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Research Article

Characterization and Stability of Multi-Euler-Lagrange Quadratic Functional Equations

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The aim of the current article is to characterize and to prove the stability of multi-Euler-Lagrange quadratic mappings. In other words, it reduces a system of equations defining the multi-Euler-Lagrange quadratic mappings to an equation, say, the multi-Euler-Lagrange quadratic functional equation. Moreover, some results corresponding to known stability (Hyers, Rassias, and Găvruta) outcomes regarding the multi-Euler-Lagrange quadratic functional equation are presented in quasi-β-normed and Banach spaces by using the fixed point methods. Lastly, an example for the nonstable multi-Euler-Lagrange quadratic functional equation is indicated.

1. Introduction

The celebrated Ulam challenge [1] arises from this question that how we can find an exact solution near to an approximate solution of an equation. This phenomenon of mathematics is called the *stability* of functional equations which has many applications in nonlinear analysis. The mentioned question has been partially solved by Hyers [2], Aoki [3], and Rassias [4] for the linear, additive, and linear (unbounded Cauchy difference) mappings, respectively. Next, many Hyers-Ulam stability problems for miscellaneous functional equations were studied by authors in the spirit of Rassias approach (see for instance [5–14] and other resources).

During the last two decades, stability problems for multivariable mappings were studied and extended by a number of authors. One of the mappings is the multiquadratic mapping, studied, for example, in [15–17]. Recall that a multivariable mapping $f : V^n \longrightarrow W$ is said to be *multiquadratic* [11] if it fulfills the famous quadratic equation

$$Q(u+v)+Q(u-v)=2Q(u)+2Q(v), \tag{1}$$

in each component. Note that equation (1) is a suitable tool for obtaining some characterizations in the setting of inner product spaces and in fact plays a prominent role. In other words, any square norm on an inner product space fulfills

$$\|u+v\|^2+\|u-v\|^2=2\|u\|^2+\|v\|^2, \tag{2}$$

which is called the *parallelogram equality*. However, some functional equations have been applied to characterize inner product spaces and are available in [18, 19] and references therein. In addition, the quadratic functional equation was used to characterize inner product spaces in [20, 21].

A lot of information about equation (1) and some equations which are equivalent to it (in particular, about their solutions and stability) and more applications can be found for instance in [22–24]. Park was the first author who studied the stability of multiquadratic in the setting of Banach algebras [16]. After that, some authors introduced various versions of multiquadratic mappings and investigated the Hyers-Ulam stability of such mappings in Banach spaces and non-Archimedean spaces; these results are available for instance in [15, 25–29]. As for an unification of the multiquadratic mappings, Zhao et al. [17] were the first authors who described the structure of multiquadratic mappings, and in fact, they showed that $f : V^n \longrightarrow W$ is a multiquadratic mapping if and only if the equation

$$\sum_{t\in\{-1,1\}^n} f(v_1 + tv_2) = 2^n \sum_{i_1,\cdots,i_n\in\{1,2\}} f\left(v_{1i_1}, \cdots, v_{ni_n}\right) \quad (3)$$

holds, where $v_i = (x_{1i}, \cdots, v_{ni}) \in V^n$ and $i \in \{1, 2\}$.

Rassias [30] introduced the following notion of a generalized Euler-Lagrange-type quadratic mapping and investigated its generalized stability.

Definition 1. Suppose that V and W are linear spaces. A nonlinear mapping $\mathfrak{Q} : V \longrightarrow W$ satisfying the functional equation

$$\mathfrak{Q}(au + bv) + \mathfrak{Q}(bu - av) = \left(a^2 + b^2\right)[\mathfrak{Q}(u) + \mathfrak{Q}(v)] \quad (4)$$

is called 2-dimensional quadratic, where $u, v \in V$ and a, b are the fixed reals with $a^2 + b^2 > 1$.

It is easily seen that the Euler-Lagrange equality

$$(au + bv)^2 + (bu - av)^2 = \left(a^2 + b^2\right)\left(u^2 + v^2\right) \quad (5)$$

is valid for $\mathfrak{Q}$, defined in Definition 1 with any fixed reals a, b, and hence, (4) is also called Euler-Lagrange quadratic functional equation; we refer to [31] for Euler-Lagrange type cubic functional equation and its stability. Note that equation (4) is a general form of (1) in the case that $a = b = 1$, and so the function $\mathfrak{Q}(v) = v^2$ satisfies (4). Next, Xu [32] extended the definition above to several variable mappings and presented the next definition.

Definition 2. Let V and W be vector spaces. A mapping $f : V^n \longrightarrow W$ is said to be the n -Euler-Lagrange quadratic or multi-Euler-Lagrange quadratic if the mapping

$$v \mapsto f(v_1, \cdots, v_{i-1}, v, v_{i+1}, \cdots, v_n) \quad (6)$$

satisfies (4), for all $i \in \{1, \cdots, n\}$ and all $v_i \in V$.

In this article, we include a characterization of multi-Euler-Lagrange quadratic mappings and show that every multi-Euler-Lagrange quadratic mapping can be described as an equation (namely, the multi-Euler-Lagrange quadratic equation). Under the quadratic condition (2-power condition) in each variable, every multivariable mappings satisfying the mentioned earlier equation is multi-Euler-Lagrange quadratic (Theorem 5). Furthermore, we bring two Hyers-Ulam stability results for multi-Euler-Lagrange quadratic functional equations in quasi-β-normed and Banach spaces which their proof is based according to some known fixed point methods; see [33, 34] for more stability results in quasi-β-Banach spaces setting. Finally, we indicate an example to show that the multi-Euler-Lagrange quadratic functional equation is nonstable in the case of singularity.

2. Characterization of Multi-Euler-Lagrange Quadratic Mappings

Throughout, we consider the following known notations:

(i) $\mathbb{N}$=the set of all natural numbers

(ii) $\mathbb{Z}$= the set of all integer numbers

(iii) $\mathbb{Q}$= the set of all rational numbers

(iv) $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$

(v) $\mathbb{R}_+ := [0,\infty)$

Let V be a linear space over $\mathbb{Q}$. Given $n \in \mathbb{N}$, $p \in \mathbb{N}_0$, $s = (s_1, \cdots, s_n) \in \mathbb{Q}^n$, and $v = (v_1, \cdots, v_n) \in V^n$. We write $sv := (s_1v_1, \cdots, s_nv_n)$ and $pv := (pv_1, \cdots, pv_n)$ which belong to V^n. Here and subsequently, V is linear space over $\mathbb{Q}$ and $v_i^n = (v_{i1}, v_{i2}, \cdots, v_{\text{in}}) \in V^n$, in which $i \in \{1, 2\}$. Furthermore, for given the fixed elements $a_i^n = (a_{i1}, a_{i2}, \cdots, a_{\text{in}}) \in \mathbb{Z}^n$ such that $a_{ij} \neq 0, \pm 1$, where $i = 1, 2$ and $j = 1, \cdots, n$ (here and the rest of the paper). We will write a_i^n and v_i^n simply a_i and v_i, respectively, when no confusion can arise.

For $v_1, v_2 \in V^n$ and $a_1, a_2 \in \mathbb{Z}^n$, set

$$\begin{aligned} A_j^{+1} &= \sum_{i=1}^{2} a_{ij} v_{ij}, \\ A_j^{-1} &= \sum_{i=1}^{2} (-1)^{i+1} a_{3-i,j} v_{ij}, (j \in \{1, \cdots, n\}). \end{aligned} \quad (7)$$

In continuation, we show that the equation

$$\begin{aligned} &\sum_{t_1,\cdots t_n\in\{-1,+1\}} f\left(A_1^{t_1}, \cdots, A_n^{t_n}\right) \\ &= \prod_{j=1}^{n}\left(a_{1j}^2 + a_{2j}^2\right) \sum_{l_1,\cdots,l_n\in\{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_n n}\right) \end{aligned} \quad (8)$$

is a general form of (4) for the multivariable case. In other words, we prove that every multi-Euler-Lagrange quadratic mapping fulfills (1) and vice versa. For doing this, we need some definitions and the upcoming lemma.

Definition 3. Let V and W be vector spaces over $\mathbb{Q}$ and $f : V^n \longrightarrow W$ be a multivariable mapping.

(i) We say f satisfies (has) the 2-power (quadratic) condition in the jth component if

$$\begin{aligned} &f\left(x_1, \cdots, x_{j-1}, a^* x_j, x_{j+1}, \cdots, x_n\right) \\ &= (a^*)^2 f\left(x_1, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_n\right), \end{aligned} \quad (9)$$

for all $x_1, \cdots, x_n \in V$, where $a^* \in \{a_{1j}, a_{2j}\}$ for all $j \in \{1, \cdots, n\}$

(ii) If $f(x_1, \cdots, x_n) = 0$ when the fixed x_j is zero, then we say that f has zero functional equation in the

jth variable. Moreover, if $f(x_1, \cdots, x_n) = 0$ for any $(x_1, \cdots, x_n) \in V^n$ with at least one x_j is zero, we say f has zero functional equation

We consider two hypotheses as follows:
(H1) f has the quadratic condition in all variables.
(H2) f has zero functional equation.

Remark 4. It is clear that if a mapping $f : V^n \longrightarrow W$ satisfies the quadratic condition in the jth variable, then it has zero functional equation in the same variable. Therefore, if f fulfills (H1), then it satisfies (H2).

Theorem 5. *For a mapping $f : V^n \longrightarrow W$, the following assertions are equivalent:*

(i) f is multi-Euler-Lagrange quadratic

(ii) f fulfills (8) and H1

Proof. (i) ⇒ (ii) In view of [30], one can show that f satisfies H1. By induction on n, we now proceed the rest of this implication so that f satisfies equation (8). Obviously, f satisfies equation (4) for $n = 1$. The induction hypothesis is

$$\sum_{t_1,\cdots t_n \in \{-1,+1\}} f\left(A_1^{t_1}, \cdots, A_n^{t_n}\right) = \prod_{j=1}^{n}\left(a_{1j}^2 + a_{2j}^2\right) \sum_{l_1,\cdots,l_n \in \{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_n n}\right). \tag{10}$$

Then

$$\begin{aligned}
\sum_{t_1,\cdots,t_{n+1} \in \{-1,1\}} f\left(A_1^{t_1}, \cdots, A_{n+1}^{t_{n+1}}\right)
&= \sum_{t_1,\cdots,t_n \in \{-1,1\}} f\left(A_1^{t_1}, \cdots, A_{n+1}^{+1}\right) + \sum_{t_1,\cdots,t_n \in \{-1,1\}} f\left(A_1^{t_1}, \cdots, A_{n+1}^{-1}\right) \\
&= \left(a_{1,n+1}^2 + a_{2,n+1}^2\right)\left(\sum_{t_1,\cdots,t_n \in \{-1,1\}} f\left(A_1^{t_1}, \cdots, A_n^{t_n}, v_{1,n+1}\right) + \sum_{t_1,\cdots,t_n \in \{-1,1\}} f\left(A_1^{t_1}, \cdots, A_n^{t_n}, v_{2,n+1}\right)\right) \\
&= \left(a_{1,n+1}^2 + a_{2,n+1}^2\right) \prod_{j=1}^{n}\left(a_{1j}^2 + a_{2j}^2\right) \cdot \left(\sum_{l_1,\cdots,l_n \in \{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_n n}, v_{1,n+1}\right) + \sum_{l_1,\cdots,l_n \in \{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_n n}, v_{2,n+1}\right)\right) \\
&= \prod_{j=1}^{n+1}\left(a_{1j}^2 + a_{2j}^2\right) \sum_{l_1,\cdots,l_{n+1} \in \{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_{n+1}, n+1}\right).
\end{aligned} \tag{11}$$

(ii) ⇒ (i) Let $j \in \{1,\cdots,n\}$ be arbitrary and fixed. Taking $v_{2k} = 0$ for all $k \in \{1, \cdots, n\} \setminus \{j\}$ in (8) and applying Remark 4, the left side will be as follows:

$$\begin{aligned}
& f\left(a_{11}v_{11}, \cdots, a_{1,j-1}v_{1,j-1}, A_j^{+1}, a_{1,j+1}v_{1,j+1}, \cdots, a_{1n}v_{1n}\right) \\
&\quad + f\left(a_{21}v_{11}, \cdots, a_{2,j-1}v_{1,j-1}, A_j^{+1}, a_{2,j+1}v_{1,j+1}, \cdots, a_{2n}v_{1n}\right) \\
&\quad + f\left(a_{11}v_{11}, \cdots, a_{1,j-1}v_{1,j-1}, A_j^{-1}, a_{1,j+1}v_{1,j+1}, \cdots, a_{1n}v_{1n}\right) \\
&\quad + f\left(a_{21}v_{11}, \cdots, a_{2,j-1}v_{1,j-1}, A_j^{-1}, a_{2,j+1}v_{1,j+1}, \cdots, a_{2n}v_{1n}\right) \\
&= a_{11}^2 a_{21}^2 a_{12}^2 a_{22}^2 \cdots a_{1,j-1}^2 a_{2,j-1}^2 a_{1,j+1}^2 a_{2,j+1}^2 \cdots a_{1n}^2 a_{2n}^2 \\
&\quad \cdot \left[f\left(v_{11}, \cdots, v_{1,j-1}, A_j^{+1}, v_{1,j+1}, \cdots, v_{1n}\right) + f\left(v_{11}, \cdots, v_{1,j-1}, A_j^{-1}, v_{1,j+1}, \cdots, v_{1n}\right)\right].
\end{aligned} \tag{12}$$

Once again, the same replacements convert the right side of (8) to

$$\prod_{\substack{k=1 \\ k\neq j}}^{n-1}\left(a_{1k}^2 + a_{2k}^2\right)\left(a_{1j}^2 + a_{2j}^2\right)\left[f\left(v_{11}, \cdots, v_{1,j-1}, v_{1j}, v_{1,j+1}, \cdots, v_{1n}\right) + f\left(v_{11}, \cdots, v_{1,j-1}, v_{2j}, v_{1,j+1}, \cdots, v_{1n}\right)\right]. \tag{13}$$

It follows from (12) and (13) that f is Euler-Lagrange (a_{1j}, a_{2j})-quadratic in the jth component, and this completes the proof. □

We should note that Theorem 5 necessitates that the mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined through $f(x_1, \cdots, x_n) = C\prod_{j=1}^{n} x_j^2$ fulfills equation (8). Hence, this equation can be called the multi-Euler-Lagrange quadratic functional equation.

3. Stability and Nonstability Results

The goals of this section are to prove miscellaneous result stability of multi-Euler-Lagrange quadratic equation (14) such as Hyers and Găvruta stability. Here, we mention a special case of equation (8) in which $a_1 = (a, \cdots, a)$ and $a_2 = (b, \cdots, b)$, and so (8) converts to

$$\sum_{t_1,\cdots t_n \in \{(a,b),(b,a)\}} f\left(A_1^{t_1}, \cdots, A_n^{t_n}\right) = \left(a^2 + b^2\right)^n \sum_{l_1,\cdots,l_n \in \{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_n n}\right), \tag{14}$$

in which

$$A_j^{(a,b)} = av_{1j} + bv_{2j}, \text{and } A_j^{(b,a)} = bv_{1j} - av_{2j}, \tag{15}$$

and $m = a^2 + b^2$ (used here and from now on) for all $j \in \{1, \cdots, n\}$.

For a set E, a function $d : E \times E \longrightarrow [0,\infty]$ is said to be a generalized metric on E provided that d fulfills the statements below, for all $u, v, w \in E$.

(i) $d(u, v) = 0$ if and only if $u = v$

(ii) $d(u, v) = d(v, u)$

(iii) $d(u, w) \le d(u, v) + d(v, w)$

The next theorem from [35] is one of fundamental results in fixed point theory and useful to achieve our first purpose in this section.

Theorem 6. *Suppose that (Ω, d) is a complete generalized metric space and $\mathcal{J} : \Omega \longrightarrow \Omega$ is a mapping such that its Lipschitz constant is $L < 1$. Then, for each element $x \in \Omega$, one of following cases can be happen:*

(i) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$for all $n \ge 0$ or

(ii) There is an $n_0 \in \mathbb{N}$ such that $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ for all $n \ge n_0$, and the sequence $\{\mathcal{J}^n x\}$ is convergent to a fixed point x^* of $\mathcal{J}$ which belongs to the set $\Lambda = \{x \in \Omega : d(\mathcal{J}^{n_0} x, x) < \infty\}$. Moreover, $d(x, x^*) \le (1/(1 - L)) d(x, \mathcal{J} x)$ for all $x \in \Lambda$

In the sequel, for any mapping $f : V^n \longrightarrow W$, we define the operator $\mathbf{D} f : V^n \times V^n \longrightarrow W$ via

$$\mathbf{D} f(v_1, v_2) \coloneqq \sum_{t_1, \cdots t_n \in \{(a,b),(b,a)\}} f\left(A_1^{t_1}, \cdots, A_n^{t_n}\right) - m^n \sum_{l_1, \cdots, l_n \in \{1,2\}} f\left(v_{l_1 1}, \cdots, v_{l_n n}\right), \tag{16}$$

for the fixed nonzero integers a, b where $A_j^{(a,b)}$ and $A_j^{(b,a)}$ are defined in (15) for all $j = 1, \cdots, n$.

In the incoming stability result for equation (14), $\|\mathbf{D} f(v_1, v_2)\|$ is controlled by a small positive number ε. We recall that for $i = 1, 2$, we consider $v_i = (v_{i1}, \cdots, v_{in}) \in V^n$.

Theorem 7. *Given $\varepsilon > 0$. Let V and W be a linear space and a complete normed space, respectively. Suppose that a mapping $f : V^n \longrightarrow W$ fulfilling H2 and*

$$\|\mathbf{D} f(v_1, v_2)\| \le \varepsilon, \tag{17}$$

for all $v_1, v_2 \in V^n$. Then, there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (14) such that

$$\|f(v) - \mathcal{Q}(v)\| \le \frac{m^n + 1}{m^{2n} - 1} \varepsilon, \tag{18}$$

for all $v \in V^n$. In addition,

$$\mathcal{Q}(v) = \lim_{l \longrightarrow \infty} \left(\frac{1}{m^{2n}}\right)^l f\left(m^l v\right), \tag{19}$$

for all $v \in V^n$.

Proof. Putting $v_2 = 0$ in (17) and using the assumption H2, we have

$$\left\|\tilde{f}(v_1) - m^n f(v_1)\right\| \le \varepsilon, \tag{20}$$

for all $v_1 \in V^n$, where

$$\tilde{f}(v_1) = \sum_{a_{l_1 1}, \cdots, a_{l_n n} \in \{a,b\}} f\left(a_{l_1 1} v_{11}, \cdots, a_{l_n n} v_{1n}\right). \tag{21}$$

Set $v_1 = v$ for simply and for the rest of the proof, all the equations and inequalities are valid for all $v \in V^n$. Once more, by replacing (v_1, v_2) instead of $(av_1, bv_1) = (av, bv)$ in (17), we get

$$\left\|f(mv) - m^n \tilde{f}(v)\right\| \le \varepsilon. \tag{22}$$

Multiplying both sides of (20) by m^n and plugging to (22), we obtain

$$\begin{aligned} \|f(mv) - m^{2n} f(v)\| &\le \left\|f(mv) - m^n \tilde{f}(v)\right\| \\ &\quad + \left\|m^n \tilde{f}(v) - m^{2n} f(v)\right\| \\ &\le (m^n + 1)\varepsilon, \end{aligned} \tag{23}$$

and thus

$$\|f(mv) - m^{2n} f(v)\| \le (m^n + 1)\varepsilon. \tag{24}$$

Let $\Omega \coloneqq \{f : V^n \longrightarrow W | f \text{ satisfies } (H2)\}$. For each $f, g \in \Omega$, we define the function d on Ω as follows:

$$d(g, h) \coloneqq \inf \left\{C \in [0,\infty] : \|g(v) - h(v)\| \le C_{g,h} \varepsilon, \text{for all } v \in V^n\right\}. \tag{25}$$

Similar to the proof of ([36], Theorem 2.2), it is seen that (Ω, d) is a complete generalized metric space. Define $\mathcal{J} : \Omega \longrightarrow \Omega$ through

$$\mathcal{J} f(v) \coloneqq \frac{1}{m^{2n}} f(mv), \tag{26}$$

for all $v \in V^n$. Take $g, h \in \Omega$ and $C_{g,h} \in [0,\infty]$ with $d(g, h) \le C_{g,h}$. Then, $\|g(v) - h(v)\| \le C_{g,h} \varepsilon$, and hence

$$\|\mathcal{J} g(v) - \mathcal{J} h(v)\| \le \frac{1}{m^{2n}} \|g(mv) - h(mv)\| \le \frac{1}{m^{2n}} C_{g,h} \varepsilon. \tag{27}$$

Therefore, $d(\mathcal{J}g, \mathcal{J}h) \le (1/m^{2n})C_{g,h}$. This shows that $d(\mathcal{J}g, \mathcal{J}h) \le (1/m^{2n})d(g,h)$ and in fact $\mathcal{J}$ is a strictly contractive operator such that its Lipschitz is $1/m^{2n}$. It concludes from (24) that

$$\|\mathcal{J}f(v) - f(v)\| \le \left\| \frac{1}{m^{2n}} f(mv) - f(v) \right\| \le \frac{m^n + 1}{m^{2n}} \varepsilon. \quad (28)$$

Hence,

$$d(\mathcal{J}f, f) \le \frac{m^n + 1}{m^{2n}} < \infty. \quad (29)$$

An application of Theorem 6 for the space (Ω, d), the operator $\mathcal{J}$, $n_0 = 0$, and $x = f$, shows that the sequence $(\mathcal{J}^l f)_{l\in\mathbb{N}}$ is convergent in (Ω, d) and its limit; $\mathcal{Q}$ is a fixed point of $\mathcal{J}$. Indeed, $\mathcal{Q}(v) = \lim_{l\longrightarrow\infty}\mathcal{J}^l f(v)$, and

$$\mathcal{Q}(v) = \frac{1}{m^{2n}} \mathcal{Q}(mv), (v \in V^n). \quad (30)$$

In other words, by induction on l, it is easily verified that for each $v \in V^n$, we have

$$\mathcal{J}^l f(v) \coloneqq \left(\frac{1}{m^{2n}}\right)^l f\left(m^l v\right), \quad (31)$$

and (19) follows. Note that clearly $f \in \Lambda$, and hence, part (iii) of Theorem 6 and (29) necessitate that

$$d(f, \mathcal{Q}) \le \frac{1}{1 - (1/m^{2n})} d(\mathcal{J}f, f) \le \frac{m^n + 1}{m^{2n} - 1}, \quad (32)$$

which proves (18). In addition,

$$\begin{aligned} \|D\mathcal{Q}(v_1, v_2)\| &= \lim_{l\longrightarrow\infty} \left(\frac{1}{m^{2n}}\right)^l \left\| Df\left(m^l v_1, m^l v_2\right) \right\| \\ &\le \lim_{l\longrightarrow\infty} \left(\frac{1}{m^{2n}}\right)^l \varepsilon = 0, \end{aligned} \quad (33)$$

for all $v_1, v_2 \in V^n$. The last relation shows that $D\mathcal{Q}(v_1, v_2) = 0$ for all $v_1, v_2 \in V^n$ and means that $\mathcal{Q}$ fulfills (14). Let us finally suppose that $\mathfrak{Q} : V^n \longrightarrow W$ is another solution of equation (14) satisfies H2 such that inequality (18) holds. Then, $\mathfrak{Q}$ satisfies (30), and so it is a fixed point of $\mathcal{J}$. Furthermore, by (18), we get

$$d(f, \mathfrak{Q}) \le \frac{m^n + 1}{m^{2n} - 1} < \infty, \quad (34)$$

and consequently, $\mathfrak{Q} \in \Lambda$. It now follows from part (ii) of Theorem 6 that $\mathfrak{Q} = \mathcal{Q}$. This finishes the proof. □

Remark 8. In the proof of Theorem 7, if we put $v_1 = 0$, we can not reach to (20) unless it is assumed that f is even in each component. Recall from [33] that $f : V^n \longrightarrow W$ is even in the kth component if

$$f(x_1, \cdots, x_{k-1}, -x_k, x_{k+1}, \cdots, x_n) = f(x_1, \cdots, x_{k-1}, x_k, x_{k+1}, \cdots, x_n). \quad (35)$$

In other words, this condition is redundant, and we do not need it.

Hereafter, we concentrate our mind on the quasi-β-normed spaces.

Definition 9. Let β be a fix real number with $0 < \beta < 1$ and $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Suppose that E is a vector space over $\mathbb{K}$. A quasi-β-norm is a real-valued function on E fulfilling the next conditions for all $x, y \in E$ and $t \in \mathbb{K}$.

(i) $\|x\| \ge 0$ and moreover $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|tx\| = |t|^\beta \|x\|$

(iii) There exists a real number $M \ge 1$ such that $\|x + y\| \le M(\|x\| + \|y\|)$

When $\beta = 1$, the norm above is a quasinorm. Recall that M is the modulus of concavity of the norm $\|\cdot\|$. Moreover, if $\|\cdot\|$ is a quasi-β-norm on E, the pair $(E, \|\cdot\|)$ is said to be a *quasi- β -normed space*. Similar to normed spaces, a complete quasi-β-normed space is called a *quasi-β-Banach space*. For $0 < p \le 1$, if $\|x + y\|^p \le \|x\|^p + \|y\|^p$, for all $x, y \in E$, then the quasi-β-norm $\|\cdot\|$ is called a (β, p)-norm. In this case, every quasi-β-Banach space is said to be a (β, p)-Banach space. A result of the Aoki-Rolewicz theorem [37] shows that every quasinorm can be equivalent to a p-norm, for some p.

A main tool of this section is the upcoming fixed point lemma which has been proved in ([38], Lemma 3.1).

Lemma 10. *Given the fixed $j \in \{-1, 1\}$ and $a, t \in \mathbb{N}$ with $a \ge 2$. Suppose that V is a linear space and W is a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_W$. If $\phi : V \longrightarrow [0,\infty)$ is a function such that there exists an $L < 1$ with $\phi(a^j v) < L a^{jt\beta}\phi(v)$ for all $v \in V$ and $g : V \longrightarrow W$ is a mapping satisfying*

$$\left\| g(av) - a^t g(v) \right\|_W \le \phi(v), \quad (36)$$

for all $v \in V$, then there exists a uniquely determined mapping $G : V \longrightarrow W$ such that $G(av) = a^t G(v)$ and

$$\|g(v) - G(v)\|_W \le \frac{1}{a^{t\beta}\left|1 - L^j\right|} \phi(v), (v \in V). \quad (37)$$

Furthermore, for each $v \in V$, we have $G(v) = \lim_{l\longrightarrow\infty} (g(a^{jl} v)/a^{jlt})$.

In the next theorem, we prove the Găvruta stability of (14) in quasi-β-normed spaces.

Theorem 11. *Given $j \in \{-1, 1\}$. Let V be a vector space over $\mathbb{Q}$ and W be a (β, p) -Banach space. Assume that $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a function such that $\varphi(m^j v_1, m^j v_2) \le m^{2nj\beta} L\varphi(v_1, v_2)$ for all $v_1, v_2 \in V^n$, where $0 < L < 1$. If a mapping $f : V^n \longrightarrow W$ satisfying H2 and*

$$\|Df(v_1, v_2)\|_W \le \varphi(v_1, v_2), (v_1, v_2 \in V^n), \tag{38}$$

then there is a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (14) so that

$$\|f(v) - \mathcal{Q}(v)\|_W \le \frac{1}{|1 - L^j|} \frac{1}{m^{2n\beta}} \tilde{\varphi}(v), (v \in V^n), \tag{39}$$

where

$$\tilde{\varphi}(v) = M\left[m^{n\beta}\varphi(v, 0) + \varphi(av, bv)\right], \tag{40}$$

whereas M is the modulus of concavity of the norm $\|\cdot\|_W$.

Proof. Setting $v_2 = 0$ in (38) and applying H2, we have

$$\left\|\tilde{f}(v) - m^n f(v)\right\|_W \le \varphi(v, 0), \tag{41}$$

for all $v_1 \coloneqq v \in V^n$, where $\tilde{f}(v) = \tilde{f}(v_1)$ is defined in (21). Interchanging (v_1, v_2) into $(av_1, bv_1) = (av, bv)$ in (38), we obtain

$$\left\|f(mv) - m^n \tilde{f}(v)\right\|_W \le \varphi(av, bv), \tag{42}$$

for all $v \in V^n$. Multiplying both sides of (41) by $m^{n\beta}$, we get

$$\left\|m^n \tilde{f}(v) - m^{2n} f(v)\right\|_W \le m^{n\beta}\varphi(v, 0), \tag{43}$$

for all $v \in V^n$. It follows from (42), (43), and part (iii) of Definition 9 that

$$\left\|f(mv) - m^{2n} f(v)\right\|_W \le \tilde{\varphi}(v), \tag{44}$$

for all $v \in V^n$, where $\tilde{\varphi}(v)$ is defined in (40). By Lemma 10, there exists a mapping $\mathcal{Q} : V^n \longrightarrow W$ which is unique such that $\mathcal{Q}(mv) = m^{2n}\mathcal{Q}(v)$ and

$$\|f(v) - \mathcal{Q}(v)\|_W \le \frac{1}{|1 - L^j|} \frac{1}{m^{2n\beta}} \tilde{\varphi}(v), (v \in V^n). \tag{45}$$

Lastly, we show that $\mathcal{Q}$ fulfilling (14). Note that Lemma 10 implies that for each $v \in V^n$, $\mathcal{Q}(v) = \lim_{l\longrightarrow\infty}(f(m^{jl}v)/m^{2njl})$. For each $v_1, v_2 \in V^n$ and $l \in \mathbb{N}$, by (38), we find

$$\begin{aligned} \left\|\frac{Df\left(m^{jl}v_1, m^{jl}v_2\right)}{m^{2njl}}\right\|_W &\le m^{-2njl\beta}\varphi\left(m^{jl}v_1, m^{jl}v_2\right) \\ &\le m^{-2njl\beta}\left(m^{2nj\beta}L\right)^l \varphi(v_1, v_2) \\ &= L^l \varphi(v_1, v_2). \end{aligned} \tag{46}$$

Taking $l \longrightarrow \infty$ in the last relation, we observe that $D\mathcal{Q}(v_1, v_2) = 0$ for all $v_1, v_2 \in V^n$, and therefore, $\mathcal{Q}$ fulfills (14). □

The following corollary is a consequence of Theorem 11 when the norm of $\|Df(v_1, v_2)\|$ is controlled by sum of variable norms of v_1 and v_2 with positive powers.

Corollary 12. *Let V be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_V$, and W be a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_W$. Let θ and λ be positive numbers with $\lambda \ne 2n(\beta/\alpha)$. If a mapping $f : V^n \longrightarrow W$ satisfying*

$$\|Df(v_1, v_2)\|_W \le \theta \sum_{k=1}^{2}\sum_{l=1}^{n} \|v_{kl}\|_V^\lambda, \tag{47}$$

for all $v_1, v_2 \in V^n$, then there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (14) such that

$$\|f(v) - \mathcal{Q}(v)\|_W \le \begin{cases} \dfrac{\theta\Lambda}{m^{2n\beta} - m^{\alpha\lambda}} \displaystyle\sum_{l=1}^{n} \|v_{1l}\|_V^\lambda, & \lambda \in \left(0, 2n\dfrac{\beta}{\alpha}\right), \\ \dfrac{m^{\alpha\lambda}\Lambda\theta}{m^{2n\beta}\left(m^{\alpha\lambda} - m^{2n\beta}\right)} \displaystyle\sum_{l=1}^{n} \|v_{1l}\|_V^\lambda, & \lambda \in \left(2n\dfrac{\beta}{\alpha}, \infty\right), \end{cases} \tag{48}$$

for all $v = v_1 \in V^n$, where $\Lambda = M[m^{n\beta} + |a|^{\alpha\lambda} + |b|^{\alpha\lambda}]$.

Proof. Taking $\varphi(v_1, v_2) = \theta\sum_{k=1}^{2}\sum_{l=1}^{n}\|v_{kl}\|_V^\lambda$, the result concludes from Theorem 11. □

We bring an elementary lemma without proof as follows.

Lemma 13. *If a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous and satisfies (1), then it has the form $g(x) = cx^2$, for all $x \in \mathbb{R}$, where $c = g(1)$.*

It is easily seen that when $a = b = 1$ in (14), then this equation and (3) are the same. In the upcoming result, we extend Lemma 13 for multivariable functions. In fact, we use it to make a counterexample.

Proposition 14. *Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous which satisfies (3). Then, f has the form*

$$f(r_1, \cdots, r_n) = cr_1^2 \cdots r_n^2, (r_1, \cdots, r_n \in \mathbb{R}), \tag{49}$$

where c is a constant in $\mathbb{R}$.

Proof. We first recall from Theorem 2 in [17] that f is a n-quadratic mapping. By induction on n, we proceed the proof. For $n = 1$, (49) holds by Lemma 13. Assume that (49) is valid for a $n \in \mathbb{N}$, and $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is a continuous $(n + 1)$-quadratic function. Fix the variables $r_1, \cdots, r_n$ in $\mathbb{R}$.

Then, the function $r \mapsto f(r_1, \cdots, r_n, r)$ is quadratic and continuous, and hence, by Lemma 13, f has the form

$$f(r_1, \cdots, r_n, r) = cr^2, (r \in \mathbb{R}), \tag{50}$$

where c is a constant in $\mathbb{R}$. One should note that c depends on $r_1, \cdots, r_n$, and hence

$$c = c(r_1, \cdots, r_n). \tag{51}$$

Letting $r = 1$ in (50) and applying (51), we have

$$c = c(r_1, \cdots, r_n) = f(r_1, \cdots, r_n, 1). \tag{52}$$

It is known that f is $(n+1)$-quadratic and c is an n-quadratic function. Therefore, by the induction assumption, there exists a real number c_0 so that

$$c = c(r_1, \cdots, r_n) = c_0 r_1^2 \cdots r_n^2. \tag{53}$$

It now follows from (50) and (53) that (49) holds for $n+1$.

Here, we present a nonstable example for the multiquadratic mappings on $\mathbb{R}^n$ (see [8]). Indeed, for the case $\alpha = \beta = a = b = 1$, we show that the assumption $\lambda \neq 2n$ can not be eliminated in Corollary 12. □

Example 1. Given $n \in \mathbb{N}$ and $\delta > 0$. Set $\mu := ((2^{2n} - 1)/2^{4n}(2^n + 4^n))\delta$. The function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined via

$$\psi(r_1, \cdots, r_n) = \begin{cases} \mu \prod_{j=1}^{n} r_j^2, \text{for all } r_j \text{ with } |r_j| < 1, \\ \mu, \text{otherwise}. \end{cases} \tag{54}$$

Consider $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ as a function defined by

$$f(r_1, \cdots, r_n) = \sum_{l=0}^{\infty} \frac{\psi(2^l r_1, \cdots, 2^l r_n)}{2^{2nl}}, (r_j \in \mathbb{R}). \tag{55}$$

Obviously, f is a nonnegative function and moreover is an even function in all components. Additionally, ψ is bounded by μ and continuous. Since f is a uniformly convergent series of continuous functions, it is continuous and bounded. In other words, we get $f(r_1, \cdots, r_n) \leq (2^{2n}/(2^{2n}-1))\mu$ for all $(r_1, \cdots, r_n) \in \mathbb{R}^n$. For $i \in \{1, 2\}$, take $x_i = (x_{i1}, \cdots, x_{in})$. We shall prove that

$$|Df(x_1, x_2)| \leq \delta \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n}, \tag{56}$$

for all $x_1, x_2 \in \mathbb{R}^n$. Clearly, (56) holds for $x_1 = x_2 = 0$. Let $x_1, x_2 \in \mathbb{R}^n$ with

$$\sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n} < \frac{1}{2^{2n}}. \tag{57}$$

Inequality (57) necessitates that there is $N \in \mathbb{N}$ such that

$$\frac{1}{2^{2n(N+1)}} < \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n} < \frac{1}{2^{2nN}}, \tag{58}$$

and so $x_{ij}^{2n} < \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n} < 1/2^{2nN}$. It follows the last relation that $2^N |x_{ij}| < 1$ for all $i = 1, 2$ and $j = 1, \cdots, n$. Hence, $2^{N-1}|x_{ij}| < 1$. Let $y_1, y_2 \in \{x_{ij} | i = 1, 2, j = 1, \cdots, n\}$. Then $2^{N-1}|y_1 \pm y_2| < 1$. It is known that ψ is multiquadratic function on $(-1, 1)^n$, and hence, $D\psi(2^l x_1, 2^l x_2) = 0$ for all $l \in \{0, 1, 2, \cdots, N-1\}$. Now, the last equality and relation (58) imply that

$$\begin{aligned} \frac{|Df(2^l x_1, 2^l x_2)|}{\sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n}} &\leq \sum_{l=N}^{\infty} \frac{|D\psi(2^l x_1, 2^l x_2)|}{2^{2nl} \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n}} \\ &\leq \sum_{l=0}^{\infty} \frac{\mu(2^n + 4^n)}{2^{2n(l+N)} \sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n}} \\ &\leq \mu(2^n + 4^n) \sum_{l=0}^{\infty} \frac{1}{2^{2nl}} \\ &\leq \mu(2^n + 4^n) 2^{2n} \frac{2^{2n}}{2^{2n} - 1} \\ &= \mu(2^n + 4^n) \frac{2^{4n}}{2^{2n} - 1} = \delta, \end{aligned} \tag{59}$$

for all $x_1, x_2 \in \mathbb{R}^n$. Hence, (56) is valid for case (57). If $\sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n} \geq 1/2^{2n}$, then

$$\frac{|Df(2^l x_1, 2^l x_2)|}{\sum_{i=1}^{2} \sum_{j=1}^{n} x_{ij}^{2n}} \leq 2^{2n} \frac{2^{2n}}{2^{2n} - 1} \mu(2^n + 4^n) = \delta. \tag{60}$$

Therefore, f satisfies in (56) for all $x_1, x_2 \in \mathbb{R}^n$. Assume that there exists a number $b \in [0, \infty)$ and a multiquadratic function $\mathcal{Q} : \mathbb{R}^n \longrightarrow \mathbb{R}$ for which the inequality $|f(r_1, \cdots, r_n) - \mathcal{Q}(r_1, \cdots, r_n)| < b \prod_{j=1}^{n} r_j^2$ is valid for all $(r_1, \cdots, r_n) \in \mathbb{R}^n$. An application of Proposition 14 shows that there is a constant $c \in \mathbb{R}$ such that $\mathcal{Q}(r_1, \cdots, r_n) = c \prod_{j=1}^{n} r_j^2$, and hence

$$f(r_1, \cdots, r_n) \leq (|c| + b) \prod_{j=1}^{n} r_j^2, ((r_1, \cdots, r_n) \in \mathbb{R}^n). \tag{61}$$

Furthermore, choose $N \in \mathbb{N}$ such that $N\mu > |c| + b$. Take $r = (r_1, \cdots, r_n) \in \mathbb{R}^n$ in which $r_j \in (0, 1/2^{N-1})$ for all $j \in \{1, \cdots, n\}$, then $2^l r_j \in (0, 1)$ for all $l = 0, 1, \cdots, N-1$. Therefore

$$\begin{aligned} f(r_1, \cdots, r_n) &= \sum_{l=0}^{\infty} \frac{\psi(2^l r_1, \cdots, 2^l r_2)}{2^{2nl}} \geq \sum_{l=0}^{N-1} \frac{\mu 2^{2nl} \prod_{j=1}^{n} r_j^2}{2^{2nl}} \\ &= N\mu \prod_{j=1}^{n} r_j^2 > (|c| + b) \prod_{j=1}^{n} r_j^2, \end{aligned} \tag{62}$$

which is a contradiction with (61).

We close the paper by an alternative stability result for equation (14) as follows.

Corollary 15. *Let V be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_V$ and W be a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_W$. Suppose $\lambda_{il} > 0$ for $i \in \{1,2\}$ and $l \in \{1, \cdots, n\}$ with $\lambda = \lambda^* + \lambda^\bullet \neq 2n(\beta/\alpha)$, where $\lambda^* = \sum_{l=1}^{n} \lambda_{1l}$ and $\lambda^\bullet = \sum_{l=1}^{n} \lambda_{2l}$. If a mapping $f : V^n \longrightarrow W$ fulfilling the inequality*

$$\|Df(v_1, v_2)\|_W \leq \theta \prod_{i=1}^{2} \prod_{l=1}^{n} \|v_{il}\|_V^{\lambda_{il}}, \tag{63}$$

for all $v_1, v_2 \in V^n$, then there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (14) so that

$$\|f(v) - \mathcal{Q}(v)\|_W \leq \begin{cases} \dfrac{\theta\Omega}{m^{2n\beta} - m^{\alpha\lambda}} \displaystyle\prod_{l=1}^{n} \|v_{1l}\|_V^{2\lambda_{1l}}, & \lambda \in \left(0, 2n\dfrac{\beta}{\alpha}\right), \\ \dfrac{m^{\alpha\lambda}\Omega\theta}{m^{2n\beta}\left(m^{\alpha\lambda} - m^{2n\beta}\right)} \displaystyle\prod_{l=1}^{n} \|v_{1l}\|_V^{2\lambda_{1l}}, & \lambda \in \left(2n\dfrac{\beta}{\alpha}, \infty\right), \end{cases} \tag{64}$$

for all $v = v_1 \in V^n$, where $\Omega = M|a|^{\alpha\lambda^*}|b|^{\alpha\lambda^\bullet}$.

Proof. Setting $\varphi(v_1, v_2) = \theta\prod_{i=1}^{2}\prod_{l=1}^{n}\|v_{il}\|_V^{\lambda_{il}}$ in Theorem 11, one can obtain the desired results. □

4. Conclusion

In this paper, by using Euler-Lagrange type quadratic functional equations, we have defined the multi-Euler-Lagrange quadratic mappings and have studied the structure of such mappings. Indeed, we have described the multi-Euler-Lagrange quadratic mapping as an equation. In continuation, we have shown that some fixed point theorems can be applied to prove the Hyers-Ulam stability version of multi-Euler-Lagrange quadratic functional equations in the setting of quasi-β-normed and Banach spaces. In the last part, we have brought an example which shows that such functional equations can be nonstable in the some cases.

The current work provides guidelines for further research and proposals for new directions and open problems with relevant discussions. Here, we give some questions and information on the connections between the fixed point theory and the Hyers-Ulam stability.

(1) Which equation can describe the multi-Euler-Lagrange cubic mappings defined in [31]? Are these mappings stable on various Banach spaces? Can the known fixed point methods be useful to prove their Hyers-Ulam stability?

(2) Definition of the multiadditive-quartic mappings by using [14] as a system of n functional equations. The characterization of such mappings and discussion about their stability via a fixed point approach

(3) Applying the functional equations indicated in [5, 12, 13, 34], we can generalize such mappings and equations to multiple variables

Data Availability

All results are obtained without any software and found by manual computations. In other words, the manuscript is in the pure mathematics (mathematical analysis) category.

Conflicts of Interest

There do not exist any competing interests regarding this article.

Authors' Contributions

A.B proposed the topic. H.M and A.M prepared the first draft. Lastly, A.B edited and finalized the manuscript.

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