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Research Article

Generalized Lucas Tau Method for the Numerical Treatment of the One and Two-Dimensional Partial Differential Heat Equation

Y. H. Youssri[1] **W. M. Abd-Elhameed**[1] **and S. M. Sayed**[2]

[1]*Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt*
[2]*Department of Mathematics, Faculty of Science, Helwan University, Cairo 11795, Egypt*

Correspondence should be addressed to Y. H. Youssri; youssri@aucegypt.edu

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This paper is dedicated to proposing two numerical algorithms for solving the one- and two-dimensional heat partial differential equations (PDEs). In these algorithms, generalized Lucas polynomials (GLPs) involving two parameters are utilized as basis functions. The two proposed numerical schemes in one and two- dimensions are based on solving the corresponding integral equation to the heat equation, and after that employing, respectively, the tau and collocation methods to convert the heat equations subject to their underlying conditions into systems of linear algebraic equations that can be treated efficiently via suitable numerical procedures. In this article, the convergence analysis is examined for the proposed generalized Lucas expansion. Five illustrative problems are numerically solved via the two proposed numerical schemes to show the applicability and accuracy of the presented algorithms. Our obtained results compare favourably with the exact solutions.

1. Introduction

Many mathematical models of real-world problems give rise to partial differential equations (*PDEs*) of initial and boundary conditions. *PDEs* are frequently represented as mathematical equations that connect various amounts and their derivatives, e.g., heat transition, a particle's movement in a straight line, the movement of a rocket, a molecule's vibration, and a change in a substance's molecular composition, etc. Every one of these issues is represented by hyperbolic, elliptic, or parabolic partial differential equation (*PPDE*) and might be homogeneous, in one, two, or three dimensions, with non-local boundary conditions in addition to the initial conditions found in the prose. A parabolic *PDE* is used to solve a variety of scientific problems, including ocean acoustic propagation as well as heat diffusion. The hyperbolic *PDE* indicates the wave transformation and sound waves of an elastic string, whereas the elliptic *PDE* describes the Laplace equation.

Fibonacci and Lucas polynomial sequences are crucial and they play vital roles in various disciplines. These sequences are employed to find approximate solutions of different types of DEs. For instance, Fibonacci polynomials were used to treat multi-term fractional DEs in [1]. In [2], Lucas polynomials are employed for the numerical treatment of sinh-Gordon equation. The authors in [3] developed a matrix method using Fibonacci polynomials for the treatment of the generalized pantograph equations with functional arguments. Another approach based on mixed Fibonacci and Lucas polynomials is followed in [4] to obtain numerical solutions of Sobolev equation in two dimensions. Lucas polynomials are employed in [5] to obtain numerical solutions of multidimensional Burgers-type equations. Lucas polynomials were also employed in [6] to solve the fractional-order electro-hydrodynamics flow model.

The Fibonacci and Lucas sequences can be generalized. For example, the authors in [7, 8] introduced two generalized families of Fibonacci and Lucas polynomials. In addition, they employed such generalized sequences to treat some fractional differential equations.

It is well-known that the heat equation is a parabolic *PDE* that describes the distribution of heat. There are two types of heat equations: non-homogeneous and homogeneous. Non-homogeneous heat equations have source terms in the partial differential equations, whereas homogeneous heat equations do not have source terms. Many authors have researched theoretically and numerically the heat equations. For example, the authors in [9] obtained a numerical solution of the one-dimensional heat equation by using the Chebyshev wavelets method. In [10], the authors treated the same equation using a high-order compact boundary value method. The authors in [11] treated the heat equation using radial basis functions. In [12] a modified Crank-Nicolson scheme Richardson extrapolation is followed to treat the one-dimensional heat equation. Recently, the Chebyshev collocation algorithm is followed in [13] to treat the same equation.

A *PDE* governs the temperature of a rod that is frequently defined as [14]:

$$u_t(\xi, t) = K\, u_{\xi\xi}(\xi, t),\ 0 \le \xi \le L,\ t \ge 0, \tag{1}$$

where $u(\xi, t)$ is the temperature of a rod at position ξ at time t and K is the thermal conductivity of the material, which measures the rod's ability to conduct heat.

The solution's domain is a semi-infinite wire of length L that extends endlessly in time. In practice, the result is found only for a limited time. The solution with equation (1) necessitates the requirements of an initial condition at $t = 0$ as well as boundary conditions at $\xi = 0$, and $\xi = L$.

Initial condition:

$$u(\xi, 0) = g(\xi), \quad 0 \le \xi \le L. \tag{2}$$

Boundary conditions:

$$u(0, t) = S_1(t), \quad t \ge 0, \tag{3}$$

$$u(L, t) = S_2(t), \quad t \ge 0. \tag{4}$$

It is essential to refer here that (1) is called the homogeneous heat equation, whereas the non-homogeneous heat equation is given as:

$$u_t(\xi, t) = K\, u_{\xi\xi}(\xi, t) + g(\xi, t), \quad 0 \le \xi \le L, \quad t \ge 0, \tag{5}$$

where $g(\xi, t)$ is referred to as the heat source.

It is worth mentioning that the heat equation (1) governed by (2)-(4) can be extended to higher-dimensional heat equations. These types of equations were treated analytically and numerically by many authors. For example, the Adomian decomposition method was utilized for handling the two-dimensional heat equation in [14]. In addition, the collocation method together with the finite differences was employed to solve the same type of equations in [15]. Some analytical and numerical studies of a two-dimensional nonlinear heat equation with a source term were presented in [16]. Some other forms of the heat equations were handled in other contributions. For example, the authors in [17] applied the finite difference method of lines to treat the heat equation in three space variables. An Adomian decomposition method is applied to the treatment of a non-linear heat equation in [18]. For some other contributions relating to the heat equation, on can be referred to [19–23].

There are numerous methods that have a significant impact on numerical analysis in general, see for example [24–26]. Among these methods are the spectral methods, which play important roles in dealing with *PDEs* [27, 28], ordinary differential equations (*ODEs*) [29, 30], and fractional differential equations (*FDEs*) [31–34]. The basic idea behind spectral methods is that the proposed approximate solution is written as linear combinations of many basic functions, which may be orthogonal or otherwise. The popular spectral approaches are Galerkin, collocation, and tau. In the context of numerical *DEs*, each version has its own significance. Several authors have made extensive use of the latter methods. The Galerkin approach was followed to treat some types of differential equations. For example, the authors in [35] applied the Galerkin method to obtain spectral solutions of BVPs of even-orders, where the authors in [36] obtained approximate solutions of the fractional telegraph equation via implementing a spectral Legendre-Galerkin algorithm. Regarding the collocation method, it is an advantageous method from its capability for treating any type of differential equations governed by any underlying conditions. For example, it is followed in [37] to treat the initial value problems of any order with the aid of the operational matrices of some orthogonal polynomials. The tau method is different from the tau method in that no restrictions on choosing the basis functions. This of course makes its application to different types of *DEs* is easier than the application of the Galerkin method. So, as a result, it is used for solving several types of differential equations.

The structure of this paper is as follows: Section 2 presents an overview of generalized Lucas polynomials and some of their fundamental properties. In Section 3, a numerical method based on the spectral tau method is applied to solve the one-dimensional partial differential heat equation. An extension to solve the two-dimensional heat equation is proposed in Section 4 based on the application of the collocation method. Section 5 examines the convergence and error analysis of the proposed *GLPs* expansion. Numerical outcomes and comparisons are presented in Section 6 to demonstrate the validity of our proposed methods. Section 7 is made up of a brief outline paper.

2. An Overview on Generalized Lucas Polynomials

The purpose of this section is to give an overview of the (*GLPs*). Furthermore, some of the basic formulas of these polynomials are presented.

The *GLPs* may be constructed with the aid of the following recursive formula:

$$\phi_j^{a,b}(\varepsilon) = a\,\varepsilon\,\phi_{j-1}^{a,b}(\varepsilon) + b\,\phi_{j-2}^{a,b}(\varepsilon), \quad \phi_0^{a,b}(\varepsilon) = 2, \quad \phi_1^{a,b}(\varepsilon) = a\,\varepsilon, \quad j \ge 2, \tag{6}$$

They also may be generated by the following Binet's formula:

$$\phi_j^{a,b}(\varepsilon) = \frac{\left(a\,\varepsilon - \sqrt{a^2\,\varepsilon^2 + 4\,b}\right)^j + \left(a\,\varepsilon + \sqrt{a^2\,\varepsilon^2 + 4\,b}\right)^j}{2^j}, \quad j \geq 0. \tag{7}$$

The first few ones of the $\phi_j^{a,b}(\varepsilon)$ are given as follows:

$$\begin{aligned} &\phi_0^{a,b}(\varepsilon) = 2, \phi_1^{a,b}(\varepsilon) = a\,\varepsilon, \\ &\phi_2^{a,b}(\varepsilon) = a^2\,\varepsilon^2 + 2\,b, \phi_3^{a,b}(\varepsilon) = a^3\,\varepsilon^3 + 3\,a\,b\,\varepsilon. \end{aligned} \tag{8}$$

It is important to point out that this kind of polynomials was employed in [8] to deal with some types of fractional *DEs*.

Some celebrated polynomials can be obtained as special cases of the *GLPs* as a result of the existence of two parameters. In fact, the Lucas polynomials $L_i(\varepsilon)$, Fermat-Lucas polynomials $\mathscr{F}_i(\varepsilon)$, Pell-Lucas polynomials $Q_i(\varepsilon)$, Chebyshev polynomials of the first kind $T_i(\varepsilon)$, and Dickson polynomials of the first kind $D_i^\alpha(\varepsilon)$ are special ones of the *GLPs* . Explicitly, we have

$$\begin{aligned} &L_i(\varepsilon) = \phi_i^{1,1}(\varepsilon), \quad \mathscr{F}_i(\varepsilon) = \phi_i^{3,-2}(\varepsilon), \\ &Q_i(\varepsilon) = \phi_i^{2,1}(\varepsilon), \\ &D_i^\alpha(\varepsilon) = \phi_i^{1,-\alpha}(\varepsilon). \end{aligned} \tag{9}$$

The *GLPs* have the following analytic formula ([8]):

$$\phi_j^{a,b}(\varepsilon) = j \sum_{r=0}^{\left[\frac{j}{2}\right]} \frac{\binom{j-r}{r} b^r}{j-r} (a\,\varepsilon)^{j-2r}, \quad j \geq 1, \tag{10}$$

where [z] denotes the well-known floor function, which can also be written as:

$$\phi_j^{a,b}(\varepsilon) = j \sum_{k=0}^{j} \frac{2\,\delta_{j+k} \binom{j+k/2}{j-k/2} b^{j-k/2}}{j+k} (a\,\varepsilon)^k, \tag{11}$$

where

$$\delta_n = \begin{cases} 0, & \text{if } n\ odd, \\ 1, & \text{if } n\ even. \end{cases} \tag{12}$$

3. Numerical Treatment of the One-Dimensional Heat Equation

This section focuses on treating the one-dimensional partial differential heat equation We will analyze a numerical solution of the following one-dimensional linear non-homogeneous heat equation ([14]):

$$u_t(\xi, t) = K\,u_{\xi\xi}(\xi, t) + g(\xi, t), \quad 0 \leq \xi \leq L, \quad t \geq 0, \tag{13}$$

governed by the non-homogeneous boundary conditions:

$$u(0, t) = S_1(t), \quad u(L, t) = S_2(t), \quad t \geq 0, \tag{14}$$

and the initial conditions:

$$u(\xi, 0) = f_1(\xi), \quad 0 \leq \xi \leq L. \tag{15}$$

3.1. Integral Equation Corresponding to (13)-(15). Our strategy to solve the one-dimensional heat equation (13) governed by the conditions (14) and (15) is to treat with its corresponding integral equation.

Now, integrating Eq. (13) with respect to the variable t taking into the consideration the initial condition in (15), we get

$$u(\xi, t) = K \int_0^t u_{\xi\xi}(\xi, \varepsilon)\,d\varepsilon + \int_0^t g(\xi, \varepsilon)\,d\varepsilon + f_1(\xi), \tag{16}$$

governed by the non-homogeneous boundary conditions:

$$u(0, t) = S_1(t), \quad u(L, t) = S_2(t), \quad t \geq 0. \tag{17}$$

3.2. Spectral Tau Treatment for the Heat Equation. The objective of the current section is to propose a spectral tau algorithm for numerically solving the corresponding integral form to the linear one-dimensional heat type equation. First, we consider the two families of basis functions $\{\phi_j^{a,b}(\xi)\}_{j\geq 0}$ and $\{\phi_i^{a,b}(t)\}_{i\geq 0}$. Consider the next two spaces:

$$\begin{aligned} &P = \left\{\varepsilon \in \theta^2(\Omega) \colon \varepsilon(0, t) = \varepsilon(L, t) = 0\,; 0 < t \leq \tau\right\}, \\ &P_M = \operatorname{span}\left\{\phi_j^{a,b}(\xi)\,\phi_i^{a,b}(t) \colon j, i = 0, 1, \cdots, M\right\}, \end{aligned} \tag{18}$$

where $\theta^2(\Omega)$; $\Omega = (0, L) \times (0, \tau]$ is the Sobolev space [38].

Now, the following approximation can be assumed for $u(\xi, t)$:

$$u_M(\xi, t) = \sum_{j=0}^{M} \sum_{i=0}^{M} c_{ji}\,\phi_j^{a,b}(\xi)\,\phi_i^{a,b}(t). \tag{19}$$

To use the spectral tau approach to Eq. (16) implies that we first compute the residual of Eq. (13). It is given by

$$\begin{aligned} \mathbf{R}(\xi, t) = &\sum_{j=0}^{M} \sum_{i=0}^{M} c_{ji}\,\phi_j^{a,b}(\xi)\,\phi_i^{a,b}(t) \\ &- \sum_{j=0}^{M} \sum_{i=0}^{M} c_{ji}\partial_{\xi\xi}\,\phi_j^{a,b}(\xi) \int_0^t \phi_i^{a,b}(\varepsilon)\,d\varepsilon - g_2(\xi, t). \end{aligned} \tag{20}$$

The analytic form of $\phi_j^{a,b}(\xi)$ in (11) allows us to express explicitly $D^2\phi_j^{a,b}(\xi)$ and $\int_0^t \phi_i^{a,b}(\varepsilon)\, d\varepsilon$ in the following forms:

$$D^2\phi_j^{a,b}(\xi) = \begin{cases} 0, & \text{if } j = 0, \\ \displaystyle\sum_{k=0}^{j} \frac{2\, j\, k\, (k-1)\, a^k\, b^{j-k/2}\, \delta_{j+k} \begin{pmatrix} j+k/2 \\ j-k/2 \end{pmatrix}}{(j+k)} \xi^{k-2}, & \text{if } j \geq 1, \end{cases} \tag{21}$$

$$D_t^{-1}\, \phi_i^{(a,b)}(t) = \int_0^t \phi_i^{a,b}(\varepsilon) d\varepsilon = \begin{cases} 2t, & \text{if } i = 0, \\ \displaystyle\sum_{k=0}^{i} \frac{2\, i\, a^k\, b^{i-k/2}\, \delta_{i+k} \begin{pmatrix} i+k/2 \\ i-k/2 \end{pmatrix}}{(i+k)\,(k+1)} t^{k+1}, & \text{if } i \geq 1. \end{cases} \tag{22}$$

Based on the two Formulas (21) and (22), the residual in (20) can be rewritten as

$$\mathrm{R}(\xi, t) = \sum_{j=0}^{M} \sum_{i=0}^{M} 4\, j\, i\, c_{ji} \sum_{k=0}^{j} \sum_{k=0}^{i} \cdot \frac{a^{2k}\, b^{j+i/2-k}\, \delta_{j+k}\, \delta_{i+k} \begin{pmatrix} j+k/2 \\ j-k/2 \end{pmatrix} \begin{pmatrix} i+k/2 \\ i-k/2 \end{pmatrix}}{(j+k)\,(i+k)} \cdot \left((\xi\, t)^k - \frac{k\,(k-1)\, \xi^{k-2}\, t^{k+1}}{k+1} \right) - g_2(\xi, t), \tag{23}$$

and therefore, the following system of equations can be acquired after the spectral tau technique is applied (see, [7]).

$$\int_0^\tau \int_0^L \mathrm{R}(\xi, t) \phi_j^{a,b}(\xi) \phi_i^{a,b}(t)\, d\xi\, dt = 0, \quad 0 \leq j, i \leq M-1. \tag{24}$$

In addition, the boundary conditions (14) give:

$$\sum_{j=0}^{M} \sum_{i=0}^{M} c_{ji}\, \phi_j^{a,b}(0)\, \phi_i^{a,b}\left(\frac{k+1}{M+2}\tau\right) = S_1\left(\frac{k+1}{M+2}\tau\right), \quad 0 \leq k \leq M-1, \tag{25}$$

$$\sum_{j=0}^{M} \sum_{i=0}^{M} c_{ji}\, \phi_j^{a,b}(L)\, \phi_i^{a,b}\left(\frac{k+1}{M+2}\tau\right) = S_2\left(\frac{k+1}{M+2}\tau\right), \quad 0 \leq k \leq M. \tag{26}$$

Eqs. (24), (25), and (26) create a system of linear equations in the dimension $(M+1)^2$ with unknown expansion coefficients c_{ji}. The solution of this system can be found via the Gaussian elimination method.

4. Treatment of the Two-Dimensional Heat Equation

The distribution of heat flow in a two-dimensional space is governed by the following initial boundary value problem (see, [39, 40])

$$u_t(\xi, \eta, t) = \bar{K}\left(u_{\xi\xi}(\xi, \eta, t) + u_{\eta\eta}(\xi, \eta, t)\right); \qquad (\xi, \eta, t) \in \vartheta, \tag{27}$$

subject to the boundary conditions (BCs):

$$\begin{aligned} u(0, \eta, t) &= u(L_1, \eta, t) = 0, \\ u(\xi, 0, t) &= u(\xi, L_2, t) = 0, \end{aligned} \tag{28}$$

and the initial condition (IC):

$$u(\xi, \eta, 0) = g(\xi, \eta), \tag{29}$$

where $u \equiv u(\xi, \eta, t)$ is the temperature of any point located at the position (ξ, η) of a rectangular plate at any time t, $\bar{K}$ is the thermal diffusivity, and $\vartheta = (0, L_1) \times (0, L_2) \times (0, T)$.

We suggest the following approximate spectral solution

$$u_M(\xi, \eta, t) = \sum_{n=0}^{M} \sum_{m=0}^{M} \sum_{\ell=0}^{M} c_{\ell,m,n}\, \phi_\ell^{(a,b)}(\xi)\, \phi_m^{(a,b)}(\eta)\, \phi_n^{(a,b)}(t). \tag{30}$$

By integrating (27) with respect to the variable t and making use of the IC (29), we get

$$u(\xi, \eta, t) = \bar{K} \int_0^t \left(u_{\xi\xi}(\xi, \eta, t) + u_{\eta\eta}(\xi, \eta, t)\right) d\tau + g(\xi, \eta). \tag{31}$$

Now, making use of (21) and (22), we can approximate $\int_0^t (u_{\xi\xi}(\xi, \eta, t) + u_{\eta\eta}(\xi, \eta, t)) d\tau$ in the form:

$$\int_0^t \left(u_{\xi\xi}(\xi, \eta, t) u_{\eta\eta}(\xi, \eta, t)\right) d\tau \approx \sum_{n=0}^{M} \sum_{m=0}^{M} \sum_{\ell=0}^{M} c_{\ell,m,n}\, D_\xi^2\, \phi_\ell^{(a,b)}(\xi)\, D_\eta^2\, \phi_m^{(a,b)}(\eta)\, D_t^{-1}\, \phi_n^{(a,b)}(t), \tag{32}$$

where $D_\xi^2\, \phi_\ell^{(a,b)}(\xi)$, $D_\eta^2\, \phi_m^{(a,b)}(\eta)$ can be expressed by (21), and $D_t^{-1}\, \phi_n^{(a,b)}(t)$ can be expressed by (22).

Our strategy to solve numerically (27)-(29) is to utilize the spectral collocation method. For the residual of (31) is given by

$$\begin{aligned} R(\xi,\eta,t) = & \sum_{n=0}^{M}\sum_{m=0}^{M}\sum_{\ell=0}^{M} c_{\ell,m,n}\phi_{\ell}^{(a,b)}(\xi)\phi_{m}^{(a,b)}(\eta)\,\phi_{n}^{(a,b)}(t) \\ & - \bar{K}\sum_{n=0}^{M}\sum_{m=0}^{M}\sum_{\ell=0}^{M} c_{\ell,m,n}D_{\xi}^{2}\phi_{\ell}^{(a,b)}(\xi)D_{\eta}^{2}\phi_{m}^{(a,b)} \\ & \cdot(\eta)D_{t}^{-1}\phi_{n}^{(a,b)}(t) - g(\xi,\eta) \end{aligned} \tag{33}$$

We choose the following Riemann nodes $P_{ijk}=(\xi_i,\eta_j,t_k)$, with

$$\begin{aligned} \xi_i &= \frac{i+1}{M+2}L_1, \\ \eta_j &= \frac{j+1}{M+2}L_2, \\ t_k &= \frac{k+1}{M+2}T \end{aligned} \tag{34}$$

Hence, the application of the spectral collocation method implies that ([41]),

$$R\left(P_{ijk}\right)=0\,; \quad 0\le i,j\le M, \quad 0\le k\le M-4, \tag{35}$$

and the use of the BCs leads to the following constraints:

$$\begin{aligned} u\left(0,\eta_j,t_k\right) &= 0, \quad 0\le j,k\le M, \\ u\left(L_1,\eta_j,t_k\right) &= 0, \quad 0\le j,k\le M, \\ u(\xi_i,0,t_k) &= 0, \quad 0\le i,k\le M, \\ u(\xi_i,L_2,t_k) &= 0, \quad 0\le i,k\le M. \end{aligned} \tag{36}$$

Now, the above-mentioned equations build a system of algebraic equations of dimension d, where $d=(M+1)^2(M-3)+4(M+1)^2=(M+1)^3$.

Thanks to the Gaussian elimination technique, we get the proposed approximate solution $u_M(\xi,\eta,t)$.

5. Error Analysis and Convergence of the Proposed GLPs Expansion

The goal of this section is to investigate the error analysis and convergence of the GLPs expansion that is used to solve the one-dimensional heat equation (13) governed by the underlying conditions (14) and (15). In the sequel, the next two lemmas are useful.

Lemma 1. *Let $L>0$ and $\xi\in[0,L]$. For the GLPs, the following inequity is valid:*

$$\left|\phi_j^{a,b}(\xi)\right| \le 2\left(a^3+3\,b\,L\right)^{j-1}, \quad j\ge 1. \tag{37}$$

where a and b are positive values.

Proof. We prove by mathematical induction. The inequality is satisfied for $j=1$, since

$$\left|\phi_1^{a,b}(\xi)\right| = |a\,\xi| \le 2. \tag{38}$$

We now assume that (37) is satisfied for $j=k$

$$\left|\phi_k^{a,b}(\xi)\right| \le 2\left(a^3+3\,b\,L\right)^{k-1}. \tag{39}$$

Finally, we demonstrate that validity of (37) for $j=k+1$. Now, we have

$$\begin{aligned} \left|\phi_{k+1}^{a,b}(\xi)\right| &= \left|a\,\xi\,\phi_k^{a,b}(\xi)+b\,\phi_{k-1}^{a,b}(\xi)\right| \\ &\le 2\left|\phi_k^{a,b}(\xi)\right| + |b|\left|\phi_{k-1}^{a,b}(\xi)\right| \\ &= 2\left(a^3+3\,b\,L\right)^{k-1} + 2\,|b|\left(a^3+3\,b\,L\right)^{k-2} \\ &= 2\left(a^3+3\,b\,L\right)^{k}\left[\left(a^3+3\,b\,L\right)^{-1}+|b|\left(a^3+3\,b\,L\right)^{-2}\right] \\ &\le 2\left(a^3+3\,b\,L\right)^{k}. \end{aligned} \tag{40}$$

This ends the proof of Lemma 1. □

Lemma 2. *For all $L>0$, for every positive integer ν, and $\xi\in[0,L]$, the following inequity is valid for the GLPs:*

$$\left|D^{\nu}\phi_j^{a,b}(\xi)\right| \le \frac{13}{4}\,a^2\left(a^3+3\,b\,L\right)^{(j-1)\,\nu}, \tag{41}$$

Proof. By induction on j, we will get started. Assume that the inequality (41) holds for $(j-1)$ and $(j-2)$, and we have to prove that (41) itself holds. Now, our assumption implies that we have the following two inequalities:

$$\left|D^{\nu}\phi_{j-1}^{a,b}(\xi)\right| \le \frac{13}{4}\,a^2\left(a^3+3\,b\,L\right)^{(j-2)\,\nu}, \tag{42}$$

$$\left|D^{\nu}\phi_{j-2}^{a,b}(\xi)\right| \le \frac{13}{4}\,a^2\left(a^3+3\,b\,L\right)^{(j-3)\,\nu}. \tag{43}$$

In virtue of the recurrence relation (6) and the Inequalities (42) and (43), we get

$$\begin{aligned}\left|D^{\nu}\phi_j^{a,b}(\xi)\right| &= \left|\frac{a\,\xi^{1-\nu}}{\Gamma(2-\nu)}\,\phi_{j-1}^{a,b}(\xi) + a\,\xi\,D^{\nu}\phi_{j-1}^{a,b}(\xi) + b\,D^{\nu}\phi_{j-2}^{a,b}(\xi)\right| \\ &\le \frac{2\,L^{-\nu}}{\Gamma(2-\nu)}\,2\left(a^3+3\,b\,L\right)^{(j-2)} \\ &\quad + 2\left(\frac{13}{4}\,a^2\left(a^3+3\,b\,L\right)^{(j-2)\,\nu}\right) \\ &\quad + |b|\left(\frac{13}{4}\,a^2\left(a^3+3\,b\,L\right)^{(j-3)\,\nu}\right) \\ &= \frac{13\,a^2}{4}\left(a^3+3\,b\,L\right)^{(j-1)\,\nu} \\ &\quad \cdot\left[\frac{16\,L^{-\nu}}{13\,a^2\,\Gamma(2-\nu)}\left(a^3+3\,b\,L\right)^{(j-2-j\,\nu+\nu)}\right. \\ &\quad \left. + 2\left(a^3+3\,b\,L\right)^{-\nu} + |b|\left(a^3+3\,b\,L\right)^{-2\,\nu}\right] \\ &= \frac{13\,a^2}{4}\left(a^3+3\,b\,L\right)^{(j-1)\nu} \\ &\quad \cdot\left[\frac{16}{13\,a^2\,L^{\nu}\,\Gamma(2-\nu)\,(a^3+3\,b\,L)^{j\,\nu-\nu-j+2}}\right. \\ &\quad \left. + \frac{2}{(a^3+3\,b\,L)^{\nu}} + \frac{|b|}{(a^3+3\,b\,L)^{2\,\nu}}\right] \\ &\le \frac{13\,a^2}{4}\left(a^3+3\,b\,L\right)^{(j-1)\nu}.\end{aligned} \tag{44}$$

Lemma 2 is now proved. □

Theorem 3. *let $\phi_j^{a,b}(\xi)$ and $\phi_i^{a,b}(t)$ belong to the space P, and let $|(\phi_s^{a,b})^{(k)}(0)| \le \ell_s^k$, $k \ge 0$, $s = i, j$. Let $u(\xi, t)$ be expanded as*

$$u(\xi,t) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_{ji}\,\phi_j^{a,b}(\xi)\phi_i^{a,b}(t). \tag{45}$$

We have the following:

(1) $|c_{ji}| \le |a|^{-j-ji}\ell_j^j\ell_i^i \cosh\left(2|a|^{-1}|b|^{1/2}\ell_j\right)\cosh\left(2|a|^{-1}|b|^{1/2}\ell_i\right)/j!i!$, which ℓ_j, ℓ_i are positive constants.

(2) The Series Comes to a Point of Absolute Convergence.

Proof. The first part of Theorem 3 can be demonstrated by following the same steps that were used in [8]). Now, we prove the remaining part of the theorem. Based on the first part, we have

$$\begin{aligned}|u(\xi,t)| &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left|c_{ij}\phi_j^{a,b}(\xi)\phi_j^{a,b}(t)\right| \\ &\le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left|\frac{|a|^{-j-i}\ell_j^j\ell_i^i\cosh\left(2|a|^{-1}|b|^{1/2}\ell_j\right)\cosh\left(2|a|^{-1}|b|^{1/2}\ell_j\right)}{j!i!}\phi_j^{a,b}(\xi)\phi_j^{a,b}(t)\right|\end{aligned} \tag{46}$$

In virtue of Lemma 1, we get

$$\begin{aligned}&|u(\xi,t)| \\ &\le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left|\frac{|a|^{-j-i}\,\ell_j^j\,\ell_i^i\,\cosh\left(2\,|a|^{-1}\,|b|^{1/2}\,\ell_j\right)\,\cosh\left(2\,|a|^{-1}\,|b|^{1/2}\,\ell_i\right)}{j!i!}\right. \\ &\quad \left.\cdot\left(4\left(a^3+3\,b\,L\right)^{j+i-2}\right)\right| \\ &\le 4\,e^{\left|a^{-1}\,\ell_j\,\left(a^3+3\,b\,L\right)\right| + \left|a^{-1}\,\ell_i\,\left(a^3+3\,b\,L\right)\right|},\end{aligned} \tag{47}$$

then the series comes to a point of absolute convergence. □

Theorem 4. *Let $u(\xi, t)$ that belongs to the space P satisfy the presumptions of Theorem 3, one obtains*

$$|e_M| \le \frac{4\,A\,e^{\zeta}\,e^{\beta}\left[\zeta^{M+1}+\beta^{M+1}\right]}{(M+1)!}, \tag{48}$$

where the constants ζ and β are given as:

$$\begin{aligned}&\zeta = |a|^{-1}\,\ell_j\left(a^3+3\,b\,L\right),\ \beta = |a|^{-1}\,\ell_i\left(a^3+3\,b\,L\right), \text{and} \\ &A = \left(2\,|a|^{-1}\,|b|^{\frac{1}{2}}\,\ell_j\right)^3\left(2\,|a|^{-1}\,|b|^{\frac{1}{2}}\,\ell_i\right)^2.\end{aligned} \tag{49}$$

Proof. If we consider

$$|e_M(\xi,t)| = |u(\xi,t) - u_M(\xi,t)|, \tag{50}$$

then, we have

$$\begin{aligned}|e_M(\xi,t)| &= \left|\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_{ji}\,\phi_j^{a,b}(\xi)\,\phi_i^{a,b}(t) - \sum_{j=0}^{M}\sum_{i=0}^{M} c_{ji}\,\phi_j^{a,b}(\xi)\,\phi_i^{a,b}(t)\right| \\ &\le \left|\sum_{j=0}^{M}\sum_{i=M+1}^{\infty} c_{ji}\,\phi_j^{a,b}(\xi)\,\phi_i^{a,b}(t)\right| \\ &\quad + \left|\sum_{j=M+1}^{\infty}\sum_{i=0}^{\infty} c_{ji}\,\phi_j^{a,b}(\xi)\,\phi_i^{a,b}(t)\right| \\ &\le \sum_{j=0}^{M}\sum_{i=M+1}^{\infty}|c_{ji}|\left|\phi_j^{a,b}(\xi)\right|\left|\phi_i^{a,b}(t)\right| \\ &\quad + \sum_{j=M+1}^{\infty}\sum_{i=0}^{\infty}|c_{ji}|\left|\phi_j^{a,b}(\xi)\right|\left|\phi_i^{a,b}(t)\right|.\end{aligned} \tag{51}$$

From Theorem 3, we get

$$\begin{aligned}|e_M(\xi,t)| \le & \sum_{j=0}^{M}\sum_{i=M+1}^{\infty} \\ & \cdot \frac{|a|^{-j-i}\,\ell_j^j\,\ell_i^i\,\cosh\left(2|a|^{-1}|b|^{1/2}\ell_j\right)\,\cosh\left(2|a|^{-1}|b|^{1/2}\ell_i\right)}{j!\,i!} \\ & \cdot\left|\phi_j^{a,b}(\xi)\right|\left|\phi_i^{a,b}(t)\right| + \sum_{j=M+1}^{\infty}\sum_{i=0}^{\infty} \\ & \cdot \frac{|a|^{-j-i}\,\ell_j^j\,\ell_i^i\,\cosh\left(2|a|^{-1}|b|^{1/2}\ell_j\right)\,\cosh\left(2|a|^{-1}|b|^{1/2}\ell_i\right)}{j!\,i!} \\ & \cdot\left|\phi_j^{a,b}(\xi)\right|\left|\phi_i^{a,b}(t)\right| \\ \le & A\sum_{j=0}^{M}\sum_{i=M+1}^{\infty}\frac{|a|^{-j-i}\,\ell_j^j\,\ell_i^i}{j!\,i!}\left|\phi_j^{a,b}(\xi)\right|\left|\phi_i^{a,b}(t)\right| \\ & + A\sum_{j=M+1}^{\infty}\sum_{i=0}^{\infty}\frac{|a|^{-j-i}\,\ell_j^j\,\ell_i^i}{j!\,i!}\left|\phi_j^{a,b}(\xi)\right|\left|\phi_i^{a,b}(t)\right|.\end{aligned} \tag{52}$$

Based on Lemma 1, we can write

$$\begin{aligned}|e_M| \le & A\sum_{j=0}^{M}\sum_{i=M+1}^{\infty}\frac{4\,|a|^{-j-i}\,\ell_j^j\,\ell_i^i}{j!\,i!}\left(a^3+3\,b\,L\right)^{j+i-2} \\ & + A\sum_{j=M+1}^{\infty}\sum_{i=0}^{\infty}\frac{4\,|a|^{-j-i}\,\ell_j^j\,\ell_i^i}{j!\,i!}\left(a^3+3bL\right)^{j+i-2} \\ \le & A\sum_{j=0}^{M}\frac{2\,|a|^{-j}\,\ell_j^j}{j!}\left(a^3+3bL\right)^j\sum_{i=M+1}^{\infty}\frac{2\,|a|^{-i}\,\ell_i^i}{i!}\left(a^3+3bL\right)^i \\ & + A\sum_{j=M+1}^{\infty}\frac{2\,|a|^{-j}\,\ell_j^j}{j!}\left(a^3+3bL\right)^j \\ & \cdot\sum_{i=0}^{\infty}\frac{2\,|a|^{-i}\,\ell_i^i}{i!}\left(a^3+3\,b\,L\right)^i \\ \le & 4\,A\sum_{j=0}^{M}\frac{\left(|a|^{-1}\,\ell_j\left(a^3+3\,b\,L\right)\right)^j}{j!} \\ & \cdot\sum_{i=M+1}^{\infty}\frac{\left(|a|^{-1}\,\ell_i\left(a^3+3\,b\,L\right)\right)^i}{i!} \\ & + 4A\sum_{j=M+1}^{\infty}\frac{\left(|a|^{-1}\,\ell_j\left(a^3+3\,b\,L\right)\right)^j}{j!} \\ & \cdot\sum_{i=0}^{\infty}\frac{\left(|a|^{-1}\,\ell_i\left(a^3+3\,b\,L\right)\right)^i}{i!} \\ \le & 4A\sum_{j=0}^{M}\frac{(\zeta)^j}{j!}\sum_{i=M+1}^{\infty}\frac{(\beta)^i}{i!} + 4A\sum_{j=M+1}^{\infty}\frac{(\zeta)^j}{j!}\sum_{i=0}^{\infty}\frac{(\beta)^i}{i!} \\ \le & 4Ae^{\zeta}e^{\beta}\left[\frac{\Gamma(M+1,\zeta)}{\Gamma(M+1)}\,\frac{\gamma(M+1,\beta)}{\Gamma(M+1)} + \frac{\gamma(M+1,\zeta)}{\Gamma(M+1)}\right] \\ \le & 4\,Ae^{\zeta}e^{\beta}\left[\frac{\gamma(M+1,\zeta)}{\Gamma(M+1)} + \frac{\gamma(M+1,\beta)}{\Gamma(M+1)}\right]\end{aligned}$$

Table 1: The *MAEs* for Example 5 using the *GLTM*.

t	$M = 2$	$M = 6$
0.1	3.54×10^{-16}	$9.22\ 10^{-12}$
0.3	3.33×10^{-16}	1.57×10^{-11}
0.5	3.89×10^{-16}	2.23×10^{-11}
0.7	5.00×10^{-16}	3.17×10^{-11}
0.9	0	4.40×10^{-11}
1	0	5.05×10^{-11}

$$\le \frac{4\,A\,e^{\zeta}\,e^{\beta}}{\Gamma(M+1)}\left[\int_0^{\zeta}x^M\,e^{-x}\,dx + \int_0^{\beta}x^M\,e^{-x}\,dx\right], \tag{53}$$

and consequently, this leads to

$$|e_M| \le \frac{4\,A\,e^{\zeta}\,e^{\beta}\left[\zeta^{M+1}+\beta^{M+1}\right]}{(M+1)!}, \tag{54}$$

Because of simple inequity: $e^{-x}\le 1;\ x\ge 0$.

Note: $\gamma(.,.)$, $\Gamma(.,.)$, and $\Gamma(.)$ denote, respectively, lower incomplete gamma, upper incomplete gamma, and gamma functions (see, [42]). □

6. Numerical Outcomes and Comparisons

This section presents some examples to demonstrate the accuracy and performance of the following two proposed methods:

(i) The generalized Lucas tau method (*GLTM*) that employed for treating the one-dimensional heat equation.

(ii) The generalized Lucas collocation method (*GLCM*) that employed for treating the two-dimensional heat equation.

The error is represented by E in the maximum norm, that is, in one dimension E is computed by the formula:

$$E = \max\,|u_M(\xi,t) - u(\xi,t)|,\ 0\le\xi\le L,\ t\ge 0, \tag{55}$$

We refer here that Mathematica software was used to perform all of the numerical data.

Example 5 (see [9, 12]). *Consider the following heat equation:*

$$u_t(\xi,t) = u_{\xi\xi}(\xi,t),\quad 0<\xi<1,\quad 0<t<1, \tag{56}$$

with the following initial boundary conditions:

$$u(0,t)=0,\ u(1,t)=e^{-t}\,\sin(1),\ u(\xi,0)=\sin\xi. \tag{57}$$

The exact solution for Eq. (56) is: $u(\xi,t)=e^{-t}\sin\xi$. In Table 1, the maximum absolute errors (*MAEs*) obtained

Table 2: Comparison of the *MAEs* of Example 5.

t	*GLTM*	$M=4$ Method in [9]	Method in [12]
0.1	4.71×10^{-13}	1.07×10^{-5}	2.4079×10^{-7}
0.3	4.55×10^{-13}	3.45×10^{-6}	3.0662×10^{-7}
0.5	4.45×10^{-13}	5.13×10^{-6}	2.6652×10^{-7}
0.7	4.33×10^{-13}	7.45×10^{-6}	2.2036×10^{-7}
0.9	4.16×10^{-13}	9.47×10^{-6}	1.8072×10^{-7}
1	4.08×10^{-13}	1.02×10^{-5}	1.6355×10^{-7}

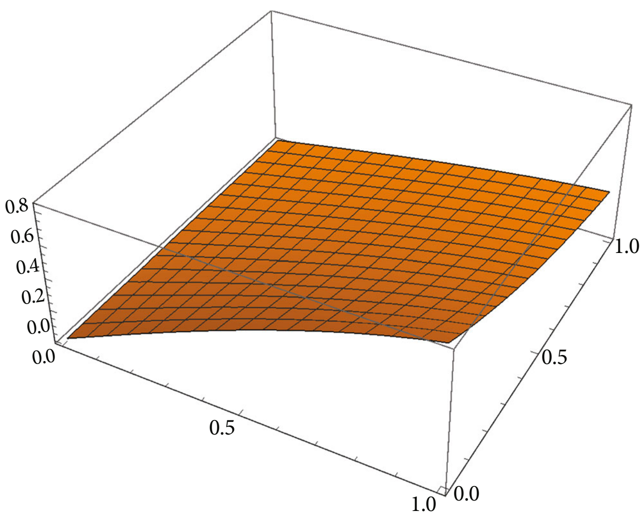

Figure 1: Exact solution for Example 5.

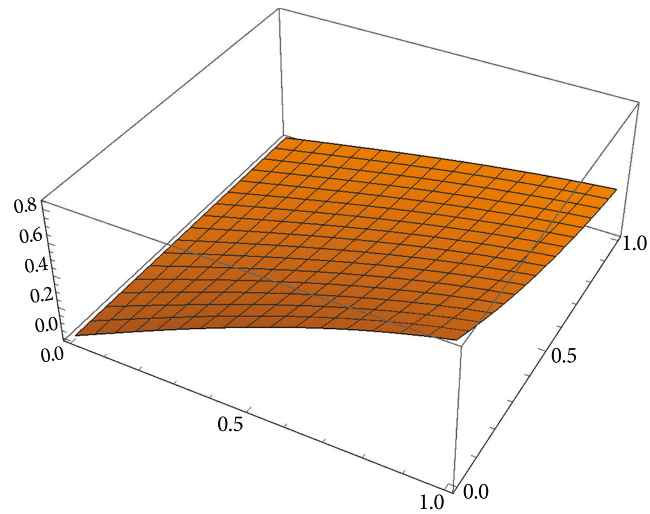

Figure 2: Approximate solution for Example 5.

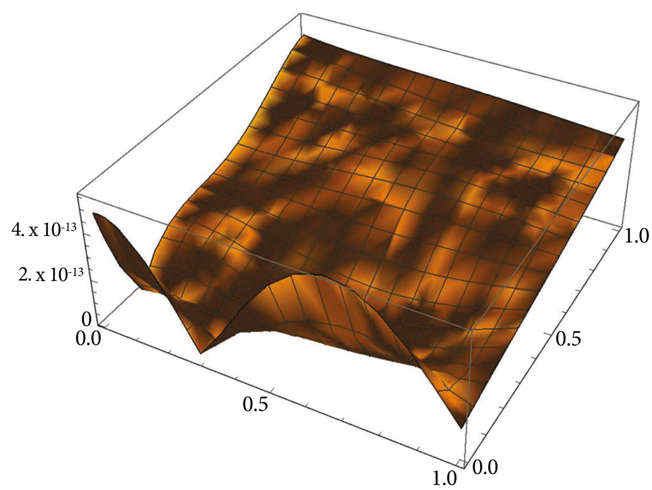

Figure 3: Absolute error for Example 5.

from the application of the *GLTM* are listed for $M=2$ and $M=6$, while in Table 2, we compare the errors resulted from the application of the *GLTM* for the case corresponding to $M=4$ and $a=b=1$ with the best errors resulted from the application of the methods developed in [9, 12]. It is noticed from the obtained results in Table 2 that *GLTM* is more accurate than the two methods that developed in [9, 12]. Figure 1 displays the exact solution. Figure 2 shows the approximate solution for the case corresponding to a = b =1, whereas Figure 3 shows the resulting *AEs* if the *GLTM* is applied.

Example 6 (see [9]). *Consider the heat equation:*

$$u_t(\xi,t)=u_{\xi\xi}(\xi,t)+\left(2t+t^2\right)\ \sin\xi,\quad 0<\xi<1,\quad 0<t<1,\tag{58}$$

governed by the following conditions:

$$u(0,t)=0, u(1,t)=t^2\ \sin(1), u(\xi,0)=0.\tag{59}$$

The exact solution of Eq. (58) is: $u(\xi,t)=t^2\ \sin\xi$. A comparison of the *MAEs* of Example (55) resulting from the *GLTM* for the choices: $M=4$, $(a=1/2, b=1)$, and (a = b =1) and the method applied in [9] is shown in Table 3. From the results in Table 3, it is evident that the *GLTM* is more accurate than the method developed in [9]. Furthermore, the exact solution, approximate solution (for the case corresponding to $M=4, a=b=1$), and

Table 3: Comparison of the *MAEs* of Example 6 for $M=4$.

t	*GLTM* Error $(a=1/2, b=1)$	Error $(a=b=1)$	Method in [9] Error
0.1	1.34×10^{-10}	1.19×10^{-12}	1.40×10^{-3}
0.3	1.55×10^{-10}	1.24×10^{-12}	5.78×10^{-3}
0.5	1.79×10^{-10}	1.25×10^{-12}	5.67×10^{-3}
0.7	1.04×10^{-10}	1.21×10^{-12}	1.10×10^{-3}
0.9	2.40×10^{-10}	1.12×10^{-12}	1.63×10^{-3}
1	2.60×10^{-10}	1.05×10^{-12}	2.27×10^{-3}

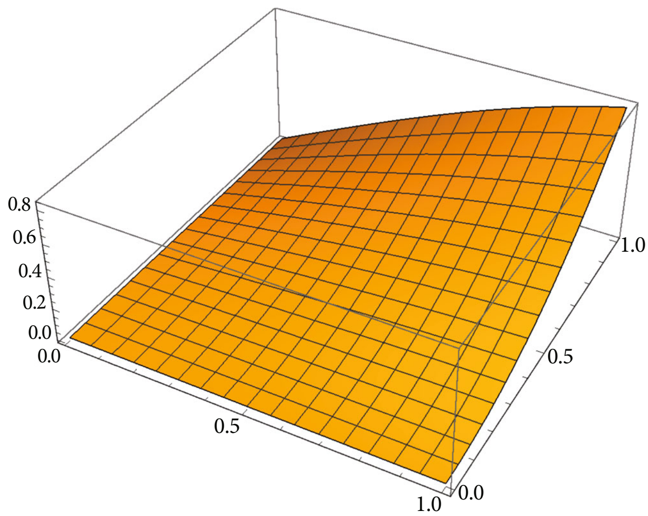

Figure 4: Exact solution for Example 6.

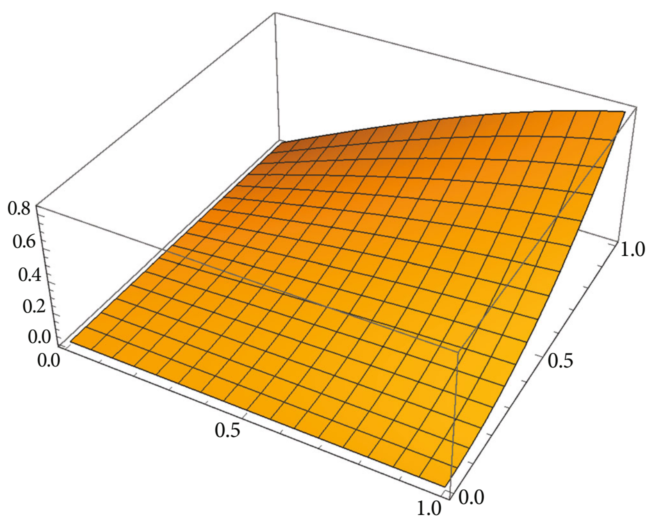

Figure 5: Approximate solution for Example 6.

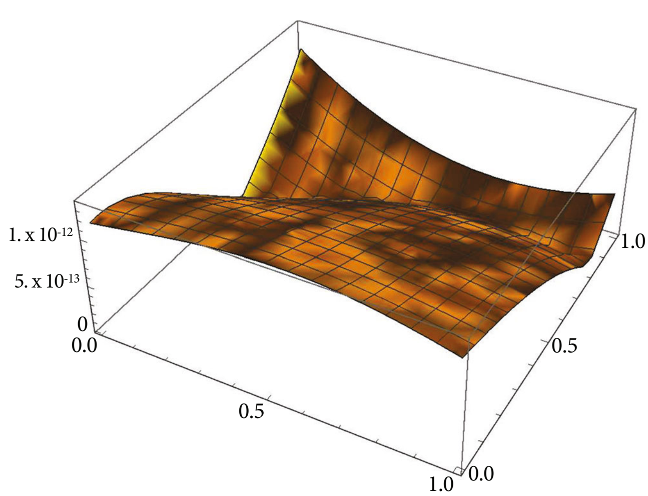

Figure 6: Absolute error for Example 6.

Table 4: Comparison of the *MAEs* of Example 7 for $M = 4$.

t	*GLTM* $(a = 1/2, b = 1)$	*GLTM* $(a = b = 1)$	*GLTM* $(a = 3, b = 1)$	Method in [9]
0.1	5.06×10^{-5}	4.29×10^{-8}	3.87×10^{-10}	6.79×10^{-3}
0.3	5.43×10^{-5}	4.47×10^{-8}	3.86×10^{-10}	3.76×10^{-4}
0.5	5.87×10^{-5}	4.37×10^{-8}	5.76×10^{-10}	2.44×10^{-4}
0.7	6.39×10^{-5}	4.21×10^{-8}	6.53×10^{-10}	3.17×10^{-4}
0.9	6.97×10^{-5}	3.70×10^{-8}	7.69×10^{-10}	3.14×10^{-3}
1	7.29×10^{-5}	3.36×10^{-8}	8.52×10^{-10}	3.32×10^{-3}

Table 5: Comparison of *MAEs* of Example 7.

M	*GLTM*	CN [10]	CBVM [10]
5	4.0×10^{-8}	1.1×10^{-1}	2.8×10^{-2}
10	1.5×10^{-4}	3.0×10^{-2}	3.8×10^{-3}

Table 6: The *MAEs* of Example 7 for $M = 2$.

t	0.1	0.3	0.5	0.7	0.9	1
	9.49×10^{-14}	0	0	0	5.68×10^{-14}	8.53×10^{-14}

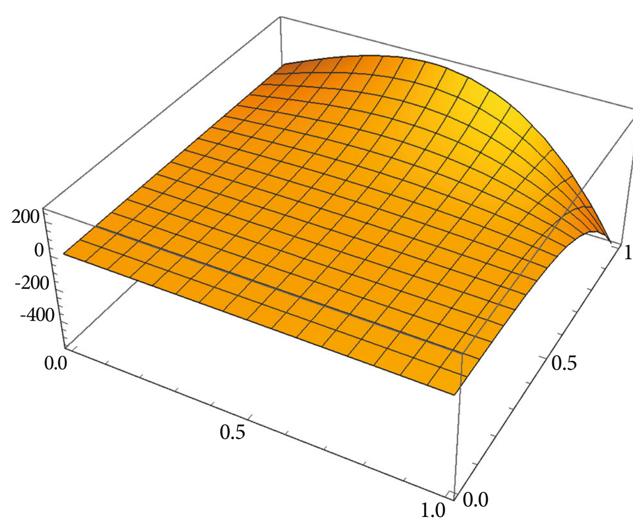

Figure 7: Exact solution for Example 7.

absolute errors of the *GLTM* are displayed, respectively, in Figures 4–6.

Example 7 (see [9, 10]. *The following homogeneous heat equation:*

$$u_t(\xi, t) = u_{\xi\xi}(\xi, t), \quad 0 < \xi < 1, \quad 0 < t < 1, \tag{60}$$

governed by the following conditions:

$$u(0, t) = u(1, t) = 0, u(\xi, 0) = \sin(\pi\xi). \tag{61}$$

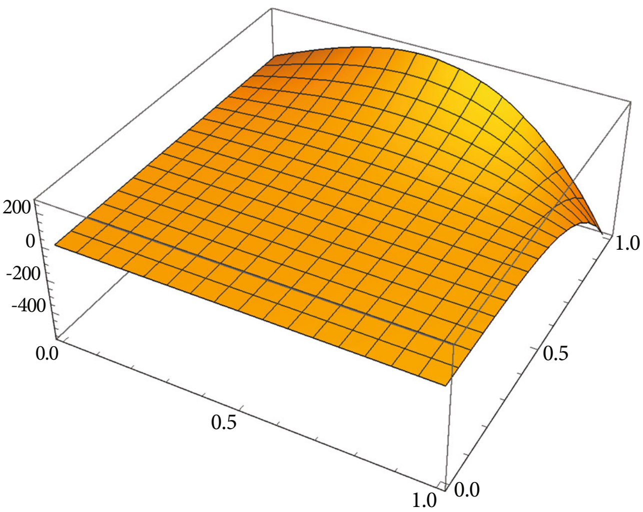

FIGURE 8: Approximate solution for Example 7.

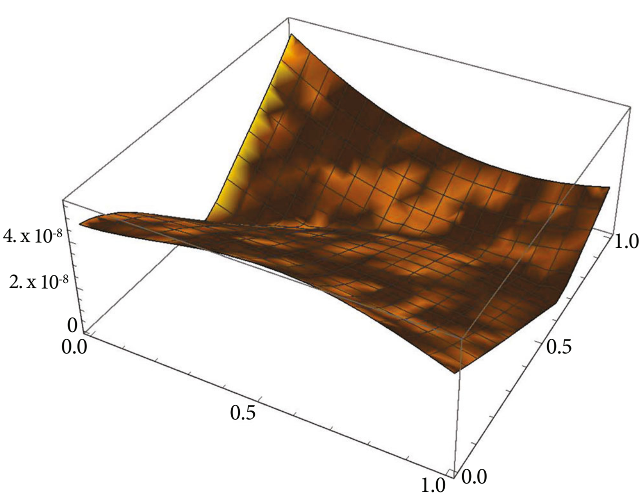

FIGURE 9: Absolute error for Example 7.

TABLE 7: Comparison of the *MAEs* of Example 8 for $M = 4$.

t	$GLTM$ $(a = 1/2, b = 1)$	$GLTM$ $(a = b = 1)$	$GLTM$ $(a = 3, b = 1)$	Method in [9]
0.1	1.96×10^{-13}	1.73×10^{-13}	2.89×10^{-14}	8.44×10^{-3}
0.3	1.94×10^{-13}	1.73×10^{-13}	9.24×10^{-15}	8.10×10^{-3}
0.5	1.98×10^{-13}	1.74×10^{-13}	4.07×10^{-14}	7.43×10^{-3}
0.7	2.00×10^{-13}	1.74×10^{-13}	4.51×10^{-14}	9.81×10^{-3}
0.9	1.87×10^{-13}	1.74×10^{-13}	1.94×10^{-14}	1.07×10^{-3}
1	1.89×10^{-13}	1.74×10^{-13}	0	1.15×10^{-3}

The exact solution of (60) is: $u(\xi, t) = e^{-\pi^2 t} \sin(\pi \xi)$. For $M = 4$, and the three choices: $(a = 1/2, b = 1)$, (a = b = 1), and (a = 3, b = 1), we compare the solutions behavior for GLTM and method in [9] as shown in Table 4. In Table 5, the *MAE* for various values of M and a = b = 1 is listed, which illustrates that the *GLTM* is more accurate than the method

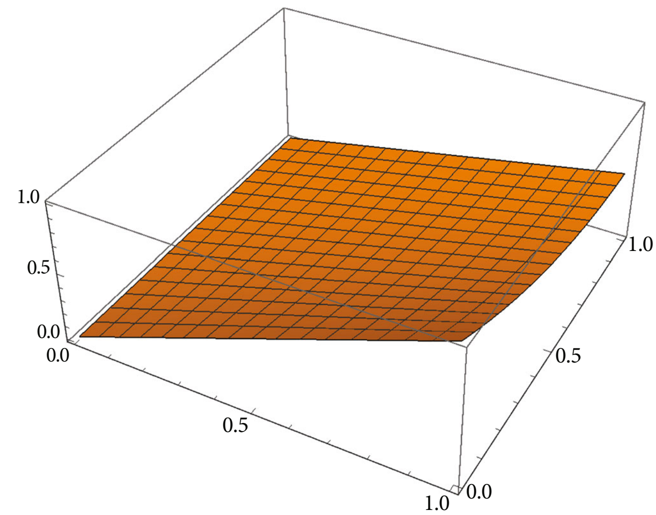

FIGURE 10: Exact solution for Example 8.

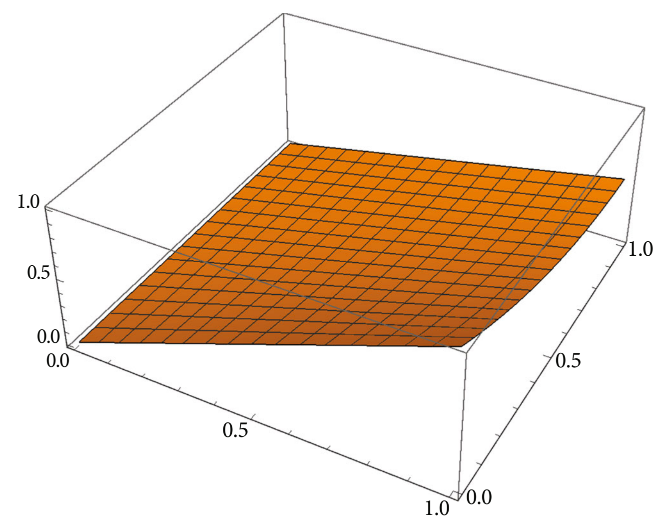

FIGURE 11: Approximate solution for Example 8.

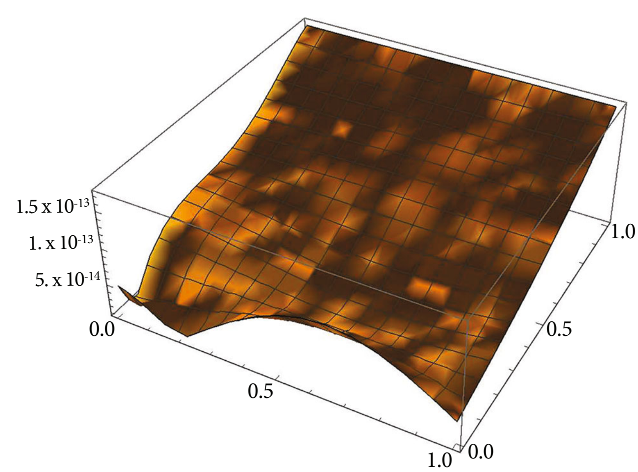

FIGURE 12: Absolute error for Example 8.

Table 8: The maximum pointwise error at different times of Example 9.

$\xi = \eta$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
$\pi/4$	6.94097×10^{-6}	1.88675×10^{-5}	2.63317×10^{-5}	3.11073×10^{-5}
$\pi/2$	1.22752×10^{-3}	3.33673×10^{-3}	4.65679×10^{-3}	5.50135×10^{-3}
$3\pi/4$	1.49306×10^{-2}	4.05855×10^{-2}	5.66416×10^{-2}	6.69141×10^{-2}

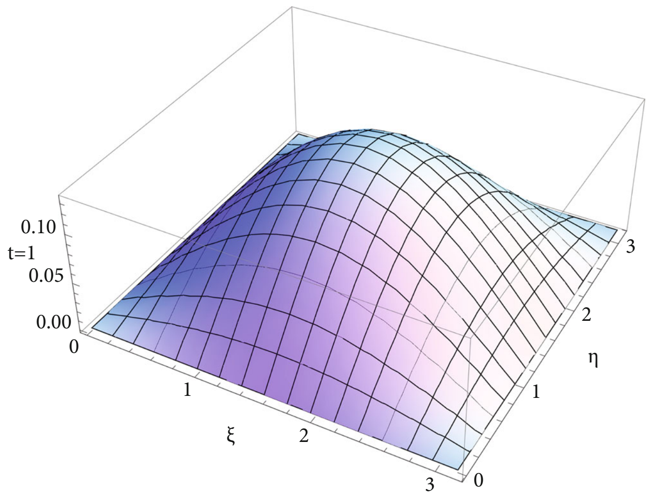

Figure 13: Approximate solution for Example 9 at t =1.

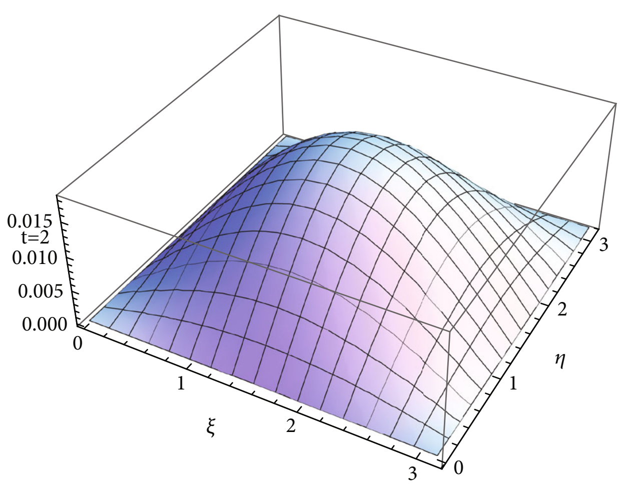

Figure 14: Approximate solution for Example 9 at t = 2.

developed in [9]. Moreover, the approximate solution closes to the exact solution for $M = 2$ and a = b =1 as shown in Table 6. For the three cases correspond to a = b =1, Figures 7 and 8 show the difference between the exact and the approximate solutions. Finally, Figure 9 plotted the absolute error when $M = 4$.

Example 8 (see [9]). *Consider the following homogeneous heat equation:*

$$u_t(\xi, t) = u_{\xi\xi}(\xi, t), \quad 0 < \xi < 1, \quad 0 < t < 1, \tag{62}$$

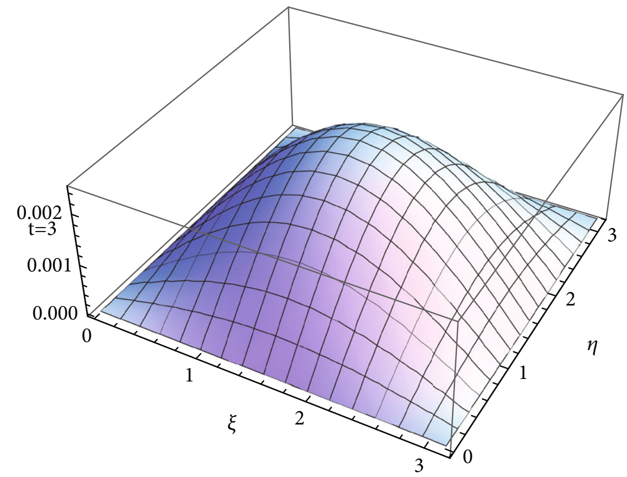

Figure 15: Approximate error for Example 9 at t = 3.

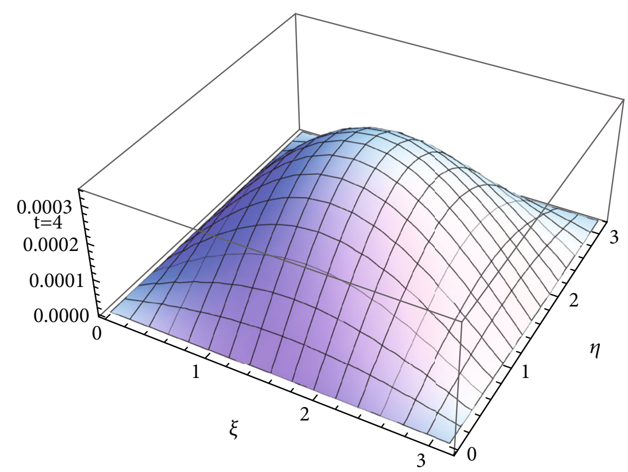

Figure 16: Approximate error for Example 9 at t = 4.

governed by the following conditions:

$$u(0, t) = 0, u(1, t) = \sinh(1)\, e^{-t}, u(\xi, 0) = \sinh \xi. \tag{63}$$

The exact solution of Eq. (62) is given by: $u(\xi, t) = e^{-t} \sinh(\xi)$. We present in Table 7 a comparison between the resulting error from the application of the *GLTM* for the case corresponding to $M = 4$, and for the three choices: $(a = 1/2, b = 1)$, $(a = b = 1)$, and $(a = 3, b = 1)$ with those obtained by the application of the method presented in [9]. In Figures 10–12, the exact solution, approximate solution, and absolute errors for the case corresponding to $a = b = 1$, resulted from the $GLTM$ are, respectively, displayed.

Example 9 (see [39, 40]). *Consider the following two-dimensional heat equation:*

$$u_t(\xi, \eta, t) = u_{\xi\xi}(\xi, \eta, t) + u_{\eta\eta}(\xi, \eta, t), \quad 0 < \xi, \eta < \pi, \quad 0 < t < T, \tag{64}$$

with the boundary conditions:

$$u(0, \eta, t) = u(\pi, \eta, t) = 0, u(\xi, 0, t) = u(\xi, \pi, t) = 0, \tag{65}$$

and the initial condition:

$$u(\xi, \eta, 0) = \sin(\xi) \sin(\eta). \tag{66}$$

The exact solution of Eq. (64) is $u(\xi, \eta, t) = e^{-2t} \sin(\xi) \sin(\eta)$. For different times (t = 1, t = 2, t = 3, and t = 4), we show the maximum pointwise error in Table 8 and the approximate solution in Figures 13, 14, 15, and 16, respectively.

7. Conclusions

In this paper, the generalized Lucas polynomials were utilized along with certain suitable spectral methods for obtaining numerical solutions of one- and two-dimensional heat equations. Two numerical approaches are followed for solving such equations. We showed that the proposed methods are superior if compared to some other methods. We have obtained more precise errors if the retained modes of the approximate expansions are small. Some estimations concerned with the generalized Lucas polynomials were proved and they served to investigate the convergence analysis of the suggested approximate expansion in one dimension. As future work, we plan to use the generalized Lucas polynomials to solve some other types of differential equations. In addition, we plan to use the generalized Lucas polynomials to solve some other types of heat equations.

Data Availability

No data is associated with this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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