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Research Article

On Numerical Radius Bounds Involving Generalized Aluthge Transform

Tao Yan,[1] Javariya Hyder,[2] Muhammad Saeed Akram,[3] Ghulam Farid,[4] and Kamsing Nonlaopon[5]

[1]*School of Computer Science, Chengdu University, Chengdu, China*
[2]*Department of Mathematics, Khwaja Fareed University of Engineering and Information Technology, Rahim Yar Khan, Pakistan*
[3]*Department of Mathematics, Faculty of Science, Ghazi University, 32200 Dera Ghazi Khan, Pakistan*
[4]*Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan*
[5]*Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand*

Correspondence should be addressed to Kamsing Nonlaopon; nkamsi@kku.ac.th

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In this paper, we establish some upper bounds of the numerical radius of a bounded linear operator S defined on a complex Hilbert space with polar decomposition $S = U|S|$, involving generalized Aluthge transform. These bounds generalize some bounds of the numerical radius existing in the literature. Moreover, we consider particular cases of generalized Aluthge transform and give some examples where some upper bounds of numerical radius are computed and analyzed for certain operators.

1. Introduction

In mathematical analysis, inequalities play a vital role in studying the properties of operators in the form of their upper and lower bounds. Mathematical inequalities provide the best way to describe as well as propose solutions to real-world problems in almost all fields of science and engineering. The boundedness property of different kinds of operators studied in the subjects of analysis, precisely in mathematical and functional analysis, is the key factor in developing the theory and applications. For example, upper and lower bounds are utilized to define the operator norm, which plays significantly in solving related problems. The study of the numerical radius of an operator defined on the Hilbert space is in the focus of researchers in these days in studying perturbation, convergence, iterative solution methods, and integrative methods, etc, see [1–9]. In this regard, the numerical radius inequality stated in (3) is studied extensively by various mathematicians, see [10–21]. Actually, it is interesting for the researchers to get refinements and generalizations of this inequality [22–27]. The goal of this paper is to study generalizations of numerical radius bounds under certain additional conditions. Henceforth, we define the preliminary notions to proceed with the findings of this work.

The polar decomposition is an important feature in the theory of operators. It is defined by $A = UB$, where U is the unitary matrix, and B is the symmetric positive semidefinite matrix. It is interesting to see that when A is nonsingular and symmetric, then B is a good symmetric positive definite approximation to A and $1/2(A + B)$ is the best symmetric positive semidefinite approximation to A, see [6]. Let $\mathscr{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on complex Hilbert space. Let $S = U|S|$ be the unique polar decomposition of $S \in \mathscr{B}(\mathscr{H})$, where U is a partial isometry and $|S|$ is the square root of an operator which is defined as $|S| = \sqrt{S^* S}$. The numerical range of an operator S is defined as

$$W(S) = \{\langle Sx, x\rangle : \|x\| = 1, x \in \mathscr{H}\}, \tag{1}$$

where $W(S)$ denotes the numerical range. The numerical radius of an operator is the radius of the smallest circle centered at the origin and contains the numerical range, i.e.,

$$w(S) = \sup \{|\lambda|: \lambda \in W(S)\}. \tag{2}$$

The numerical radius defines a norm on $\mathscr{B}(\mathscr{H})$ which is equivalent to the usual operator norm, satisfying the following inequality:

$$\frac{1}{2}\|S\| \le w(S) \le \|S\|. \tag{3}$$

If $S^2 = 0$, then the first inequality becomes equality and if S is normal then the second inequality becomes equality. Many authors worked on numerical radius inequalities and developed a number of numerical radius bounds [10, 13–18].

In [17], Kittaneh gave an upper bound of numerical radius as follows:

$$w(S) \le \frac{1}{2}\left(\|S\| + \left\|S^2\right\|^{1/2}\right), \tag{4}$$

and showed that this bound is sharper than the upper bound given in (3).

In [25], Aluthge introduced a transform of an operator $S \in \mathscr{B}(\mathscr{H})$ which is called Aluthge transform that is defined as

$$\Delta(S) = |S|^{1/2} U |S|^{1/2}. \tag{5}$$

In [26], Yamazaki developed an upper bound of the numerical radius involving Aluthge transform as follows:

$$w(S) \le \frac{1}{2}(\|S\| + w(\Delta S)), \tag{6}$$

and proved that it is sharper than the bound given in (4).

In [27], Okubo introduced a new generalization of Aluthge transform, called λ-Aluthge transform defined by

$$\Delta_\lambda S = |S|^\lambda U |S|^{1-\lambda}; \lambda \in [0, 1]. \tag{7}$$

In [23], Abu-Omar and Kittaneh further generalized the bound given in (6) using λ-Aluthge transform as follows:

$$w(S) \le \frac{1}{2}(\|S\| + w(\Delta_\lambda S)). \tag{8}$$

In [19], Bhunia et al. found some bounds of the numerical radius for $S \in \mathscr{B}(\mathscr{H})$. Later, Bag et al. [24] working along the same lines succeeded to get the following upper bounds of the numerical radius:

$$w^2(S) \le \frac{1}{2}\|S\|\|\Delta_\lambda S\| + \frac{1}{4}\|S^* S + SS^*\|, \tag{9}$$

$$w^2(S) \le \frac{1}{4}\left(w\left((\Delta_\lambda S)^2\right) + \|S\|\|\Delta_\lambda S\| + \|S^* S + SS^*\|\right), \tag{10}$$

$$\begin{aligned} w^2(S) \le \sum_{n=1}^{\infty} \frac{1}{4^n}\Big(&\|\Delta_\lambda^{n-1} S\|\|\Delta_\lambda^n S\| + \Big\|\left(\Delta_\lambda^{n-1} S\right)^*\left(\Delta_\lambda^{n-1} S\right) \\ &+ \left(\Delta_\lambda^{n-1} S\right)\left(\Delta_\lambda^{n-1} S\right)^*\Big\|\Big), \end{aligned} \tag{11}$$

$$\begin{aligned} w^4(S) \le{}& \frac{1}{16}\left(w\left((\Delta_\lambda S)^2\right) + \|S\|\|\Delta_\lambda S\|\right)^2 \\ &+ \frac{1}{8} w\left(S^2 P + PS^2\right) + \frac{1}{16}\|P\|^2, \end{aligned} \tag{12}$$

where $P = S^* S + SS^*$ and $\lambda \in [0, 1]$.

In [22], Shebrawi and Bakherad presented a new form of Aluthge transform so called generalized Aluthge transform defined by

$$\Delta_{f,g} S = f(|S|) U g(|S|), \tag{13}$$

where f and g are nonnegative continuous functions such that $f(|S|)g(|S|) = |S|$, $(|S| \ge 0)$. They proved the following upper bound of the numerical radius by using generalized Aluthge transform

$$w(S) \le \frac{1}{2}\left(\|S\| + w\left(\Delta_{f,g} S\right)\right), \tag{14}$$

which is a generalization of the upper bound shown in (6) and (8).

Our aim is to study the upper bounds of the numerical radius by applying generalized Aluthge transform defined in (13) by imposing further certain conditions on continuous functions. The first contribution of this paper is that we develop upper bounds of the numerical radius using generalized Aluthge transform, which extends and generalizes some already existing bounds. Specifically, we extend the inequalities (9)–(12) for generalized Aluthge transform under certain conditions on f and g. As a consequence, the upper bounds of numerical radius involving Aluthge transform and λ-Aluthge transform appear as a special case of our bounds. Another contribution of the paper is that we have presented examples of generalized Aluthge transform in addition to the classical Aluthge transform and λ-Aluthge transform, which are used for computing bounds of numerical radius. More precisely, we have considered five choices of continuous functions f and g in (13) and used them to compute upper bounds of numerical radius for certain operators.

2. Main Results

We start this section by attaining the generalized Aluthge transform $\Delta_{f,g}$ defined in (13) under the following additional conditions:

(i) $g(|S|)f(|S|) = |S|$

(ii) $f(|S|)$ and $g(|S|)$ both are positive operators

Now, we give some results that will be used repeatedly to achieve our goal.

Lemma 1 [26]. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, we have*

$$w(S) = \sup_{\theta\in\mathbb{R}} \|H_\theta\| = \sup_{\theta\in\mathbb{R}} \left\| \operatorname{Re}\left(e^{\iota\theta} S\right) \right\|, \tag{15}$$

where $H_\theta = (\operatorname{Re}(e^{\iota\theta} S)) = (e^{\iota\theta} S + e^{-\iota\theta} S^*)/2$ for all $\theta \in \mathbb{R}$.

Lemma 2 [23]. *Let $M_1, M_2, N_1, N_2 \in \mathscr{B}(\mathscr{H})$. Then,*

$$\begin{aligned} & r(M_1 N_1 + M_2 N_2) \\ & \quad \le \frac{1}{2}(w(N_1 M_1) + w(N_2 M_2)) \\ & \quad + \frac{1}{2}\sqrt{w(N_1 M_1) - w(N_2 M_2) + 4\|N_1 M_2\|\|N_2 M_1\|}, \end{aligned} \tag{16}$$

where r denotes the spectral radius.

Next, we give the numerical radius bound by using the generalized Aluthge transform.

Theorem 3. *Let $S \in \mathscr{B}(\mathscr{H})$. Then, we have*

$$w^2(S) \le \frac{1}{2}\|g(|S|)\|\left\|\Delta_{f,g} S\right\|\|f(|S|)\| + \frac{1}{4}\|S^* S + SS^*\|. \tag{17}$$

Proof. Since

$$H_\theta = \frac{1}{2}\left(e^{\iota\theta} S + e^{-\iota\theta} S^*\right) \text{ for all } \theta \in \mathbb{R}, \tag{18}$$

therefore,

$$\begin{aligned} H_\theta{}^2 &= \frac{1}{4}\left(e^{\iota\theta} S + e^{-\iota\theta} S^*\right)^2 \\ &= \frac{1}{4}\left(e^{2\iota\theta} S^2 + e^{-2\iota\theta} S^{*2} + SS^* + S^* S\right) \\ &= \frac{1}{4}\left(e^{2\iota\theta} U|S|U|S| + e^{-2\iota\theta}|S|U^*|S|\, U^* + SS^* + S^* S\right) \\ &= \frac{1}{4}\left(e^{2\iota\theta} U g(|S|) f(|S|) U g(|S|) f(|S|)\right. \\ &\quad \left. + e^{-2\iota\theta} f(|S|) g(|S|) U^* f(|S|) g(|S|) U^* + SS^* + S^* S\right) \\ &= \frac{1}{4}\left(e^{2\iota\theta} U g(|S|)\left(\Delta_{f,g} S\right) f(|S|)\right. \\ &\quad \left. + e^{-2\iota\theta} f(|S|)\left(\Delta_{f,g} S\right)^* g(|S|) U^* + SS^* + S^* S\right). \end{aligned} \tag{19}$$

The third equality is obtained by putting $S = U\,|\,S\,|$ and $S^* = |S\,|\,U^*$ in second equality, the fourth equality holds because $f(|S|)g(|S|) = |S|$ and $g(|S\,|)f(|S\,|) = |S\,|$, and the fifth equality holds because $\Delta_{f,g} S = f(|S\,|)Ug(|S\,|)$ and $(\Delta_{f,g} S)^* = g(|S\,|)U^* f(|S\,|)$. Since $\|ZZ^*\| = \|Z\|^2$ for any $Z \in \mathscr{B}(\mathscr{H})$, therefore

$$\begin{aligned} \|H_\theta\|^2 &= \frac{1}{4}\left(\left\| e^{2\iota\theta} U g(|S|)\left(\Delta_{f,g} S\right) f(|S|)\right.\right. \\ &\quad \left.\left. + e^{-2\iota\theta} f(|S|)\left(\Delta_{f,g} S\right)^* g(|S|) U^* + SS^* + S^* S \right\|\right) \\ &\le \frac{1}{4}\left(\|g(|S|)\|\left\|\Delta_{f,g} S\right\|\|f(|S|)\|\right. \\ &\quad \left. + \|f(|S|)\|\left\|\left(\Delta_{f,g} S\right)^*\right\|\|g(|S|)\| + \|SS^* + S^* S\|\right) \\ &= \frac{1}{4}\left(2\|g(|S|)\|\left\|\Delta_{f,g} S\right\|\|f(|S|)\| + \|SS^* + S^* S\|\right). \end{aligned} \tag{20}$$

The first inequality holds because $\|S_1 S_2\| \le \|S_1\|\|S_2\|$, $\|S_1 + S_2\| \le \|S_1\| + \|S_2\|$ for any $S_1, S_2 \in \boldsymbol{\mathscr{B}}(\boldsymbol{\mathscr{H}})$, U is partial isometry and $|e^{2\iota\theta}| = 1$ and the second equality holds by using the fact that $\|S\| = \|S^*\|$.

Now, by taking supremum of the last inequality over $\theta \in \mathbb{R}$ and then using Lemma 1, we get

$$w^2(S) \le \frac{1}{2}\|g(|S|)\|\left\|\Delta_{f,g} S\right\|\|f(|S|)\| + \frac{1}{4}\|SS^* + S^* S\|, \tag{21}$$

as required. □

The following result is another generalized bound of numerical radius for bounded linear operators on $\mathscr{H}$.

Theorem 4. *Let $S \in \mathscr{B}(\mathscr{H})$. Then,*

$$\begin{aligned} w^2(S) \le \frac{1}{4}&\left(w\left(\left(\Delta_{f,g} S\right)^2\right) + \|g(|S|)\|\left\|\Delta_{f,g} S\right\|\|f(|S|)\|\right. \\ &\left. + \left\|S^* S + SS^*\right\|\right). \end{aligned} \tag{22}$$

Proof. Let S be any bounded linear operator with polar decomposition $S = U|S|$. Since

$$H_\theta = \frac{1}{2}\left(e^{\iota\theta} S + e^{-\iota\theta} S^*\right) \text{ for all } \theta \in \mathbb{R}, \tag{23}$$

therefore,

$$H_\theta{}^2 = \frac{1}{4}\left(e^{\iota\theta} S + e^{-\iota\theta} S^*\right)^2 = \frac{1}{4}\left(e^{2\iota\theta} S^2 + e^{-2\iota\theta} S^{*2} + SS^* + S^* S\right). \tag{24}$$

Using the properties of operator norm $\|\cdot\|$ on $\mathscr{B}(\mathscr{H})$, we have

$$\begin{aligned} \|H_\theta\|^2 &\le \frac{1}{4}\left(\left\| e^{2\iota\theta} U g(|S|)\left(\Delta_{f,g} S\right) f(|S|)\right.\right. \\ &\quad \left.\left. + e^{-2\iota\theta} f(|S|)\left(\Delta_{f,g} S\right)^* g(|S|) U^* \right\| + \|SS^* + S^* S\|\right) \\ &= \frac{1}{4}\left(r\left(e^{2\iota\theta} U g(|S|)\left(\Delta_{f,g} S\right) f(|S|)\right.\right. \\ &\quad \left.\left. + e^{-2\iota\theta} f(|S|)\left(\Delta_{f,g} S\right)^* g(|S|) U^*\right) + \|SS^* + S^* S\|\right) \\ &= \frac{1}{4}\left(r(M_1 N_1 + M_2 N_2) + \|SS^* + S^* S\|\right), \end{aligned} \tag{25}$$

where $M_1 = e^{\iota\theta} U g(|S|)(\Delta_{f,g}S)$, $N_1 = f(|S|)$, $M_2 = e^{-2\iota\theta} f(|S|)(\Delta_{f,g}S)^*$, $N_2 = g(|S|)U^*$. The first equality above holds for hermitian operator $A \in \mathscr{B}(\mathscr{H})$ satisfying $r(A) = \|A\|$. Now, an application of Lemma 2 together with $w(S) = w(S^*)$ and $w(\alpha S) = |\alpha| w(S)$ yields

$$\begin{aligned}
\|H_\theta\|^2 &\le \frac{1}{4}\Big(w\Big((\Delta_{f,g}S)^2\Big) \\
&\quad + \frac{1}{2}\sqrt{4\left\|(f(|S|))^2(\Delta_{f,g}S)^*\right\|\left\|(g(|S|))^2\Delta_{f,g}S\right\|} \\
&\quad + \|SS^* + S^*S\|\Big) \\
&\le \frac{1}{4}\Big(w\Big((\Delta_{f,g}S)^2\Big) \\
&\quad + \frac{1}{2}\sqrt{4\|f(|S|)\|^2\left\|(\Delta_{f,g}S)^*\right\|\|g(|S|)\|^2\left\|\Delta_{f,g}S\right\|} \\
&\quad + \|SS^* + S^*S\|\Big) \\
&= \frac{1}{4}\Big(w\Big((\Delta_{f,g}S)^2\Big) + \|f(|S|)\|\|g(|S|)\|\left\|\Delta_{f,g}S\right\| \\
&\quad + \|SS^* + S^*S\|\Big).
\end{aligned} \tag{26}$$

The last equality holds by using the fact $\|S\| = \|S^*\|$. Now, we take supremum over $\theta \in \mathbb{R}$ to get

$$\begin{aligned}
\sup_{\theta\in\mathbb{R}}\|H_\theta\|^2 &\le \sup_{\theta\in\mathbb{R}}\Big(\frac{1}{4}\Big(w\Big((\Delta_{f,g}S)^2\Big) \\
&\quad + \|g(|S|)\|\left\|\Delta_{f,g}S\right\|\|f(|S|)\| + \|SS^* + S^*S\|\Big)\Big).
\end{aligned} \tag{27}$$

By using Lemma 1 in above inequality, we obtain

$$\begin{aligned}
w^2(S) &\le \frac{1}{4}\Big(w\Big((\Delta_{f,g}S)^2\Big) + \|g(|S|)\|\left\|\Delta_{f,g}S\right\|\|f(|S|)\| \\
&\quad + \|SS^* + S^*S\|\Big),
\end{aligned} \tag{28}$$

as required. □

The following inequality is another generalized bound of numerical radius.

Theorem 5. *Let $S \in \mathscr{B}(\mathscr{H})$. Then we have*

$$\begin{aligned}
w^4(S) &\le \frac{1}{16}\Big(w\Big((\Delta_{f,g}S)^2\Big) + \|g(|S|)\|\left\|\Delta_{f,g}S\right\|\|f(|S|)\|\Big)^2 \\
&\quad + \frac{1}{8}w\big(S^2P + PS^2\big) + \frac{1}{16}\|P\|^2,
\end{aligned} \tag{29}$$

where $P = S^*S + SS^*$.

Proof. Since

$$H_\theta = \frac{1}{2}\Big(e^{\iota\theta}S + e^{-\iota\theta}S^*\Big) \text{ for all } \theta \in \mathbb{R}, \tag{30}$$

therefore,

$$\begin{aligned}
{H_\theta}^2 &= \frac{1}{4}\Big(e^{\iota\theta}S + e^{-\iota\theta}S^*\Big)^2 \\
&= \frac{1}{4}\Big(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2} + SS^* + S^*S\Big){H_\theta}^4 \\
&= \frac{1}{16}\Big(\Big(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\Big) + P\Big)^2 \\
&= \frac{1}{16}\Big(\Big(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\Big)^2 \\
&\quad + \Big(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\Big)P + P\Big(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\Big) + P^2\Big) \\
&= \frac{1}{16}\Big(\Big(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\Big)^2 \\
&\quad + \Big(e^{2\iota\theta}S^2P + e^{-2\iota\theta}S^{*2}P + Pe^{2\iota\theta}S^2 + Pe^{-2\iota\theta}S^{*2}\Big) + P^2\Big) \\
&= \frac{1}{16}\Big(\Big(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\Big)^2 + e^{2\iota\theta}\big(S^2P + PS^2\big) \\
&\quad + e^{-2\iota\theta}\Big(S^{*2}P + PS^{*2}\Big) + P^2\Big) \\
&= \frac{1}{16}\Big(\Big(e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\Big)^2 \\
&\quad + 2\Big(\mathrm{Re}\,\Big(e^{2\iota\theta}\big(S^2P + PS^2\big)\Big)\Big) + P^2\Big),
\end{aligned} \tag{31}$$

where

$$\mathrm{Re}\,\Big(e^{2\iota\theta}\big(S^2P + PS^2\big)\Big) = \frac{e^{2\iota\theta}\big(S^2P + PS^2\big) + e^{-2\iota\theta}\big(S^2P + PS^2\big)^*}{2}. \tag{32}$$

In third equality $P = S^*S + SS^*$. Now, by using the properties of operator norm $\|\cdot\|$ on $\mathscr{B}(\mathscr{H})$, we have

$$\begin{aligned}
\|H_\theta\|^4 &\le \frac{1}{16}\Big(\left\|e^{2\iota\theta}S^2 + e^{-2\iota\theta}S^{*2}\right\|^2 \\
&\quad + 2\left\|\mathrm{Re}\,\Big(e^{2\iota\theta}\big(S^2P + PS^2\big)\Big)\right\| + \|P\|^2\Big) \\
&= \frac{1}{16}\Big(\left\|e^{2\iota\theta}U|S|U|S| + e^{-2\iota\theta}|S|U^*|S|U^*\right\|^2 \\
&\quad + 2\left\|\mathrm{Re}\,\Big(e^{2\iota\theta}\big(S^2P + PS^2\big)\Big)\right\| + \|P\|^2\Big) \\
&= \frac{1}{16}\Big(\Big\|e^{2\iota\theta}Ug(|S|)(\Delta_{f,g}S)f(|S|) \\
&\quad + e^{-2\iota\theta}f(|S|)(\Delta_{f,g}S)^*g(|S|)U^*\Big\|^2 \\
&\quad + 2\left\|\mathrm{Re}\,\Big(e^{2\iota\theta}\big(S^2P + PS^2\big)\Big)\right\| + \|P\|^2\Big)
\end{aligned}$$

$$= \frac{1}{16}\left(r^2\left(e^{2\iota\theta}Ug(|S|)(\Delta_{f,g}S)f(|S|)\right.\right.$$

$$\left. + e^{-2\iota\theta}f(|S|)(\Delta_{f,g}S)^* g(|S|)U^*\right)$$

$$\left. + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right)$$

$$= \frac{1}{16}\left(r^2(M_1N_1 + M_2N_2)\right.$$

$$\left. + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right), \quad (33)$$

where $M_1 = e^{\iota\theta}Ug(|S|)(\Delta_{f,g}S)$, $N_1 = f(|S|)$, $M_2 = e^{-2\iota\theta}f(|S|)((\Delta_{f,g}S)^*)$, and $N_2 = g(|S|)U^*$. The first equality obtained by using $S = U|S|$ and $S^* = |S|U^*$ in first inequality, the second equality obtained by using $f(|S|)g(|S|) = |S|$ and $g(|S|)f(|S|) = |S|$ in third equality, and the fifth equality holds for hermitian operator satisfying $r(A) = \|A\|$. Now, by using Lemma 2 together with $w(S) = w(S^*)$ and $w(\alpha S) = |\alpha| w(S)$, it yields

$$\begin{aligned}
\|H_\theta\|^4 \le{}& \frac{1}{16}\left(\left(w\left((\Delta_{f,g}S)^2\right)\right.\right.\\
&\left. + \frac{1}{2}\sqrt{4\left\|(f(|S|))^2 e^{-2\iota\theta}(\Delta_{f,g}S)^*\right\|\left\|(g|S|))^2 e^{\iota\theta}\Delta_{f,g}S\right\|}\right)^2\\
&\left. + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right)\\
\le{}& \frac{1}{16}\left(\left(w\left((\Delta_{f,g}S)^2\right)\right.\right.\\
&\left. + \frac{1}{2}\sqrt{4\|f(|S|)\|^2\left\|(\Delta_{f,g}S)^*\right\|\|g(|S|)\|^2\left\|\Delta_{f,g}S\right\|}\right)^2\\
&\left. + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right)\\
={}& \frac{1}{16}\left(\left(w\left((\Delta_{f,g}S)^2\right) + \|f(|S|)\|\|g(|S|)\|\left\|\Delta_{f,g}S\right\|\right)^2\right.\\
&\left. + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right).
\end{aligned} \quad (34)$$

The last equality holds by using the fact $\|S\| = \|S^*\|$. Now, we take supremum over $\theta \in \mathbb{R}$ to get

$$\begin{aligned}
\sup_{\theta\in\mathbb{R}}\|H_\theta\|^4 \le \sup_{\theta\in\mathbb{R}}\frac{1}{16}&\left(\left(w\left((\Delta_{f,g}S)^2\right) + \|f(|S|)\|\|g(|S|)\|\left\|\Delta_{f,g}S\right\|\right)^2\right.\\
&\left. + 2\left\|\operatorname{Re}\left(e^{2\iota\theta}\left(S^2P + PS^2\right)\right)\right\| + \|P\|^2\right).
\end{aligned} \quad (35)$$

Applying Lemma 1 on above inequality, we obtain

$$\begin{aligned}
w^4(S) \le{}& \frac{1}{16}\left(w\left((\Delta_{f,g}S)^2\right) + \|f(|S|)\|\|g(|S|)\|\left\|\Delta_{f,g}S\right\|\right)^2\\
&+ \frac{1}{8}w\left(S^2P + PS^2\right) + \frac{1}{16}\|P\|^2.
\end{aligned} \quad (36)$$

as required. □

To give the next bound of numerical radius, first, we define iterated generalized Aluthge transform. The iterated generalized Aluthge transform is defined as

$$\Delta^k_{f,g}S = \Delta\left(\Delta^{k-1}_{f,g}S\right); \forall k \in \mathbb{N}, \quad (37)$$

where f and g both are nonnegative and continuous functions.

By using Theorem 4 repeatedly, we can obtain numerical radius bound in terms of iterated generalized Aluthge transform.

Theorem 6. *Let $S \in \mathcal{B}(\mathcal{H})$ be such that the sequence $\{\|\Delta^n_{f,g}S\|\}_{n=1}^{\infty}$ is convergent then*

$$\begin{aligned}
w^2(S) \le \sum_{k=1}^{\infty}\frac{1}{4^k}&\left(\left\|f\left(|\Delta^{k-1}_{f,g}S|\right)\right\|\left\|g\left(|\Delta^{k-1}_{f,g}S|\right)\right\|\left\|\Delta^k_{f,g}S\right\|\right.\\
&+ \left\|\left(\Delta^{k-1}_{f,g}S\right)^*\left(\Delta^{k-1}_{f,g}S\right) + \left(\Delta^{k-1}_{f,g}S\right)\left(\Delta^{k-1}_{f,g}S\right)^*\right\|.
\end{aligned} \quad (38)$$

Proof. In order to prove the theorem, it is sufficient to prove the following assertion

$$\begin{aligned}
w^2(S) \le \sum_{k=1}^{n}\frac{1}{4^k}&\left(\left\|f\left(\left|\Delta^{k-1}_{f,g}S\right|\right)\right\|\left\|\Delta^k_{f,g}S\right\|\left\|g\left(\left|\Delta^{k-1}_{f,g}S\right|\right)\right\|\right.\\
&\left. + \left\|\left(\Delta^{k-1}_{f,g}S\right)^*\left(\Delta^{k-1}_{f,g}S\right) + \left(\Delta^{k-1}_{f,g}S\right)\left(\Delta^{k-1}_{f,g}S\right)^*\right\|\right)\\
&+ \frac{1}{4^n}w^2\left(\Delta^n_{f,g}S\right) \text{for all } n \in \mathbb{N}.
\end{aligned} \quad (39)$$

We use mathematical induction to prove the above assertion. An application of Theorem 4 gives

$$\begin{aligned}
w^2(S) \le{}& \frac{1}{4}\left(\|f(|S|)\|\|g(|S|)\|\left\|\Delta_{f,g}S\right\| + \|S^*S + SS^*\|\right)\\
&+ \frac{1}{4}w\left((\Delta_{f,g}S)^2\right).
\end{aligned} \quad (40)$$

The use of the inequality $w(S^2) \le w^2(S)$ gives

$$\begin{aligned}
w^2(S) \le{}& \frac{1}{4}\left(\|f(|S|)\|\left\|\Delta_{f,g}S\right\|\|g(|S|)\| + \|S^*S + SS^*\|\right)\\
&+ \frac{1}{4}w^2\left(\Delta_{f,g}S\right).
\end{aligned} \quad (41)$$

Table 1: Bounds (14), (17), (22), and (29) for different choices of f and g in (13).

(f, g)	Bound (14)	Bound (17)	Bound (22)	Bound (29)
$\left(e^{\|S\|}, \|S\|e^{-\|S\|}\right)$	22.7288	56.6787	40.1743	40.0320
$\left(e^{\|S\|^{1/2}}, \|S\|e^{-\|S\|^{1/2}}\right)$	3.7695	4.1934	3.6238	3.2146
$\left(\|S\|^{1/2}, \|S\|^{1/2}\right)$	3.4881	3.7078	3.3353	3.0645
$\left(e^{\|S\|^{1/3}}, \|S\|e^{-\|S\|^{1/3}}\right)$	3.3468	3.6354	3.2707	3.0389

Table 2: Bounds (14), (17), (22), and (29) for different choices of f and g in (13).

(f, g)	Bound (14)	Bound (17)	Bound (22)	Bound (29)
$\left(\|S\|^{1/3}, \|S\|^{2/3}\right)$	2.62245	2.37007943	2.27893615	2.1589862
$\left(\|S\|^{1/2}, \|S\|^{1/2}\right)$	2.5	2.2912878	2.1794494	2.0963298
$\left(e^{\|S\|^{1/3}}, \|S\|e^{-\|S\|^{1/3}}\right)$	2.5	2.29120815	2.1794075	2.09630510
$\left(e^{\|S\|^{1/2}}, \|S\|e^{-\|S\|^{1/2}}\right)$	2.5	2.29120547	2.17940617	2.09630427

Thus the preliminary induction step holds. Now, suppose that

$$\begin{aligned} w^2(S) \le \sum_{k=1}^{m} \frac{1}{4^k} \Big(&\left\| f\left(\left|\Delta_{f,g}^{k-1} S\right|\right) \right\| \left\| \Delta_{f,g}^{k} S \right\| \left\| g\left(\left|\Delta_{f,g}^{k-1} S\right|\right) \right\| \\ &+ \left\| \left(\Delta_{f,g}^{k-1} S\right)^* \left(\Delta_{f,g}^{k-1} S\right) + \left(\Delta_{f,g}^{k-1} S\right)\left(\Delta_{f,g}^{k-1} S\right)^* \right\| \Big) \\ &+ \frac{1}{4^m} w^2\left(\Delta_{f,g}^{m} S\right) \text{ for some } m \in \mathbb{N}. \end{aligned} \tag{42}$$

Then, another application of Theorem 4 yields

$$\begin{aligned} w^2(S) \le \sum_{k=1}^{m} \frac{1}{4^k} \Big(&\left\| f\left(\left|\Delta_{f,g}^{k} S\right|\right) \right\| \left\| \Delta_{f,g}^{k+1} S \right\| \left\| g\left(\left|\Delta_{f,g}^{k} S\right|\right) \right\| \\ &+ \left\| \left(\Delta_{f,g}^{k} S\right)^* \left(\Delta_{f,g}^{k} S\right) + \left(\Delta_{f,g}^{k} S\right)\left(\Delta_{f,g}^{k} S\right)^* \right\| \Big) \\ &+ \frac{1}{4} \Big(\left\| f\left(\left|\Delta_{f,g}^{m} S\right|\right) \right\| \left\| \Delta_{f,g}^{m+1} S \right\| \left\| g\left(\left|\Delta_{f,g}^{m} S\right|\right) \right\| \\ &+ \left\| \left(\Delta_{f,g}^{m} S\right)^* \left(\Delta_{f,g}^{m} S\right) + \left(\Delta_{f,g}^{m} S\right)\left(\Delta_{f,g}^{m} S\right)^* \right\| \Big) \\ &+ \frac{1}{4^{m+1}} w\left(\left(\Delta_{f,g}^{m+1} S\right)^2\right). \end{aligned} \tag{43}$$

Simplifying and using the inequality $w(S^2) \le w^2(S)$ gives

$$\begin{aligned} w^2(S) \le \sum_{k=1}^{m+1} \frac{1}{4^k} \Big(&\left\| f\left(\left|\Delta_{f,g}^{k} S\right|\right) \right\| \left\| \Delta_{f,g}^{k+1} S \right\| \left\| g\left(\left|\Delta_{f,g}^{k} S\right|\right) \right\| \\ &+ \left\| \left(\Delta_{f,g}^{k} S\right)^* \left(\Delta_{f,g}^{k} S\right) + \left(\Delta_{f,g}^{k} S\right)\left(\Delta_{f,g}^{k} S\right)^* \right\| \Big) \\ &+ \frac{1}{4^{m+1}} w^2\left(\Delta_{f,g}^{m+1} S\right). \end{aligned} \tag{44}$$

Hence, the assertion (39) holds for all $n \in \mathbb{N}$.

Now, using the inequality $w(S) \le \|S\|$ in (39) and then using the hypothesis, we get the desired inequality (38). □

Remark 7. It is easy to observe from the Theorems 3–6 that the upper bounds (17)–(38) are generalized bounds. Indeed, if we take $f(|S|) = |S|^{\lambda}$ and $g(|S|) = |S|^{1-\lambda}$ for $\lambda \in [0, 1]$, in the bounds (17)–(38), then we obtain bounds (9)–(12).

3. Examples

In this section, we shall consider some choices of f and g in generalized Aluthge transform (13) and use them to compute upper bounds of numerical radius for some matrices.

Example 8. Given $S = \begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$. Then, $S = U|S|$ is a polar decomposition of S, where $|S| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{29} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $U = \begin{pmatrix} 0 & 5/\sqrt{29} & 0 \\ 0 & 0 & 1 \\ 0 & 2/\sqrt{29} & 0 \end{pmatrix}$ is partial isometry. The bounds (14), (17), (22), and (29) are computed for some choices of f and g in (13) for given S in Table 1.

The numerical radius of S is

$$w(S) = 2.9154. \tag{45}$$

Example 9. Let $S = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix}$. Then, $S = U \mid S \mid$ is a polar decomposition of S, where $|S| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, and $U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is partial isometry.

The bounds (14), (17), (22), and (29) are computed for some choices of f and g in (13) for given S in Table 2.

The numerical radius of S is

$$w(S) = 2. \tag{46}$$

4. Conclusion

Summarizing the investigation carried out, we note that generalized Aluthge transform (13) with additional conditions (i) and (ii) is useful in achieving the generalized upper bounds for numerical radius. It is proved in Theorems 3, Theorem 4, Theorem 5, and Theorem 6 that bounds (17), (22), (29), and (38) are upper bounds of numerical radius that generalize the upper bounds (9), (10), (11), and (12) of numerical radius already existing in the literature. Theoretical investigations are supported by examples in which computations are carried out for finding bounds (14), (17), (22), and (29) of numerical radius for some choices of the pair f, g in the generalized Aluthge transform $\Delta_{f,g}$. Examples 8 and 9 demonstrate that generalized Aluthge transform provides a wide range of transforms that may be used as a tool to compute the upper bounds for numerical radius. These results might be helpful in studying perturbation, convergence, iterative solution methods, and integrative methods, which is the subject of future work. In the future, we also have a plan to investigate the lower bounds of numerical radius.

Data Availability

There is no need of any data for this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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