Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2913587, 9 pages
https://doi.org/10.1155/2022/2913587

Research Article

A Nonlinear Fractional Problem with a Second Kind Integral Condition for Time-Fractional Partial Differential Equation

Benbrahim Abdelouahab,[1] Taki-Eddine Oussaeif,[1] Adel Ouannas,[1] Khaled M. Saad,[2] Hadi Jahanshahi,[3] Ahmed Diar,[1] Awad M. Aljuaid,[4] and Ayman A. Aly[5]

[1]*Department of Mathematics and Computer Science, University of Larbi Ben M'hidi, Oum El Bouaghi, Algeria*
[2]*Department of Mathematics, Faculty of Applied Science, Taiz University, Taiz, Yemen*
[3]*Department of Mechanical Engineering, University of Manitoba, Winnipeg, Canada R3T 5V6*
[4]*Department of Industrial Engineering, College of Engineering, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia*
[5]*Department of Mechanical Engineering, College of Engineering, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia*

Correspondence should be addressed to Khaled M. Saad; khaledma_sd@hotmail.com

Received 14 September 2021; Revised 28 December 2021; Accepted 4 April 2022; Published 30 April 2022

Academic Editor: Hemant Kumar Nashine

The aim of this research is to demonstrate the existence and the uniqueness of the weak solution for a semilinear fractional parabolic problem with the special case of the second integral boundary condition. For this aim, we split the proof into two parts; to study the main linear problem part, we used the variable separation method, and concerning the semilinear problem part, we apply an iterative method and a priori estimate for the study of the weak solution.

1. Introduction

Fractional calculus is a mathematical analysis branch that explores the various different possibilities of describing the differentiation operator power of real or complex numbers. Fractional differential equations are extraordinary differential equations [1]. Many natural phenomena and modern problems of physics, mechanics, biology and technology, chemistry, engineering, etc. can be modeled by fractional partial differential equations (FPDEs). For details, see ([2–12]) and the references therein.

The first who drew attention to these problems with an integral one-space variables condition is Cannon [13], which gold of the study of heat conduction in a bar heated thin is demonstrated by using the potential method, and the importance of the problems with integral conditions has been pointed out by Samarskii [14]. The basic physical meaning of integral conditions (total energy, average temperature, the total mass of impurities, total flux, moments, etc.) has served as the main reason for the growing interest in this kind of problem.

In modern physics and technology, many problems are found using nonlocal conditions for partial differential equations, which are defined using integral conditions. So, the first type of integral condition is given by the following:

$$\int_0^1 k(x,t)u(x,t)dx = E(t), \tag{1}$$

where k is a given function.

Or second type

$$\begin{aligned} u(l,t) &= \int_0^1 k(x,t)u(x,t)dx, \quad \forall t \in (0,T), \\ \frac{\partial u}{\partial x}(l,t) &= \int_0^1 k(x,t)u(x,t)dx, \quad \forall t \in (0,T), \end{aligned} \tag{2}$$

can be used when it is impossible to directly measure the quantity sought on the border. Its total value or its average is known. We find a lot of research studied this kind of integral like [15–17].

Many researchers have widely studied the solvability "existence and uniqueness" of solutions of fractional and integer partial differential equation problem by many method; see, for example, [18–23].

There are a few works that study the nonlinear fractional partial differential equation with initial and boundary conditions, not to mention the nonlocal problem like [24].

In this work, we developed the study of fractional problems to partial differential equations on the one hand and beyond this in the nonlinear direction, which simulates heat diffusion in the complex phenomena. We also relied on the modeling of Neumann's condition, i.e., heat flow by an integral condition, which is closer to reality in this situation.

Motivated by this, in our article, we treat for the existence and the uniqueness of the weak solution of the mixed semilinear problem for a fractional partial differential equation with an integral condition of the second type, where we start by studying the nonlocal linear problem by the variable separation method, to demonstrate the existence and the uniqueness of the weak solution of the semilinear problem; we then apply an iterative method based on the results obtained by the linear problem.

2. Preliminaries

Definition 1 (see [25, 26]). For any $0<\alpha<1$, the Caputo and Riemann Liouville derivatives are defined, respectively, as follows:

(1) The left Caputo derivatives:

$$ {}_0^C D_t^\alpha u(\mathrm{x},\mathrm{t}) := \frac{1}{\Gamma(\alpha-1)}\int_0^t \frac{\partial u(\mathrm{x},\mathrm{t})}{\partial_\Gamma}\frac{1}{(\mathrm{t}-\tau)^\alpha}d\tau \tag{3}$$

(2) The left Riemann-Liouville derivatives:

$$ {}_0^R D_t^\alpha u(\mathrm{x},\mathrm{t}) := \frac{1}{\Gamma(\alpha-1)}\frac{\partial}{\partial t}\int_0^t \frac{u(\mathrm{x},\mathrm{t})}{\partial\tau}\frac{1}{(\tau-\mathrm{t})^\alpha}d\tau \tag{4}$$

(3) The right Caputo derivatives:

$$ {}_0^R D_t^\alpha u(\mathrm{x},\mathrm{t}) := \frac{-1}{\Gamma(\alpha-1)}\int_0^t \frac{\partial u(\mathrm{x},\mathrm{t})}{\partial\tau}\frac{1}{(\tau-\mathrm{t})^\alpha}d\tau \tag{5}$$

(4) The right Riemann-Liouville derivatives:

$$ {}_t^R D_t^\alpha u(\mathrm{x},\mathrm{t}) := \frac{1}{\Gamma(\alpha-1)}\frac{\partial}{\partial t}\int_0^t \frac{u(\mathrm{x},\mathrm{t})}{(\tau-\mathrm{t})^\alpha}d\tau \tag{6}$$

Proposition 2 (see [25, 26]). *For $n=1$, we have*

$$\begin{aligned} {}_0^R D_t^\alpha u(x,t) &= {}_0^c D_t^\alpha u(x,t) + \frac{u(0)}{\Gamma(1-\alpha)t^\alpha}, \\ {}_0^R D_t^\alpha u(x,t) &= {}_0^c D_t^\alpha u(x,t) + \frac{u(T)}{\Gamma(1-\alpha)(T-t)^\alpha}. \end{aligned} \tag{7}$$

If $u(x,0)=0$, then we have

$$ {}_0^R D_t^\alpha u(x,t) = {}_0^c D_t^\alpha u(x,t). \tag{8}$$

Definition 3 (see [27]). For any real $\sigma>0$ and finite interval $[a,b]$ of the real axis R, we define the seminorm

$$|u|^2_{l\,H^\alpha(\Omega)} := \left\| {}^R D_t^\alpha u \right\|^2_{L_2(\Omega)}, \tag{9}$$

and norm

$$\|u\|_{lH^\alpha(\Omega)} := \left(\|u\|^2_{L_2(\Omega)} + |u|^2_{l\,\mathrm{H}_0^\alpha(\Omega)} \right)^{1/2}. \tag{10}$$

We then define ${}^l H_0^\alpha(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{lH_0^\alpha}(\Omega)$.

Definition 4 (see [27]). For any real $\sigma>0$, we define the seminorm

$$|u|^2_{r\,H_0^\alpha(\Omega)} := \left\| {}_t^R D_T^\alpha u \right\|^2_{L2(\Omega)}, \tag{11}$$

and norm

$$\|u\|_{rH_0^\sigma(\Omega)} := \left(\|u\|^2_{L2(\Omega)} + |u|^2_{r\,H_0^\sigma(\Omega)} \right)^{1/2}. \tag{12}$$

We then define ${}^R H_0^\alpha(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{rH_0^\alpha(\Omega)}$.

Definition 5. For any real $\sigma>0$, we define the seminorm

$$|u|_{cH^\alpha(\Omega)} = \left| \frac{\left({}^R D_t^\alpha u, {}_t^R D^\alpha u \right)_{L^2(\Omega)}}{\cos(\alpha\pi)} \right|^{1/2}, \tag{13}$$

and norm

$$\|u\|_{cH^\alpha(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + |u|^2_{c\mathrm{H}^\alpha(\Omega)} \right)^{1/2}. \tag{14}$$

Lemma 6 (see [27, 28]). *For any real $\sigma\in\mathbb{R}_+$ if $u\in{}^l H^\alpha(\Omega)$ and $v\in C_0^\infty(\Omega)$, then*

$$\left({}^R D_t^\alpha u(t), v(t) \right)_{L^2(\Omega)} = \left(u(t), {}_T^R D^\alpha v(t) \right)_{L^2(\Omega)}. \tag{15}$$

Lemma 7 (see [27, 28]). *For* $0 < \sigma < 2, \sigma \neq 1, u \in H_0^{\sigma/2}(\Omega)$ *on a*

$$ {}^{R}D_t^{\sigma} u(t) = {}^{R}D_t^{\sigma/2R} D_t^{\sigma/t} u(t). \tag{16}$$

Lemma 8 see ([27, 28]). *For* $\sigma \in \mathbb{R}_+, \sigma \neq n + 1/2$, *the semi-norms* $|\cdot|_{lH^\sigma(\Omega)}, |\cdot|_{rH^\sigma(\Omega)}$ *and* $|\cdot|_{cH^\sigma(\Omega)}$ *are equivalent. Then, we pose*

$$|\cdot|_{lH^\sigma(\Omega)} \cong |\cdot|_{rH^\sigma(\Omega)}. \tag{17}$$

Lemma 9 see ([27, 28]). *For any real* $\sigma > 0$, *the space* ${}^{R}H_0^{\sigma}(\Omega)$ *with respect to the norm (12) is complete.*

3. Formulation of the Nonlinear Problem (18)

Let $Q = \{(x, t) \in \mathbb{R}^2 \text{with } 0 < x < 1 \text{and } 0 < t < T\}$.

We would like to deal with the following nonlinear problem:

$$\begin{cases} {}_0^cD_t^{\alpha} u(x,t) - a\dfrac{\partial^2 u(x,t)}{\partial x^2} + bu(x,t) = f\left(x, t, u, \dfrac{\partial u}{\partial x}\right), & \forall (x,t) \in \mathrm{Q}, \\ u(\mathrm{x}, 0) = \varphi(x), & \forall x \in (0,1), \\ u(0,t) = 0, & \forall t \in (0,1), \\ \dfrac{\partial u}{\partial x}(1,t) = \displaystyle\int_0^1 u(x,t)dx, & \forall t \in (0,\mathrm{T}), \end{cases} \tag{18}$$

where $0 < \alpha < 1$ and $a, b \in \mathbb{R}_+^*$.

In the remainder of this section, we assume $f \in L^2(Q)$ to be Lipschitz; i.e., there is a constant $k > 0$ as

$$\begin{aligned} &\| f(x,t,u,u_x) - f(x,t,v,v_x)\|_{L^2(Q)} \\ &\quad \leq k\left(\|u - v\|_{\mathrm{L}^2(\mathrm{Q})} + \|u_x - v_x\|_{\mathrm{L}^2(\mathrm{Q})} \right), \quad \forall u, u_x, v,_x \in L^2(Q). \end{aligned} \tag{19}$$

4. Study of the Associated Linear Problem

4.1. Position of the Associated Linear Problem (20). In $Q = (0,1) \times (0,T)$, with $T < \infty$, we examine the following associated linear problem:

$$\begin{cases} {}_0^cD_t^{\alpha} u(x,t) - a\dfrac{\partial^2 u(x,t)}{\partial x^2} + bu(x,t) = f(x,t), & \forall (x,t) \in \mathrm{Q}, \\ u(x, 0) = \varphi(x), & \forall x \in (0,1), \\ u(0,t) = 0, & \forall t \in (0,\mathrm{T}), \\ \dfrac{\partial u}{\partial x}(1,t) = \displaystyle\int_0^1 u(x,t)dx, & \forall t \in (0,\mathrm{T}), \end{cases} \tag{20}$$

which can be written in this operational form:

$$Lu = {}_0^cD_t^{\alpha} u(x,t) - a\frac{\partial^2 u(x,t)}{\partial x^2} + bu(x,t) = f(x,t). \tag{21}$$

With the initial condition

$$\ell u = u(x,0) = \varphi(x), \quad x \in (0,1), \tag{22}$$

Dirichlet boundary condition

$$u(0,t) = 0, \quad t \in (0,T), \tag{23}$$

and the integral condition of the second kind

$$\partial u \frac{(1,t)}{\partial x} = \int_0^1 u(x,t)dx, \quad t \in (0,T). \tag{24}$$

4.2. Resolution of Problem (20) by the Variable Separation Method. Let the following associated homogeneous linear problem:

$$\begin{cases} {}_0^cD_t^{\sigma} u(u,t) - a\dfrac{\partial^2 u(x,t)}{\partial x^2} + bu(x,t) = 0, & \forall (x,t) \in Q, \\ u(x,0) = \varphi(x), & \forall x \in (0,1), \\ u(0,t) = 0, & \forall t \in (0,\mathrm{T}), \\ \dfrac{\partial u(1,t)}{\partial x} = \displaystyle\int_0^1 u(x,t)dx, & \forall t \in (0,\mathrm{T}). \end{cases} \tag{25}$$

We can certainly show that problem (25) admits a unique solution, for that we put

$$u(x,t) = X(x)Y(\mathrm{t}). \tag{26}$$

Substituting (26) into (25), we obtain

$$\begin{cases} {}_0^cD_t^{\alpha} Y.X - aX^{''}Y + bXY = 0, \\ X(x)Y(0) = \varphi(x), \\ X(0)\mathrm{Y}(t), \\ X'(1)\mathrm{Y}(t) = \displaystyle\int_0^1 X(x)Y(t). \end{cases} \tag{27}$$

Therefore, we get for $\lambda > 0$

$$\frac{{}_0^cD_t^{\alpha} Y}{Y} = a\frac{X^{''}}{X} - b = -\lambda. \tag{28}$$

We begin by showing a Sturm-Liouville problem:

$$\begin{cases} X^{''}(x) + \dfrac{\mu}{a}X(x) = 0, \\ X(0) = 0, \\ X'(1) = \displaystyle\int_0^1 X(x)dx. \end{cases} \tag{29}$$

With $\mu = b - \lambda$, it has a solution which is given by

$$X(x) = A\cos\sqrt{\frac{\mu}{a}x + B\sin\sqrt{\frac{\mu}{a}}x}. \tag{30}$$

Or A and B are two real arbitrary.
Using the Dirichlet condition, we find

$$A = 0. \tag{31}$$

It follows that

$$X(x) = B\sin\sqrt{\frac{\mu}{a}}x. \tag{32}$$

Now, we use the integral condition, and we get

$$X'(1) = \mathrm{B}\sqrt{\frac{\mu}{a}}\cos\sqrt{\frac{\mu}{a}} = \int_0^1 X(x)dx = \int_0^1 B\sin\sqrt{\frac{\mu}{a}}xdx - \frac{B}{\sqrt{\mu/a}}\cos\sqrt{\frac{\mu}{a}}x\bigg|_0^1 = -\frac{B}{\sqrt{\mu/a}}\cos\sqrt{\frac{\mu}{a}} + \frac{B}{\sqrt{\mu/a}}. \tag{33}$$

Hence, we can assert that the eigenvalue is given by the following equation:

$$\sqrt{\mu} = \sqrt{a} \quad \text{across}\,\frac{a}{\mu + a}. \tag{34}$$

Our next goal is to determine the explicit form of $Y(t)$. According to the superposition theorem, we pose

$$u(x,t) = \sum_{n\geq 1} Xn(x)\cdot Yn(t). \tag{35}$$

Replacing (35) by (20) leads to

$$\sum_{n\geq 1}\left({}_0^c\mathrm{D}_t^\alpha \mathrm{Y}_\mathrm{n} + (b-\mu_n)\mathrm{Y}_\mathrm{n}\right)\cdot\sin\sqrt{\frac{\mu_n}{a}}x = \sum_{n\geq 1}\sin\left(\sqrt{\frac{\mu_n}{a}}\mathrm{x}\right)\cdot f_n(t), \quad \forall n\in\mathbb{N}, \tag{36}$$

which implies

$${}_0^cD_t^\alpha Y_n(t) + (\mathrm{b}-\mu_n)\mathrm{Y}_n(t) = f_n(t). \tag{37}$$

As

$$u(x,0) = \sum_{n\geq 1}\sin\left(\sqrt{\frac{\mu_n}{a}}x\right)Y_n(0) = \varphi(x) = \sum_{n\geq 0}\varphi_n\cdot\sin\left(\sqrt{\frac{\mu_n}{a}}x\right), \tag{38}$$

then

$$\varphi_n\int_0^1 \varphi(x)\cdot\sin\left(\sqrt{\frac{\mu_n}{a}x}\right)dx, \tag{39}$$

hence

$$Y_n(0) = \varphi_n. \tag{40}$$

We can then solve in a simple way the fractional ordinary differential problem (37)–(40) using the Laplace transform method; so it comes

$$L\left({}_0^cD_t^\alpha Y_n(t)\right) = \frac{sF(s) - (Y_n(0))}{s^{1-\alpha}}, \tag{41}$$

then

$$\varphi_n = \frac{sF(s) - Y_n(0)}{s^{1-\alpha}} + (b) - \mu_n F(s) = L(f_n(t)). \tag{42}$$

Therefore,

$$F(s) = \frac{1}{s^\alpha + b - \mu_n}\left[s^{\alpha-1}\varphi_n + L(f_n(t))\right]. \tag{43}$$

Finally, we obtain

$$Y_n(t) = L^{-1}\left[\frac{1}{s^\alpha + b - \mu_n}\left(s^{\alpha-1}\varphi_\mathrm{n} + L(f_n(t))\right)\right]. \tag{44}$$

Hence, we obtain the solution of (25) in the following explicit form:

$$\sum_{n\geq 0}\left(B_n\sin\left(\sqrt{\frac{\mu_n}{a}x}\right)\right)\cdot\left(L^{-1}\left[\frac{1}{s^\alpha + b - \mu_n}\left(s^{\alpha-1}\varphi_\mathrm{n} + L(f_n(t))\right)\right]\right). \tag{45}$$

5. Solvability of the Weak Solution of the Nonlinear Problem (18)

The key point to discuss in this section is to study the existence and uniqueness of the problem's weak solution (18). The key idea is to apply an iterative method and estimate a priori.

In this way, we define an auxiliary problem with a homogeneous equation:

$$\begin{cases} {}_O^cD_t^\alpha u(x,t) - a\dfrac{\partial^2 u(x,t)}{\alpha x^2} + by(x,t) = G(x,t,y,yx), & \forall (x,t) \in Q, \\ u(x,0) = \varphi(x), & \forall x \in (0,1), \\ u(0,t) = 0, & \forall t \in (0,\mathrm{T}), \\ \dfrac{\partial u(1,t)}{\partial x} = \displaystyle\int_0^1 u(x,t)dx, & \forall t \in (0,\mathrm{T}). \end{cases} \tag{46}$$

If u is a solution of (18) and v is a solution to (46), then $y = u - v$ satisfies

$$\mathscr{L}y = {}_0^cD_t^\alpha y(x,t) - a\frac{\partial^2 y(x,t)}{\partial x^2} + by(x,t) = G(x,t,y,y_x), \tag{47}$$

$$y(x,0) = 0, \quad \forall x \in (0,1), \tag{48}$$

$$y(0,t) = 0, \quad \forall t \in (0,\mathrm{t}), \tag{49}$$

$$\frac{\partial y(1,t)}{\partial} = 0, \quad \forall t \in (0,\mathrm{t}), \tag{50}$$

or $G(x,t,y,y_x) = f(x,t,u+v,u_x+v_x)$. From now on, we make the assumption: the function G is Lipschitz, that is, there is a positive constant k as

$$\begin{aligned} &\left\|G(x,t,y_1,(y_1)_x) - G(x,t,y_2,(y_2)_x)\right\|_{L^2(Q)} \\ &\quad \le k\left(\|y_1 - y_2\|_{L^2(Q)} + \left\|(y_1)_x - (y_2)_x\right\|_{L^2(Q)}\right), \quad \forall y_1,(y_1)_x, y_2,(y_2)_x \in L^2(Q). \end{aligned} \tag{51}$$

In order to get the desired results, it is necessary to propose the concept of the studied solution.

Let $\vartheta = \vartheta(x,t)$ arbitrary function of V as

$$V = \left\{\vartheta \in C^1(Q), \vartheta(0,t) = \frac{\partial}{\partial x}\vartheta(1,t) = 0, t \in [0,T]\right\}. \tag{52}$$

Multiply 9 by ϑ and integrate it on Q_τ; we find

$$\begin{aligned} &\int_{Q_\tau} {}_0^cD_t^\alpha y(x,t)\cdot\vartheta(x,t)dxdt - \int_{Q_\tau} a\frac{\partial^2 y}{\partial x^2}(x,t)\cdot\vartheta(x,t)dxdt \\ &\quad + \int_{Q_\tau} by(x,t)\cdot\vartheta(x,t)dxdt = \int_{Q_\tau} G(x,t,y,y_x)\cdot\vartheta(x,t)dxdt. \end{aligned} \tag{53}$$

By the use of integration by parts and the conditions on y, ϑ, we get

$$\begin{aligned} &\int_{Q_\tau} {}_0^RD_t^\alpha y(x,t)\cdot\vartheta(x,t)dxdt + \int_{Q_\tau} a\frac{\partial y}{\partial x}(x,t)\cdot\frac{\partial\vartheta}{\partial x}(x,t)dxdt \\ &\quad + \int_{Q_\tau} by(x,t)\cdot\vartheta(x,t)dxdt = \int_{Q_\tau} G(x,t,y,y_x)\cdot\vartheta(x,t)dxdt. \end{aligned} \tag{54}$$

Then, (54) establishes the following formula:

$$A(y,\vartheta) = \int_{Q_\tau} G(x,t,y,y_x)\cdot\vartheta(x,t)dxdt, \tag{55}$$

or

$$\begin{aligned} A(y,\vartheta) = &\int_{Q_\tau} {}_0^RD_t^\alpha y(x,t)\cdot\vartheta(x,t)dxdt \\ &+ \int_{Q_\tau} a\frac{\partial y}{\partial x}(x,t)\cdot\frac{\partial\vartheta}{\partial x}(x,t)dxdt \\ &+ \int_{Q_\tau} by(x,t)\cdot\vartheta(x,t)dxdt. \end{aligned} \tag{56}$$

Definition 10. We say the solution to problems (47)–(50) is weak; any function $y \in L^2(0,T;H^1(0,1))$ such as (33), (49), and (50) are achieved.

Now, we build a recurring sequence defined as follows: from $y^{(n-1)}$ we can define $(y^{(n)})_{n\in\mathbb{N}}$, where the first element is given by $y^{(0)} = 0$; then, for $n = 1,2,3,\cdots$, the following problem is solved:

$$\begin{cases} {}_0^RD_t^\alpha y^{(n)}(x,t) - a\dfrac{\partial^2 y^{(n)}(x,t)}{\partial x^2} + by^{(n)}(x,t) = G\left(x,t,y^{(n-1)},(y)_x^{(n-1)}\right), \\ y^{(n)}(x,0) = 0, \\ y^{(n)}(0,t) = 0, \\ \dfrac{\partial y^{(n)}(1,t)}{\partial x} = 0. \end{cases} \tag{57}$$

For any n fixed, thanks to the study of (20) which we gave the solution explicitly using the variable separation method; then, problem (57) has a unique solution $y^{(n)}(x,t)$.

Now, suppose $z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t)$; so we get a new problem:

$$\begin{cases} {}_0^RD_t^\alpha z^{(n)}(x,t) - a\dfrac{\partial^2 z^{(n)}(x,t)}{\partial x^2} + bz^{(n)}(x,t) = p^{(n-1)}(x,t), \\ z^{(n)}(x.0) = 0, \\ z^{(n)}(0,t) = 0, \\ \dfrac{\partial z^{(n)}(1,t)}{\partial x} = 0, \end{cases} \tag{58}$$

or

$$p^{(n-1)}(x,t) = G\left(x,t,y^{(n)},y_x^{(n)}\right) - G\left(x,t,y^{(n-1)},y_x^{(n-1)}\right). \tag{59}$$

Lemma 11. *We assume that (51) be satisfied. So we obtain for problem (58) the following a priori estimate:*

$$\left\| z^{(n)} \right\|_{L^2\left(0,T,H^1(0,1)\right)} \le c \left\| z^{(n-1)} \right\|_{L^2\left(0,T,H^1(0,1)\right)}, \tag{60}$$

or

$$c = \sqrt{\frac{k^2}{2\varepsilon \min\left(a, b-(\varepsilon/2)\right)}}. \tag{61}$$

Proof. We multiply by $z^{(n)}$ this equation

$${}_0^R D_t^\alpha z^{(n)}(x,t) - a\frac{\partial^2 z^{(n)}(x,t)}{\partial x^2} + b z^{(n)}(x,t) = p^{(n-1)}(x,t), \tag{62}$$

and we integrate it on Q_τ; we get

$$\begin{aligned} &\int_{Q_\tau} {}_0^R D_t^\alpha z^{(n)}(x,t)\cdot z^{(n)}(x,,t)dxdt \\ &\quad - \int_{Q_\tau} a\frac{\partial^2 z^{(n)}(x,t)}{\partial x^2}\cdot z^{(n)}(x,t)dxdt \\ &\quad + \int_{Q_\tau} b\left(z^{(n)}(x,t)\right)^2 dxdt = \int_{Q_\tau} p^{(n-1)}(x,t)\cdot z^{(n)}(x,t)dxdt. \end{aligned} \tag{63}$$

Performing integration by part and using lemma 2, 3, 4, and 5 by the same way in the articles [15–24], we obtain

$$\begin{aligned} &\int_0^1 \left({}_0^R D_t^{\alpha/2} z^{(n)}(x,\tau)\right)^2 dx + \int_{Q_\tau} a\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt \\ &\quad + \int_{Q_\tau} b\left(z^{(n)}(x,t)\right)^2 dxdt = \int_{Q_\tau} p^{(n-1)}(x,t)\cdot z^{(n)}(x,t)dxdt. \end{aligned} \tag{64}$$

Using Cauchy with $\varepsilon-$ inequality, we get

$$\begin{aligned} &\int_0^1 \left({}_0^R D_t^{\alpha/2} z^{(n)}(x,\tau)\right)^2 dx + \int_{Q_t} a\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt \\ &\quad + \int_{Q_t} b\left(z^{(n)}(x,t)\right)^2 dxdt \le \frac{1}{2\varepsilon}\int_{Q_t} a\left(p^{(n-1)}\right)^2 dxdt \\ &\quad + \int_{Q_t} \left(z^{(n)}(x,t)\right)^2 dxdt. \end{aligned} \tag{65}$$

Exploiting the fact that

$$\left|p^{n-1}(x,t)\right|^2 = \left|G\left(x,t,y^{(n)},y_x^{(n)}\right) - G\left(x,t,y^{(n-1)},y_x^{(n-1)}\right)\right|^2, \tag{66}$$

we conclude from (51) that

$$\begin{aligned} \int_{Q_\tau} \left|p^{n-1}(x,t)\right|^2 &\le \int_{Q_\tau} k^2\left(\left|y^{(n)} - y^{(n-1)}\right| + \left|y_x^{(n)} - y_x^{(n-1)}\right|\right)^2 \\ &= \int_{Q_\tau} k^2\left(\left|z^{(n-1)}\right|^2 + \left|z_x^{(n-1)}\right|^2\right) \\ &\le k^2 \left\| z^{(n-1)}(x,t)\right\|^2_{L^2\left(0,T;H^1(0,1)\right)}. \end{aligned} \tag{67}$$

Eliminating the first term from (64), we get

$$\begin{aligned} &b\int_{Q_\tau} \left(z^{(n)}(x,\tau)\right)^2 dx + a\int_{Q_\tau}\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt \\ &\quad \le \int_0^1 \left({}_0^R D_t^{\alpha/2} z^{(n)}(x,\tau)\right)^2 dx + \int_{Q_\tau} a\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt \\ &\quad + \int_{Q_\tau} b\left(z^{(n)}(x,t)\right)^2 dxdt \le \frac{1}{2\varepsilon}\int_{Q_\tau}\left(p^{(n-1)}\right)^2 dxdt, \end{aligned} \tag{68}$$

which enables us to deduce

$$\begin{aligned} &\left\| z^{(n)}(x,t)\right\|^2_{L^2\left(0,T;H^1(0,1)\right)} \\ &\quad \le \frac{k^2}{2\varepsilon \min\left(a, b-(\varepsilon/2)\%\right)}\left\| z^{(n-1)}(x,t)\right\|^2_{L^2\left(0,T;H^1(0,1)\right)}. \end{aligned} \tag{69}$$

□

Our next concern will be the convergence of the sequence $\left(y^{(n)}\right)_n$, so we are thus looking for the convergence of the series $\sum_{n=1}^{\infty} z^{(n)}$.

For this, we use the criterion of convergence of series which gives us

$$\sqrt{\frac{k^2}{2\varepsilon \min\left(a, b-(\varepsilon/2)\right)}} < 1, \tag{70}$$

Then,

$$k < \sqrt{2\varepsilon \min\left(a, b-(\varepsilon/2)\right)}. \tag{71}$$

As $z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t)$ and $y^{(0)}(x,t) = 0$, we have

$$\sum_{i=0}^{n-1} z^{(i)} = \sum_{i=0}^{n-1}\left(y^{(i+1)}(x,t) - y^{(i)}(x,t)\right) = y^{(1)} - y^{(0)} + y^{(2)} - y^{(1)} + \cdots + y^{(n)} - y^{(n-1)} = y^{(n)}. \tag{72}$$

Note that we have actually proved that the sequence $(y^{(n)})_n$ defined by

$$y^{(n)}(x,t) = \sum_{i=0}^{n-1} z^{(i)}, \tag{73}$$

is convergent to an element $y \in L^2(0, T, H^1(0,1))$. Having disposed of this preliminary step, the task is now to show that $\lim_{n\longrightarrow\infty} y^{(n)}(x,t) = y(x,t)$ is a solution of problems (47)–(50).

In order to get this result, it will be necessary to check that y satisfied

$$A(y, \vartheta) = \int_{Q_\tau} G(x,t,y,y_x)\cdot\vartheta(x,t)dxdt. \tag{74}$$

From (57), we get

$$\begin{aligned} A\left(y^{(n)}, \vartheta\right) = &\int_{Q_\tau} {}_0^R D_t^\alpha y^{(n)}(x,t)\cdot\vartheta(x,t)dxdt \\ &+ \int_{Q_\tau} a\frac{\partial y^{(n)}}{\partial x}(x,t)\cdot\frac{\partial\vartheta}{\partial x}(x,t)dxdt \\ &+ \int_{Q_\tau} by^{(n)}(x,t)\cdot\vartheta(x,t)dxdt. \end{aligned} \tag{75}$$

As A is linear, we get

$$\begin{aligned} A\left(y^{(n)}, \vartheta\right) &= A\left(y^{(n)} - y, \vartheta\right) + A(y, \vartheta) \\ &= \int_{Q_\tau} {}_0^R D_t^\alpha\left(y^{(n)} - y\right)(x,t)\cdot\vartheta(x,t)dxdt \\ &\quad + \int_{Q_\tau} a\frac{\partial\left(y^{(n)} - y\right)}{\partial x}(x,t)\cdot\frac{\partial\vartheta}{\partial x}(x,t)dxdt \\ &\quad \cdot \int_{Q_\tau} b\left(y^{(n)} - y\right)(x,t)\cdot\vartheta(x,t)dxdt \\ &\quad + \int_{Q_\tau} {}_0^R D_t^\alpha y(x,t)\cdot\vartheta(x,t)dxdt + \int_{Q_\tau} a\frac{\partial y}{\partial x}(x,t) \\ &\quad \cdot\frac{\partial\vartheta}{\partial x}(x,t)dxdt + \int_{Q_\tau} by(x,t)\cdot\vartheta(x,t)dxdt. \end{aligned} \tag{76}$$

Applying the Cauchy Schwartz inequality on $A(y^{(n)} - y, \vartheta)$, we get

$$\begin{aligned} &\int_{Q_\tau} {}_0^R D_t^\alpha\left(y^{(n)} - y\right)(x,t)\cdot\vartheta(x,t)dxdt + \int_{Q_\tau} a\frac{\partial\left(y^{(n)} - y\right)}{\partial x}(x,t) \\ &\cdot\frac{\partial\vartheta}{\partial x}(x,t)dxdt + \int_{Q_\tau} b\left(y^{(n)} - y\right)(x,t)\cdot\vartheta(x,t)dxdt \\ &\quad\le \max\left(1, a\gamma, b\right)\|\vartheta_x\|_{L^2(Q)} \\ &\cdot\left[\left\|{}_0^R D_t^\alpha\left(y^{(n)} - y\right)\right\|_{L^2\left(0,T,H^1(0,1)\right)} + \left\|\left(y^{(n)} - y\right)_x\right\|_{L^2\left(0,T,H^1(0,1)\right)}\right], \end{aligned} \tag{77}$$

where γ is Poincare's inequality constant. On the other hand, as

$$y^{(n)} \longrightarrow y \quad \text{dans} \quad \mathrm{L}^2\left(0, \mathrm{T}, \mathrm{H}^1(0,1)\right) \cong \mathrm{H}^1(\mathrm{Q}). \tag{78}$$

So

$$\begin{aligned} y^{(n)} &\longrightarrow y \quad \text{dans} \quad \mathrm{L}^2(\mathrm{Q}), \\ \mathrm{y}_\mathrm{x}^{(\mathrm{n})} &\longrightarrow \mathrm{y}_\mathrm{x} \quad \text{in} \quad \mathrm{L}^2(\mathrm{Q}). \end{aligned} \tag{79}$$

Letting $n \longrightarrow +\infty$, we find

$$\lim_{n\longrightarrow+\infty} A\left(y^{(n)} - y, \vartheta\right) = 0. \tag{80}$$

From (80) and going to the limit in (76), we obtain

$$\lim_{n\longrightarrow+\infty} A\left(y^{(n)}, \vartheta\right) = A(y, \vartheta). \tag{81}$$

Theorem 12. *If the condition (51) is satisfied and*

$$k < \sqrt{2\varepsilon\min\left(a, b - (\varepsilon/2)\right)}. \tag{82}$$

So problems (47)–(50) admit a weak solution which belongs to $L^2(0, T; H^1(0,1))$.

It remains to be proven that problems (47)–(50) admit a unique solution.

Theorem 13. *According to (51), the solution of problems (47)–(50) is unique.*

Proof. Suppose that y_1 and y_2 in $L^2(0, T; \%H^1(0,1))$ are two solutions of (47)–(50), then $Z = y_1 - y_2$ satisfied $Z \in L^2(0, T; H^1(0,1))$ and

$$\begin{aligned} \mathscr{L}Z = {}_0^R D_t^\alpha Z(x,t) - a\frac{\partial^2 Z(x,t)}{\partial x^2} + bZ(x,t) = \psi(x,t), &\quad \forall(x,t) \in \bar{Q}, \\ Z(x,0) = 0, &\quad \forall \mathrm{x} \in (0,1), \\ Z(0,t) = 0 &\quad \forall \mathrm{t} \in (0,\mathrm{t}), \\ \frac{\partial Z(1,t)}{\partial X} = 0 &\quad \forall \mathrm{t} \in (0,\mathrm{t}), \end{aligned} \tag{83}$$

Or

$$\psi(x,t) = G\left(x,t,y_1,(y_1)_x\right) - G\left(x,t,y_2,(y_2)_x\right). \quad (84)$$

Similar analysis to that in the proof of Lemma 11, it shows that

$$\|Z\|_{L^2\left(0,T;H^1(0,1)\right)} \le \|Z\|_{L^2\left(0,T;H^1(0,1)\right)}. \quad (85)$$

Or c is the same constant of Lemma 11. As $c<1$, so according to (85), it comes that

$$(1-c)\|Z\|_{L^2\left(0,T;H^1(0,1)\right)} \le 0. \quad (86)$$

We conclude that $y_1 = y_2$ in $L^2(0,T;H^1(0,1))$ is the desired conclusion. □

6. Conclusion

The study of fractional nonlinear parabolic problems is of great theoretical interest, because of new modeling with fractional derivatives which plays an important role in the description in the part of modeling in mathematics. Also, the use of the condition of Neumann in terms of integral condition which means the flux of the solution is an average gives a very good model of boundary condition. Then, there remains the numerical part as perspective in the fractional problem especially when there is a nonlinear term and integral condition of the second type.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The research was supported by the Taif University Researchers Supporting Project number (TURSP-2020/77), Taif, Saudi Arabia.

References

[1] D. Zwillinger, *Handbook of Differential Equations*, Elsevier Science, 2014.

[2] B. Ahmad and J. J. Nieto, "Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations," *Abstract and Applied Analysis*, vol. 2009, Article ID 494720, 9 pages, 2009.

[3] M. Belmekki, J. J. Nieto, and R. Rodriguez-Lopez, "Existence of periodic solution for a nonlinear fractional differential equation," *Boundary Value Problems*, vol. 2009, Article ID 324561, 18 pages, 2009.

[4] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[5] K. B. Oldham and J. Spainer, *The Fractional Calculus*, Academic Press, New York-London, 1974.

[6] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[7] H. Weitzner and G. M. Zaslavsky, "Some applications of fractional equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, pp. 935–945, 2010.

[8] D. Baleanu, O. G. Mustafa, and R. P. Agarwal, "On the solution set for a class of sequential fractional differential equations," *Journal of Physics A: Mathematical and Theoretical*, vol. 43, no. 38, article 385209, 2010.

[9] R. R. Nigmatullin, "To the theoretical explanation of the "universal response"," *physica status solidi (b)*, vol. 123, no. 2, pp. 739–745, 1984.

[10] R. R. Nigmatullin, "On the theory of relaxation for systems with "remnant" memory," *physica status solidi (b)*, vol. 124, pp. 389–393, 1984.

[11] Y. Z. Povstenko, "Thermoelasticity that uses fractional heat conduction equation," *Journal of Mathematical Sciences*, vol. 162, no. 2, pp. 296–305, 2009.

[12] Y. Z. Povstenko, "Theory of thermoelasticity based on the space-time-fractional heat conduction equation," *Physica Scripta*, vol. T136, article 014017, 2009.

[13] J. R. Cannon, "The solution of the heat equation subject to the specification of energy," *Quarterly of Applied Mathematics*, vol. 21, no. 2, pp. 155–160, 1963.

[14] A. A. Samarskii, "On some problems of current interest of the theory of differential equations," *Differentsial'nye Uravneniya [Differential Equations]*, vol. 16, no. 11, pp. 1221–1228, 1980.

[15] T. E. Oussaeif and A. Bouziani, "Existence and uniqueness of solutions to parabolic fractional differential equations with integral conditions," *Electronic Journal of Differential Equations*, vol. 2014, no. 179, pp. 1–10, 2014.

[16] B. Abdelfatah, O. Taki-Eddine, and B. A. Leila, "A mixed problem with an integral two-space-variables condition for parabolic equation with the Bessel operator," *Journal of Mathematics*, vol. 2013, Article ID 457631, 8 pages, 2013.

[17] D. Sofiane, B. Abdelfatah, and O. Taki-Eddine, "Study of solution for a parabolic integrodifferential equation with the second kind integral condition," *international Journal of analysis and Applications*, vol. 16, no. 4, pp. 569–593, 2018.

[18] N. J. Ford, J. Xiao, and Y. Yan, "A finite element method for time fractional partial differential equations," *Fractional Calculus and Applied Analysis*, vol. 14, no. 3, pp. 454–474, 2011.

[19] O. Zigen, "Existence and uniqueness of the solutions for a class of nonlinear fractional order partial differential equations with delay," *Computers and Mathematics with Applications*, vol. 61, no. 4, pp. 860–870, 2011.

[20] V. Daftardar-Gejji and H. Jafari, "Boundary value problems for fractional diffusion-wave equation," *Australian Journal of Mathematical Analysis and Applications*, vol. 3, pp. 1–8, 2006.

[21] R. Imad and O. Taki-Eddine, "Solvability of a solution and controllability of partial fractional differential systems," *Journal of Interdisciplinary Mathematics*, vol. 24, no. 5, pp. 1175–1200, 2021.

[22] R. Imad, O. Taki-Eddine, and B. Abdelouahab, "Solvability of a solution and controllability for nonlinear fractional differential equations," *bulletin of the institute of mathematics*, vol. 15, no. 3, pp. 237–249, 2020.

[23] B. Antara, O. Taki-Eddine, and R. Imad, "Unique solvability of a Dirichlet problem for a fractional parabolic equation using

energy-inequality method," *Methods of Functional Analysis and Topology*, vol. 26, no. 3, pp. 216–226, 2020.

[24] O. Taki-Eddine and B. Abdelfatah, "A priori estimates for weak solution for a time-fractional nonlinear reaction-diffusion equations with an integral condition," *Chaos, Solitons & Fractals*, vol. 103, pp. 79–89, 2017.

[25] A. A. Kilbas and H. M. Srivastava, *Theory and applications of fractional differential equations*, vol. 204, North-Holland Mathematics Studies, 2006.

[26] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integral and Derivative: Theory and Applications*, Gordan and Breach, Switzerland, 1993.

[27] X. J. Li and C. J. Xu, "A space-time spectral method for the time fractional diffusion equation," *SIAM Journal on Numerical Analysis*, vol. 47, no. 3, pp. 2108–2131, 2009.

[28] X. J. Li and C. J. Xu, "Existence and uniqueness of the weak solution of the space-time fractional diffusion equation and a spectral method approximation," *Communications in Computational Physics*, vol. 8, no. 5, pp. 1016–1051, 2010.