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Research Article

Fractional Version of Hermite-Hadamard and Fejér Type Inequalities for a Generalized Class of Convex Functions

Lei Geng[1] **Muhammad Shoaib Saleem**[2] **Kiran Naseem Aslam,**[2] **and Rahat Bano**[2]

[1]*School of Science, Henan Institute of Technology, Xinxiang 453003, China*
[2]*Department of Mathematics, University of Okara, Okara, Pakistan*

Correspondence should be addressed to Lei Geng; genglei@hait.edu.cn

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In the present paper, we deal with some fractional integral inequalities for strongly reciprocally (p, h)-convex functions. We established fractional version of Hermite-Hadamard and Fejér type inequalities for strongly reciprocally (p, h)-convex functions. Our results extend and generalize many exiting results of literate.

1. Introduction

The convex functions nowadays are widely used in many branches of mathematics like optimization theory, functional analysis, and modeling theory [1, 2]. The interesting geometry of convex functions makes its study distinct from other functions. From geometric point of view, a function $\psi(x)$ is convex provided that the line segment connecting any two points of its graph lies on or above the graph of function. The new inequalities in analysis are always appreciable [3, 4].

Since the classical convexity is not enough to attain certain goals in applied mathematics, so the classical convexity has been generalized in many directions. For recent generalizations, one can see [5, 6].

For the class of convex functions, various inequalities have been developed [7, 8], but the most famous inequality is Hermite-Hadamard's inequality. It is stated as follows:

Let $\psi : M \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function, and let d_1, $d_2 \in M$ with $d_1 d_2$; then, the following double inequality holds:

$$\psi\left(\frac{d_1 + d_2}{2}\right) \leq \frac{1}{d_2 - d_1}\int_{d_1}^{d_2} \psi(x)dx \leq \frac{\psi(d_1) + \psi(d_2)}{2}. \quad (1)$$

In [9], Fejér gave the weighted version of Hermite-Hadamard inequality (1) as follows:

Let $\psi : [d_1, d_2] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $w : [d_1, d_2] \longrightarrow \mathbb{R}$ a non-negative, integrable, and symmetric function about $(d_1 + d_2)/2$; then, the following inequality holds:

$$\psi\left(\frac{d_1 + d_2}{2}\right)\int_{d_1}^{d_2} w(x)dx \leq \int_{d_1}^{d_2} \psi(x)w(x)dx \leq \frac{\psi(d_1) + \psi(d_2)}{2}\int_{d_1}^{d_2} w(x)dx. \quad (2)$$

The aim of present paper is to establish fractional version of Hermite-Hadamard and Fejér type inequalities for a more generalized class of functions. The present paper is organized as follows: The §2 is concerned with some preliminary material. In Section 3, we give some basic results for strongly reciprocally (p,h)-convex functions, and in Sections 4 and 5, we develop Hermite-Hadamard and Fejér type inequalities, respectively, for strongly reciprocally (p,h)-convex function. Moreover, the last section is devoted for fractional integral inequalities.

2. Preliminaries

In this section, we give a brief review of some preliminaries.

Definition 1 (p-convex set see [10, 11]). A set $M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\}$ is p -convex set, if

$$\left[(jx^p + (1-j)y^p)^{1/p}\right] \in M, \tag{3}$$

for all $x, y \in M$, $j \in [0, 1]$, where $p = 2u + 1$ or $p = a/b, a = 2v + 1, b = 2w + 1$ and $u, v, w \in N$.

Definition 2 (p-convex function see [10]). Let $M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\}$ be a p -convex set. A function $\psi : M = [d_1, d_2] \longrightarrow \mathbb{R}$ is called p-convex function, if

$$\psi\left[(jx^p + (1-j)y^p)^{1/p}\right] \le j\psi(x) + (1-j)\psi(y) \in M, \tag{4}$$

holds for all $x, y \in M$ and $j \in [0, 1]$.

Definition 3 (Strongly convex function see [4]). A function $\psi : M = [d_1, d_2] \longrightarrow \mathbb{R}$ is called strongly convex function with modulus μ on M, where $\mu \ge 0$, if

$$\psi(jx + (1-j)y) \le j\psi(x) + (1-j)\psi(y) - \mu j(1-j)(y-x)^2, \tag{5}$$

holds for all $x, y \in M$ and $j \in [0, 1]$.

Definition 4 (Strongly p-convex function see [12]). A function $\psi : M = [d_1, d_2] \longrightarrow \mathbb{R}$ is called strongly p-convex function, if

$$\psi\left[(jx^p + (1-j)y^p)^{1/p}\right] \le j\psi(x) + (1-j)\psi(y) - \mu j(1-j)(y^p - x^p)^2, \tag{6}$$

holds for all $x, y \in M$ and $j \in [0, 1]$.

Definition 5 (Harmonic convex set see [11, 13]). A set $M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\}$ is said to be harmonic convex set, if

$$\frac{xy}{jx + (1-j)y} \in M, \tag{7}$$

for all $x, y \in M$ and $j \in [0, 1]$.

Definition 6 (Harmonic convex function see [11, 14]). Let $M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\}$ be the harmonic convex set. A function $\psi : M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is harmonic convex function, if

$$\psi\left(\frac{xy}{jx + (1-j)y}\right)(1-j)\psi(x) + j\psi(y), \tag{8}$$

holds for all $x, y \in M$ and $j \in [0, 1]$.

Definition 7 (p-harmonic convex set see [11, 15]). A set $M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\}$ is p -harmonic convex set, if

$$\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \in M, \tag{9}$$

for all $x, y \in M$ and $j \in [0, 1]$.

Definition 8 (p-harmonic convex function see [11, 15]). Let $M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\}$ be p-harmonic convex set. A function $\psi : M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is p-harmonic convex, if

$$\psi\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \le (1-j)\psi(x) + j\psi(y), \tag{10}$$

holds for all $x, y \in M$ and $j \in [0, 1]$.

Definition 9 (Strongly reciprocally convex function see [16]). Let M be an interval and let $\mu \in (0,\infty)$. A function $\psi : M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is said to be strongly reciprocally convex function with modulus μ on M, if

$$\psi\left(\frac{xy}{jx + (1-j)y}\right) \le (1-j)\psi(x) + j\psi(y) - \mu j(1-j)\left(\frac{1}{x} - \frac{1}{y}\right)^2, \tag{11}$$

holds for all $x, y \in M$ and $j \in [0, 1]$.

Definition 10 (Strongly reciprocally p-convex function see [17]). Let M be a p -harmonic convex set and let $\mu \in (0,\infty)$. A function $\psi : M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is said to be strongly reciprocally p-convex function with modulus μ on M, if

$$\psi\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \le (1-j)\psi(x) + j\psi(y) - \mu j(1-j)\left(\frac{1}{x^p} - \frac{1}{y^p}\right)^2, \tag{12}$$

holds for all $x, y \in M$ and $j \in [0, 1]$.

Definition 11 (h-convex function see [18]). Choose the functions $\psi, h : M = [d_1, d_2] \longrightarrow \mathbb{R}$ that are non-negative; then, ψ is called h-convex function, if

$$\psi(jx + (1-j)y) \le h(j)\psi(x) + h(1-j)\psi(y), \tag{13}$$

for all $x, y \in M$ and $j \in [0, 1]$.

Now, we are ready to introduce a new class of convex functions by generalizing the concept of strongly reciprocally p-convex functions, which we will call strongly reciprocally (p,h)-convex functions.

Definition 12 (Strongly reciprocally (p, h)-convex function). Let M be a p-harmonic convex set and let $\mu \in (0,\infty)$. A function $\psi : M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ is said to be strongly reciprocally (p, h)-convex function with modulus μ on M, if

$$\psi\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \le h(1-j)\psi(x) + h(j)\psi(y) - \mu j(1-j)\left(\frac{1}{x^p} - \frac{1}{y^p}\right)^2, \tag{14}$$

hold for all $x, y \in M$ and $j \in [0, 1]$.

Throughout the paper, for convenience, we represent the class of strongly reciprocally (p, h) convex functions by $SR(p, h)$.

Remark 13. Inserting $h(j) = j$ in Definition 12, we obtain Definition 10, and inserting $h(j) = j$ and $p = 1$, Definition 12 reduces to Definition 9.

Similarly, from Definition 12, Definitions 6 and 8 can be obtained by inserting $h(j) = j$, $\mu = 0$, $p = 1$, and $h(j) = j$ with $\mu = 0$, respectively.

3. Basic Results

This section collects some basic and straight forward facts based on algebraic operations.

The following proposition is concerned about the addition of two functions from $SR(p, h)$.

Proposition 14. *Let $\psi, \sigma : M \longrightarrow \mathbb{R}$ be two strongly reciprocally (p,h)-convex functions with modulus μ on M; then, $\psi + \sigma : M \longrightarrow \mathbb{R}$ is also strongly reciprocally (p,h)-convex function with modulus μ^* on M, where $\mu^* = 2\mu$.*

Proof. We will start by definition of strongly reciprocally (p,h)-convexity of ψ and σ:

$$\begin{aligned}(\psi+\sigma)\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p} &= \psi\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] + \sigma\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \\ &\le h(j)\psi(x) + h(1-j)\psi(y) - \mu j(1-j)\left(\frac{1}{y^p} - \frac{1}{x^p}\right)^2 \\ &\quad + h(j)\sigma(x) + h(1-j)\sigma(y) - \mu j(1-j)\left(\frac{1}{y^p} - \frac{1}{x^p}\right)^2,\end{aligned} \tag{15}$$

which in turns implies that

$$\begin{aligned}(\psi+\sigma)\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p} &\le h(j)(\psi+\sigma)(x) + h(1-j)(\psi+\sigma)(y) - 2\mu j(1-j)\left(\frac{1}{y^p} - \frac{1}{x^p}\right)^2 \\ &= h(j)(\psi+\sigma)(x) + h(1-j)(\psi+\sigma)(y) - \mu^* j(1-j)\left(\frac{1}{y^p} - \frac{1}{x^p}\right)^2,\end{aligned} \tag{16}$$

where $\mu^* = 2\mu$ and $\mu \ge 0$. This completes the proof. □

Our next result is concerned with the scalar multiplication of strongly reciprocally (p,h)-convex function.

Proposition 15. *Let $\psi : M \longrightarrow \mathbb{R}$ be a strongly reciprocally (p,h)-convex function; then, for any $\lambda \ge 0$, $\lambda\psi : M \longrightarrow \mathbb{R}$ is also strongly reciprocally (p,h)-convex function with modulus ν^* on M, where $\nu^* = \lambda\mu$.*

Proof. Let $\lambda \ge 0$, $\psi \in SR(p, h)$, we obtain

$$\begin{aligned}\lambda\psi\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] + \lambda\left[\psi\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right]\right] &\le \lambda\left[h(j)\psi(x) + h(1-j)\psi(y) - \mu j(1-j)\left(\frac{1}{y^p} - \frac{1}{x^p}\right)^2\right] \\ &= h(j)\lambda\psi(x) + h(1-j)\lambda\psi(y) - \lambda\mu j(1-j)\left(\frac{1}{y^p} - \frac{1}{x^p}\right)^2 \\ &= h(j)\lambda\psi(x) + h(1-j)\lambda\psi(y) - \nu^* j(1-j)\left(\frac{1}{y^p} - \frac{1}{x^p}\right)^2,\end{aligned} \tag{17}$$

where $\nu^* = \lambda\mu$ and $\mu \ge 0$. This completes the proof. □

Proposition 16. *Let $\psi_i : M \longrightarrow \mathbb{R}$, where $1 \le i \le n$ be in $SR(p, h)$ with modulus μ; then, for $\lambda_i \ge 0$ where $1 \le i \le n$, the function $\psi : M \longrightarrow \mathbb{R}$, where $\psi = \sum_{i=1}^{n} \lambda_i \psi_i$ is also in $SR(p, h)$ with modulus $\gamma \ge 0$, where $\gamma = \sum_{i=1}^{n} \lambda_i \mu$.*

Proof. Let M is a p-harmonic convex set. Then, $\forall x, y \in M$ and $j \in [0, 1]$, we have

$$\begin{aligned}\psi\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] &= \sum_{i=1}^{n} \lambda_i \psi_i\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \\ &\le \sum_{i=1}^{n} \lambda_i\left[h(j)\psi_i(x) + h(1-j)\psi_i(y) - \mu j(1-j)\left(\frac{1}{y^p} - \frac{1}{x^p}\right)^2\right] \\ &= h(j)\sum_{i=1}^{n} \lambda_i \psi_i(x) + h(1-j)\sum_{i=1}^{n} \lambda_i \psi_i(y) \\ &\quad - \sum_{i=1}^{n} \lambda_i\left[\mu j(1-j)\left(\frac{1}{y^p} - \frac{1}{x^p}\right)^2\right] \\ &= h(j)f(x) + h(1-j)\psi(y) - \gamma j(1-j)\left(\frac{1}{y^p} - \frac{1}{x^p}\right)^2,\end{aligned} \tag{18}$$

where $\gamma = \sum_{i=1}^{n} \lambda_i \mu$. This completes the proof. □

Proposition 17. *Let $\psi_i : M \longrightarrow \mathbb{R}$, where $1 \le i \le n$ be strongly reciprocally (p,h)-convex functions with modulus μ; then $\psi = \max\{\psi_i, i = 1, 2, \cdots, n\}$, is also strongly reciprocally (p,h)-convex functions with modulus μ.*

Proof. Let M is p-harmonic convex set. Then $\forall x, y \in M$ and $j \in [0, 1]$, we have

$$\begin{aligned}\psi\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] &= \max\left\{\psi_i\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right], i = 1, 2, 3, \cdots, n\right\} \\ &= \psi_c\left[\left(\frac{x^p y^p}{jx^p + (1-j)y^p}\right)^{1/p}\right] \\ &\le h(j)\psi_c(x) + h(1-j)\psi_c(y) - \mu j(1-j)\left(\frac{1}{y^p} - \frac{1}{x^p}\right)^2 \\ &= h(j)\max\{\psi_i(x)\} + h(1-j)\max\{\psi_i(y)\} \\ &\quad - \mu j(1-j)\left(\frac{1}{y^p} - \frac{1}{x^p}\right)^2 \\ &= h(j)\psi(x) + h(1-j)\psi(y) - \mu j(1-j)\left(\frac{1}{y^p} - \frac{1}{x^p}\right)^2.\end{aligned} \tag{19}$$

This completes the proof. □

Our next intension is to develop Hermite-Hadamard's inequality for this generalization.

4. Hermite-Hadamard Type Inequality

Theorem 18. *Let $M \subset \mathbb{R} \setminus \{0\}$ be an interval. If $\psi : M \longrightarrow \mathbb{R}$ be a strongly reciprocally (p,h)-convex function with modulus $\mu \geq 0$ and $\psi \in L[d_1, d_2]$, then for $h(1/2) \neq 0$, we have*

$$\begin{aligned}\frac{1}{2h(1/2)}\left[\psi\left(\frac{2d_1^p d_2^p}{d_1^p + d_2^p}\right)^{1/p} + \frac{\mu}{12}\left(\frac{d_2^p - d_1^p}{d_1^p d_2^p}\right)^2\right] &\leq \frac{p\left(d_1^p d_2^p\right)}{d_2^p - d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx \\ &\leq \int_0^1 [h(1-j)\psi(d_1) + h(j)\psi(d_2)]\, dj - \frac{\mu}{6}\left(\frac{d_2^p - d_1^p}{d_1^p d_2^p}\right)^2.\end{aligned} \tag{20}$$

Proof. Since, $\psi \in SR(p, h)$, and allowing $j = 1/2$ yields

$$\psi\left[\left(\frac{2x^p y^p}{x^p + y^p}\right)^{1/p}\right] \leq h\left(\frac{1}{2}\right)\psi(x) + h\left(\frac{1}{2}\right)\psi(y) - \mu\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{x^p} - \frac{1}{y^p}\right)^2. \tag{21}$$

Let $x = [((d_1^p d_2^p)/(jd_1^p + (1-j)d_2^p))^{1/p}]$ and $y = [((d_1^p d_2^p)/(jd_2^p + (1-j)d_1^p))^{1/p}]$ and integrating (21) w.r.t j over $[0, 1]$, we obtain

$$\begin{aligned}\psi\left[\left(\frac{2d_1^p d_2^p}{d_1^p + d_2^p}\right)^{1/p}\right] &\leq h\left(\frac{1}{2}\right)\psi\left[\left(\frac{d_1^p d_2^p}{jd_1^p + (1-j)d_2^p}\right)^{1/p}\right] \\ &\quad + h\left(\frac{1}{2}\right)\psi\left[\left(\frac{d_1^p d_2^p}{jd_2^p + (1-j)d_1^p}\right)^{1/p}\right] \\ &\quad - \frac{\mu}{4}\left(\frac{d_2^p - d_1^p}{d_1^p d_2^p}\right)^2 (1-2j)^2 \int_0^1 \psi\left[\left(\frac{2d_1^p d_2^p}{d_1^p + d_2^p}\right)^{1/p}\right] dj \\ &\leq \int_0^1 h\left(\frac{1}{2}\right)\psi\left[\left(\frac{d_1^p d_2^p}{jd_1^p + (1-j)d_2^p}\right)^{1/p}\right] dj \\ &\quad + \int_0^1 h\left(\frac{1}{2}\right)\psi\left[\left(\frac{d_1^p d_2^p}{jd_2^p + (1-j)d_1^p}\right)^{1/p}\right] dj \\ &\quad - \frac{\mu}{4}\left(\frac{d_2^p - d_1^p}{d_1^p d_2^p}\right)^2 \int_0^1 (1-2j)^2 dj\, \psi\left[\left(\frac{2d_1^p d_2^p}{d_1^p + d_2^p}\right)^{1/p}\right] \\ &\leq 2h\left(\frac{1}{2}\right)\frac{p\left(d_1^p d_2^p\right)}{d_2^p - d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx - \frac{\mu}{12}\left(\frac{d_2^p - d_1^p}{d_1^p d_2^p}\right)^2,\end{aligned} \tag{22}$$

and

$$\begin{aligned}\psi\left[\left(\frac{2d_1^p d_2^p}{d_1^p + d_2^p}\right)^{1/p}\right] + \frac{\mu}{12}\left(\frac{d_2^p - d_1^p}{d_1^p d_2^p}\right)^2 &\leq 2h\left(\frac{1}{2}\right)\frac{p\left(d_1^p d_2^p\right)}{d_2^p - d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx\,\frac{1}{2h(1/2)} \\ &\quad \times \left[\psi\left[\left(\frac{2d_1^p d_2^p}{d_1^p + d_2^p}\right)^{1/p}\right] + \frac{\mu}{12}\left(\frac{d_2^p - d_1^p}{d_1^p d_2^p}\right)^2\right] \\ &\leq \frac{p\left(d_1^p d_1^p\right)}{d_2^p - d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx,\end{aligned} \tag{23}$$

which gives one side of (20).

For the right side of (20), since $\psi \in SR(p, h)$, so by setting $x = d_1$ and $y = d_2$, we obtain the following result:

$$\begin{aligned}\psi\left[\left(\frac{d_1^p d_2^p}{jd_1^p + (1-j)d_2^p}\right)^{1/p}\right] &\leq h(1-j)\psi(d_1) + h(j)\psi(d_2) \\ &\quad - \mu j(1-j)\left(\frac{1}{d_1^p} - \frac{1}{d_2^p}\right)^2.\end{aligned} \tag{24}$$

Integrating (24) w.r.t j over $[0, 1]$, we obtain

$$\begin{aligned}\int_0^1 \psi\left[\left(\frac{d_1^p d_2^p}{jd_1^p + (1-j)d_2^p}\right)^{1/p}\right] dj &\leq \int_0^1 h(1-j)\psi(d_1)dj + \int_0^1 h(j)\psi(d_2)dj \\ &\quad - \mu\left(\frac{d_2^p - d_1^p}{d_1^p d_2^p}\right)^2 \int_0^1 j(1-j)dj,\end{aligned} \tag{25}$$

and

$$\frac{p\left(d_1^p d_2^p\right)}{d_2^p - d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx \leq \int_0^1 [h(1-j)\psi(d_1) + h(j)\psi(d_2)]dj - \frac{\mu}{6}\left(\frac{d_2^p - d_1^p}{d_1^p d_2^p}\right)^2, \tag{26}$$

which is right side of (20), which completes the proof. □

Remark 19.

(1) Inserting $h(j) = j$ and $p = 1$ in Theorem 18, we obtained Hermite-Hadamard inequality for strongly reciprocally convex function [16] (Theorem 3.1)

(2) Insertion $h(j) = j$, $p = 1$, and $\mu = 0$ in Theorem 18 yields Hermite-Hadamard inequality for harmonic convex functions [14] (Theorem 2.4)

Now, we develop Fejér type inequality for this new class of convex functions.

5. Fejér Type Inequality

Theorem 20. *Let $M \subset \mathbb{R} \setminus \{0\}$ be an interval. If $\psi : M \longrightarrow \mathbb{R}$ be a strongly reciprocally (p,h)-convex function with modulus*

$\mu \geq 0$, then for $h(1/2) \neq 0$, we have

$$\frac{1}{2h(1/2)}\left[\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]\int_{d_1}^{d_2}\frac{w(x)}{x^{1+p}}dx+\frac{\mu}{\left(2d_1^pd_2^p\right)^2}\int_{d_1}^{d_2}\frac{\left(2d_1^pd_2^p-\left(d_1^p+d_2^p\right)x^p\right)^2w(x)}{x^{1+3p}}dx\right]$$
$$\leq\int_{d_1}^{d_2}\frac{\psi(x)w(x)}{x^{1+p}}dx\leq[\psi(d_1)+\psi(d_2)]\int_{d_1}^{d_2}h\left(\frac{d_1^p\left(d_2^p-x^p\right)}{x^p\left(d_2^p-d_1^p\right)}\right)\frac{w(x)}{x^{1+p}}dx$$
$$-\frac{\mu}{d_1^pd_2^p}\int_{d_1}^{d_2}\frac{\left(x^p-d_1^p\right)\left(d_2^p-x^p\right)w(x)}{x^{1+3p}}dx, \tag{27}$$

holds for $d_1, d_2 \in M$ with $d_1 \leq d_2$ and $\psi \in L[d_1, d_2]$, where $w : M \longrightarrow \mathbb{R}$ is a non-negative integrable function that satisfies

$$w\left(\frac{d_1^pd_2^p}{x^p}\right)^{1/p}=w\left[\left(\frac{d_1^pd_2^p}{d_1^p+d_2^p-x^p}\right)^{1/p}\right]. \tag{28}$$

Proof. Since $\psi \in SR(p, h)$, and allowing $j = 1/2$ yields

$$\psi\left[\left(\frac{2x^py^p}{x^p+y^p}\right)^{1/p}\right]\leq h\left(\frac{1}{2}\right)\psi(x)+h\left(\frac{1}{2}\right)\psi(y)-\mu\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{x^p}-\frac{1}{y^p}\right)^2. \tag{29}$$

Inserting $x = [(d_1^pd_2^p/jd_1^p + (1-j)d_2^p)^{1/p}]$ and $y = [(d_1^pd_2^p/jd_2^p + (1-jd_1^p)^{1/p}]$ in (29), we obtain

$$\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]\leq h\left(\frac{1}{2}\right)\psi\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]+h\left(\frac{1}{2}\right)\psi\left[\left(\frac{d_1^pd_2^p}{jd_2^p+(1-j)d_1^p}\right)^{1/p}\right]$$
$$-\frac{\mu}{4}\left(\frac{jd_1^p+(1-j)d_2^p}{d_1^pd_2^p}-\frac{jd_2^p+(1-j)d_1^p}{d_1^pd_2^p}\right)^2. \tag{30}$$

Since, w is a non-negative symmetric and integrable function, so

$$\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]w\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]$$
$$\leq h\left(\frac{1}{2}\right)\psi\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]w\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]$$
$$+h\left(\frac{1}{2}\right)\psi\left[\left(\frac{d_1^pd_2^p}{jd_2^p+(1-j)d_1^p}\right)^{1/p}\right]w\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]$$
$$-\frac{\mu}{4}\left(\frac{jd_1^p+(1-j)d_2^p}{d_1^pd_2^p}-\frac{jd_2^p+(1-j)d_1^p}{d_1^pd_2^p}\right)^2w\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]. \tag{31}$$

Integrating inequality (31) w.r.t j over $[0, 1]$, we obtain

$$\int_0^1\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]w\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]dj$$
$$\leq\int_0^1h\left(\frac{1}{2}\right)\psi\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]w\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]dj$$
$$+\int_0^1h\left(\frac{1}{2}\right)\psi\left[\left(\frac{d_1^pd_2^p}{jd_2^p+(1-j)d_1^p}\right)^{1/p}\right]w\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]dj$$
$$-\frac{\mu}{4}\int_0^1\left(\frac{jd_1^p+(1-j)d_2^p}{d_1^pd_2^p}-\frac{jd_2^p+(1-j)d_1^p}{d_1^pd_2^p}\right)^2w\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]dj, \tag{32}$$

and

$$\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]\int_{a_1}^{b_1}\frac{w(x)}{x^{1+p}}dx+\frac{\mu}{\left(2d_1^pd_2^p\right)^2}\int_{d_1}^{d_2}$$
$$\times\frac{\left(2d_1^pd_2^p-\left(d_1^p+d_2^p\right)x^p\right)^2w(x)}{x^{1+3p}}dx$$
$$\leq 2h\left(\frac{1}{2}\right)\int_{a_1}^{b_1}\frac{\psi(x)w(x)}{x^{1+p}}dx, \tag{33}$$

so,

$$\frac{1}{2h(1/2)}\left[\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]\int_{d_1}^{d_2}\frac{w(x)}{x^{1+p}}dx+\frac{\mu}{\left(2d_1^pd_2^p\right)^2}\int_{d_1}^{d_2}\frac{\left(2d_1^pd_2^p-\left(d_1^p+d_2^p\right)x^p\right)^2w(x)}{x^{1+3p}}dx\right]$$
$$\leq\int_{d_1}^{d_2}\frac{\psi(x)w(x)}{x^{1+p}}dx, \tag{34}$$

which is left side of (27). □

Finally, for the right side of (27), since $\psi \in SR(p, h)$ and by setting $x = d_1$ and $y = d_2$, we obtain the following result:

$$\psi\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]\leq h(1-j)\psi(d_1)+h(j)\psi(d_2)-\mu j(1-j)\left(\frac{1}{d_1^p}-\frac{1}{d_2^p}\right)^2. \tag{35}$$

Since w is a non-negative symmetric and integrable function, so

$$\psi\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]w\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]$$
$$\leq h(1-j)\psi(d_1)w\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]$$
$$+h(j)\psi(d_2)w\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]$$
$$-\mu j(1-j)\left(\frac{1}{d_1^p}-\frac{1}{d_2^p}\right)^2w\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]. \tag{36}$$

Integrating inequality (36) w.r.t j over $[0, 1]$, we obtain

$$\begin{aligned}\int_0^1 \psi\left[\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right] w\left[\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right] dj \\ \leq \int_0^1 h(1-j)\psi(d_1) w\left[\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right] dj \\ + \int_0^1 h(j)\psi(d_2) w\left[\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right] dj \\ - \mu\int_0^1 j(1-j)\left(\frac{1}{d_1^p}-\frac{1}{d_2^p}\right)^2 w\left[\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right] dj,\end{aligned} \tag{37}$$

and

$$\begin{aligned}\int_{d_1}^{d_2} \frac{\psi(x)w(x)}{x^{1+p}} dx \leq [\psi(d_1)+\psi(d_2)]\int_{a_1}^{b_1} h\left(\frac{d_1^p(d_2^p-x^p)}{x^p(d_2^p-d_1^p)}\right)\frac{w(x)}{x^{1+p}} dx \\ - \frac{\mu}{d_1^p d_2^p}\int_{d_1}^{d_2} \frac{(x^p-d_1^p)(d_2^p-x^p)w(x)}{x^{1+3p}} dx,\end{aligned} \tag{38}$$

which is the right side of inequality (27); this completes the proof.

Remark 21.

(1) Inserting $h(j)=j$ and $p=1$ in Theorem 20, we obtained Fejér type inequality for strongly reciprocally convex function [16] (Theorem 3.7)

(2) Insertion $h(j)=j$, in Theorem 20, yields Fejér type inequality for strongly reciprocally p-convex functions [17] (Theorem 3.5)

6. Fractional Integral Inequalities

Fractional integral inequalities are important to study means [19–22]. This section is devoted for some fractional integral inequalities for functions whose derivatives are in $SR(p, h)$. A source for results of the desired type is the following lemma.

Lemma 22 see ([23], Lemma 2.1). *Let $\psi : M=[d_1, d_2] \subseteq \mathbb{R}$ is differentiable defined on the interior M of M. If $\psi' \in L[d_1, d_2]$ and $\lambda \in [0, 1]$, then*

$$\begin{aligned}(1-\lambda)\psi\left[\left(\frac{2d_1^p d_2^p}{d_1^p+d_2^p}\right)^{1/p}\right] + \lambda\left(\frac{\psi(d_1)+\psi(d_2)}{2}\right) - \frac{p(d_1^p d_2^p)}{d_2^p-d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx \\ = \frac{(d_2^p-d_1^p)}{2p(d_1^p d_2^p)}\left[\int_0^{1/2}(2j-\lambda)\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\psi'\left[\left(\frac{a_1^p b_1^p}{ja_1^p+(1-j)b_1^p}\right)^{1/p}\right]dj \right. \\ \left. + \int_{1/2}^1 (2j-2+\lambda)\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\psi'\left[\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]dj\right].\end{aligned} \tag{39}$$

As a first application of Lemma 22, we prove the following result.

Theorem 23. *Let $M=[d_1, d_2] \subset \mathbb{R}\setminus\{0\}$ be a p -harmonic convex set, and let $\psi : M=[d_1, d_2] \subseteq \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ be differentiable defined on the interior M of M. If $M' \in L[d_1, d_2]$ and $|\psi'|^q$ is strongly reciprocally (p,h)-convex function on M, $q \geq 1$, and $\lambda \in [0, 1]$, then*

$$\begin{aligned}\left|(1-\lambda)\psi\left[\left(\frac{2d_1^p d_2^p}{d_1^p+d_2^p}\right)^{1/p}\right] + \lambda\left(\frac{\psi(d_1)+\psi(d_2)}{2}\right) - \frac{p(d_1^p d_2^p)}{d_2^p-d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx\right| \\ \leq \frac{(d_2^p-d_1^p)}{2p(d_1^p d_2^p)}\left[k_1(p,d_1,d_2)^{1-(1/q)}\left[k_{15}(p,d_1,d_2)|\psi'(d_1)|^q + k_{17}(p,d_1,d_2)|\psi'(d_2)|^q \right.\right. \\ \left. + k_7(p,d_1,d_2)\mu\right]^{1/q} + k_2(p,d_2,d_1)^{1-(1/q)}\left[k_{18}(p,d_2,d_1)|\psi'(d_1)|^q \right. \\ \left.\left. + k_{16}(p,d_2,d_1)|\psi'(d_2)|^q + k_8(p,d_2,d_1)\mu\right]^{1/q}\right],\end{aligned} \tag{40}$$

where

$$k_1(p,d_1,d_2) = \int_0^{1/2}|2j-\lambda|\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}dj, \tag{41}$$

$$k_2(p,d_2,d_1) = \int_{1/2}^1|2j-2+\lambda|\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}dj, \tag{42}$$

$$k_{15}(p,d_1,d_2) = \int_0^{1/2}h(1-j)|2j-\lambda|\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}dj, \tag{43}$$

$$k_{16}(p,d_2,d_1) = \int_{1/2}^1 h(j)|2j-2+\lambda|\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}dj, \tag{44}$$

$$k_{17}(p,d_1,d_2) = \int_0^{1/2}h(j)|2j-\lambda|\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}dj, \tag{45}$$

$$k_{18}(p,d_2,d_1) = \int_{1/2}^1 h(1-j)|2j-2+\lambda|\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}dj, \tag{46}$$

$$k_7(p,d_1,d_2) = \int_0^{1/2}j(j-1)|2j-\lambda|\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\left(\frac{1}{d_2^p}-\frac{1}{d_1^p}\right)^2 dj, \tag{47}$$

$$k_8(p,d_2,d_1) = \int_{1/2}^1 j(j-1)|2j-2+\lambda|\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\left(\frac{1}{d_2^p}-\frac{1}{d_1^p}\right)^2 dj. \tag{48}$$

Proof. Using Lemma 22 and power mean inequality, we have

$$\begin{aligned}
&\left|(1-\lambda)\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]+\lambda\left(\frac{\psi(d_1)+\psi(d_2)}{2}\right)-\frac{p(d_1^pd_2^p)}{d_2^p-d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\
&\leq\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\left[\int_0^{1/2}\left|(2j-\lambda)\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\right|\times\left|\psi'\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]\right|dj\right.\\
&\quad\left.+\int_{1/2}^1\left|(2j-2+\lambda)\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\right|\times\left|\psi'\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]\right|dj\right].\\
&\leq\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\left[\left(\int_0^{1/2}|(2j-\lambda)|\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}dj\right)^{1-(1/q)}\right.\\
&\quad\times\left(\int_0^{1/2}|(2j-\lambda)|\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\times\left|\psi'\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}\\
&\quad+\left(\int_{1/2}^1|(2j-2+\lambda)|\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}dt\right)^{1-(1/q)}\\
&\quad\times\left(\int_{1/2}^1|(2j-2+\lambda)|\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\right)\\
&\quad\left.\times\left|\psi'\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}\right].
\end{aligned}\tag{49}$$

Since $|\psi'(x)|^q\in SR(p,h)$, so

$$\begin{aligned}
&\left|(1-\lambda)\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]+\lambda\left(\frac{\psi(d_1)+\psi(d_2)}{2}\right)-\frac{p(d_1^pd_2^p)}{d_2^p-d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\
&\leq\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\left[\left(\int_0^{1/2}|(2j-\lambda)|\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}dj\right)^{1-(1/q)}\right.\\
&\quad\times\left(\int_0^{1/2}|(2j-\lambda)|\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\right.\\
&\quad\left.\times\left[h(1-j)|\psi'(d_1)|^q+h(j)|\psi'(d_2)|^q-\mu j(1-j)\left(\frac{1}{d_2^p}-\frac{1}{d_1^p}\right)^2\right]dj\right)^{1/q}\\
&\quad+\left(\int_{1/2}^1|(2j-2+\lambda)|\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}dj\right)^{1-(1/q)}\\
&\quad\times\left(\int_{1/2}^1|(2j-2+\lambda)|\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\right.\\
&\quad\left.\left.\times\left[h(1-j)|\psi'(d_1)|^q+h(j)|\psi'(d_2)|^q-\mu j(1-j)\left(\frac{1}{d_2^p}-\frac{1}{d_1^p}\right)^2\right]dj\right)^{1/q}\right]\\
&=\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\left[k_1(p,d_1,d_2)^{1-(1/q)}\left[k_{15}(p,d_1,d_2)|\psi'(d_1)|^q\right.\right.\\
&\quad\left.+k_{17}(p,d_1,d_2)|\psi'(d_2)|^q+k_7(p,d_1,d_2)\mu\right]^{1/q}\\
&\quad+k_2(p,d_2,d_1)^{1-(1/q)}\left[k_{18}(p,d_2,d_1)|\psi'(d_1)|^q+k_{16}(p,d_2,d_1)|\psi'(d_2)|^q\right.\\
&\quad\left.\left.+k_8(p,d_2,d_1)\mu\right]^{1/q}\right].
\end{aligned}\tag{50}$$

This completes the proof. □

Remark 24. Inserting $\mu=0$ and $h(j)=j$ in Theorem 23, we obtained [23] Theorem 2.2.

For $q=1$, Theorem 23 reduces to the following result.

Corollary 25. *Let $M=[d_1,d_2]\subset\mathbb{R}\setminus\{0\}$ be a p-harmonic convex set, and let $\psi:M=[d_1,d_2]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ be differentiable defined on the interior M of M. If $\psi'\in L[d_1,d_2]$ and $|\psi'|^q$ is in $SR(p,h)$ on M and $\lambda\in[0,1]$, then*

$$\begin{aligned}
&\left|(1-\lambda)\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]+\lambda\left(\frac{\psi(d_1)+\psi(d_2)}{2}\right)-\frac{p(d_1^pd_2^p)}{d_2^p-d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\
&\leq\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\left[(k_{15}(p,d_1,d_2)+k_{18}(p,d_2,d_1))|\psi'(d_1)|\right.\\
&\quad+(k_{16}(p,d_2,d_1)+k_{17}(p,d_1,d_2))|\psi'(d_2)|+(k_7(p,d_1,d_2)\\
&\quad\left.+k_8(p,d_2,d_1))\mu\right],
\end{aligned}\tag{51}$$

where $k_{15},k_{16},k_{17},k_{18},k_7,$ and k_8 are given by (43) to (48).

Remark 26. Inserting $h(j)=j$ and $\mu=0$ in Corollary 25, we obtained [23] Corollary 2.3.

Using Lemma 22, we can prove the following result.

Theorem 27. *Let $M=[d_1,d_2]\subset\mathbb{R}\setminus\{0\}$ be a p-harmonic convex set, and let $\psi:M=[d_1,d_2]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ be differentiable defined on the interior M of M. If $M'\in L[d_1,d_2]$ and $|\psi'|^q$ is in $SR(p,h)$ on M, $r,q>1$, $(1/r)+(1/q)=1$ and $\lambda\in[0,1]$, then*

$$\begin{aligned}
&\left|(1-\lambda)\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]+\lambda\left(\frac{\psi(d_1)+\psi(d_2)}{2}\right)-\frac{p(d_1^pd_2^p)}{d_2^p-d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\
&\leq\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\times\left(\frac{\lambda^{r+1}+(1-\lambda)^{r+1}}{2(r+1)}\right)^{1/r}\\
&\quad\times\left[\left(k_{19}(q,p;d_1,d_2)|\psi'(d_1)|^q+k_{21}(q,p;d_1,d_2)|\psi'(d_2)|^q+k_{13}(q,p;d_1,d_2)\mu\right)^{1/q}\right.\\
&\quad\left.+\left(k_{22}(q,p;d_2,d_1)|\psi'(d_1)|^q+k_{20}(q,p;d_2,d_1)|\psi'(d_2)|^q+k_{14}(q,p;d_2,d_1)\mu\right)^{1/q}\right],
\end{aligned}\tag{52}$$

where

$$k_{19}(q,p;d_1,d_2)=\int_0^{1/2}h(1-j)\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{q+(q/p)}dj,\tag{53}$$

$$k_{20}(q,p;d_2,d_1)=\int_{1/2}^1h(j)\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{q+(q/p)}dj,\tag{54}$$

$$k_{21}(q,p;d_1,d_2)=\int_0^{1/2}h(j)\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{q+(q/p)}dj,\tag{55}$$

$$k_{22}(q,p;d_2,d_1)=\int_{1/2}^1h(1-j)\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{q+(q/p)}dj,\tag{56}$$

$$k_{13}(q,p;d_1,d_2)=\int_0^{1/2} j(j-1)\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{q+(q/p)}\left(\frac{1}{b_1^p}-\frac{1}{a_1^p}\right)^2 dj, \tag{57}$$

$$k_{14}(q,p;d_2,d_1)=\int_{1/2}^{1} j(j-1)2j-2+\lambda\left(\frac{d_1^p d_2^p}{jd_1^p+(1-j)d_2^p}\right)^{q+(q/p)}\left(\frac{1}{d_2^p}-\frac{1}{d_1^p}\right)^2 dj. \tag{58}$$

Proof. Using Lemma 22, we have

$$\begin{aligned}&\left|(1-\lambda)\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]+\lambda\left(\frac{\psi(d_1)+\psi(d_2)}{2}\right)-\frac{p(d_1^pd_2^p)}{d_2^p-d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\&\le\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\left[\int_0^{1/2}\left|(2j-\lambda)\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\right|\times\left|\psi'\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]\right|dj\right.\\&\left.+\int_{1/2}^{1}\left|(2j-2+\lambda)\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\right|\times\left|\psi'\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]\right|dj\right].\end{aligned} \tag{59}$$

Applying Holder's integral inequality,

$$\begin{aligned}&\left|(1-\lambda)\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]+\lambda\left(\frac{\psi(d_1)+\psi(d_2)}{2}\right)-\frac{p(d_1^pd_2^p)}{d_2^p-d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\&\le\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\left[\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\left(\int_0^{1/2}\left|\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\right.\right.\right.\\&\left.\left.\times\psi'\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}\\&+\left(\int_{1/2}^{1}|(2j-2+\lambda)|^r dj\right)^{1/r}\left(\int_{1/2}^{1}\left|\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1+(1/p)}\right.\right.\\&\left.\left.\left.\times\psi'\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}\right]\\&=\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\left[\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\left(\int_0^{1/2}\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{q+(q/p)}\right.\right.\\&\left.\times\left|\psi'\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}\\&+\left(\int_{1/2}^{1}|(2j-2+\lambda)|^r dj\right)^{1/r}\\&\left.\times\left(\int_{1/2}^{1}\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{q+(q/p)}\left|\times\psi'\left[\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{1/p}\right]\right|^q dj\right)^{1/q}\right].\end{aligned} \tag{60}$$

Since $|\psi'(x)|^q\in SR(p,h)$, so

$$\begin{aligned}&\left|(1-\lambda)\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]+\lambda\left(\frac{\psi(d_1)+\psi(d_2)}{2}\right)-\frac{p(d_1^pd_2^p)}{d_2^p-d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\&\le\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\left[\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\times\left(\int_0^{1/2}\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{q+(q/p)}\right.\right.\\&\left.\times\left[h(1-j)|\psi'(d_1)|^q+h(j)|\psi'(d_2)|^q-\mu j(1-j)\left(\frac{1}{d_2^p}-\frac{1}{d_1^p}\right)^2\right]dj\right)^{1/q}\\&+\left(\int_{1/2}^{1}|(2j-2+\lambda)|^r dj\right)^{1/r}\left(\int_{1/2}^{1}\left(\frac{d_1^pd_2^p}{jd_1^p+(1-j)d_2^p}\right)^{q+(q/p)}\right.\\&\left.\left.\times\left[h(1-j)|\psi'(d_1)|^q+h(j)|\psi'(d_2)|^q-\mu j(1-j)\left(\frac{1}{d_2^p}-\frac{1}{d_1^p}\right)^2\right]dj\right)^{1/q}\right]\\&\le\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\left[\left(\int_0^{1/2}|(2j-\lambda)|^r dj\right)^{1/r}\right.\\&\times\left(k_{19}(q,p;d_1,d_2)|\psi'(d_1)|^q+k_{21}(q,p;d_1,d_2)|\psi'(d_2)|^q\right.\\&\left.\times+k_{13}(q,p;d_1,d_2)\mu\right)^{1/q}+\left(\int_0^{1/2}|(2j-2+\lambda)|^r dj\right)^{1/r}\\&\times\left(k_{22}(q,p;d_2,d_1)|\psi'(d_1)|^q+k_{20}(q,p;d_2,d_1)|\psi'(d_2)|^q\right.\\&\left.\left.+k_{14}(q,p;d_2,d_1)\mu\right)^{1/q}\right]\\&\le\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\times\left(\frac{\lambda^{r+1}+(1-\lambda)^{r+1}}{2(r+1)}\right)^{1/r}\times\left[\left(\left(k_{19}(q,p;d_1,d_2)|\psi'(d_1)|^q\right.\right.\right.\\&\left.+k_{21}(q,p;d_1,d_2)|\psi'(d_2)|^q+k_{13}(q,p;d_1,d_2)\mu\right)^{1/q}\\&+\left(k_{22}(q,p;d_2,d_1)|\psi'(d_1)|^q+k_{20}(q,p;d_2,d_1)|\psi'(d_2)|^q\right.\\&\left.\left.+k_{14}(q,p;d_2,d_1)\mu\right)^{1/q}\right)\right].\end{aligned} \tag{61}$$

Which is the required result. □

Remark 28. Inserting $h(j)=j$ and $\mu=0$ in Theorem 27, we obtained [23] Theorem 2.5.

For $\lambda=0$, Theorem 27 reduces to the following result.

Corollary 29. *Let $M=[d_1,d_2]\subset\mathbb{R}\setminus\{0\}$ be a p-harmonic convex set, and let $\psi: M=[d_1,d_2]\subseteq\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}$ be differentiable defined on the interior M of M. If $\psi'\in L[d_1,d_2]$ and $|\psi'|^q$ is in $SR(p,h)$ on M, $r,q>1$, $(1/r)+(1/q)=1$ and $\lambda\in[0,1]$, then*

$$\begin{aligned}&\left|\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]-\frac{p(d_1^pd_2^p)}{d_2^p-d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\&\le\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\times\left(\frac{1}{2(r+1)}\right)^{1/r}\left[\left(k_{19}(q,p;d_1,d_2)|\psi'(d_1)|^q\right.\right.\\&\left.+k_{21}(q,p;d_1,d_2)|\psi'(d_2)|^q+k_{13}(q,p;d_1,d_2)\mu\right)^{1/q}\\&+\left(k_{22}(q,p;d_2,d_1)|\psi'(d_1)|^q+k_{20}(q,p;d_2,d_1)|\psi'(d_2)|^q\right.\\&\left.\left.+k_{14}(q,p;d_2,d_1)\mu\right)^{1/q}\right],\end{aligned} \tag{62}$$

where k_{19}, k_{20}, k_{21}, k_{22}, k_{13}, and k_{14} are given by (53)-(58).

Remark 30. Inserting $h(j) = j$ and $\mu = 0$ in Corollary 29, we obtained [23] Corollary 3.5.

For $\lambda = 1$, Theorem 27 reduces to the following result.

Corollary 31. *Let $M = [d_1, d_2] \subset \mathbb{R} \setminus \{0\}$ be a p -harmonic convex set, and let $\psi : M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be differentiable defined on the interior M of M. If $M' \in L[a_1, b_1]$ and $|f'|^q$ is in $SR(p, h)$ on M, $r, q > 1$, $(1/r) + (1/q) = 1$ and $\lambda \in [0, 1]$ then,*

$$\begin{aligned}&\left|\frac{\psi(d_1)+\psi(d_2)}{2}-\frac{p(d_1^pd_2^p)}{d_2^p-d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\&\le\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\times\left(\frac{1}{2(r+1)}\right)^{1/r}\left[\left(k_{19}(q,p;d_1,d_2)|\psi'(d_1)|^q\right.\right.\\&\left.+k_{21}(q,p;d_1,d_2)|\psi'(d_2)|^q+k_{13}(q,p;d_1,d_2)\mu\right)^{1/q}\\&+\left(k_{22}(q,p;d_2,d_1)|\psi'(d_1)|^q+k_{20}(q,p;d_2,d_1)|\psi'(d_2)|^q\right.\\&\left.\left.+k_{14}(q,p;d_2,d_1)\mu\right)^{1/q}\right],\end{aligned}\tag{63}$$

where $k_{19}, k_{20}, k_{21}, k_{22}, k_{13}$, and k_{14} are given by (53)-(58).

Remark 32. Inserting $h(j) = j$ and $\mu = 0$ in Corollary 31, we obtained [23] Corollary 3.6.

For $\lambda = 1/3$, Theorem 27 reduces to the following result.

Corollary 33. *Let $M = [d_1, d_2] \subset \mathbb{R} \setminus \{0\}$ be a p-harmonic convex set, and let $\psi : M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be differentiable defined on the interior M of M. If $\psi' \in L[d_1, d_2]$ and $|\psi'|^q$ is in $SR(p, h)$ on M, $r, q > 1$, $(1/r) + (1/q) = 1$ and $\lambda \in [0, 1]$ then,*

$$\begin{aligned}&\left|\frac{1}{6}\left[\psi(d_1)+4\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]+\psi(d_2)\right]-\frac{p(d_1^pd_2^p)}{d_2^p-d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\&\le\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\times\left(\frac{1+2^{r+1}}{6.3^r(r+1)}\right)^{1/r}\left[\left(k_{19}(q,p;d_1,d_2)|\psi'(d_1)|^q\right.\right.\\&\left.+k_{21}(q,p;d_1,d_2)|\psi'(d_1)|^q+k_{13}(q,p;d_1,d_2)\mu\right)^{1/q}\\&+\left(k_{22}(q,p;d_2,d_1)|\psi'(d_1)|^q+k_{20}(q,p;d_2,d_1)|\psi'(d_2)|^q\right.\\&\left.\left.+k_{14}(q,p;d_2,d_1)\mu\right)^{1/q}\right],\end{aligned}\tag{64}$$

where $k_{19}, k_{20}, k_{21}, k_{22}, k_{13}$, and k_{14} are given by (53)-(58).

Remark 34. Inserting $h(j) = j$ and $\mu = 0$ in Corollary 33, we obtained [23] Corollary 3.7.

For $\lambda = 1/2$, Theorem 27) reduces to the following result.

Corollary 35. *Let $M = [d_1, d_2] \subset \mathbb{R} \setminus \{0\}$ be a p-harmonic convex set, and let $\psi : M = [d_1, d_2] \subseteq \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be differentiable defined on the interior M of M. If $\psi' \in L[d_1, d_2]$ and $|\psi'|^q$ is in $SR(p, h)$ on M, $r, q > 1$, $(1/r) + (1/q) = 1$ and $\lambda \in [0, 1]$ then,*

$$\begin{aligned}&\left|\frac{1}{4}\left[\psi(d_1)+2\psi\left[\left(\frac{2d_1^pd_2^p}{d_1^p+d_2^p}\right)^{1/p}\right]+\psi(d_2)\right]-\frac{p(d_1^pd_2^p)}{d_2^p-d_1^p}\int_{d_1}^{d_2}\frac{\psi(x)}{x^{1+p}}dx\right|\\&\le\frac{(d_2^p-d_1^p)}{2p(d_1^pd_2^p)}\times\left(\frac{2}{4.2^r(r+1)}\right)^{1/r}\left[\left(\left(k_{19}(q,p;d_1,d_2)|\psi'(d_1)|^q\right.\right.\right.\\&\left.+k_{21}(q,p;d_1,d_2)|\psi'(d_2)|^q+k_{13}(q,p;d_1,d_2)\mu\right)^{1/q}\\&+\left(k_{22}(q,p;d_2,d_1)|\psi'(d_1)|^q+k_{20}(q,p;d_2,d_1)|\psi'(d_2)|^q\right.\\&\left.\left.+k_{14}(q,p;d_2,d_1)\mu\right)^{1/q}\right]\end{aligned}\tag{65}$$

where $k_{19}, k_{20}, k_{21}, k_{22}, k_{13}$, and k_{14} are given by (53)–(58).

Remark 36. Inserting $h(j) = j$ and $\mu = 0$ in Corollary 35, we obtained [23] Corollary 3.8.

7. Conclusion

Convexity plays very important role in pure and applied mathematics. In many problems, the classical definition of convexity is not enough, so the definition of convex functions is generalized in various directions. In this paper, we developed a new generalization called strongly reciprocally (*p*,*h*)-convex function. We have also developed Hermite-Hadamard and Fejér type inequality for this generalization. Moreover, we conclude that the results proved in this article are the generalization of results in [14, 16, 17, 23].

Data Availability

All data is included within this paper.

Conflicts of Interest

The authors have no conflict of interests.

Authors' Contributions

The authors contributed equally in this paper.

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