

## Retraction

# Retracted: A New Approach to Model Machine Learning by Using Complex Bipolar Intuitionistic Fuzzy Information

### Journal of Function Spaces

Received 23 January 2024; Accepted 23 January 2024; Published 24 January 2024

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

### References

- [1] N. Jan, R. Maqsood, A. Nasir, M. S. Alhilal, A. Alabrah, and N. Al-Aidroos, "A New Approach to Model Machine Learning by Using Complex Bipolar Intuitionistic Fuzzy Information," *Journal of Function Spaces*, vol. 2022, Article ID 3147321, 17 pages, 2022.

## Research Article

# A New Approach to Model Machine Learning by Using Complex Bipolar Intuitionistic Fuzzy Information

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Received 22 November 2021; Accepted 20 December 2021; Published 19 January 2022

Academic Editor: Muhammad Gulzar

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Many industries are developing robust models, capable of analyzing huge and complex data by using machine learning (ML) while delivering faster and more accurate results on vast scales. ML is a subfield of artificial intelligence, which is broadly defined as the capability of a machine to imitate intelligent human behavior. ML tools enable organizations to swiftly identify profitable opportunities and potential risks. Besides these uses, ML also has a wide range of applications in our daily lives. So, the development in ML is most important in this age of digital system to solve more complex problems. In order to further develop ML and diminish the uncertainties to improve accuracy, an innovative concept of complex bipolar intuitionistic fuzzy sets (CBIFSs) is introduced in this article. Further, the Cartesian product of two CBIFSs is defined. Moreover, the complex bipolar intuitionistic fuzzy relations (CBIFRs) and their types with suitable examples are defined. In addition, some important results and properties are also presented. The proposed modeling techniques are used to study different ML factors and their interrelationship, so that the functionality of ML might be enhanced. Furthermore, the advantages and benefits of proposed methods are described by their side to side comparison with preexisting frameworks in the literature.

## 1. Introduction

Uncertainty is the main thing found in each decision of humans. An increasing sense of uncertainty reflects a changing environment that will impact the choices we make. Recognizing and accommodating these changes provide the opportunity to increase decision-making effectiveness. To reduce these uncertainties and ambiguities, in 1965, a new invention by Zadeh [1] is introduced which is capable of modeling uncertainty and ambiguity easily named as fuzzy set (FS). The characteristic of fuzzy sets is that the range of truth value of the membership is the closed interval  $[0, 1]$  of real numbers and membership grades explain the effectiveness of any element. Fuzzy set theory deals with the statement of less or greater form. But crisp set theory only deals with yes or no statement. Klir and Folger [2] intro-

duced the crisp relation which only explains the yes or no situation of any object. This structure gives us limited information and does not help in human decision-making. Application of fuzzy set is found in artificial intelligence [3], social sciences [4], control decisioning [5], expert systems [6], and management sciences [7]. Mendel [8] introduced the fuzzy relations (FRs) for the first time which describe the quality level of any object. Torra [9] presented the hesitant FSs, Zadeh [10] proposed the FSs as a basis for possibility theory, Negoita and Ralescu [11] applied the FSs to system analysis, and Laengle et al. [12] proposed a bibliometric analysis of FSs. Atanassov [13] introduced a new concept of intuitionistic fuzzy sets (IFs) which deals with membership grades as well as nonmembership grades. Both of the grades attain the values from the unit interval  $[0, 1]$ . Membership and nonmembership grades show effectiveness and ineffectiveness

of any object. Burillo and Bustince [14] introduced the concepts of intuitionistic fuzzy relations (IFRs) for IFSs. These sets discuss the relationship in the environment of intuitionistic fuzzy set theory through the membership and nonmembership grades. De et al. [15] applied IFSs in medical diagnosis, Szmidt and Kacprzyk [16] found the distances between IFSs, De et al. [17] defined some operations on IFSs, Gerstenkorn and Mańko [18] gave the correlation of IFSs, and Xue et al. [19] proposed an application of uncertain database retrieval with measure-based belief function attribute values under IFS.

After this, a more advanced form of FS was introduced by Ramot et al. [20] known as complex fuzzy set (CFS) that deals with a multivariable problem and periodicity. It consists of an amplitude term and phase term, i.e.,  $m(x) = \alpha(x) e^{(2\pi)ig(x)}$ , where  $m(x)$  is the complex membership function and  $\alpha(x) \in [0, 1]$  is the amplitude term and  $g(x) \in [0, 1]$  shows the phase term. They also introduced the complex fuzzy relations (CFRs) which explain the relations between CFSs. Zhang et al. [21] explained the  $\delta$ -equalities of the complex fuzzy set. Ramot et al. [22] introduced the complex fuzzy logic. Alkouri and Salleh [23] proposed a new concept of complex intuitionistic fuzzy set (CIFFS) which consists of membership and nonmembership in a complex form and both attains the values from the unit interval  $[0, 1]$ . Jan et al. [24] came up with an innovated idea of complex intuitionistic fuzzy relations (CIFRs). It is an extended form of CFRs. Rani and Garg [25] proposed an application of distance measures between CFSs. Ngan et al. [26] represented CIFFSs by quaternion numbers. Nasir et al. [27] gave an application of cybersecurity against the loopholes in an industrial control system by using interval valued complex intuitionistic fuzzy relations.

An innovation in fuzzy algebra was brought up by Zhang [28] who developed the bipolar fuzzy set (BFS) and bipolar fuzzy relation (BFR). BFSs are more extended form of fuzzy set. In this membership taken in the form of mappings, one is positive mapping that attains values from the interval  $[0, 1]$  and second is negative mapping that attains values from the interval  $[-1, 0]$ . Positive mapping shows possibility, and negative mapping shows the impossibility of any element. Recently, Lee and Hur [29] also described the bipolar fuzzy relations (BFRs). Dudziak and Pe [30] explain the equivalent bipolar fuzzy relations. Lee et al. [31] defined a comparison between interval valued fuzzy sets, intuitionistic fuzzy sets, and bipolar fuzzy sets. Bosc and Pivert [32] worked on the division of bipolar fuzzy relations. Alkouri et al. [33] introduced the concept of complex bipolar fuzzy set (CBFS) which is capable of solving problems with periodicity. Singh [34] introduced bipolar  $\delta$ -equal complex fuzzy lattice with its application. Ezhilmaran and Sankar [35] introduced the concept of intuitionistic bipolar fuzzy set and relations which explain the possibility and impossibility of membership and nonmembership.

In this article, a new structure based on FSs, named as complex bipolar intuitionistic fuzzy set (CBIFFS) and Cartesian product of two CBIFFSs, is introduced. Additionally, it defines the complex bipolar intuitionistic fuzzy relation (CBIFR) and its several types such as reflexive, symmetric, transitive, equivalence, partial order, linear order, strict

order, inverse, and equivalence classes and many more with suitable examples. Some authentic results have also been proven. The innovative structure of CBIFFSs is superior to all preexisting structure, i.e., FS, CFS, IFS, CIFS, BFS, and BIFS. The benefit of this newly introduced structure is that it explains the membership and nonmembership with the properties of possibility and impossibility. It covers all the predefined structures in a way that if nonmembership is equal to zero, it converted into CFBRs. If the phase term and nonmembership are simultaneously removed, it changes into BFRs. If only the phase term is removed, then we get a structure with only amplitude terms, i.e., BIFRs. This article also proposes an application of effective working of ML, which is an important part of a digital system. Every now and then, each organization, institute, industry, and business changes their working setup into a digital system. ML is used to make machines work like humans. The application analyzed the impacts of factors of ML on each other with the help of CBIFRs. In future, this innovative structure of CBIFRs would be used in various fields of sciences like economics, statistics, technology, chemistry, geology, computer science, and physics.

The arrangement of the remaining sections is as follows: Section 2 presents the basic concepts used in this article. Section 3 consists of the newly defined framework of fuzzy algebra with suitable examples, i.e., CBIFRs. Some results also have been proved. Section 4 explains an application of effective working of ML by using CBIFRs. Section 5 compares the CBIFRs with preexisting frameworks to show the superiority of CBIFRs. Section 6 concludes the paper.

## 2. Preliminaries

In this section, we discussed the preexisting structures of fuzzy algebra like fuzzy set (FS), complex fuzzy set (CFS), intuitionistic fuzzy set (IFS), complex intuitionistic fuzzy set (CIFFS), bipolar fuzzy set (BFS), complex bipolar fuzzy set (CBFS), and intuitionistic bipolar fuzzy set (IBFS).

*Definition 1* (see [1]). On a nonempty set  $P$ , a fuzzy set (FS)  $G$  on  $P$  with mappings  $\mu : P \rightarrow [0, 1]$  can be defined as

$$G = \{g, \mu(g) ; g \in P\}. \quad (1)$$

And  $\mu(g)$  is the membership degree of  $g$ .

*Definition 2* (see [20]). On a nonempty set  $P$ , a complex fuzzy set (CFS)  $G$  with a mapping  $\alpha, \partial : P \rightarrow [0, 1]$  can be expressed as

$$G = \left\{ g, \alpha(g) e^{(2\pi)i\partial(g)} ; g \in P \right\}, \quad (2)$$

where  $\alpha(g), \partial(g)$  are the amplitude and phase terms of the membership degree of  $g$  and  $2\pi$  is just a notion which represents the cycle of a circle.

*Definition 3* (see [13]). On a nonempty set  $P$ , an intuitionistic fuzzy set (IFS)  $G$  with a real valued mapping  $\mu, \sigma : P$

$\rightarrow [0, 1]$  can be defined as

$$G = \{g, \mu(g), \sigma(g) ; g \in P\}, \quad (3)$$

where  $\mu(g), \sigma(g)$  are the membership and nonmembership degrees of  $g$  with a condition  $0 \leq \mu(g) + \sigma(g) \leq 1$ .

*Definition 4* (see [23]). On a nonempty set  $P$ , a complex intuitionistic fuzzy set (CIFs)  $G$  can be defined as

$$G = \left\{ g, \alpha_\mu(g)e^{(2\pi)i\partial_\mu(g)}, \alpha_\sigma(g)e^{(2\pi)i\partial_\sigma(g)} ; g \in P \right\}, \quad (4)$$

where  $\alpha_\mu, \alpha_\sigma$  are the amplitude terms of membership and nonmembership degrees,  $\partial_\mu, \partial_\sigma$  are the phase terms of membership and nonmembership degrees of  $p$ , and  $0 \leq \alpha_\mu, \alpha_\sigma, \partial_\mu, \partial_\sigma \leq 1$ .

*Definition 5* (see [28]). On a nonempty set  $P$ , a bipolar fuzzy set (BFS)  $G$  on  $P$  with mappings  $\mu^+ : P \rightarrow [0, 1]$  and  $\mu^- : P \rightarrow [-1, 0]$  can be defined as

$$G = \{g, \mu^+(g), \mu^-(g) ; g \in P\}, \quad (5)$$

where conditions  $\mu^+(g)$  and  $\mu^-(g)$  are positive membership mapping and negative membership mapping, respectively.

*Definition 6* (see [34]). On a nonempty set  $P$ , a complex bipolar fuzzy set (CBFS)  $G$  can be defined as

$$G = \left\{ g, \alpha^+(g)e^{(2\pi)i\partial^+(g)}, \alpha^-(g)e^{(2\pi)i\partial^-(g)} ; g \in P \right\}, \quad (6)$$

where  $\alpha^+(g), \alpha^-(g) : P \rightarrow [0, 1]$  are the positive and negative amplitude terms of membership and nonmembership mapping and  $\partial^+(g), \partial^-(g) : P \rightarrow [0, 1]$  are the positive and negative phase terms of membership and nonmembership mapping.

*Definition 7* (see [35]). On a nonempty set  $P$ , a bipolar intuitionistic fuzzy set (BIFS)  $G$  can be defined as

$$G = \{g, (\mu^+(g), \mu^-(g)), (\sigma^+(g), \sigma^-(g)) ; g \in P\}, \quad (7)$$

where  $\mu^+(g), \sigma^+(g)$ , and  $\mu^-(g), \sigma^-(g)$  are positive and negative membership mappings and nonmembership mappings with conditions  $\mu^+(g), \sigma^+(g) : P \rightarrow [0, 1]$ ,  $\mu^-(g), \sigma^-(g) : P \rightarrow [-1, 0]$ ,  $0 \leq \mu^+(g) + \sigma^+(g) \leq 1$ , and  $-1 \leq \mu^-(g) + \sigma^-(g) \leq 0$ .

*Definition 8* (see [35]). Take two BIFSs  $H = \{h, (\mu^+(h), \mu^-(h)), (\sigma^+(h), \sigma^-(h)) ; h \in P\}$  and  $I = \{i, (\mu^+(i), \mu^-(i)), (\sigma^+(i), \sigma^-(i)) ; i \in P\}$ . Then, their Cartesian product is

$$H \times I = \{(h, i), (\mu^+(h, i), \mu^-(h, i)), (\sigma^+(h, i), \sigma^-(h, i)) ; h, i \in P\}, \quad (8)$$

where  $\mu^+(h, i) = \min \{\mu^+(h), \mu^+(i)\}$ ,  $\mu^-(h, i) = \max \{\mu^-(h), \mu^-(i)\}$ ,  $\sigma^+(h, i) = \max \{\sigma^+(h), \sigma^+(i)\}$ , and  $\sigma^-(h, i) = \min \{\sigma^-(h), \sigma^-(i)\}$ .

*Example 1.* The Cartesian product of two BIFSs  $H = \{(b, (0.54, -0.70), (0.32, -0.11)), (c, (0.32 - 0.09), (0.41 - 0.43))\}$  and  $I = \{(j, (0.11, -0.73), (0.91, -0.03)), (k, (0.42, -0.43), (0.21, -0.29))\}$  is taken as

$$H \times I = \left\{ \begin{aligned} &((b, j), (0.11, -0.70), (0.91, -0.11)), ((b, k), (0.42, -0.43), (0.32, -0.29)) \\ &((c, j), (0.11, -0.09), (0.91, -0.43)), ((c, k), (0.32, -0.09), (0.41, -0.43)) \end{aligned} \right\}. \quad (9)$$

### 3. Main Results

In this section, define some advanced structures which are complex bipolar intuitionistic fuzzy set (CBIFS), Car-

tesian product of two CBIFSs, and CBIF relation and their types.

*Definition 9.* On a nonempty set  $P$ , a complex bipolar intuitionistic fuzzy set (CBIFS)  $G$  can be expressed as

$$G = \left\{ g, \left( \alpha_\mu^+(g)e^{(2\pi)i\partial_\mu^+(g)}, \alpha_\mu^-(g)e^{(2\pi)i\partial_\mu^-(g)} \right), \left( \alpha_\sigma^+(g)e^{(2\pi)i\partial_\sigma^+(g)}, \alpha_\sigma^-(g)e^{(2\pi)i\partial_\sigma^-(g)} \right) ; g \in P \right\}. \quad (10)$$

Here,  $\alpha_\mu^+(g)$ ,  $\partial_\mu^+(g)$  and  $\partial_\mu^-(g)$ ,  $\partial_\mu^-(g)$  are the amplitude and phase terms of positive and negative membership mappings.  $\alpha_\sigma^+(g)$ ,  $\alpha_\sigma^-(g)$  and  $\partial_\sigma^+(g)$ ,  $\partial_\sigma^-(g)$  are the amplitude and phase terms of positive and negative nonmembership mappings with conditions that  $0 \leq \alpha_\mu^+(g)$ ,  $\partial_\mu^+(g)$ ,  $\alpha_\sigma^+(g)$ ,  $\alpha_\sigma^-(g) \leq$

$1$ ,  $-1 \leq \partial_\mu^-(g)$ ,  $\partial_\mu^-(g)$ ,  $\partial_\sigma^+(g)$ ,  $\partial_\sigma^-(g) \leq 0$ ,  $0 \leq \alpha_\mu^+(g) + \alpha_\sigma^+(g) \leq 1$ ,  $0 \leq \partial_\mu^+(g) + \partial_\sigma^+(g) \leq 1$ ,  $-1 \leq \alpha_\mu^-(g) + \alpha_\sigma^-(g) \leq 0$ , and  $-1 \leq \partial_\mu^-(g) + \partial_\sigma^-(g) \leq 0$ .

*Example 2.* The set

$$I = \left\{ \left( i_1, \left( \begin{array}{cc} 0.21e^{0.32(2\pi)i} & 0.67e^{0.51(2\pi)i} \\ -0.55e^{-0.09(2\pi)i} & -0.37e^{-0.62(2\pi)i} \end{array} \right), \left( \begin{array}{cc} 0.11e^{0.27(2\pi)i} & 0.81e^{0.71(2\pi)i} \\ -0.44e^{-0.01(2\pi)i} & -0.11e^{-0.90(2\pi)i} \end{array} \right) \right\} \quad (11)$$

is a CBIFS.

*Definition 10.* Take two CBIFSs on  $P$  as

$$\begin{aligned} G &= \left\{ \left( p, \left( \alpha_\mu^+(p)e^{(2\pi)i\partial_\mu^+(p)}, \alpha_\mu^-(p)e^{i(2\pi)\partial_\mu^-(p)} \right), \left( \alpha_\sigma^+(p)e^{(2\pi)i\partial_\sigma^+(p)}, \alpha_\sigma^-(p)e^{(2\pi)i\partial_\sigma^-(p)} \right) \right); p \in P \right\}, \\ H &= \left\{ \left( q, \left( \alpha_\mu^+(q)e^{(2\pi)i\partial_\mu^+(q)}, \alpha_\mu^-(q)e^{(2\pi)i\partial_\mu^-(q)} \right), \left( \alpha_\sigma^+(q)e^{(2\pi)i\partial_\sigma^+(q)}, \alpha_\sigma^-(q)e^{(2\pi)i\partial_\sigma^-(q)} \right) \right); q \in P \right\}. \end{aligned} \quad (12)$$

Then, their Cartesian product is defined as follows:

$$G \times H = \left\{ \left( p, q, \left( \alpha_\mu^+(p, q)e^{(2\pi)i\partial_\mu^+(p, q)}, \alpha_\mu^-(p, q)e^{(2\pi)i\partial_\mu^-(p, q)} \right), \left( \alpha_\sigma^+(p, q)e^{(2\pi)i\partial_\sigma^+(p, q)}, \alpha_\sigma^-(p, q)e^{(2\pi)i\partial_\sigma^-(p, q)} \right) \right); p, q \in P \right\}, \quad (13)$$

where  $\alpha_\mu^+(p, q) = \min \{ \alpha_\mu^+(p), \alpha_\mu^+(q) \}$ ,  $\partial_\mu^+(p, q) = \min \{ \partial_\mu^+(p), \partial_\mu^+(q) \}$ ,  $\alpha_\mu^-(p, q) = \max \{ \alpha_\mu^-(p), \alpha_\mu^-(q) \}$ ,  $\partial_\mu^-(p, q) = \max \{ \partial_\mu^-(p), \partial_\mu^-(q) \}$ ,  $\alpha_\sigma^+(p, q) = \max \{ \alpha_\sigma^+(p), \alpha_\sigma^+(q) \}$ ,  $\partial_\sigma^+(p, q) = \max$

$\{ \partial_\sigma^+(p), \partial_\sigma^+(q) \}$ ,  $\alpha_\sigma^-(p, q) = \min \{ \alpha_\sigma^-(p), \alpha_\sigma^-(q) \}$ , and  $\partial_\sigma^-(p, q) = \min \{ \partial_\sigma^-(p), \partial_\sigma^-(q) \}$ .

*Example 3.* Take a CBIFS:

$$M = \left\{ \left( m_1, \left( \begin{array}{cc} 0.04e^{0.51(2\pi)i} & 0.25e^{0.42(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} & -0.37e^{-0.52(2\pi)i} \end{array} \right), \left( \begin{array}{cc} 0.15e^{0.05(2\pi)i} & 0.64e^{0.90(2\pi)i} \\ -0.63e^{-0.61(2\pi)i} & -0.32e^{-0.10(2\pi)i} \end{array} \right) \right), \right. \\ \left. \left( m_3, \left( \begin{array}{cc} 0.21e^{0.32(2\pi)i} & 0.55e^{0.41(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} & -0.69e^{-0.32(2\pi)i} \end{array} \right), \left( \begin{array}{cc} 0.38e^{0.43(2\pi)i} & 0.49e^{0.51(2\pi)i} \\ -0.47e^{-0.39(2\pi)i} & -0.31e^{-0.44(2\pi)i} \end{array} \right) \right), \right. \\ \left. \left( m_5, \left( \begin{array}{cc} 0.46e^{0.33(2\pi)i} & 0.37e^{0.42(2\pi)i} \\ -0.47e^{-0.78(2\pi)i} & -0.43e^{-0.19(2\pi)i} \end{array} \right) \right) \right\}. \quad (14)$$

Then, its self-Cartesian product is

$$M \times M = \left\{ \begin{aligned} & \left( (m_1, m_1), \begin{pmatrix} 0.04e^{0.51(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.25e^{0.42(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{pmatrix} \right), \left( (m_1, m_2), \begin{pmatrix} 0.04e^{0.05(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.64e^{0.90(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_1, m_3), \begin{pmatrix} 0.04e^{0.32(2\pi)i} \\ 0.00e^{-0.09(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.55e^{0.42(2\pi)i} \\ -0.69e^{-0.52(2\pi)i} \end{pmatrix} \right), \left( (m_1, m_4), \begin{pmatrix} 0.04e^{0.43(2\pi)i} \\ 0.00e^{-0.39(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.49e^{0.51(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_1, m_5), \begin{pmatrix} 0.04e^{0.33(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.37e^{0.42(2\pi)i} \\ -0.43e^{-0.52(2\pi)i} \end{pmatrix} \right), \left( (m_2, m_1), \begin{pmatrix} 0.04e^{0.05(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.64e^{0.90(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_2, m_2), \begin{pmatrix} 0.15e^{0.05(2\pi)i} \\ -0.63e^{-0.61(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.64e^{0.90(2\pi)i} \\ -0.32e^{-0.10(2\pi)i} \end{pmatrix} \right), \left( (m_2, m_3), \begin{pmatrix} 0.15e^{0.05(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.64e^{0.90(2\pi)i} \\ -0.69e^{-0.32(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_2, m_4), \begin{pmatrix} 0.15e^{0.05(2\pi)i} \\ -0.47e^{-0.39(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.64e^{0.90(2\pi)i} \\ -0.32e^{-0.44(2\pi)i} \end{pmatrix} \right), \left( (m_2, m_5), \begin{pmatrix} 0.15e^{0.05(2\pi)i} \\ -0.47e^{-0.61(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.64e^{0.90(2\pi)i} \\ -0.43e^{-0.19(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_3, m_1), \begin{pmatrix} 0.04e^{0.32(2\pi)i} \\ 0.00e^{-0.09(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.55e^{0.42(2\pi)i} \\ -0.69e^{-0.52(2\pi)i} \end{pmatrix} \right), \left( (m_3, m_2), \begin{pmatrix} 0.15e^{0.05(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.64e^{0.90(2\pi)i} \\ -0.69e^{-0.32(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_3, m_3), \begin{pmatrix} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.55e^{0.41(2\pi)i} \\ -0.69e^{-0.32(2\pi)i} \end{pmatrix} \right), \left( (m_3, m_4), \begin{pmatrix} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.55e^{0.51(2\pi)i} \\ -0.69e^{-0.44(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_3, m_5), \begin{pmatrix} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.55e^{0.42(2\pi)i} \\ -0.69e^{-0.32(2\pi)i} \end{pmatrix} \right), \left( (m_4, m_1), \begin{pmatrix} 0.04e^{0.43(2\pi)i} \\ 0.00e^{-0.39(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.49e^{0.51(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_4, m_2), \begin{pmatrix} 0.15e^{0.05(2\pi)i} \\ -0.47e^{-0.39(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.64e^{0.90(2\pi)i} \\ -0.32e^{-0.44(2\pi)i} \end{pmatrix} \right), \left( (m_4, m_3), \begin{pmatrix} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.55e^{0.51(2\pi)i} \\ -0.69e^{-0.44(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_4, m_4), \begin{pmatrix} 0.38e^{0.43(2\pi)i} \\ -0.47e^{-0.39(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.49e^{0.51(2\pi)i} \\ -0.31e^{-0.44(2\pi)i} \end{pmatrix} \right), \left( (m_4, m_5), \begin{pmatrix} 0.38e^{0.33(2\pi)i} \\ -0.47e^{-0.39(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.49e^{0.51(2\pi)i} \\ -0.43e^{-0.44(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_5, m_1), \begin{pmatrix} 0.04e^{0.33(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.37e^{0.42(2\pi)i} \\ -0.43e^{-0.52(2\pi)i} \end{pmatrix} \right), \left( (m_5, m_2), \begin{pmatrix} 0.15e^{0.05(2\pi)i} \\ -0.47e^{-0.61(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.64e^{0.90(2\pi)i} \\ -0.43e^{-0.19(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_5, m_3), \begin{pmatrix} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.55e^{0.42(2\pi)i} \\ -0.69e^{-0.32(2\pi)i} \end{pmatrix} \right), \left( (m_5, m_4), \begin{pmatrix} 0.38e^{0.33(2\pi)i} \\ -0.47e^{-0.39(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.49e^{0.51(2\pi)i} \\ -0.43e^{-0.44(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_5, m_5), \begin{pmatrix} 0.46e^{0.33(2\pi)i} \\ -0.47e^{-0.78(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.37e^{0.42(2\pi)i} \\ -0.43e^{-0.19(2\pi)i} \end{pmatrix} \right) \end{aligned} \right\} \tag{15}$$

**Definition 11.** Any subset of the Cartesian product of two CBIFSs is called complex bipolar intuitionistic fuzzy relation (CBIFR) and denoted by  $R$ .

**Example 4.** The relation from (15)

$$R = \left\{ \begin{aligned} & \left( (m_1, m_1), \begin{pmatrix} 0.04e^{0.51(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.25e^{0.42(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_1, m_2), \begin{pmatrix} 0.04e^{0.05(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.64e^{0.90(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_2, m_1), \begin{pmatrix} 0.04e^{0.05(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.64e^{0.90(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_2, m_2), \begin{pmatrix} 0.15e^{0.05(2\pi)i} \\ -0.63e^{-0.61(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.64e^{0.90(2\pi)i} \\ -0.32e^{-0.10(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_3, m_3), \begin{pmatrix} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.55e^{0.41(2\pi)i} \\ -0.69e^{-0.32(2\pi)i} \end{pmatrix} \right), \\ & \left( (m_3, m_4), \begin{pmatrix} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{pmatrix}, \begin{pmatrix} 0.55e^{0.51(2\pi)i} \\ -0.69e^{-0.44(2\pi)i} \end{pmatrix} \right), \end{aligned} \right\} \tag{16}$$

is a CBIFR.

**Definition 12.** A relation  $R$  is said to be complex bipolar intuitionistic reflexive fuzzy relation (CBI-reflexive-FR) on CBIFS  $G$ , if

$$\forall (p, \left( \alpha_{\mu}^+(p)e^{i(2\pi)\partial_{\mu}^+(p)}, \alpha_{\mu}^-(p)e^{i(2\pi)\partial_{\mu}^-(p)} \right), \left( \alpha_{\sigma}^+(p)e^{i(2\pi)\partial_{\sigma}^+(p)}, \alpha_{\sigma}^-(p)e^{i(2\pi)\partial_{\sigma}^-(p)} \right)) \in G. \tag{17}$$

Then,

$$\left( (p, p), \left( \alpha_{\mu}^+(p, p)e^{i(2\pi)\partial_{\mu}^+(p, p)}, \alpha_{\mu}^-(p, p)e^{i(2\pi)\partial_{\mu}^-(p, p)} \right), \left( \alpha_{\sigma}^+(p, p)e^{i(2\pi)\partial_{\sigma}^+(p, p)}, \alpha_{\sigma}^-(p, p)e^{i(2\pi)\partial_{\sigma}^-(p, p)} \right) \right) \in R. \tag{18}$$

**Definition 13.** A relation  $R$  is said to be complex bipolar intuitionistic symmetric fuzzy relation (CBI-symmetric-FR) on CBIFS  $G$ , if

$$\forall \left\{ \left( p, \left( \alpha_{\mu}^{+}(p)e^{i(2\pi)\partial_{\mu}^{+}(p)}, \alpha_{\mu}^{-}(p)e^{i(2\pi)\partial_{\mu}^{-}(p)} \right), \left( \alpha_{\sigma}^{+}(p)e^{i(2\pi)\partial_{\sigma}^{+}(p)}, \alpha_{\sigma}^{-}(p)e^{i(2\pi)\partial_{\sigma}^{-}(p)} \right) \right), \right. \\ \left. \left( q, \left( \alpha_{\mu}^{+}(q)e^{i(2\pi)\partial_{\mu}^{+}(q)}, \alpha_{\mu}^{-}(q)e^{i(2\pi)\partial_{\mu}^{-}(q)} \right), \left( \alpha_{\sigma}^{+}(q)e^{i(2\pi)\partial_{\sigma}^{+}(q)}, \alpha_{\sigma}^{-}(q)e^{i(2\pi)\partial_{\sigma}^{-}(q)} \right) \right) \right\} \in G. \quad (19)$$

Then,

$$\left( (p, q), \left( \alpha_{\mu}^{+}(p, q)e^{i(2\pi)\partial_{\mu}^{+}(p, q)}, \alpha_{\mu}^{-}(p, q)e^{i(2\pi)\partial_{\mu}^{-}(p, q)} \right), \left( \alpha_{\sigma}^{+}(p, q)e^{i(2\pi)\partial_{\sigma}^{+}(p, q)}, \alpha_{\sigma}^{-}(p, q)e^{i(2\pi)\partial_{\sigma}^{-}(p, q)} \right) \right) \in R. \quad (20)$$

This implies

$$\forall \left\{ \left( p, \left( \alpha_{\mu}^{+}(p)e^{i(2\pi)\partial_{\mu}^{+}(p)}, \alpha_{\mu}^{-}(p)e^{i(2\pi)\partial_{\mu}^{-}(p)} \right), \left( \alpha_{\sigma}^{+}(p)e^{i(2\pi)\partial_{\sigma}^{+}(p)}, \alpha_{\sigma}^{-}(p)e^{i(2\pi)\partial_{\sigma}^{-}(p)} \right) \right), \right. \\ \left. \left( q, \left( \alpha_{\mu}^{+}(q)e^{i(2\pi)\partial_{\mu}^{+}(q)}, \alpha_{\mu}^{-}(q)e^{i(2\pi)\partial_{\mu}^{-}(q)} \right), \left( \alpha_{\sigma}^{+}(q)e^{i(2\pi)\partial_{\sigma}^{+}(q)}, \alpha_{\sigma}^{-}(q)e^{i(2\pi)\partial_{\sigma}^{-}(q)} \right) \right), \right. \\ \left. \left( r, \left( \alpha_{\mu}^{+}(r)e^{i(2\pi)\partial_{\mu}^{+}(r)}, \alpha_{\mu}^{-}(r)e^{i(2\pi)\partial_{\mu}^{-}(r)} \right), \left( \alpha_{\sigma}^{+}(r)e^{i(2\pi)\partial_{\sigma}^{+}(r)}, \alpha_{\sigma}^{-}(r)e^{i(2\pi)\partial_{\sigma}^{-}(r)} \right) \right) \right\} \in G. \quad (22)$$

Then,

$$\left\{ \left( (p, q), \left( \alpha_{\mu}^{+}(p, q)e^{i(2\pi)\partial_{\mu}^{+}(p, q)}, \alpha_{\mu}^{-}(p, q)e^{i(2\pi)\partial_{\mu}^{-}(p, q)} \right), \left( \alpha_{\sigma}^{+}(p, q)e^{i(2\pi)\partial_{\sigma}^{+}(p, q)}, \alpha_{\sigma}^{-}(p, q)e^{i(2\pi)\partial_{\sigma}^{-}(p, q)} \right) \right), \right. \\ \left. \left( (q, r), \left( \alpha_{\mu}^{+}(q, r)e^{i(2\pi)\partial_{\mu}^{+}(q, r)}, \alpha_{\mu}^{-}(q, r)e^{i(2\pi)\partial_{\mu}^{-}(q, r)} \right), \left( \alpha_{\sigma}^{+}(q, r)e^{i(2\pi)\partial_{\sigma}^{+}(q, r)}, \alpha_{\sigma}^{-}(q, r)e^{i(2\pi)\partial_{\sigma}^{-}(q, r)} \right) \right) \right\} \in R. \quad (23)$$

implies that

$$\left( (p, r), \left( \alpha_{\mu}^{+}(p, r)e^{i(2\pi)\partial_{\mu}^{+}(p, r)}, \alpha_{\mu}^{-}(p, r)e^{i(2\pi)\partial_{\mu}^{-}(p, r)} \right), \left( \alpha_{\sigma}^{+}(p, r)e^{i(2\pi)\partial_{\sigma}^{+}(p, r)}, \alpha_{\sigma}^{-}(p, r)e^{i(2\pi)\partial_{\sigma}^{-}(p, r)} \right) \right) \in R. \quad (24)$$

*Definition 15.* A relation  $R$  is said to be complex bipolar intuitionistic irreflexive fuzzy relation (CBI-irreflexive-FR) on CBIFS  $G$ , if

$$\forall \left( p, \left( \alpha_{\mu}^{+}(p)e^{i(2\pi)\partial_{\mu}^{+}(p)}, \alpha_{\mu}^{-}(p)e^{i(2\pi)\partial_{\mu}^{-}(p)} \right), \left( \alpha_{\sigma}^{+}(p)e^{i(2\pi)\partial_{\sigma}^{+}(p)}, \alpha_{\sigma}^{-}(p)e^{i(2\pi)\partial_{\sigma}^{-}(p)} \right) \right) \in G. \quad (25)$$

Then,

$$\left( (p, p), \left( \alpha_{\mu}^{+}(p, p)e^{i(2\pi)\partial_{\mu}^{+}(p, p)}, \alpha_{\mu}^{-}(p, p)e^{i(2\pi)\partial_{\mu}^{-}(p, p)} \right), \left( \alpha_{\sigma}^{+}(p, p)e^{i(2\pi)\partial_{\sigma}^{+}(p, p)}, \alpha_{\sigma}^{-}(p, p)e^{i(2\pi)\partial_{\sigma}^{-}(p, p)} \right) \right) \notin R. \quad (26)$$

*Definition 16.* A relation  $R$  is said to be complex bipolar intuitionistic antisymmetric fuzzy relation (CBI-antisymmetric-FR) on CBIFS  $G$ , if

$$\forall \left\{ \left( p, \left( \alpha_{\mu}^{+}(p)e^{i(2\pi)\partial_{\mu}^{+}(p)}, \alpha_{\mu}^{-}(p)e^{i(2\pi)\partial_{\mu}^{-}(p)} \right), \left( \alpha_{\sigma}^{+}(p)e^{i(2\pi)\partial_{\sigma}^{+}(p)}, \alpha_{\sigma}^{-}(p)e^{i(2\pi)\partial_{\sigma}^{-}(p)} \right) \right), \right. \\ \left. \left( q, \left( \alpha_{\mu}^{+}(q)e^{i(2\pi)\partial_{\mu}^{+}(q)}, \alpha_{\mu}^{-}(q)e^{i(2\pi)\partial_{\mu}^{-}(q)} \right), \left( \alpha_{\sigma}^{+}(q)e^{i(2\pi)\partial_{\sigma}^{+}(q)}, \alpha_{\sigma}^{-}(q)e^{i(2\pi)\partial_{\sigma}^{-}(q)} \right) \right) \right\} \in G. \quad (27)$$

Then,

$$\left( (p, q), \left( \begin{matrix} \alpha_{\mu}^+(p, q)e^{i(2\pi)\partial_{\mu}^+(p,q)} \\ \alpha_{\mu}^-(p, q)e^{i(2\pi)\partial_{\mu}^-(p,q)} \end{matrix} \right), \left( \begin{matrix} \alpha_{\sigma}^+(p, q)e^{i(2\pi)\partial_{\sigma}^+(p,q)} \\ \alpha_{\sigma}^-(p, q)e^{i(2\pi)\partial_{\sigma}^-(p,q)} \end{matrix} \right) \right) \in R. \tag{28}$$

This implies

$$\left( (q, p), \left( \begin{matrix} \alpha_{\mu}^+(q, p)e^{i(2\pi)\partial_{\mu}^+(q,p)} \\ \alpha_{\mu}^-(q, p)e^{i(2\pi)\partial_{\mu}^-(q,p)} \end{matrix} \right), \left( \begin{matrix} \alpha_{\sigma}^+(q, p)e^{i(2\pi)\partial_{\sigma}^+(q,p)} \\ \alpha_{\sigma}^-(q, p)e^{i(2\pi)\partial_{\sigma}^-(q,p)} \end{matrix} \right) \right) \in R. \tag{29}$$

Then,

$$\left( (p, q), \left( \begin{matrix} \alpha_{\mu}^+(p, q)e^{i(2\pi)\partial_{\mu}^+(p,q)} \\ \alpha_{\mu}^-(p, q)e^{i(2\pi)\partial_{\mu}^-(p,q)} \\ \alpha_{\sigma}^+(p, q)e^{i(2\pi)\partial_{\sigma}^+(p,q)} \\ \alpha_{\sigma}^-(p, q)e^{i(2\pi)\partial_{\sigma}^-(p,q)} \end{matrix} \right) \right) = \left( (q, p), \left( \begin{matrix} \alpha_{\mu}^+(q, p)e^{i(2\pi)\partial_{\mu}^+(q,p)} \\ \alpha_{\mu}^-(q, p)e^{i(2\pi)\partial_{\mu}^-(q,p)} \\ \alpha_{\sigma}^+(q, p)e^{i(2\pi)\partial_{\sigma}^+(q,p)} \\ \alpha_{\sigma}^-(q, p)e^{i(2\pi)\partial_{\sigma}^-(q,p)} \end{matrix} \right) \right). \tag{30}$$

**Definition 17.** A relation  $R$  is said to be complex bipolar intuitionistic complete fuzzy relation (CBI-complete-FR) on CBIFS  $G$ , if

$$\forall \left\{ \left( p, \left( \begin{matrix} \alpha_{\mu}^+(p)e^{i(2\pi)\partial_{\mu}^+(p)} \\ \alpha_{\mu}^-(p)e^{i(2\pi)\partial_{\mu}^-(p)} \end{matrix} \right), \left( \begin{matrix} \alpha_{\sigma}^+(p)e^{i(2\pi)\partial_{\sigma}^+(p)} \\ \alpha_{\sigma}^-(p)e^{i(2\pi)\partial_{\sigma}^-(p)} \end{matrix} \right) \right), \left( q, \left( \begin{matrix} \alpha_{\mu}^+(q)e^{i(2\pi)\partial_{\mu}^+(q)} \\ \alpha_{\mu}^-(q)e^{i(2\pi)\partial_{\mu}^-(q)} \end{matrix} \right), \left( \begin{matrix} \alpha_{\sigma}^+(q)e^{i(2\pi)\partial_{\sigma}^+(q)} \\ \alpha_{\sigma}^-(q)e^{i(2\pi)\partial_{\sigma}^-(q)} \end{matrix} \right) \right) \right\} \in G. \tag{31}$$

Then,

$$\left( (p, q), \left( \begin{matrix} \alpha_{\mu}^+(p, q)e^{i(2\pi)\partial_{\mu}^+(p,q)} \\ \alpha_{\mu}^-(p, q)e^{i(2\pi)\partial_{\mu}^-(p,q)} \end{matrix} \right), \left( \begin{matrix} \alpha_{\sigma}^+(p, q)e^{i(2\pi)\partial_{\sigma}^+(p,q)} \\ \alpha_{\sigma}^-(p, q)e^{i(2\pi)\partial_{\sigma}^-(p,q)} \end{matrix} \right) \right) \in R, \tag{32}$$

or

$$\left( (q, p), \left( \begin{matrix} \alpha_{\mu}^+(q, p)e^{i(2\pi)\partial_{\mu}^+(q,p)} \\ \alpha_{\mu}^-(q, p)e^{i(2\pi)\partial_{\mu}^-(q,p)} \end{matrix} \right), \left( \begin{matrix} \alpha_{\sigma}^+(q, p)e^{i(2\pi)\partial_{\sigma}^+(q,p)} \\ \alpha_{\sigma}^-(q, p)e^{i(2\pi)\partial_{\sigma}^-(q,p)} \end{matrix} \right) \right) \in R. \tag{33}$$

**Definition 18.** A relation  $R$  is said to be complex bipolar intuitionistic equivalence fuzzy relation (CBI-equivalence-FR) on CBIFS  $G$ , if

- (i) CBI-reflexive-FR
- (ii) CBI-symmetric-FR
- (iii) CBI-transitive-FR

**Example 5.** From the Cartesian product in equation (15), take a relation as

$$R = \left\{ \left( \begin{matrix} (m_1, m_1), \left( \begin{matrix} 0.04e^{0.51(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.25e^{0.42(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{matrix} \right) \right), \right. \\ \left( \begin{matrix} (m_1, m_2), \left( \begin{matrix} 0.04e^{0.05(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.64e^{0.90(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{matrix} \right) \right), \\ \left( \begin{matrix} (m_2, m_1), \left( \begin{matrix} 0.04e^{0.05(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.64e^{0.90(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{matrix} \right) \right), \\ \left( \begin{matrix} (m_2, m_2), \left( \begin{matrix} 0.15e^{0.05(2\pi)i} \\ -0.63e^{-0.61(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.64e^{0.90(2\pi)i} \\ -0.32e^{-0.10(2\pi)i} \end{matrix} \right) \right), \\ \left( \begin{matrix} (m_3, m_3), \left( \begin{matrix} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.55e^{0.41(2\pi)i} \\ -0.69e^{-0.32(2\pi)i} \end{matrix} \right) \right), \\ \left( \begin{matrix} (m_3, m_4), \left( \begin{matrix} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.55e^{0.51(2\pi)i} \\ -0.69e^{-0.44(2\pi)i} \end{matrix} \right) \right), \\ \left( \begin{matrix} (m_4, m_3), \left( \begin{matrix} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.55e^{0.51(2\pi)i} \\ -0.69e^{-0.44(2\pi)i} \end{matrix} \right) \right), \\ \left( \begin{matrix} (m_4, m_4), \left( \begin{matrix} 0.38e^{0.43(2\pi)i} \\ -0.47e^{-0.39(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.49e^{0.51(2\pi)i} \\ -0.31e^{-0.44(2\pi)i} \end{matrix} \right) \right), \\ \left( \begin{matrix} (m_5, m_5), \left( \begin{matrix} 0.46e^{0.33(2\pi)i} \\ -0.47e^{-0.78(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.37e^{0.42(2\pi)i} \\ -0.43e^{-0.19(2\pi)i} \end{matrix} \right) \right) \right\}, \tag{34}$$

which is an CBI-equivalence-FR.

**Definition 19.** A relation  $R$  is said to be complex bipolar intuitionistic partial order fuzzy relation (CBI-partial order-FR) on CBIFS  $G$ , if

- (i) CBI-reflexive-FR
- (ii) CBI-antisymmetric-FR
- (iii) CBI-transitive-FR

**Example 6.** Take a relation from (15):

$$R = \left\{ \left( \begin{matrix} (m_1, m_1), \left( \begin{matrix} 0.04e^{0.51(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.25e^{0.42(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{matrix} \right) \right), \right. \\ \left( \begin{matrix} (m_1, m_2), \left( \begin{matrix} 0.04e^{0.05(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.64e^{0.90(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{matrix} \right) \right), \\ \left( \begin{matrix} (m_1, m_5), \left( \begin{matrix} 0.04e^{0.33(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.37e^{0.42(2\pi)i} \\ -0.43e^{-0.52(2\pi)i} \end{matrix} \right) \right), \\ \left( \begin{matrix} (m_2, m_2), \left( \begin{matrix} 0.15e^{0.05(2\pi)i} \\ -0.63e^{-0.61(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.64e^{0.90(2\pi)i} \\ -0.32e^{-0.10(2\pi)i} \end{matrix} \right) \right), \\ \left( \begin{matrix} (m_2, m_5), \left( \begin{matrix} 0.15e^{0.05(2\pi)i} \\ -0.47e^{-0.61(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.64e^{0.90(2\pi)i} \\ -0.43e^{-0.19(2\pi)i} \end{matrix} \right) \right), \\ \left( \begin{matrix} (m_3, m_3), \left( \begin{matrix} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.55e^{0.41(2\pi)i} \\ -0.69e^{-0.32(2\pi)i} \end{matrix} \right) \right), \\ \left( \begin{matrix} (m_4, m_4), \left( \begin{matrix} 0.38e^{0.43(2\pi)i} \\ -0.47e^{-0.39(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.49e^{0.51(2\pi)i} \\ -0.31e^{-0.44(2\pi)i} \end{matrix} \right) \right), \\ \left( \begin{matrix} (m_5, m_5), \left( \begin{matrix} 0.46e^{0.33(2\pi)i} \\ -0.47e^{-0.78(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.37e^{0.42(2\pi)i} \\ -0.43e^{-0.19(2\pi)i} \end{matrix} \right) \right) \right\} \tag{35}$$

is a CBI-partial order-FR.



**Definition 20.** A relation  $R$  is said to be complex bipolar intuitionistic strict order fuzzy relation (CBI-strict order-FR) on CBIFS  $G$ , if

- (i) CBI-irreflexive-FR
- (ii) CBI-transitive-FR

**Example 7.** Take a relation  $R$  from (15):

$$R = \left\{ \begin{array}{l} \left( (m_1, m_2), \left( \begin{array}{l} 0.04e^{0.05(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{array} \right), \left( \begin{array}{l} 0.64e^{0.90(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{array} \right) \right), \\ \left( (m_1, m_3), \left( \begin{array}{l} 0.04e^{0.32(2\pi)i} \\ 0.00e^{-0.09(2\pi)i} \end{array} \right), \left( \begin{array}{l} 0.55e^{0.42(2\pi)i} \\ -0.69e^{-0.52(2\pi)i} \end{array} \right) \right), \\ \left( (m_2, m_3), \left( \begin{array}{l} 0.15e^{0.05(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{array} \right), \left( \begin{array}{l} 0.64e^{0.90(2\pi)i} \\ -0.69e^{-0.32(2\pi)i} \end{array} \right) \right), \\ \left( (m_4, m_2), \left( \begin{array}{l} 0.15e^{0.05(2\pi)i} \\ -0.47e^{-0.39(2\pi)i} \end{array} \right), \left( \begin{array}{l} 0.64e^{0.90(2\pi)i} \\ -0.32e^{-0.44(2\pi)i} \end{array} \right) \right), \\ \left( (m_4, m_3), \left( \begin{array}{l} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{array} \right), \left( \begin{array}{l} 0.55e^{0.51(2\pi)i} \\ -0.69e^{-0.44(2\pi)i} \end{array} \right) \right) \end{array} \right\} \quad (36)$$

is a strict order relation.

**Definition 21.** A relation  $R$  is said to be complex bipolar intuitionistic linear order fuzzy relation (CBI-linear order-FR) on CBIFS  $G$ , if

- (i) CBI-reflexive-FR
- (ii) CBI-antisymmetric-FR
- (iii) CBI-transitive-FR
- (iv) CBI-complete-FR

**Definition 22.** A relation  $R$  is said to be complex bipolar intuitionistic strict order fuzzy relation (CBI-strict order-FR) on CBIFS  $G$ , if

- (i) CBI-reflexive-FR
- (ii) CBI-transitive-FR

**Definition 23.** A relation  $R_1 \circ R_2$  is said to be complex bipolar intuitionistic composite fuzzy relation (CBI-composite-FR) for CBIFRs  $R_1$  and  $R_2$  on CBIFS  $G$ , if

$$\forall \left\{ \begin{array}{l} \left( p, \left( \alpha_{\mu}^{+}(p)e^{i(2\pi)\partial_{\mu}^{+}(p)}, \alpha_{\mu}^{-}(p)e^{i(2\pi)\partial_{\mu}^{-}(p)} \right), \left( \alpha_{\sigma}^{+}(p)e^{i(2\pi)\partial_{\sigma}^{+}(p)}, \alpha_{\sigma}^{-}(p)e^{i(2\pi)\partial_{\sigma}^{-}(p)} \right) \right), \\ \left( q, \left( \alpha_{\mu}^{+}(q)e^{i(2\pi)\partial_{\mu}^{+}(q)}, \alpha_{\mu}^{-}(q)e^{i(2\pi)\partial_{\mu}^{-}(q)} \right), \left( \alpha_{\sigma}^{+}(q)e^{i(2\pi)\partial_{\sigma}^{+}(q)}, \alpha_{\sigma}^{-}(q)e^{i(2\pi)\partial_{\sigma}^{-}(q)} \right) \right), \\ \left( r, \left( \alpha_{\mu}^{+}(r)e^{i(2\pi)\partial_{\mu}^{+}(r)}, \alpha_{\mu}^{-}(r)e^{i(2\pi)\partial_{\mu}^{-}(r)} \right), \left( \alpha_{\sigma}^{+}(r)e^{i(2\pi)\partial_{\sigma}^{+}(r)}, \alpha_{\sigma}^{-}(r)e^{i(2\pi)\partial_{\sigma}^{-}(r)} \right) \right) \end{array} \right\} \in G. \quad (37)$$

Then,

$$\begin{aligned} \left( (p, q), \left( \begin{array}{l} \alpha_{\mu}^{+}(p, q)e^{i(2\pi)\partial_{\mu}^{+}(p, q)} \\ \alpha_{\mu}^{-}(p, q)e^{i(2\pi)\partial_{\mu}^{-}(p, q)} \end{array} \right), \left( \begin{array}{l} \alpha_{\sigma}^{+}(p, q)e^{i(2\pi)\partial_{\sigma}^{+}(p, q)} \\ \alpha_{\sigma}^{-}(p, q)e^{i(2\pi)\partial_{\sigma}^{-}(p, q)} \end{array} \right) \right) \in R_1, \\ \left( (q, r), \left( \begin{array}{l} \alpha_{\mu}^{+}(q, r)e^{i(2\pi)\partial_{\mu}^{+}(q, r)} \\ \alpha_{\mu}^{-}(q, r)e^{i(2\pi)\partial_{\mu}^{-}(q, r)} \end{array} \right), \left( \begin{array}{l} \alpha_{\sigma}^{+}(q, r)e^{i(2\pi)\partial_{\sigma}^{+}(q, r)} \\ \alpha_{\sigma}^{-}(q, r)e^{i(2\pi)\partial_{\sigma}^{-}(q, r)} \end{array} \right) \right) \in R_2 \end{aligned} \quad (38)$$

imply that

$$R_1 \circ R_2 = \left\{ (p, r), \left( \begin{array}{l} \alpha_{\mu}^{+}(p, r)e^{i(2\pi)\partial_{\mu}^{+}(p, r)} \\ \alpha_{\mu}^{-}(p, r)e^{i(2\pi)\partial_{\mu}^{-}(p, r)} \end{array} \right), \left( \begin{array}{l} \alpha_{\sigma}^{+}(p, r)e^{i(2\pi)\partial_{\sigma}^{+}(p, r)} \\ \alpha_{\sigma}^{-}(p, r)e^{i(2\pi)\partial_{\sigma}^{-}(p, r)} \end{array} \right) \right\}. \quad (39)$$

**Definition 24.** A relation  $R$  is said to be complex bipolar intuitionistic converse fuzzy relation (CBI-converse-FR) on CBIFS  $G$ , if

$$\forall \left\{ \begin{array}{l} \left( p, \left( \alpha_{\mu}^{+}(p)e^{i(2\pi)\partial_{\mu}^{+}(p)}, \alpha_{\mu}^{-}(p)e^{i(2\pi)\partial_{\mu}^{-}(p)} \right), \left( \alpha_{\sigma}^{+}(p)e^{i(2\pi)\partial_{\sigma}^{+}(p)}, \alpha_{\sigma}^{-}(p)e^{i(2\pi)\partial_{\sigma}^{-}(p)} \right) \right), \\ \left( q, \left( \alpha_{\mu}^{+}(q)e^{i(2\pi)\partial_{\mu}^{+}(q)}, \alpha_{\mu}^{-}(q)e^{i(2\pi)\partial_{\mu}^{-}(q)} \right), \left( \alpha_{\sigma}^{+}(q)e^{i(2\pi)\partial_{\sigma}^{+}(q)}, \alpha_{\sigma}^{-}(q)e^{i(2\pi)\partial_{\sigma}^{-}(q)} \right) \right) \end{array} \right\} \in G. \quad (40)$$

Then,

$$\forall \left( (p, q), \left( \alpha_{\mu}^{+}(p, q)e^{i(2\pi)\partial_{\mu}^{+}(p,q)}, \alpha_{\mu}^{-}(p, q)e^{i(2\pi)\partial_{\mu}^{-}(p,q)} \right), \left( \alpha_{\sigma}^{+}(p, q)e^{i(2\pi)\partial_{\sigma}^{+}(p,q)}, \alpha_{\sigma}^{-}(p, q)e^{i(2\pi)\partial_{\sigma}^{-}(p,q)} \right) \right) \in R. \tag{41}$$

This implies

$$R^{-1} = \left\{ (q, p), \left( \alpha_{\mu}^{+}(q, p)e^{i(2\pi)\partial_{\mu}^{+}(q,p)}, \alpha_{\mu}^{-}(q, p)e^{i(2\pi)\partial_{\mu}^{-}(q,p)} \right), \left( \alpha_{\sigma}^{+}(q, p)e^{i(2\pi)\partial_{\sigma}^{+}(q,p)}, \alpha_{\sigma}^{-}(q, p)e^{i(2\pi)\partial_{\sigma}^{-}(q,p)} \right) \right\}. \tag{42}$$

*Definition 25.* The complex bipolar intuitionistic fuzzy equivalence class of  $p$  modulo  $R$  is defined as

$$R[p] = \left\{ \left( q, \left( \alpha_{\mu}^{+}(q)e^{i(2\pi)\partial_{\mu}^{+}(q)}, \alpha_{\mu}^{-}(q)e^{i(2\pi)\partial_{\mu}^{-}(q)} \right), \left( \alpha_{\sigma}^{+}(q)e^{i(2\pi)\partial_{\sigma}^{+}(q)}, \alpha_{\sigma}^{-}(q)e^{i(2\pi)\partial_{\sigma}^{-}(q)} \right) \right); \left( (p, q), \left( \alpha_{\mu}^{+}(p, q)e^{i(2\pi)\partial_{\mu}^{+}(p,q)}, \alpha_{\mu}^{-}(p, q)e^{i(2\pi)\partial_{\mu}^{-}(p,q)} \right), \left( \alpha_{\sigma}^{+}(p, q)e^{i(2\pi)\partial_{\sigma}^{+}(p,q)}, \alpha_{\sigma}^{-}(p, q)e^{i(2\pi)\partial_{\sigma}^{-}(p,q)} \right) \right) \in R \right\}, \tag{43}$$

for  $(p, (\alpha_{\mu}^{+}(p)e^{i(2\pi)\partial_{\mu}^{+}(p)}, \alpha_{\mu}^{-}(p)e^{i(2\pi)\partial_{\mu}^{-}(p)}), (\alpha_{\sigma}^{+}(p)e^{i(2\pi)\partial_{\sigma}^{+}(p)}, \alpha_{\sigma}^{-}(p)e^{i(2\pi)\partial_{\sigma}^{-}(p)}))$  and a CBI-equivalence-FR  $R$ .

*Example 8.* Take a CBIFS:

$$M = \left\{ \left( m_1, \left( \begin{matrix} 0.04e^{0.51(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.25e^{0.42(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{matrix} \right) \right), \left( m_2, \left( \begin{matrix} 0.15e^{0.05(2\pi)i} \\ -0.63e^{-0.61(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.64e^{0.90(2\pi)i} \\ -0.32e^{-0.10(2\pi)i} \end{matrix} \right) \right), \right. \\ \left. \left( m_3, \left( \begin{matrix} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.55e^{0.41(2\pi)i} \\ -0.69e^{-0.32(2\pi)i} \end{matrix} \right) \right) \right\}. \tag{44}$$

Take an equivalence relation from the Cartesian product of  $M \times M$  which is

$$R = \left\{ \left( (m_1, m_1), \left( \begin{matrix} 0.04e^{0.51(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.25e^{0.42(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{matrix} \right) \right), \right. \\ \left( (m_1, m_2), \left( \begin{matrix} 0.04e^{0.05(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.64e^{0.90(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{matrix} \right) \right), \\ \left( (m_2, m_1), \left( \begin{matrix} 0.04e^{0.05(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.64e^{0.90(2\pi)i} \\ -0.37e^{-0.52(2\pi)i} \end{matrix} \right) \right), \\ \left( (m_2, m_2), \left( \begin{matrix} 0.15e^{0.05(2\pi)i} \\ -0.63e^{-0.61(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.64e^{0.90(2\pi)i} \\ -0.32e^{-0.10(2\pi)i} \end{matrix} \right) \right), \\ \left( (m_3, m_3), \left( \begin{matrix} 0.21e^{0.32(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.55e^{0.41(2\pi)i} \\ -0.69e^{-0.32(2\pi)i} \end{matrix} \right) \right), \right\}. \tag{45}$$

Then, modulo classes of each element are defined as

$$R[m_1] = \left\{ \left( m_1, \left( \begin{matrix} 0.04e^{0.51(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.25e^{0.42(2\pi)i} \\ -0.37e^{-0.62(2\pi)i} \end{matrix} \right) \right), \right. \\ \left. \left( m_2, \left( \begin{matrix} 0.04e^{0.05(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.64e^{0.90(2\pi)i} \\ -0.63e^{0.61(2\pi)i} \end{matrix} \right) \right) \right\},$$

$$R[m_2] = \left\{ \left( m_1, \left( \begin{matrix} 0.04e^{0.51(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.25e^{0.42(2\pi)i} \\ -0.37e^{-0.62(2\pi)i} \end{matrix} \right) \right), \right. \\ \left. \left( m_2, \left( \begin{matrix} 0.04e^{0.05(2\pi)i} \\ 0.00e^{-0.42(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.64e^{0.90(2\pi)i} \\ -0.63e^{0.61(2\pi)i} \end{matrix} \right) \right) \right\}, \\ R[m_3] = \left\{ \left( m_3, \left( \begin{matrix} 0.15e^{0.05(2\pi)i} \\ -0.01e^{-0.09(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.64e^{0.90(2\pi)i} \\ -0.69e^{-0.32(2\pi)i} \end{matrix} \right) \right) \right\}. \tag{46}$$

**Theorem 26.** A CBIFR  $R$  on  $G$  is a CBI-symmetric-FR  $\iff R = R^{-1}$ .

*Proof.* Necessity condition:

Assume that  $R = R^{-1}$ . Then,

$$\left( (p, q), \left( \alpha_{\mu}^{+}(p, q)e^{(2\pi)i\partial_{\mu}^{+}(p,q)}, \alpha_{\mu}^{-}(p, q)e^{(2\pi)i\partial_{\mu}^{-}(p,q)} \right), \left( \alpha_{\sigma}^{+}(p, q)e^{(2\pi)i\partial_{\sigma}^{+}(p,q)}, \alpha_{\sigma}^{-}(p, q)e^{(2\pi)i\partial_{\sigma}^{-}(p,q)} \right) \right) \in R, \\ \iff \left( (q, p), \left( \alpha_{\mu}^{+}(q, p)e^{(2\pi)i\partial_{\mu}^{+}(q,p)}, \alpha_{\mu}^{-}(q, p)e^{(2\pi)i\partial_{\mu}^{-}(q,p)} \right), \left( \alpha_{\sigma}^{+}(q, p)e^{(2\pi)i\partial_{\sigma}^{+}(q,p)}, \alpha_{\sigma}^{-}(q, p)e^{(2\pi)i\partial_{\sigma}^{-}(q,p)} \right) \right) \in R^{-1}. \tag{47}$$

Since  $R = R^{-1}$ , thus,

$$\Leftrightarrow \left( (q, p), \left( \begin{array}{c} \alpha_{\mu}^{+}(q, p)e^{(2\pi)i\partial_{\mu}^{+}(q,p)} \\ \alpha_{\mu}^{-}(q, p)e^{(2\pi)i\partial_{\mu}^{-}(q,p)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(q, p)e^{(2\pi)i\partial_{\sigma}^{+}(q,p)} \\ \alpha_{\sigma}^{-}(q, p)e^{(2\pi)i\partial_{\sigma}^{-}(q,p)} \end{array} \right) \right) \in R. \quad (48)$$

Therefore,  $R$  is a complex bipolar intuitionistic symmetric fuzzy relation on  $G$ .

Sufficient condition:

Assume that  $R$  is a complex bipolar intuitionistic symmetric fuzzy relation on  $G$ ; then,

$$\begin{aligned} & \left( (p, q), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, q)e^{(2\pi)i\partial_{\mu}^{+}(p,q)} \\ \alpha_{\mu}^{-}(p, q)e^{(2\pi)i\partial_{\mu}^{-}(p,q)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, q)e^{(2\pi)i\partial_{\sigma}^{+}(p,q)} \\ \alpha_{\sigma}^{-}(p, q)e^{(2\pi)i\partial_{\sigma}^{-}(p,q)} \end{array} \right) \right) \in R, \\ \Leftrightarrow & \left( (q, p), \left( \begin{array}{c} \alpha_{\mu}^{+}(q, p)e^{(2\pi)i\partial_{\mu}^{+}(q,p)} \\ \alpha_{\mu}^{-}(q, p)e^{(2\pi)i\partial_{\mu}^{-}(q,p)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(q, p)e^{(2\pi)i\partial_{\sigma}^{+}(q,p)} \\ \alpha_{\sigma}^{-}(q, p)e^{(2\pi)i\partial_{\sigma}^{-}(q,p)} \end{array} \right) \right) \in R. \end{aligned} \quad (49)$$

But

$$\left( (q, p), \left( \begin{array}{c} \alpha_{\mu}^{+}(q, p)e^{(2\pi)i\partial_{\mu}^{+}(q,p)} \\ \alpha_{\mu}^{-}(q, p)e^{(2\pi)i\partial_{\mu}^{-}(q,p)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(q, p)e^{(2\pi)i\partial_{\sigma}^{+}(q,p)} \\ \alpha_{\sigma}^{-}(q, p)e^{(2\pi)i\partial_{\sigma}^{-}(q,p)} \end{array} \right) \right) \in R^{-1}. \quad (50)$$

Thus,  $R = R^{-1}$  and the theorem is proven.  $\square$

**Theorem 27.** A CBIFR  $R$  on  $G$  is a CBI transitive FR  $\Leftrightarrow R \circ R \subseteq R$ .

*Proof.* Necessity condition:

Assume that  $R$  is a complex bipolar intuitionistic transitive fuzzy relation on a CBIFS  $G$ .

Let

$$\left( (p, r), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, r)e^{(2\pi)i\partial_{\mu}^{+}(p,r)} \\ \alpha_{\mu}^{-}(p, r)e^{(2\pi)i\partial_{\mu}^{-}(p,r)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, r)e^{(2\pi)i\partial_{\sigma}^{+}(p,r)} \\ \alpha_{\sigma}^{-}(p, r)e^{(2\pi)i\partial_{\sigma}^{-}(p,r)} \end{array} \right) \right) \in R \circ R. \quad (51)$$

For any

$$\begin{aligned} & \left( (p, q), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, q)e^{(2\pi)i\partial_{\mu}^{+}(p,q)} \\ \alpha_{\mu}^{-}(p, q)e^{(2\pi)i\partial_{\mu}^{-}(p,q)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, q)e^{(2\pi)i\partial_{\sigma}^{+}(p,q)} \\ \alpha_{\sigma}^{-}(p, q)e^{(2\pi)i\partial_{\sigma}^{-}(p,q)} \end{array} \right) \right) \in R, \\ & \left( (q, r), \left( \begin{array}{c} \alpha_{\mu}^{+}(q, r)e^{(2\pi)i\partial_{\mu}^{+}(q,r)} \\ \alpha_{\mu}^{-}(q, r)e^{(2\pi)i\partial_{\mu}^{-}(q,r)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(q, r)e^{(2\pi)i\partial_{\sigma}^{+}(q,r)} \\ \alpha_{\sigma}^{-}(q, r)e^{(2\pi)i\partial_{\sigma}^{-}(q,r)} \end{array} \right) \right) \in R. \end{aligned} \quad (52)$$

This implies that

$$\left( (p, r), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, r)e^{(2\pi)i\partial_{\mu}^{+}(p,r)} \\ \alpha_{\mu}^{-}(p, r)e^{(2\pi)i\partial_{\mu}^{-}(p,r)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, r)e^{(2\pi)i\partial_{\sigma}^{+}(p,r)} \\ \alpha_{\sigma}^{-}(p, r)e^{(2\pi)i\partial_{\sigma}^{-}(p,r)} \end{array} \right) \right) \in R, \quad (53)$$

since  $R \circ R \subseteq R$ .

Sufficient condition:

Conversely suppose that  $R \circ R \subseteq R$ ; then,

$$\begin{aligned} & \left( (p, q), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, q)e^{(2\pi)i\partial_{\mu}^{+}(p,q)} \\ \alpha_{\mu}^{-}(p, q)e^{(2\pi)i\partial_{\mu}^{-}(p,q)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, q)e^{(2\pi)i\partial_{\sigma}^{+}(p,q)} \\ \alpha_{\sigma}^{-}(p, q)e^{(2\pi)i\partial_{\sigma}^{-}(p,q)} \end{array} \right) \right) \in R, \\ & \left( (q, r), \left( \begin{array}{c} \alpha_{\mu}^{+}(q, r)e^{(2\pi)i\partial_{\mu}^{+}(q,r)} \\ \alpha_{\mu}^{-}(q, r)e^{(2\pi)i\partial_{\mu}^{-}(q,r)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(q, r)e^{(2\pi)i\partial_{\sigma}^{+}(q,r)} \\ \alpha_{\sigma}^{-}(q, r)e^{(2\pi)i\partial_{\sigma}^{-}(q,r)} \end{array} \right) \right) \in R. \end{aligned} \quad (54)$$

This implies

$$\left( (p, r), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, r)e^{(2\pi)i\partial_{\mu}^{+}(p,r)} \\ \alpha_{\mu}^{-}(p, r)e^{(2\pi)i\partial_{\mu}^{-}(p,r)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, r)e^{(2\pi)i\partial_{\sigma}^{+}(p,r)} \\ \alpha_{\sigma}^{-}(p, r)e^{(2\pi)i\partial_{\sigma}^{-}(p,r)} \end{array} \right) \right) \in R \circ R. \quad (55)$$

But it is assumed that  $R \circ R \subseteq R$ . Thus,

$$\left( (p, r), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, r)e^{(2\pi)i\partial_{\mu}^{+}(p,r)} \\ \alpha_{\mu}^{-}(p, r)e^{(2\pi)i\partial_{\mu}^{-}(p,r)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, r)e^{(2\pi)i\partial_{\sigma}^{+}(p,r)} \\ \alpha_{\sigma}^{-}(p, r)e^{(2\pi)i\partial_{\sigma}^{-}(p,r)} \end{array} \right) \right) \in R. \quad (56)$$

Hence,  $R$  is a complex bipolar intuitionistic transitive fuzzy relation on a CBFS  $G$ .  $\square$

**Theorem 28.** A CBIFR  $R$  on  $G$  is a CBI-equivalence-FR  $\Leftrightarrow R \circ R = R$ .

*Proof.* Necessity condition:

Assume that

$$\left( (p, q), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, q)e^{(2\pi)i\partial_{\mu}^{+}(p,q)} \\ \alpha_{\mu}^{-}(p, q)e^{(2\pi)i\partial_{\mu}^{-}(p,q)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, q)e^{(2\pi)i\partial_{\sigma}^{+}(p,q)} \\ \alpha_{\sigma}^{-}(p, q)e^{(2\pi)i\partial_{\sigma}^{-}(p,q)} \end{array} \right) \right) \in R. \quad (57)$$

Then, by the definition of symmetric relation,

$$\left( (q, p), \left( \begin{array}{c} \alpha_{\mu}^{+}(q, p)e^{(2\pi)i\partial_{\mu}^{+}(q,p)} \\ \alpha_{\mu}^{-}(q, p)e^{(2\pi)i\partial_{\mu}^{-}(q,p)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(q, p)e^{(2\pi)i\partial_{\sigma}^{+}(q,p)} \\ \alpha_{\sigma}^{-}(q, p)e^{(2\pi)i\partial_{\sigma}^{-}(q,p)} \end{array} \right) \right) \in R. \quad (58)$$

Now, by the definition of transitivity,

$$\left( (p, p), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, p)e^{(2\pi)i\partial_{\mu}^{+}(p,p)} \\ \alpha_{\mu}^{-}(p, p)e^{(2\pi)i\partial_{\mu}^{-}(p,p)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p)e^{(2\pi)i\partial_{\sigma}^{+}(p)} \\ \alpha_{\sigma}^{-}(p)e^{(2\pi)i\partial_{\sigma}^{-}(p)} \end{array} \right) \right) \in R. \quad (59)$$

Also, by the definition of the composition,

$$\left( (p, p), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, p)e^{(2\pi)i\partial_{\mu}^{+}(p,p)} \\ \alpha_{\mu}^{-}(p, p)e^{(2\pi)i\partial_{\mu}^{-}(p,p)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p)e^{(2\pi)i\partial_{\sigma}^{+}(p)} \\ \alpha_{\sigma}^{-}(p)e^{(2\pi)i\partial_{\sigma}^{-}(p)} \end{array} \right) \right) \in R \circ R. \quad (60)$$

Therefore,

$$R \subseteq R \circ R. \quad (61)$$

Sufficient condition:  
Suppose that

$$\left( (p, q), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, q)e^{(2\pi)i\partial_{\mu}^{+}(p,q)} \\ \alpha_{\mu}^{-}(p, q)e^{(2\pi)i\partial_{\mu}^{-}(p,q)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, q)e^{(2\pi)i\partial_{\sigma}^{+}(p,q)} \\ \alpha_{\sigma}^{-}(p, q)e^{(2\pi)i\partial_{\sigma}^{-}(p,q)} \end{array} \right) \right) \in R \circ R, \quad (62)$$

then there exists  $r \in P$  such that

$$\left( (p, r), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, r)e^{(2\pi)i\partial_{\mu}^{+}(p,r)} \\ \alpha_{\mu}^{-}(p, r)e^{(2\pi)i\partial_{\mu}^{-}(p,r)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, r)e^{(2\pi)i\partial_{\sigma}^{+}(p,r)} \\ \alpha_{\sigma}^{-}(p, r)e^{(2\pi)i\partial_{\sigma}^{-}(p,r)} \end{array} \right) \right) \in R, \quad (63)$$

$$\left( (r, q), \left( \begin{array}{c} \alpha_{\mu}^{+}(r, q)e^{(2\pi)i\partial_{\mu}^{+}(r,q)} \\ \alpha_{\mu}^{-}(r, q)e^{(2\pi)i\partial_{\mu}^{-}(r,q)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(r, q)e^{(2\pi)i\partial_{\sigma}^{+}(r,q)} \\ \alpha_{\sigma}^{-}(r, q)e^{(2\pi)i\partial_{\sigma}^{-}(r,q)} \end{array} \right) \right) \in R.$$

Hence, this implies that

$$\left( (p, q), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, q)e^{(2\pi)i\partial_{\mu}^{+}(p,q)} \\ \alpha_{\mu}^{-}(p, q)e^{(2\pi)i\partial_{\mu}^{-}(p,q)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, q)e^{(2\pi)i\partial_{\sigma}^{+}(p,q)} \\ \alpha_{\sigma}^{-}(p, q)e^{(2\pi)i\partial_{\sigma}^{-}(p,q)} \end{array} \right) \right) \in R, \quad (64)$$

$$\Rightarrow R \circ R \subseteq R. \quad (65)$$

Thus, equations (61) and (62) imply the following

$$R \circ R = R. \quad (66)$$

□

□

**Theorem 29.** The inverse of the CBI-partial order-FR  $R$  on  $F_p$  is again a CBI-partial order-FR  $R$  on  $G$ .

*Proof.* The CBI-partial order-FR  $R$  on a CBIFS  $G$ ,  $R$  satisfies the properties of complex bipolar intuitionistic reflexive, antisymmetric, and transitive fuzzy relations.

Since  $R$  is a CBI-partial order-FR, for any  $p \in P$ ,

$$\left( (p, p), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, p)e^{(2\pi)i\partial_{\mu}^{+}(p,p)} \\ \alpha_{\mu}^{-}(p, p)e^{(2\pi)i\partial_{\mu}^{-}(p,p)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p)e^{(2\pi)i\partial_{\sigma}^{+}(p)} \\ \alpha_{\sigma}^{-}(p)e^{(2\pi)i\partial_{\sigma}^{-}(p)} \end{array} \right) \right) \in R. \quad (67)$$

This implies that

$$\left( (p, p), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, p)e^{(2\pi)i\partial_{\mu}^{+}(p,p)} \\ \alpha_{\mu}^{-}(p, p)e^{(2\pi)i\partial_{\mu}^{-}(p,p)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p)e^{(2\pi)i\partial_{\sigma}^{+}(p)} \\ \alpha_{\sigma}^{-}(p)e^{(2\pi)i\partial_{\sigma}^{-}(p)} \end{array} \right) \right) \in R^{-1}. \quad (68)$$

Hence,  $R^{-1}$  is also a CBI-reflexive-FR.

Let

$$\left( (p, q), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, q)e^{(2\pi)i\partial_{\mu}^{+}(p,q)} \\ \alpha_{\mu}^{-}(p, q)e^{(2\pi)i\partial_{\mu}^{-}(p,q)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, q)e^{(2\pi)i\partial_{\sigma}^{+}(p,q)} \\ \alpha_{\sigma}^{-}(p, q)e^{(2\pi)i\partial_{\sigma}^{-}(p,q)} \end{array} \right) \right) \in R^{-1},$$

$$\left( (q, p), \left( \begin{array}{c} \alpha_{\mu}^{+}(q, p)e^{(2\pi)i\partial_{\mu}^{+}(q,p)} \\ \alpha_{\mu}^{-}(q, p)e^{(2\pi)i\partial_{\mu}^{-}(q,p)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(q, p)e^{(2\pi)i\partial_{\sigma}^{+}(q,p)} \\ \alpha_{\sigma}^{-}(q, p)e^{(2\pi)i\partial_{\sigma}^{-}(q,p)} \end{array} \right) \right) \in R^{-1}. \quad (69)$$

Then, by the definition of inverse,

$$\left( (p, q), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, q)e^{(2\pi)i\partial_{\mu}^{+}(p,q)} \\ \alpha_{\mu}^{-}(p, q)e^{(2\pi)i\partial_{\mu}^{-}(p,q)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, q)e^{(2\pi)i\partial_{\sigma}^{+}(p,q)} \\ \alpha_{\sigma}^{-}(p, q)e^{(2\pi)i\partial_{\sigma}^{-}(p,q)} \end{array} \right) \right) \in R,$$

$$\left( (q, p), \left( \begin{array}{c} \alpha_{\mu}^{+}(q, p)e^{(2\pi)i\partial_{\mu}^{+}(q,p)} \\ \alpha_{\mu}^{-}(q, p)e^{(2\pi)i\partial_{\mu}^{-}(q,p)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(q, p)e^{(2\pi)i\partial_{\sigma}^{+}(q,p)} \\ \alpha_{\sigma}^{-}(q, p)e^{(2\pi)i\partial_{\sigma}^{-}(q,p)} \end{array} \right) \right) \in R. \quad (70)$$

So,  $R$  is a CBI-antisymmetric-FR. Therefore,

$$\left( (p, q), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, q)e^{(2\pi)i\partial_{\mu}^{+}(p,q)} \\ \alpha_{\mu}^{-}(p, q)e^{(2\pi)i\partial_{\mu}^{-}(p,q)} \\ \alpha_{\sigma}^{+}(p, q)e^{(2\pi)i\partial_{\sigma}^{+}(p,q)} \\ \alpha_{\sigma}^{-}(p, q)e^{(2\pi)i\partial_{\sigma}^{-}(p,q)} \end{array} \right) \right) = \left( (q, p), \left( \begin{array}{c} \alpha_{\mu}^{+}(q, p)e^{(2\pi)i\partial_{\mu}^{+}(q,p)} \\ \alpha_{\mu}^{-}(q, p)e^{(2\pi)i\partial_{\mu}^{-}(q,p)} \\ \alpha_{\sigma}^{+}(q, p)e^{(2\pi)i\partial_{\sigma}^{+}(q,p)} \\ \alpha_{\sigma}^{-}(q, p)e^{(2\pi)i\partial_{\sigma}^{-}(q,p)} \end{array} \right) \right). \quad (71)$$

Hence,  $R^{-1}$  is also a CBI-antisymmetric-FR.

Let

$$\left( (p, q), \left( \begin{array}{c} \alpha_{\mu}^{+}(p, q)e^{(2\pi)i\partial_{\mu}^{+}(p,q)} \\ \alpha_{\mu}^{-}(p, q)e^{(2\pi)i\partial_{\mu}^{-}(p,q)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(p, q)e^{(2\pi)i\partial_{\sigma}^{+}(p,q)} \\ \alpha_{\sigma}^{-}(p, q)e^{(2\pi)i\partial_{\sigma}^{-}(p,q)} \end{array} \right) \right) \in R^{-1},$$

$$\left( (q, r), \left( \begin{array}{c} \alpha_{\mu}^{+}(q, r)e^{(2\pi)i\partial_{\mu}^{+}(q,r)} \\ \alpha_{\mu}^{-}(q, r)e^{(2\pi)i\partial_{\mu}^{-}(q,r)} \end{array} \right), \left( \begin{array}{c} \alpha_{\sigma}^{+}(q, r)e^{(2\pi)i\partial_{\sigma}^{+}(q,r)} \\ \alpha_{\sigma}^{-}(q, r)e^{(2\pi)i\partial_{\sigma}^{-}(q,r)} \end{array} \right) \right) \in R^{-1}. \quad (72)$$



FIGURE 1: Algorithm of the application.

Then,

$$\Rightarrow \left( (r, q), \left( \begin{array}{l} \alpha_{\mu}^{+}(r, q)e^{(2\pi)i\partial_{\mu}^{+}(r, q)} \\ \alpha_{\mu}^{-}(r, q)e^{(2\pi)i\partial_{\mu}^{-}(r, q)} \end{array} \right), \left( \begin{array}{l} \alpha_{\sigma}^{+}(r, q)e^{(2\pi)i\partial_{\sigma}^{+}(r, q)} \\ \alpha_{\sigma}^{-}(r, q)e^{(2\pi)i\partial_{\sigma}^{-}(r, q)} \end{array} \right) \right) \in R,$$

$$\left( (q, p), \left( \begin{array}{l} \alpha_{\mu}^{+}(q, p)e^{(2\pi)i\partial_{\mu}^{+}(q, p)} \\ \alpha_{\mu}^{-}(q, p)e^{(2\pi)i\partial_{\mu}^{-}(q, p)} \end{array} \right), \left( \begin{array}{l} \alpha_{\sigma}^{+}(q, p)e^{(2\pi)i\partial_{\sigma}^{+}(q, p)} \\ \alpha_{\sigma}^{-}(q, p)e^{(2\pi)i\partial_{\sigma}^{-}(q, p)} \end{array} \right) \right) \in R.$$
(73)

So,  $R$  is a CBI-transitive-FR. Therefore,

$$\left( (r, p), \left( \begin{array}{l} \alpha_{\mu}^{+}(r, p)e^{(2\pi)i\partial_{\mu}^{+}(r, p)} \\ \alpha_{\mu}^{-}(r, p)e^{(2\pi)i\partial_{\mu}^{-}(r, p)} \end{array} \right), \left( \begin{array}{l} \alpha_{\sigma}^{+}(r, p)e^{(2\pi)i\partial_{\sigma}^{+}(r, p)} \\ \alpha_{\sigma}^{-}(r, p)e^{(2\pi)i\partial_{\sigma}^{-}(r, p)} \end{array} \right) \right) \in R,$$

$$\Rightarrow \left( (p, r), \left( \begin{array}{l} \alpha_{\mu}^{+}(p, r)e^{(2\pi)i\partial_{\mu}^{+}(p, r)} \\ \alpha_{\mu}^{-}(p, r)e^{(2\pi)i\partial_{\mu}^{-}(p, r)} \end{array} \right), \left( \begin{array}{l} \alpha_{\sigma}^{+}(p, r)e^{(2\pi)i\partial_{\sigma}^{+}(p, r)} \\ \alpha_{\sigma}^{-}(p, r)e^{(2\pi)i\partial_{\sigma}^{-}(p, r)} \end{array} \right) \right) \in R^{-1}.$$
(74)

Hence,  $R^{-1}$  is also a CBI-transitive-FR.

Thus, we prove that  $R^{-1}$  is also a CBI-partial order-FR on  $G$ .  $\square$

## 4. Application

In this section, we proposed an application by using a new concept of CBIFSs and their relations.

**4.1. Machine Learning.** Machine learning (ML) is a type of artificial intelligence (AI) that allows software applications to become more accurate at predicting outcomes without being explicitly programmed to do so. Machine learning algorithms use historical data as input to predict new output values. Machine learning algorithms are used in a wide variety of applications, such as in medicine, email filtering, speech recognition, and computer vision, where it is difficult or unfeasible to develop conventional algorithms to perform the needed tasks. It gives enterprises a view of trends in customer behavior

and business operational patterns and supports the development of new products. Many of today's leading companies, such as Facebook, Google, and Uber, make machine learning a central part of their operations. Machine learning has become a significant competitive differentiator for many companies. The iterative aspect of machine learning is important because as models are exposed to new data, they are able to independently adapt. They learn from previous computations to produce reliable, repeatable decisions and results. It is a science that is not new but one that has gained fresh momentum. Figure 1 explains the algorithm of the application.

Some factors that affect the ML in different ways are discussed below:

- (i) More data: the data always becomes more accurate when there is more data on the algorithm. You should avoid subsampling because it will help in getting the best data. There is an intuitive characterization regarding the prediction error when it comes to machine learning. If the data is limited, it will not be able to support the complexity of the model required for solving the problem

$$(MD, (0.88e^{0.33(2\pi)i}, -0.65e^{-0.27(2\pi)i}), (0.39e^{0.41(2\pi)i}, -0.61e^{-0.99(2\pi)i})).$$
(75)

- (ii) Keep the given problem in mind: when you are selecting the machine learning method, it is important to keep the given problem in mind. It plays an important role in determining the level of success. Make sure that you are using the algorithm that is most suitable for the characteristics of the data. It will help in getting the most accurate results

$$(KM, (0.73e^{0.67(2\pi)i}, -0.41e^{-0.31(2\pi)i}), (0.20e^{0.17(2\pi)i}, -0.58e^{-0.09(2\pi)i})).$$
(76)

(iii) Parameters: the parameters of the method are not always easy for the nondata scientists to understand. When it comes to the modern machine learning algorithms, there are always some knobs that need tweaking. Each algorithm has multiple parameter settings. The experience and intuition can help a lot in understanding the parameters

$$(P, (0.39e^{0.70(2\pi)i}, -0.23e^{-0.41(2\pi)i}), (0.44e^{0.27(2\pi)i}, -0.70e^{-0.51(2\pi)i})). \tag{77}$$

(iv) Data Quality: the quality of the machine learning depends on the quality of the data. If the collection of the data is not proper, then it will be difficult to create machine learning models that are general and predictive. It is essential that the data is reviewed carefully. The data should allow the experts of the subject matter to get a proper insight into the data. It will also provide insight into the data generation process which can help in identifying data quality issues that are connected to the features, records, sampling, or values

$$(DQ, (0.49e^{0.51(2\pi)i}, -0.55e^{-0.19(2\pi)i}), (0.25e^{0.42(2\pi)i}, -0.37e^{-0.62(2\pi)i})). \tag{78}$$

(v) Features in the data: the impacts of predictability depend on understanding the features in the data. It is crucial for machine learning to consider the raw data in a rich feature space. Knowing the features will help a lot with the learning processes

$$(FD, (0.71e^{0.51(2\pi)i}, -0.09e^{-0.37(2\pi)i}), (0.27e^{0.21(2\pi)i}, -0.83e^{-0.12(2\pi)i})). \tag{79}$$

(vi) Objective/loss function: the success of the application depends a lot on the choice of appropriate objective/loss function. The machine learning algorithms are mostly formulated as problems that are optimized. Adjusting the objective function according to the nature of the business is essential for machine learning success. These are some of the factors that need to be remembered

$$(OF, (0.55e^{0.94(2\pi)i}, -0.73e^{-0.29(2\pi)i}), (0.30e^{0.03(2\pi)i}, -0.06e^{-0.36(2\pi)i})). \tag{80}$$

Each factor of ML describes the two stages, i.e., membership which means effectiveness and non-membership which implies ineffectiveness. The positive and negative values show possibility and impossibility of both effectiveness and ineffectiveness. If the positive value is closer to 1, it shows that the possibility of effectiveness and ineffectiveness is higher and the phase term explains the duration of possibility. If negative value is closer to zero, it

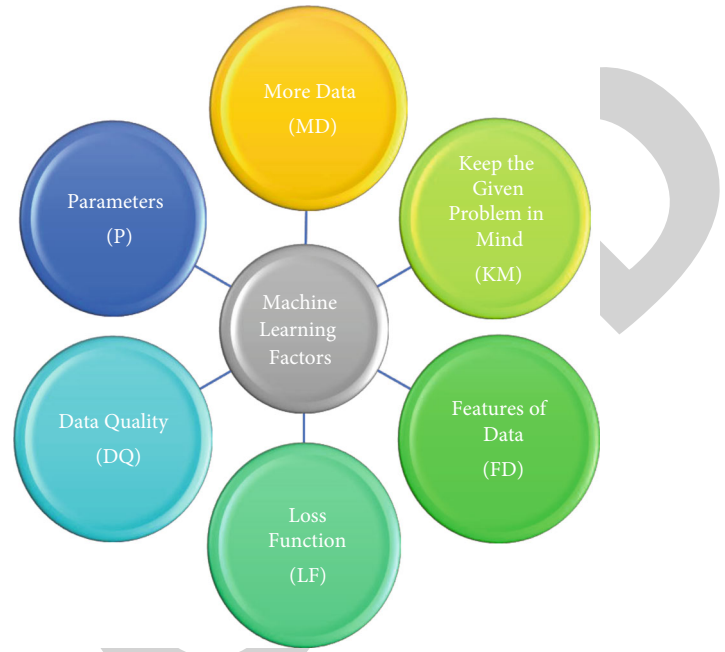


FIGURE 2: Summary of factors.

shows higher impossibility of that factor. For example, the positive and negative membership of the element  $(DQ, (0.49e^{0.51(2\pi)i}, -0.55e^{-0.19(2\pi)i}), (0.25e^{0.42(2\pi)i}, -0.37e^{-0.62(2\pi)i}))$  indicates that the possibility of effectiveness of the data quality on ML is 49% with about half time units and the possibility of ineffectiveness is about 55% for very short period of time, respectively. Similarly, we can read off the values for positive and negative nonmemberships. Here, we discussed the impact of one factor on the other.

The summary of all factors of ML is shown in Figure 2.

Then, the set of all factors with given grades is

$$N = \left\{ \begin{array}{l} \left( MD, \left( 0.88e^{0.33(2\pi)i}, \left( 0.39e^{0.41(2\pi)i}, -0.61e^{-0.39(2\pi)i} \right) \right) \right), \\ \left( KM, \left( 0.73e^{0.67(2\pi)i}, \left( 0.20e^{0.17(2\pi)i}, -0.58e^{-0.09(2\pi)i} \right) \right) \right), \\ \left( P, \left( 0.39e^{0.70(2\pi)i}, \left( 0.44e^{0.27(2\pi)i}, -0.70e^{-0.51(2\pi)i} \right) \right) \right), \\ \left( DQ, \left( 0.49e^{0.51(2\pi)i}, \left( 0.25e^{0.42(2\pi)i}, -0.37e^{-0.62(2\pi)i} \right) \right) \right), \\ \left( FD, \left( 0.71e^{0.51(2\pi)i}, \left( 0.27e^{0.21(2\pi)i}, -0.83e^{-0.12(2\pi)i} \right) \right) \right), \\ \left( OF, \left( 0.55e^{0.94(2\pi)i}, \left( 0.30e^{0.03(2\pi)i}, -0.06e^{-0.36(2\pi)i} \right) \right) \right) \end{array} \right. \tag{81}$$

Then, its Cartesian product is in Table 1.

TABLE 1: Cartesian product.

Ordered pair	Membership	Nonmembership
(MD, MD)	$(0.88e^{0.33(2\pi)i}, -0.65e^{-0.27(2\pi)i})$	$(0.39e^{0.41(2\pi)i}, -0.61e^{-0.39(2\pi)i})$
(MD, KM)	$(0.73e^{0.33(2\pi)i}, -0.41e^{-0.27(2\pi)i})$	$(0.39e^{0.41(2\pi)i}, -0.61e^{-0.39(2\pi)i})$
(MD, P)	$(0.39e^{0.33(2\pi)i}, -0.23e^{-0.27(2\pi)i})$	$(0.44e^{0.41(2\pi)i}, -0.70e^{-0.51(2\pi)i})$
(MD, DQ)	$(0.49e^{0.33(2\pi)i}, -0.55e^{-0.19(2\pi)i})$	$(0.39e^{0.42(2\pi)i}, -0.61e^{-0.62(2\pi)i})$
(MD, FD)	$(0.71e^{0.33(2\pi)i}, -0.09e^{-0.27(2\pi)i})$	$(0.39e^{0.41(2\pi)i}, -0.83e^{-0.39(2\pi)i})$
(MD, OF)	$(0.55e^{0.33(2\pi)i}, -0.65e^{-0.27(2\pi)i})$	$(0.39e^{0.41(2\pi)i}, -0.61e^{-0.39(2\pi)i})$
(KM, MD)	$(0.73e^{0.33(2\pi)i}, -0.41e^{-0.27(2\pi)i})$	$(0.39e^{0.41(2\pi)i}, -0.61e^{-0.39(2\pi)i})$
(KM, KM)	$(0.73e^{0.67(2\pi)i}, -0.41e^{-0.31(2\pi)i})$	$(0.20e^{0.17(2\pi)i}, 0.58e^{-0.09(2\pi)i})$
(KM, P)	$(0.39e^{0.67(2\pi)i}, -0.23e^{-0.31(2\pi)i})$	$(0.44e^{0.27(2\pi)i}, -0.70e^{-0.51(2\pi)i})$
(KM, DQ)	$(0.49e^{0.51(2\pi)i}, -0.41e^{-0.19(2\pi)i})$	$(0.25e^{0.42(2\pi)i}, -0.58e^{-0.62(2\pi)i})$
(KM, FD)	$(0.71e^{0.51(2\pi)i}, -0.09e^{-0.31(2\pi)i})$	$(0.27e^{0.21(2\pi)i}, -0.83e^{-0.12(2\pi)i})$
(KM, OF)	$(0.55e^{0.67(2\pi)i}, -0.41e^{-0.29(2\pi)i})$	$(0.30e^{0.17(2\pi)i}, -0.58e^{-0.36(2\pi)i})$
(P, MD)	$(0.39e^{0.33(2\pi)i}, -0.23e^{-0.27(2\pi)i})$	$(0.44e^{0.41(2\pi)i}, -0.70e^{-0.51(2\pi)i})$
(P, KM)	$(0.39e^{0.67(2\pi)i}, -0.23e^{-0.31(2\pi)i})$	$(0.44e^{0.27(2\pi)i}, -0.70e^{-0.51(2\pi)i})$
(P, P)	$(0.39e^{0.70(2\pi)i}, -0.23e^{-0.41(2\pi)i})$	$(0.44e^{0.27(2\pi)i}, -0.70e^{-0.51(2\pi)i})$
(P, DQ)	$(0.39e^{0.51(2\pi)i}, -0.23e^{-0.19(2\pi)i})$	$(0.44e^{0.42(2\pi)i}, -0.70e^{-0.62(2\pi)i})$
(P, FD)	$(0.39e^{0.51(2\pi)i}, -0.09e^{-0.37(2\pi)i})$	$(0.44e^{0.27(2\pi)i}, -0.83e^{-0.51(2\pi)i})$
(P, OF)	$(0.39e^{0.70(2\pi)i}, -0.23e^{-0.29(2\pi)i})$	$(0.44e^{0.27(2\pi)i}, -0.70e^{-0.51(2\pi)i})$
(DQ, MD)	$(0.49e^{0.33(2\pi)i}, -0.55e^{-0.19(2\pi)i})$	$(0.39e^{0.42(2\pi)i}, -0.61e^{-0.62(2\pi)i})$
(DQ, KM)	$(0.49e^{0.51(2\pi)i}, -0.41e^{-0.19(2\pi)i})$	$(0.25e^{0.42(2\pi)i}, -0.58e^{-0.62(2\pi)i})$
(DQ, P)	$(0.39e^{0.51(2\pi)i}, -0.23e^{-0.19(2\pi)i})$	$(0.44e^{0.42(2\pi)i}, -0.70e^{-0.62(2\pi)i})$
(DQ, DQ)	$(0.49e^{0.51(2\pi)i}, -0.55e^{-0.19(2\pi)i})$	$(0.25e^{0.42(2\pi)i}, -0.37e^{-0.62(2\pi)i})$
(DQ, FD)	$(0.49e^{0.51(2\pi)i}, -0.09e^{-0.19(2\pi)i})$	$(0.27e^{0.42(2\pi)i}, -0.83e^{-0.62(2\pi)i})$
(DQ, OF)	$(0.49e^{0.51(2\pi)i}, -0.55e^{-0.19(2\pi)i})$	$(0.30e^{0.42(2\pi)i}, -0.37e^{-0.62(2\pi)i})$
(FD, MD)	$(0.71e^{0.33(2\pi)i}, -0.09e^{-0.27(2\pi)i})$	$(0.39e^{0.41(2\pi)i}, -0.83e^{-0.39(2\pi)i})$
(FD, KM)	$(0.71e^{0.51(2\pi)i}, -0.09e^{-0.31(2\pi)i})$	$(0.27e^{0.21(2\pi)i}, -0.83e^{-0.12(2\pi)i})$
(FD, P)	$(0.39e^{0.51(2\pi)i}, -0.09e^{-0.37(2\pi)i})$	$(0.44e^{0.27(2\pi)i}, -0.83e^{-0.51(2\pi)i})$
(FD, DQ)	$(0.49e^{0.51(2\pi)i}, -0.09e^{-0.19(2\pi)i})$	$(0.27e^{0.42(2\pi)i}, -0.83e^{-0.62(2\pi)i})$
(FD, FD)	$(0.71e^{0.51(2\pi)i}, -0.09e^{-0.37(2\pi)i})$	$(0.71e^{0.51(2\pi)i}, -0.09e^{-0.37(2\pi)i})$
(FD, OF)	$(0.55e^{0.51(2\pi)i}, -0.09e^{-0.29(2\pi)i})$	$(0.30e^{0.42(2\pi)i}, -0.37e^{-0.62(2\pi)i})$
(OF, MD)	$(0.55e^{0.33(2\pi)i}, -0.65e^{-0.27(2\pi)i})$	$(0.39e^{0.41(2\pi)i}, -0.61e^{-0.39(2\pi)i})$
(OF, KM)	$(0.55e^{0.67(2\pi)i}, -0.41e^{-0.29(2\pi)i})$	$(0.30e^{0.17(2\pi)i}, -0.58e^{-0.36(2\pi)i})$
(OF, PM)	$(0.39e^{0.70(2\pi)i}, -0.23e^{-0.29(2\pi)i})$	$(0.44e^{0.27(2\pi)i}, -0.70e^{-0.51(2\pi)i})$
(OF, QD)	$(0.49e^{0.51(2\pi)i}, -0.55e^{-0.19(2\pi)i})$	$(0.30e^{0.42(2\pi)i}, -0.37e^{-0.62(2\pi)i})$
(OF, FD)	$(0.55e^{0.51(2\pi)i}, -0.09e^{-0.29(2\pi)i})$	$(0.30e^{0.42(2\pi)i}, -0.37e^{-0.62(2\pi)i})$
(OF, OF)	$(0.55e^{0.94(2\pi)i}, -0.73e^{-0.29(2\pi)i})$	$(0.30e^{0.03(2\pi)i}, -0.06e^{-0.36(2\pi)i})$

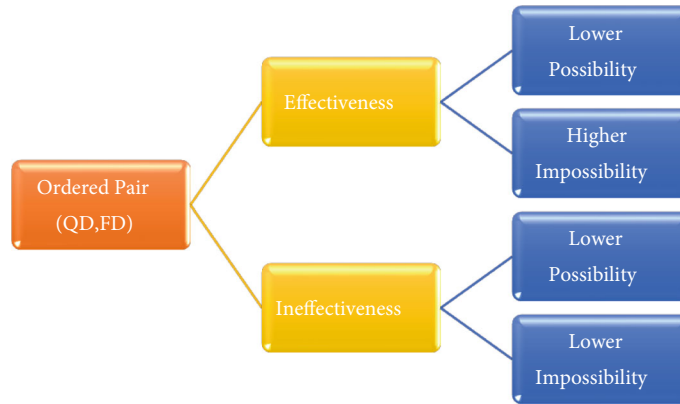


FIGURE 3: Impact of QD on FD.

TABLE 2: Comparative analysis.

Structures	Membership		Nonmembership		Multidimensional
	Possibility	Impossibility	Possibility	Impossibility	
CFRs	Yes	No	No	No	Yes
CIFRs	Yes	No	Yes	No	Yes
CBFRs	Yes	Yes	No	No	Yes
CBIFRs	Yes	Yes	Yes	Yes	Yes

Each ordered pair of Cartesian product  $N \times N$  expresses the impact of one factor on the other. Here,

$$\left( (OF, MD), \left( \begin{matrix} 0.55e^{0.33(2\pi)i} \\ -0.65e^{-0.27(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.39e^{0.41(2\pi)i} \\ -0.61e^{-0.39(2\pi)i} \end{matrix} \right) \right) \tag{82}$$

explains that the objective function impacts on more data; i.e., effectiveness has higher possibility with shorter time duration and lower impossibility with larger time duration. In the same way, possibility of ineffectiveness is lower with normal phase time and impossibility is lower with longer time duration. This means that objective function deals more data in ML effectively and makes working of machine learning more developed. Similarly, another ordered pair

$$\left( (DQ, FD), \left( \begin{matrix} 0.49e^{0.51(2\pi)i} \\ -0.09e^{-0.19(2\pi)i} \end{matrix} \right), \left( \begin{matrix} 0.27e^{0.42(2\pi)i} \\ -0.83e^{-0.62(2\pi)i} \end{matrix} \right) \right) \tag{83}$$

is explained with the help of Figure 3.

### 5. Comparative Analysis

In this section, the omnipotence of the newly proposed framework of CBIFRs is verified through a comparison of CBIFRs with preexisting structures such as CFRs, CIFRs,

and CBFRs. Since CFRs tell only membership grades and lack the nonmembership grades, it describes only possible values of the membership and does not deal with the impossibility factor of membership. CIFRs tell the membership and nonmembership grades. It also just explains the positive mapping of membership and nonmembership and has lack of negative mapping. Due to complexity, both CFRs and CIFRs deal with periodicity and the problems with multivariable. CBFRs explain the membership grades with positive and negative mappings that describe the possibility and impossibility values of the effectiveness. But the CBIFRs consist of both positive and negative mappings of membership and nonmembership. It explains the effectiveness and non-effectiveness of an ordered pair of the Cartesian product in a broader way by telling the possibility and impossibility values. CBFRs are superior to all preexisting frameworks of fuzzy set. Now, we explain the advantage of CBIFRs with CBFRs through an already discussed example.

The set of all factors with given grades is

$$N = \left\{ \begin{matrix} (MD, (0.88e^{0.33(2\pi)i}, -0.65e^{-0.27(2\pi)i})), \\ (KM, (0.73e^{0.67(2\pi)i}, -0.41e^{-0.31(2\pi)i})), \\ (P, (0.39e^{0.70(2\pi)i}, -0.23e^{-0.41(2\pi)i})), \\ (DQ, (0.49e^{0.51(2\pi)i}, -0.55e^{-0.19(2\pi)i})), \\ (FD, (0.71e^{0.51(2\pi)i}, -0.09e^{-0.37(2\pi)i})), \\ (OF, (0.55e^{0.94(2\pi)i}, -0.73e^{-0.29(2\pi)i})) \end{matrix} \right\}. \tag{84}$$

Then, its Cartesian product is



$$N \times N = \left\{ \begin{array}{l} ((MD, MD), (0.88e^{0.33(2\pi)i}, -0.65e^{-0.27(2\pi)i})), ((MD, KM), (0.73e^{0.33(2\pi)i}, -0.41e^{-0.27(2\pi)i})), \\ ((MD, P), (0.39e^{0.33(2\pi)i}, -0.23e^{-0.27(2\pi)i})), ((MD, DQ), (0.49e^{0.33(2\pi)i}, -0.55e^{-0.19(2\pi)i})), \\ ((MD, FD), (0.71e^{0.33(2\pi)i}, -0.09e^{-0.27(2\pi)i})), ((MD, OF), (0.55e^{0.33(2\pi)i}, -0.65e^{-0.27(2\pi)i})), \\ ((KM, MD), (0.73e^{0.33(2\pi)i}, -0.41e^{-0.27(2\pi)i})), ((KM, KM), (0.73e^{0.67(2\pi)i}, -0.41e^{-0.31(2\pi)i})), \\ ((KM, P), (0.39e^{0.67(2\pi)i}, -0.23e^{-0.31(2\pi)i})), ((KM, DQ), (0.49e^{0.51(2\pi)i}, -0.41e^{-0.19(2\pi)i})), \\ ((KM, FD), (0.71e^{0.51(2\pi)i}, -0.09e^{-0.31(2\pi)i})), ((KM, OF), (0.55e^{0.67(2\pi)i}, -0.41e^{-0.29(2\pi)i})), \\ ((P, MD), (0.39e^{0.33(2\pi)i}, -0.23e^{-0.27(2\pi)i})), ((P, KM), (0.39e^{0.67(2\pi)i}, -0.23e^{-0.31(2\pi)i})), \\ ((P, P), (0.39e^{0.70(2\pi)i}, -0.23e^{-0.41(2\pi)i})), ((P, DQ), (0.39e^{0.51(2\pi)i}, -0.23e^{-0.19(2\pi)i})), \\ ((P, FD), (0.39e^{0.51(2\pi)i}, -0.09e^{-0.37(2\pi)i})), ((P, OF), (0.39e^{0.70(2\pi)i}, -0.23e^{-0.29(2\pi)i})), \\ ((DQ, MD), (0.49e^{0.33(2\pi)i}, -0.55e^{-0.19(2\pi)i})), ((DQ, KM), (0.49e^{0.51(2\pi)i}, -0.41e^{-0.19(2\pi)i})), \\ ((DQ, P), (0.39e^{0.51(2\pi)i}, -0.23e^{-0.19(2\pi)i})), ((DQ, DQ), (0.49e^{0.51(2\pi)i}, -0.55e^{-0.19(2\pi)i})), \\ ((DQ, FD), (0.49e^{0.51(2\pi)i}, -0.09e^{-0.19(2\pi)i})), ((DQ, OF), (0.49e^{0.51(2\pi)i}, -0.55e^{-0.19(2\pi)i})), \\ ((FD, MD), (0.71e^{0.33(2\pi)i}, -0.09e^{-0.27(2\pi)i})), ((FD, KM), (0.71e^{0.51(2\pi)i}, -0.09e^{-0.31(2\pi)i})), \\ ((FD, P), (0.39e^{0.51(2\pi)i}, -0.09e^{-0.37(2\pi)i})), ((FD, DQ), (0.49e^{0.51(2\pi)i}, -0.09e^{-0.19(2\pi)i})), \\ ((FD, FD), (0.71e^{0.51(2\pi)i}, -0.09e^{-0.37(2\pi)i})), ((FD, OF), (0.55e^{0.51(2\pi)i}, -0.09e^{-0.29(2\pi)i})), \\ ((OF, MD), (0.55e^{0.33(2\pi)i}, -0.65e^{-0.27(2\pi)i})), ((OF, KM), (0.55e^{0.67(2\pi)i}, -0.41e^{-0.29(2\pi)i})), \\ ((OF, P), (0.39e^{0.70(2\pi)i}, -0.23e^{-0.29(2\pi)i})), ((OF, DQ), (0.49e^{0.51(2\pi)i}, -0.55e^{-0.19(2\pi)i})), \\ ((OF, FD), (0.55e^{0.51(2\pi)i}, -0.09e^{-0.29(2\pi)i})), ((OF, OF), (0.55e^{0.94(2\pi)i}, -0.73e^{-0.29(2\pi)i})) \end{array} \right\}. \quad (85)$$

Each ordered pair of the Cartesian product  $N \times N$  shows relation between factors. An ordered pair just shows the membership of impact of factors and does not explain the nonmembership, while CBIFRs explain the both. All preexisting structures of fuzzy algebra give limited information about any object. Table 2 gives the comparison of CBIFRs with other frameworks.

## 6. Conclusion

This article proposed the novel concept of complex bipolar intuitionistic fuzzy sets (CBIFs) and the Cartesian product between CBIFs. Additionally, complex bipolar intuitionistic fuzzy relations (CBIFRs) and their types are also defined, such as CBI-reflexive-FR, CBI-symmetric-FR, CBI-transitive-FR, CBI-antisymmetric-FR, CBI-irreflexive-FR, CBI-equivalence-FR, CBI-partial order-FR, CBI-linear order-FR, and many more. Further, some important results are also shown. Moreover, in order to contribute to the fields of machine learning (ML) and artificial intelligence (AI), these newly proposed concepts and novel modeling techniques are used to address the impacts of effective factors of machine learning (ML) on each other. Lastly, the advantages of proposed structures are put together by a comparative analysis, as (1) CBIFs and CBIFRs express the effectiveness and ineffectiveness with periodicity of any entity and (2) they state the possibilities or impossibilities of both effectiveness and

ineffectiveness. In future, CBIFs are used in various fields, i.e., economics, cyber security, physics, and chemistry.

## Data Availability

Data sharing is not applicable to this article as no data set was generated or analyzed during the current study.

## Conflicts of Interest

There is no conflict of interest among the authors regarding the publication of this manuscript.

## Acknowledgments

This research was supported by the Researchers Supporting Project number (RSP2022R476), King Saud University, Riyadh, Saudi Arabia.

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