-$power integrable and positive functions on $[u, v]$ such that the ratio $\phi(\xi)/\psi(\xi)$ is bounded above by ζ_2 and bounded below by $\zeta_1 \forall \xi \in [u, v]$. Let $m, n > 1$ such that $1/m + 1/n = 1$; then*

$$[(\mathbf{\Omega}\phi)(\xi ; s)]^{1/m}[(\mathbf{\Omega}\psi)(\xi ; s)]^{1/n} \leq \left[\left(\mathbf{\Omega}\zeta_2^{1/mn}\phi^{1/m}\psi^{1/n}\right)(\xi ; s)\right]^{1/m}\left[\left(\mathbf{\Omega}\zeta_1^{-(1/mn)}\phi^{1/m}\psi^{1/n}\right)(\xi ; s)\right]^{1/n}. \tag{26}$$

Proof. According to the statement of the theorem, we have

$$0 < \zeta_1(t) \leq \frac{\phi(t)}{\psi(t)} \leq \zeta_2(t), t \in [u, v]. \tag{27}$$

By considering the lower bound, the above inequality produces

$$\psi(t) \leq \frac{1}{\zeta_1^{1/m}} \phi^{1/m}(t)\psi^{1/n}(t). \tag{28}$$

By multiplying both sides of the above inequality with $(g(\xi) - g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi) - g(t))^{\alpha} ; s)g'(t)$ and integrating on $[u, \xi]$, we get

$$[(\mathbf{\Omega}\psi)(\xi ; s)]^{1/n} \leq \left[\left(\mathbf{\Omega}\zeta_1^{-(1/mn)}\phi^{1/m}\psi^{1/n}\right)(\xi ; s)\right]^{1/n}. \tag{29}$$

Also, by considering the upper bound of inequality (27), the following inequality holds:

$$\phi(t) \leq \zeta_2^{1/n}\phi^{1/m}(t)\psi^{1/n}(t). \tag{30}$$

Multiplying both sides of the above inequality with $(g(\xi) - g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi) - g(t))^{\alpha} ; s)g'(t)$ and integrating on $[u, \xi]$, we get the following inequality:

$$[(\mathbf{\Omega}\phi)(\xi ; s)]^{1/m} \leq \left[\left(\mathbf{\Omega}\zeta_2^{1/mn}\phi^{1/m}\psi^{1/n}\right)(\xi ; s)\right]^{1/m}. \tag{31}$$

The product of (29) and (31) results in inequality (26). □

Corollary 17. *Under the assumptions of Theorem 16 along with the condition that $\zeta_1(\xi) = k_1$ and $\zeta_2(\xi) = k_2$ with $0 < k_1 < k_2 \forall \xi \in [u, v]$, (26) takes the following form:*

$$[(\mathbf{\Omega}\phi)(\xi ; s)]^{1/m}[(\mathbf{\Omega}\psi)(\xi ; s)]^{1/n} \leq \left(\frac{k_2}{k_1}\right)^{1/mn}\left[\left(\mathbf{\Omega}\phi^{1/m}\psi^{1/n}\right)(\xi ; s)\right]. \tag{32}$$

Corollary 18. *Under the assumptions of above theorem and substituting $a_i = l$, $s = 0$ and $\rho > 0$ in (26), we get the following* *inequality:*

$$\begin{aligned}&\left[\left({}_{Q}\mathbf{\Omega}\phi\right)(\xi)\right]^{1/m}\left[\left({}_{Q}\mathbf{\Omega}\psi\right)(\xi)\right]^{1/n}\\&\quad\leq\left[\left({}_{Q}\mathbf{\Omega}\zeta_2^{1/mn}\phi^{1/m}\psi^{1/n}\right)(\xi)\right]^{1/m}\left[\left({}_{Q}\mathbf{\Omega}\zeta_1^{-(1/mn)}\phi^{1/m}\psi^{1/n}\right)(\xi)\right]^{1/n}.\end{aligned} \tag{33}$$

Remark 19.

(i) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (32), the inequality given in [8], Theorem 5, is deduced

(ii) By setting g, the identity function in (32) the inequality [7], Theorem 4, is deduced

$$[(\mathbf{I}\phi)(\xi;s)]^{1/m}[(\mathbf{I}\psi)(\xi;s)]^{1/n}\leq\left(\frac{k_2}{k_1}\right)^{1/mn}\left[\left(\mathbf{I}\phi^{1/m}\psi^{1/n}\right)(\xi;s)\right] \tag{34}$$

(iii) Under the assumptions of the Corollary 21 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2\forall\xi\in[u,v]$ and setting the function g to be an identity function, the inequality given in [7], Corollary 1, is deduced

(iv) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (34), the Minkowski-type inequality (6) is deduced

In the proof of our next result, we will use Young's inequality for $x,y\geq0$ with $m,n>1$ satisfying $m^{-1}+n^{-1}=1$:

$$xy\leq m^{-1}x^m+n^{-1}y^n. \tag{35}$$

Also, the following inequality will be required:

$$(x+y)^m\leq2^{m-1}(x^m+y^m);\,x,y\geq0\text{ and }m>1. \tag{36}$$

Theorem 20. *Under the assumptions of Theorem 16 the following inequality holds:*

$$\begin{aligned}(\mathbf{\Omega}(\phi\psi))(\xi;s)&\leq m^{-1}2^{m-1}\left(\mathbf{\Omega}\left(\frac{\zeta_2}{\zeta_2+1}\right)^m(\phi^m+\psi^m)\right)(\xi;s)\\&\quad+n^{-1}2^{n-1}\left(\mathbf{\Omega}\left(\frac{1}{\zeta_1+1}\right)^n(\phi^n+\psi^n)\right)(\xi;s).\end{aligned} \tag{37}$$

Proof. Taking the left side of inequality (27), we obtain the following form:

$$\psi^n(t)\leq(\zeta_1+1)^{-n}(\phi(t)+\psi(t))^n. \tag{38}$$

Multiplying both sides of inequality by $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha;s)g'(t)$ and integrating over $[u,\xi]$, the above inequality gives

$$n^{-1}(\mathbf{\Omega}\psi^n)(\xi;s)\leq n^{-1}\left(\mathbf{\Omega}(\zeta_1+1)^{-n}(\phi+\psi)^n\right)(\xi;s). \tag{39}$$

Also, by considering right side of inequality (27), we have the following inequality:

$$\phi^m(t)\leq\left(\frac{\zeta_2}{\zeta_2+1}\right)^m(\psi(t)+\phi(t))^m. \tag{40}$$

Multiplying both sides of the above inequality by $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha;s)g'(t)$, integrating over $[u,\xi]$ and multiplying resulting inequality by m^{-1}, we get

$$m^{-1}(\mathbf{\Omega}\phi^m)(\xi;s)\leq m^{-1}\left(\mathbf{\Omega}\left(\frac{\zeta_2}{\zeta_2+1}\right)^m(\psi+\phi)^m\right)(\xi;s). \tag{41}$$

By Young's inequality, we have

$$\phi(t)\psi(t)\leq m^{-1}\phi^m(t)+n^{-1}\psi^n(t). \tag{42}$$

Multiplying both sides of the above inequality by $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha;s)g'(t)$ and integrating over $[u,\xi]$, the above inequality takes the form

$$(\mathbf{\Omega}(\phi\psi))(\xi;s)\leq m^{-1}(\mathbf{\Omega}\phi^m)(\xi;s)+n^{-1}(\mathbf{\Omega}\psi^n)(\xi;s). \tag{43}$$

Applying (43) to the sum of (39) and (41), we get the following inequality:

$$\begin{aligned}(\mathbf{\Omega}(\phi\psi))(\xi;s)&\leq m^{-1}\left(\mathbf{\Omega}\left(\frac{\zeta_2}{\zeta_2+1}\right)^m(\phi+\psi)^m\right)(\xi;s)\\&\quad+n^{-1}\left(\mathbf{\Omega}\left(\frac{1}{\zeta_1+1}\right)^n(\phi+\psi)^n\right)(\xi;s).\end{aligned} \tag{44}$$

Inequality (37) follows by using (36) in (44). □

Corollary 21. *Under the assumptions of Theorem 20 together with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2\forall\xi\in[u,v]$, (37) becomes*

$$\begin{aligned}(\mathbf{\Omega}(\phi\psi))(\xi;s)&\leq m^{-1}2^{m-1}\left(\frac{k_2}{k_2+1}\right)^m(\mathbf{\Omega}(\phi^m+\psi^m))(\xi;s)\\&\quad+n^{-1}2^{n-1}\left(\frac{1}{k_1+1}\right)^n(\mathbf{\Omega}(\phi^n+\psi^n))(\xi;s).\end{aligned} \tag{45}$$

Corollary 22. *Under the assumptions of the above theorem and setting $a_i=l, s=0$ and $\rho>0$ in (37), the following* *inequality holds true:*

$$\left({}_{Q}\boldsymbol{\Omega}(\phi\psi)\right)(\xi) \le m^{-1}2^{m-1}\left({}_{Q}\boldsymbol{\Omega}\left(\frac{\zeta_2}{\zeta_2+1}\right)^{m}(\phi^m+\psi^m)\right)(\xi) + n^{-1}2^{n-1}\left({}_{Q}\boldsymbol{\Omega}\left(\frac{1}{\zeta_1+1}\right)^{n}(\phi^n+\psi^n)\right)(\xi). \tag{46}$$

Remark 23.

(i) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (45), the inequality given in [8], Theorem 6, is deduced

(ii) By setting g the identity function, (45) is deduced to the following inequality given in [7]:

$$(\mathbf{I}(\phi\psi))(\xi;s) \le m^{-1}2^{m-1}\left(\frac{k_2}{k_2+1}\right)^{m}(\mathbf{I}(\phi^m+\psi^m))(\xi;s) + n^{-1}2^{n-1}\left(\frac{1}{k_1+1}\right)^{n}(\mathbf{I}(\phi^n+\psi^n))(\xi;s) \tag{47}$$

(iii) Under the assumptions of Corollary 25 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2\forall\xi\in[u,v]$ and setting the function g the identity function, the inequality given in [7], Corollary 2, is deduced

(iv) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (47), the Minkowski-type inequality given in [9], Theorem 3.2, is deduced

Theorem 24. *Suppose the assumptions of Theorem 16 hold; then for $m\ge 1$, the following inequalities hold:*

$$\left(\boldsymbol{\Omega}\left(\frac{1}{\zeta_2}(\phi\psi)\right)\right)(\xi;s) \le \left(\boldsymbol{\Omega}\left(\frac{1}{(\zeta_1+1)(\zeta_2+1)}\right)(\phi+\psi)^2\right)(\xi;s) \le \left(\boldsymbol{\Omega}\left(\frac{1}{\zeta_1}(\phi\psi)\right)\right)(\xi;s). \tag{48}$$

Proof. Considering right side of inequality (27), we get the following inequalities:

$$\phi(t)+\psi(t) \le (\zeta_2(t)+1)\psi(t), \tag{49}$$

$$\zeta_2^{-1}(t)(\zeta_2(t)+1)\phi(t) \le \phi(t)+\psi(t). \tag{50}$$

Also, from the left side of inequality (27), we have the following inequalities:

$$\phi(t)+\psi(t) \ge (\zeta_1(t)+1)\psi(t), \tag{51}$$

$$\zeta_1^{-1}(t)(\zeta_1(t)+1)\phi(t) \ge \phi(t)+\psi(t). \tag{52}$$

Combining the inequalities (49) and (51), the following inequality holds

$$(\zeta_1(t)+1)\psi(t) \le \phi(t)+\psi(t) \le (\zeta_2(t)+1)\psi(t). \tag{53}$$

By the combining the inequalities (50) and (52), we get

$$\zeta_2^{-1}(t)(\zeta_2(t)+1)\phi(t) \le \phi(t)+\psi(t) \le \zeta_1^{-1}(t)(\zeta_1(t)+1)\phi(t). \tag{54}$$

The product of the above two inequalities yields

$$\zeta_2^{-1}(t)(\phi(t)\psi(t)) \le \left(\frac{1}{(\zeta_1(t)+1)(\zeta_2(t)+1)}\right)(\phi(t)+\psi(t))^2 \le \zeta_1^{-1}(t)(\phi(t)\psi(t)). \tag{55}$$

Now, multiplying $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^{\alpha};s)g'(t)$ with the above inequality and integrating over $[u,\xi]$, we get the required inequality (48). □

Corollary 25. *Under the assumptions of Theorem 24 and taking $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2\forall\xi\in[u,v]$, (48) takes the following form:*

$$\frac{1}{k_2}(\boldsymbol{\Omega}(\phi\psi))(\xi;s) \le \frac{1}{(k_1+1)(k_2+1)}\left(\boldsymbol{\Omega}(\phi+\psi)^2\right)(\xi;s) \le \frac{1}{k_1}(\boldsymbol{\Omega}(\phi\psi))(\xi;s). \tag{56}$$

Corollary 26. *Under the assumptions of the above theorem and considering $a_i=l, s=0$ and $\rho>0$ in (48), the following inequality holds:*

$$\left({}_{Q}\boldsymbol{\Omega}\left(\frac{1}{\zeta_2}(\phi\psi)\right)\right)(\xi) \le \left({}_{Q}\boldsymbol{\Omega}\left(\frac{1}{(\zeta_1+1)(\zeta_2+1)}\right)(\phi+\psi)^2\right)(\xi) \le \left({}_{Q}\boldsymbol{\Omega}\left(\frac{1}{\zeta_1}(\phi\psi)\right)\right)(\xi). \tag{57}$$

Remark 27.

(i) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (56), the inequality given in [8], Theorem 16 is deduced

(ii) By setting g the identity function, (56) gives the following inequality [7]:

$$\frac{1}{k_2}(\mathbf{I}(\phi\psi))(\xi;s) \le \frac{1}{(k_1+1)(k_2+1)}\left(\mathbf{I}(\phi+\psi)^2\right)(\xi;s) \le \frac{1}{k_1}(\mathbf{I}(\phi\psi))(\xi;s) \tag{58}$$

(iii) Under the assumptions of Corollary 29 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2 \forall \xi \in [u,v]$ and setting the function g to be the identity function, the inequality given in [7], Corollary 3, is deduced

(iv) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (58), the Minkowski-type inequality given in [9], Theorem 3.3, is deduced

Theorem 28. *Let the assumptions of Theorem 16 hold true. Also, let $\phi,\psi,\zeta_1,\zeta_2,f$ be $m-$power integrable and positive functions on $[u,v]$ such that $0<f(t)<\zeta_1(t)\le\phi(t)/\psi(t)\le\zeta_2(t)\forall t\in[u,v]$; then the following inequalities hold for $m\ge 1$:*

$$\begin{aligned}&\left[\left(\mathbf{\Omega}\left(\frac{\phi-f\psi}{\zeta_2-f}\right)^m\right)(\xi;s)\right]^{1/m}+\left[\left(\mathbf{\Omega}\left(\frac{\zeta_2(\phi-f\psi)}{\zeta_2-f}\right)^m\right)(\xi;s)\right]^{1/m}\\ &\le[(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m}+[(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m}\\ &\le\left[\left(\mathbf{\Omega}\left(\frac{\phi-f\psi}{\zeta_1-f}\right)^m\right)(\xi;s)\right]^{1/m}+\left[\left(\mathbf{\Omega}\left(\frac{\zeta_1(\phi-f\psi)}{\zeta_1-f}\right)^m\right)(\xi;s)\right]^{1/m}.\end{aligned}\tag{59}$$

Proof. By the assumption of the theorem, we have

$$0<f(t)<\zeta_1(t)\le\frac{\phi(t)}{\psi(t)}\le\zeta_2(t),\ t\in[u,v]. \tag{60}$$

The above inequality can be arranged as follows:

$$\zeta_1(t)-f(t)\le\frac{\phi(t)-f(t)\psi(t)}{\psi(t)}\le\zeta_2(t)-f(t). \tag{61}$$

From which we can write

$$\frac{(\phi(t)-f(t)\psi(t))^m}{(\zeta_2(t)-f(t))^m}\le\psi^m(t)\le\frac{(\phi(t)-f(t)\psi(t))^m}{(\zeta_1(t)-f(t))^m}. \tag{62}$$

The following inequality follows by multiplying $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha;s)g'(t)$ throughout the above inequality and integrating over $[u,\xi]$:

$$\left[\left(\mathbf{\Omega}\left(\frac{\phi-f\psi}{\zeta_2-f}\right)^m\right)(\xi;s)\right]^{1/m}\le[(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m}\le\left[\left(\mathbf{\Omega}\left(\frac{\phi-f\psi}{\zeta_1-f}\right)^m\right)(\xi;s)\right]^{1/m}. \tag{63}$$

Also, from (60), one can have

$$\frac{\zeta_1(t)-f(t)}{\zeta_1(t)}\le\frac{\phi(t)-f(t)\psi(t)}{\phi(t)}\le\frac{\zeta_2(t)-f(t)}{\zeta_2(t)}, \tag{64}$$

which can also be written as

$$\frac{\zeta_2(t)(\phi(t)-f(t)\psi(t))}{\zeta_2(t)-f(t)}\le\phi(t)\le\frac{\zeta_1(t)(\phi(t)-f(t)\psi(t))}{\zeta_1(t)-f(t)}. \tag{65}$$

Taking the power m, after multiplying by $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha;s)g'(t)$ throughout the above inequality and integrating over $[u,\xi]$, one can get the following inequality:

$$\begin{aligned}\left[\left(\mathbf{\Omega}\left(\frac{\zeta_2(\phi-f\psi)}{\zeta_2-f}\right)^m\right)(\xi;s)\right]^{1/m}&\le[(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m}\\ &\le\left[\left(\mathbf{\Omega}\left(\frac{\zeta_1(\phi-f\psi)}{\zeta_1-f}\right)^m\right)(\xi;s)\right]^{1/m}.\end{aligned}\tag{66}$$

The sum of (63) and (66) produces the required inequality (59). □

Corollary 29. *Under the assumptions of Theorem 28 along with the condition that $f(\xi)=m,\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2\forall\xi\in[u,v]$ in (59), the following inequality holds:*

$$\begin{aligned}&\frac{k_2+1}{k_2-m}[(\mathbf{\Omega}(\phi-m\psi)^m)(\xi;s)]^{1/m}\\ &\le[(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m}+[(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m}\\ &\le\frac{k_1+1}{k_1-m}[(\mathbf{\Omega}(\phi-m\psi)^m)(\xi;s)]^{1/m}.\end{aligned}\tag{67}$$

Corollary 30. *Under the assumptions of the above theorem with the condition that $a_i=l,s=0$ and $\rho>0$ in (59), the following inequality holds:*

$$\begin{aligned}&\left[\left({}_{\mathrm{Q}}\mathbf{\Omega}\left(\frac{\phi-f\psi}{\zeta_2-f}\right)^m\right)(\xi)\right]^{1/m}+\left[\left({}_{\mathrm{Q}}\mathbf{\Omega}\left(\frac{\zeta_2(\phi-f\psi)}{\zeta_2-f}\right)^m\right)(\xi)\right]^{1/m}\\ &\le\left[({}_{\mathrm{Q}}\mathbf{\Omega}\phi^m)(\xi;s)\right]^{1/m}+\left[({}_{\mathrm{Q}}\mathbf{\Omega}\psi^m)(\xi)\right]^{1/m}\\ &\le\left[\left(\mathbf{\Omega}\left(\frac{\phi-f\psi}{\zeta_1-f}\right)^m\right)(\xi)\right]^{1/m}+\left[\left(\mathbf{\Omega}\left(\frac{\zeta_1(\phi-f\psi)}{\zeta_1-f}\right)^m\right)(\xi)\right]^{1/m}.\end{aligned}\tag{68}$$

Remark 31.

(i) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (67), the inequality given in [8], Theorem 8, is deduced

(ii) By setting g the identity function, (67) is deduced to the following inequality [7]:

$$\frac{k_2+1}{k_2-m}[(\mathbf{I}(\phi-m\psi)^m)(\xi;s)]^{1/m} \le [(\mathbf{I}\phi^m)(\xi;s)]^{1/m} + [(\mathbf{I}\psi^m)(\xi;s)]^{1/m} \le \frac{k_1+1}{k_1-m}[(\mathbf{I}(\phi-m\psi)^m)(\xi;s)]^{1/m} \tag{69}$$

(iii) Under the assumptions of Corollary 33 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2 \forall \xi \in [u,v]$ and setting the function g to be the identity function, the inequality given in [7], Corollary 4, is deduced

(iv) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (69), the Minkowski-type inequality given [9], Theorem 3.4, is deduced

4. Reverse Minkowski-Type Fractional Integral Inequalities

In this section, we state and prove some reverse versions of Minkowski-type inequalities that are the generalizations of (2), (3), and (5).

Theorem 32. *Under the assumptions of Theorem 16, the following inequality holds for $m \ge 1$:*

$$[(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m} + [(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m} \le \left[\left(\mathbf{\Omega}\left(\left(\frac{\zeta_2}{1+\zeta_2}\right)^m (\phi+\psi)^m\right)\right)(\xi;s)\right]^{1/m} + \left[\left(\mathbf{\Omega}\left(\left(\frac{1}{1+\zeta_1}\right)^m (\phi+\psi)^m\right)\right)(\xi;s)\right]^{1/m}. \tag{70}$$

Proof. From (27), one can obtain the following inequality:

$$\psi^m(t) \le \frac{1}{(1+\zeta_1(t))^m}(\phi(t)+\psi(t))^m. \tag{71}$$

Multiplying both sides of inequality by $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha;s)g'(t)$ and integrating on $[u,\xi]$, the above inequality can take the form as follows:

$$[(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m} \le \left[\left(\mathbf{\Omega}\left(\frac{1}{(1+\zeta_1(t))^m}(\psi+\phi)^m\right)\right)(\xi;s)\right]^{1/m}. \tag{72}$$

Also, by considering inequality (27), one can have the following inequality:

$$\phi^m(t) \le \left(\frac{\zeta_2}{1+\zeta_2(t)}\right)^m (\psi(t)+\phi(t))^m. \tag{73}$$

Multiplying both sides of the above inequality by $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha;s)g'(t)$ and integrating over $[u,\xi]$, we can get

$$[(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m} \le \left[\left(\mathbf{\Omega}\left(\left(\frac{\zeta_2}{1+\zeta_2(t)}\right)^m (\psi+\phi)^m\right)\right)(\xi;s)\right]^{1/m}. \tag{74}$$

Adding (72) and (74), inequality (70) can be obtained. □

Corollary 33. *Under the assumptions of Theorem 32 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2 \forall \xi \in [u,v]$, (70) takes the following form:*

$$[(\mathbf{\Omega}\phi^m)(\xi;s)]^{1/m} + [(\mathbf{\Omega}\psi^m)(\xi;s)]^{1/m} \le \left(1+\frac{k_2-k_1}{(k_1+1)(k_2+1)}\right)[(\mathbf{\Omega}(\phi+\psi)^m)(\xi;s)]^{1/m}. \tag{75}$$

Corollary 34. *Under the assumptions of the above theorem and taking $a_i=l, s=0$, and $\rho>0$ in (70), the following inequality is obtained:*

$$\left[\left({}_{\mathrm{Q}}\mathbf{\Omega}\phi^m\right)(\xi)\right]^{1/m} + \left[\left({}_{\mathrm{Q}}\mathbf{\Omega}\psi^m\right)(\xi)\right]^{1/m} \le \left[\left({}_{\mathrm{Q}}\mathbf{\Omega}\left(\left(\frac{\zeta_2}{1+\zeta_2}\right)^m (\phi+\psi)^m\right)\right)(\xi)\right]^{1/m} + \left[\left({}_{\mathrm{Q}}\mathbf{\Omega}\left(\left(\frac{1}{1+\zeta_1}\right)^m (\phi+\psi)^m\right)\right)(\xi)\right]^{1/m}. \tag{76}$$

Remark 35.

(i) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (70), the inequality introduced by Andric et. al [8] (Theorem 3) is generated

(ii) Taking $g:[u,v]\longrightarrow\mathbb{R}$ to be an identity function, (75) gives the following inequality [7]:

$$[(\mathbf{I}\phi^m)(\xi;s)]^{1/m} + [(\mathbf{I}\psi^m)(\xi;s)]^{1/m} \le \left(1+\frac{k_2-k_1}{(k_1+1)(k_2+1)}\right)[(\mathbf{I}(\phi+\psi)^m)(\xi;s)]^{1/m} \tag{77}$$

(iii) Under the assumptions of Corollary 37 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2 \forall \xi \in [u,v]$ and setting the function g to be an identity function, we obtain the inequality as in [7], Corollary 5

(iv) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (77), Minkowski-type inequality (5) is deduced

Theorem 36. *Under the assumptions of Theorem 16, for $m \geq 1$, we have*

$$\left[(\mathbf{\Omega}\phi^m)(\xi;s)\right]^{2/m} + \left[(\mathbf{\Omega}\psi^m)(\xi;s)\right]^{2/m} \geq \left[\left(\mathbf{\Omega}\left(\frac{1+\zeta_2}{\zeta_2}\right)^m \phi^m\right)(\xi;s)\right]^{1/m} \left[\left(\mathbf{\Omega}(1+\zeta_1)^m\psi^m\right)(\xi;s)\right]^{1/m} - 2\left[(\mathbf{\Omega}\phi^m)(\xi;s)\right]^{1/m}\left[(\mathbf{\Omega}\psi^m)(\xi;s)\right]^{1/m}. \tag{78}$$

Proof. Inequalities (71) and (73) from the previous theorem can be arranged in the following forms:

$$(1+\zeta_1(t))^m\psi^m(t) \leq (\phi(t)+\psi(t))^m, \quad \left(\frac{1+\zeta_2(t)}{\zeta_2(t)}\right)^m \phi^m(t) \leq (\psi(t)+\phi(t))^m. \tag{79}$$

By multiplying with $(g(\xi)-g(t))^{\beta-1}\mathbf{M}(\omega(g(\xi)-g(t))^\alpha;s)g'(t)$ and integrating over $[u,\xi]$ and taking the power $1/m$ of the resulting inequalities, the above inequalities further take the following forms:

$$\left[\left(\mathbf{\Omega}(1+\zeta_1)^m\psi^m\right)(\xi;s)\right]^{1/m} \leq \left[(\mathbf{\Omega}(\phi+\psi)^m)(\xi;s)\right]^{1/m}, \tag{80}$$

$$\left[\left(\mathbf{\Omega}\left(\frac{1+\zeta_2}{\zeta_2}\right)^m\phi^m\right)(\xi;s)\right]^{1/m} \leq \left[(\mathbf{\Omega}(\psi+\phi)^m)(\xi;s)\right]^{1/m}. \tag{81}$$

By multiplying (80) and (81), we get the following inequality:

$$\left[\left(\mathbf{\Omega}(1+\zeta_1)^m\psi^m\right)(\xi;s)\right]^{1/m}\left[\left(\mathbf{\Omega}\left(\frac{1+\zeta_2}{\zeta_2}\right)^m\phi^m\right)(\xi;s)\right]^{1/m} \leq \left[\left((\mathbf{\Omega}(\psi+\phi)^m)(\xi;s)\right)^{1/m}\right]^2. \tag{82}$$

Applying Minkowski's inequality on the term within the square brackets at the right side of the above inequality and then using $(a+b)^2 = a^2+2ab+b^2$, the above inequality gives the required inequality (78). □

Corollary 37. *Under the assumptions of Theorem 36 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2 \forall \xi \in [u,v]$, (78) takes the following form:*

$$\left[(\mathbf{\Omega}\phi^m)(\xi;s)\right]^{2/m} + \left[(\mathbf{\Omega}\psi^m)(\xi;s)\right]^{2/m} \geq \left(\frac{2+(k_1-1)(k_2+1)}{k_2}\right)\left[(\mathbf{\Omega}\phi^m)(\xi;s)\right]^{1/m}\left[(\mathbf{\Omega}\psi^m)(\xi;s)\right]^{1/m}. \tag{83}$$

Corollary 38. *Under the assumptions of above theorem together with the condition $a_i = l$, $s=0$, and $\rho>0$, (78) results in the following inequality:*

$$\left[\left({}_{\mathbf{Q}}\mathbf{\Omega}\phi^m\right)(\xi)\right]^{2/m} + \left[\left({}_{\mathbf{Q}}\mathbf{\Omega}\psi^m\right)(\xi)\right]^{2/m} \geq \left[\left({}_{\mathbf{Q}}\mathbf{\Omega}\left(\frac{1+\zeta ss_2}{\zeta_2}\right)^m\phi^m\right)(\xi)\right]^{1/m}\left[\left({}_{\mathbf{Q}}\mathbf{\Omega}(1+\zeta_1)^m\psi^m\right)(\xi)\right]^{1/m} - 2\left[\left({}_{\mathbf{Q}}\mathbf{\Omega}\phi^m\right)(\xi)\right]^{1/m}\left[\left({}_{\mathbf{Q}}\mathbf{\Omega}\psi^m\right)(\xi)\right]^{1/m}. \tag{84}$$

Remark 39.

(i) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (78), the inequality given in [8], Theorem 4, is deduced

(ii) By setting g the identity function in (83), the following inequality is deduced [7]:

$$\left[(\mathbf{I}\phi^m)(\xi;s)\right]^{2/m} + \left[(\mathbf{I}\psi^m)(\xi;s)\right]^{2/m} \geq \left(\frac{2+(k_1-1)(k_2+1)}{k_2}\right)\left[(\mathbf{I}\phi^m)(\xi;s)\right]^{1/m}\left[(\mathbf{I}\psi^m)(\xi;s)\right]^{1/m} \tag{85}$$

(iii) Under the assumptions of Corollary 13 along with the condition that $\zeta_1(\xi)=k_1$ and $\zeta_2(\xi)=k_2$ with $0<k_1<k_2\forall\xi\in[u,v]$ and setting g the identity function, the inequality given in [7], Corollary 6, is deduced

(iv) By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$, and $\delta>0$ in (85), the Minkowski-type inequality given in [9], Theorem 2.2, is deduced

Remark 40. All results of this paper hold for the right-sided integral operator:

$$\left({}_g\Omega^{\omega,\lambda,\rho,\theta,k,n}_{v^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\phi\right)(\xi;s) = \int_\xi^v (g(t)-g(\xi))^{\beta-1} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,u,\nu}\left(\omega(g(t)-g(\xi))^\alpha;s\right)g'(t)\phi(t)dt. \tag{86}$$

5. Conclusion

A generalized integral operator with the help of a unified Mittag-Leffler function is defined, and its boundedness is proved. By giving specific values to parameters and considering suitable functions involved in the kernel of this operator, various kinds of well-known integral and fractional integral operators can be reproduced. For a fractional integral operator, we have constructed several Minkowski- and reverse Minkowski-type inequalities. The particular cases of the results of this paper are connected with many already published results.

Data Availability

All data required for this research is included within this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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