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Research Article

Existence and Uniqueness of Solutions for a Fractional Hybrid System with Nonseparated Integral Boundary Hybrid Conditions

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In this paper, we are going to investigate the existence and uniqueness of solutions of a coupled system of nonlinear fractional hybrid equations with nonseparated type integral boundary hybrid conditions. We are going to use Banach's and Leray-Schauder alternative fixed point theorems to obtain the main results. Lastly, we are giving two examples to show the effectiveness of the main results.

1. Introduction

Fractional differential equations appear naturally in a number of fields bymany fields of scince such as physics, engineering, biophysics, blood flow phenomena, aerodynamics, electron-analytical chemistry, biology, and economy; for more details, we refer the readers to [1–4] and many other references therein which give an excellent account on the study of fractional differential equations.

Hybrid fractional differential equations have also been studied by several researchers. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers (see [5, 6]).

On the other hand, coupled systems of fractional differential equations are very important to study byattract the attention of many researches, because they appear naturally in many problems (see [3, 7–10]).

In [11], Sitho et al. discussed the following existence bysome results for the following hybrid fractional integrodifferential equations:

$$\begin{aligned} &D^{\alpha}\left(\frac{x(t)-\sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t))}{f(t,x(t))}\right) \\ &= g(t,x(t))\ a.e.t \in J=[0,T],\ 0<\alpha\le 1, \\ &x(0)=0, \end{aligned} \tag{1}$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order $\alpha, 0<\alpha\le 1, I^{\phi}$ is the Riemann-Liouville fractional integral of order $\phi>0, \phi \in \{\beta_1,\beta_2,\cdots,\beta_m\}$, $f \in C(J\times\mathbb{R}, \mathbb{R}\setminus\{0\}), g\in C(J\times\mathbb{R},\mathbb{R})$, with $h_i \in C(J\times\mathbb{R},\mathbb{R})$ with $h_i(0,0)=0, i=1,2,\cdots,m$.

In [12], Hilal and Kajouni considered the following boundary value problems for hybrid differential equations with fractional order (BVPHDEF for short) involving Caputo differential operators of order $0<\alpha<1$:

$$\begin{aligned} &D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t))\ a.e.t \in J=[0,T], \\ &a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c, \end{aligned} \tag{2}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$,$g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$.

Dhage and Lakshmikantham [13] discussed the following first-order hybrid differential equation:

$$\begin{gathered} \frac{d}{dt}\left[\frac{x(t)}{f(t, x(t))}\right] = g(t, x(t))\ a.e.t \in J = [0, T], \\ x(t_0) = x_0 \in \mathbb{R}, \end{gathered} \tag{3}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathscr{C}(J \times \mathbb{R}, \mathbb{R})$. They established the existence, uniqueness results, and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proving the utilization of the theory of inequalities, the existence of extremal solutions, and comparison results.

Zhao et al. [5] discussed the following fractional hybrid differential equations involving Riemann-Liouville differential operators:

$$\begin{gathered} D^q\left[\frac{x(t)}{f(t, x(t))}\right] = g(t, x(t))\ a.e.t \in J = [0, T], \\ x(0) = 0, \end{gathered} \tag{4}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathscr{C}(J \times \mathbb{R}, \mathbb{R})$. They established the existence byof solutions and some fundamental differential inequalities are also established and the existence of extremal solutions.

Benchohra et al. [14] studied the following boundary value problems for differential equations with fractional order:

$$\begin{gathered} {}^cD^\alpha y(t) = f(t, y(t)), \quad \text{for each } t \in J = [0, T], 0 < \alpha < 1, \\ ay(0) + by(T) = c, \end{gathered} \tag{5}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and a, b, c are real constants with $a + b \neq 0$.

Hannabou and Hilal [15] considered the boundary value problem of a class of impulsive hybrid fractional coupled differential equations:

$$\begin{gathered} D^\alpha\left(\frac{u(t)}{f_1(t, u(t), v(t))}\right) = g_1(t, u(t), v(t)), t \in [0, 1], t \neq t_i\, i = 1, 2, \cdots, n, 0 < \alpha < 1, \\ u(t_i^+) = u(t_i^-) + I_i(u(t_i^-)), t_i \in (0, 1), i = 1, 2, \cdots, n, \\ D^\beta\left(\frac{u(t)}{f_2(t, u(t), v(t))}\right) = g_2(t, u(t), v(t)), t \in [0, 1], t \neq t_j, j = 1, 2, \cdots, m, 0 < \beta < 1, \\ v\left(t_j^+\right) = v\left(t_j^-\right) + I_j\left(v\left(t_j^-\right)\right), t_j \in (0, 1), j = 1, 2, \cdots, m, \\ \frac{u(0)}{f_1(0, u(0), v(0))} = \phi(u), \quad \frac{v(0)}{f_2(0, u(0), v(0))} = \psi(v). \end{gathered} \tag{6}$$

where D^α, D^β stand for Caputo fractional derivatives of order α, β, respectively, $f_i \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g_i \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $(i = 1, 2)$ and $\phi, \psi : C([0, 1], \mathbb{R}) \longrightarrow \mathbb{R}$ are continuous functions defined by $\phi(u) = \sum_{i=1}^n \lambda_i u(\xi_i)$,$\psi(v) = \sum_{j=1}^n \delta_j v(\eta_j)$, where $\xi_i, \eta_j \in (0, 1)$ for $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$, and $I_k : \mathbb{R} \longrightarrow \mathbb{R}$,$u(t_k^+) = \lim_{\varepsilon \longrightarrow 0^+} u(t_k + \varepsilon)$ and $u(t_k^-) = \lim_{\varepsilon \longrightarrow 0^-} u(t_k + \varepsilon)$ represent the right, left limits of $u(t)$ at $t = t_k$, $(k = i, j)$.

The present paper is a continuation of the work in [16] in order to study the existence and uniqueness of solutions for a coupled system of fractional hybrid equation of the following forme:

$$\begin{aligned} &{}^cD^{\beta_1}\left({}^cD^{\alpha_1} + \lambda_1\right)\left(\frac{x_1(t)}{f_1(t, x_1(t), x_2(t))}\right) \\ &\quad = \omega_1(t, x_1(t), x_2(t)), a.e. \quad t \in J = [0, 1], \\ &{}^cD^{\beta_2}\left({}^cD^{\alpha_2} + \lambda_2\right)\left(\frac{x_2(t)}{f_2(t, x_1(t), x_2(t))}\right) \\ &\quad = \omega_2(t, x_1(t), x_2(t)), a.e. \quad t \in J = [0, 1], \end{aligned} \tag{7}$$

subject to the fractional nonseparated integral boundary hybrid conditions

$$\begin{aligned} &\frac{x_1(0)}{f_1(0, x_1(0), x_2(0))} + \mu_1 \frac{x_1(1)}{f_1(1, x_1(1), x_2(1))} \\ &\quad = \sigma_{11}\int_0^1 g_1(s, x_1(s))ds, \\ &cD^{\alpha_1}\left(\frac{x_1(0)}{f_1(0, x_1(0), x_2(0))}\right) + \mu_1{}^cD^{\alpha_1}\left(\frac{x_1(1)}{f_1(1, x_1(1), x_2(1))}\right) \\ &\quad = \sigma_{21}\int_0^1 h_1(s, x_1(s))ds, \\ &cD^{2\alpha_1}\left(\frac{x_1(0)}{f_1(0, x_1(0), x_2(0))}\right) + \mu_1{}^cD^{2\alpha_1}\left(\frac{x_1(1)}{f(1, x_1(1), x_2(1))}\right) \\ &\quad = \sigma_{31}\int_0^1 k_1(s, x_1(s))ds, \\ &\frac{x_2(0)}{f_2(0, x_1(0), x_2(0))} + \mu_2 \frac{x_2(1)}{f_2(1, x_1(1), x_2(1))} \\ &\quad = \sigma_{12}\int_0^1 g_2(s, x_2(s))ds, \\ &cD^{\alpha_2}\left(\frac{x_2(0)}{f_2(0, x_1(0), x_2(0))}\right) + \mu_2{}^cD^{\alpha_2}\left(\frac{x_2(1)}{f_2(1, x_1(1), x_2(1))}\right) \\ &\quad = \sigma_{22}\int_0^1 h_2(s, x_2(s))ds, \\ &cD^{2\alpha_2}\left(\frac{x_2(0)}{f_2(0, x_1(0), x_2(0))}\right) + \mu_2{}^cD^{2\alpha_2}\left(\frac{x_2(1)}{f_2(1, x_1(1), x_2(1))}\right) \\ &\quad = \sigma_{32}\int_0^1 k_2(s, x_2(s))ds, \end{aligned} \tag{8}$$

where $0 < \alpha_i < 1, 1 < \beta_i \leq 2, \lambda_i, \mu_i, \sigma_{1i}, \sigma_{2i}, \sigma_{3i} \in \mathbb{R}^*$ with $\mu_i \neq -1$ for $i = 1, 2, {}^cD^{\beta_i}, {}^cD^{\alpha_i}$ are the Caputo fractional derivatives, $f_1, f_2 : [0\,; 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R} \setminus \{0\}$, $\omega_1, \omega_2 : [0\,; 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

and $g_1, h_1, k_1, g_2, h_2, k_2 : [0;1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are given continuous functions.

By a solution of the problems (7)–(8), we mean a function $x \in C(J, \mathbb{R})$ such that

(i) the function $t \mapsto x/f_i(t, x_1, x_2)(i = 1, 2)$ is continuous for each $(x_1, x_2) \in \mathbb{R}^2$,

(ii) (x_1, x_2) satisfies the equations in (7)–(8).

This paper is organized as follows: in the second section, we recall some notations and several known results. In the third section, we show the existence and uniqueness of solutions of problem (7)–(8), these results can be viewed as extension of the result given in [12]. In the fourth section, we give some examples to demonstrate the application of our main results.

2. Preliminaries and Notations

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let X be a Banach space of all continuous functions defined frome endowed with norm. Then, the product space is also a Banach space equipped with the norm.

We denote by$L^1(J, \mathbb{R})$ the space of Lebesgue integrable real-valued functions on J equipped with the norm $\|.\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^T |x(s)| ds. \quad (9)$$

Definition 1 (see [17]). The fractional integral of the function $h \in L^1([a, b], \mathbb{R}^+)$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds, \quad (10)$$

where Γ is the gamma function.

Definition 2 (see [17]). Let h be a fonction difened on $[a, b]$, the Riemann-Liouville fractional derivative of order α is defined by

$$({}^cD_{a^+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)} (s) ds, \quad (11)$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 3 (see [17]). Let h be a function defined on $[a, b]$, the Caputo fractional derivative of order α is defined by

$$({}^cD_{a^+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)} h^{(n)}(s) ds, \quad (12)$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 4 (see [2]). *Let $\alpha, \beta \geq 0$, then the following relations hold:*

$$\begin{aligned} I^\alpha t^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}, \\ {}^cD^\alpha t^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \end{aligned} \quad (13)$$

Lemma 5 (see [2]). *Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$. If f is a continuous function, then we have*

$$I^{\alpha c}D^\alpha f(t) = f(t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}. \quad (14)$$

Lemma 6 (Leray-Schauder alternative, see [18]). *Let $\mathcal{F} : G \longrightarrow G$ be a completely continuous operator (i.e., a map that is restricted to any bounded set in G is compact). Let $P(\mathcal{F}) = \{u \in G : u = \lambda \mathcal{F} u$ for some $0 < \lambda < 1\}$. Then, either the set $P(\mathcal{F})$ is unbounded or $\mathcal{F}$ has at least one fxed point.*

We make the following assumption:

(H_0) The function $x \mapsto x/f_i(t, x_1, x_2)(i = 1, 2)$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$.

Lemma 7. *Assume that hypothesis (H_0) holds. Then, for any $y_1, y_2 \in C(J, \mathbb{R}^2)$. The function $x \in C(J, \mathbb{R})$ is a solution of the coupled system,*

$$\begin{aligned} {}^cD^{\beta_1}({}^cD^{\alpha_1} + \lambda_1)\left(\frac{x_1(t)}{f_1(t, x_1(t), x_2(t))}\right) &= y_1(t), t \in [0, 1], \\ cD^{\beta_2}({}^cD^{\alpha_2} + \lambda_2)\left(\frac{x_2(t)}{f_2(t, x_1(t), x_2(t))}\right) &= y_2(t), t \in [0, 1], \end{aligned} \quad (15)$$

subject to the boundary condition (8), has a solution given by

$$\begin{aligned} x_2(t) = f_2(t, x_1(t), x_2(t)) \Bigg[& \frac{1}{\Gamma(\alpha_2+\beta_2)} \int_0^t (t-s)^{\alpha_2+\beta_2-1} y_2(s) ds \\ & + A_{12}(t) \int_0^1 h_2(s, x_2(s)) ds + A_{22}(t) \int_0^1 g_2(s, x_2(s)) ds \\ & + A_{32}(t) \int_0^1 k_2(s, x_2(s)) ds + \frac{A_{42}(t)}{\Gamma(\beta_2 - 1 - \alpha_2 - 1)} \\ & \cdot \int_0^1 (1-s)^{\beta_2-\alpha_2-1} y_2(s) ds + \frac{A_{51}(t)}{\Gamma(\beta_2)} \int_0^1 (1-s)^{\beta_2-1} y_2(s) ds \\ & + \frac{\mu_2 \lambda_2}{(1+\mu_2)\Gamma(\alpha_2)} \int_0^1 (1-s)^{\alpha_2-1} x_2(s) ds \\ & - \frac{\mu_2}{(1+\mu_2)\Gamma(\alpha_2+\beta_2)} \int_0^1 (1-s)^{\alpha_2+\beta_2} y_2(s) ds \Bigg], \\ & - \frac{\lambda_2}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} x_2(s) ds, \end{aligned} \quad (16)$$

where

$$\begin{aligned} A_{1i}(t) &= \frac{t^{\alpha_i}\sigma_{2i}(1-\lambda_i\Gamma(2-\alpha_i))}{\Gamma(\alpha_i+1)(1+\mu_i)} + \frac{t^{\alpha_i+1}\lambda_i\sigma_{2i}\Gamma(2-\alpha_i)}{\Gamma(2+\alpha_i)\mu_i} \\ &\quad + \left(\frac{\mu_i}{(1+\mu_i)^2\Gamma(\alpha_i+1)} - \frac{1}{(1+\mu_i)\Gamma(\alpha_i+2)}\right) \\ &\quad \cdot \Gamma(2-\alpha_i)\lambda_i\sigma_{2i} - \frac{\mu_i\sigma_{2i}}{(1+\mu_i)^2\Gamma(\alpha_i+1)}, \\ A_{2i}(t) &= \frac{t^{\alpha_i}\lambda_i\sigma_{1i}}{\Gamma(\alpha_i+1)(1+\mu_i)} - \frac{\sigma_{1i}}{1+\mu_i} + \frac{\mu_i\lambda_i\sigma_{1i}}{(1+\mu_i)^2\Gamma(\alpha_i+1)}, \\ A_{3i}(t) &= -\frac{t^{\alpha_i}\Gamma(2-\alpha_i)\sigma_{3i}}{\Gamma(\alpha_i+1)(1+\mu_i)} + \frac{t^{\alpha_i+1}\Gamma(2-\alpha_i)\sigma_{3i}}{\Gamma(\alpha_i+2)\mu_i} \\ &\quad + \frac{\mu_i\Gamma(2-\alpha_i)\sigma_{3i}}{(1+\mu_i)^2\Gamma(\alpha_i+1)} - \frac{\Gamma(2-\alpha_i)\sigma_{3i}}{(1+\mu_i)\Gamma(\alpha_i+2)}, \\ A_{4i}(t) &= \frac{t^{\alpha_i}\Gamma(2-\alpha_i)\mu_i}{\Gamma(\alpha_i+1)(1+\mu_i)} - \frac{t^{\alpha_i+1}\Gamma(2-\alpha_i)}{\Gamma(\alpha_i+2)} \\ &\quad + \Gamma(2-\alpha_i)\left(\frac{\mu_i}{(1+\mu_i)\Gamma(\alpha_i+2)} - \frac{\mu_i^2}{(1+\mu_i)^2\Gamma(\alpha_i+1)}\right), \\ A_{5i}(t) &= -\frac{t^{\alpha_i}\mu_i}{\Gamma(\alpha_i+1)(1+\mu_i)} - \frac{\mu_i^2}{(1+\mu_i)^2\Gamma(\alpha_i+1)}, \quad \text{for } i=1,2. \end{aligned} \tag{17}$$

Proof. Using Lemma 5, we obtain

$$\begin{aligned} ({}^cD^{\alpha_1}+\lambda_1)\left(\frac{x_1(t)}{f_1(t,x_1(t),x_2(t))}\right) &= I^{\beta_1}y_1(t)+a_0+a_1t, \\ {}^cD^{\alpha_1}\left(\frac{x_1(t)}{f_1(t,x_1(t),x_2(t))}\right) &= I^{\beta_1}y_1(t)+a_0+a_1t-\lambda_1\left(\frac{x_1(t)}{f_1(t,x_1(t),x_2(t))}\right), \\ \frac{x_1(t)}{f_1(t,x_1(t),x_2(t))} &= I^{\alpha_1+\beta_1}y_1(t)+I^{\alpha_1}a_0+I^{\alpha_1}a_1t-I^{\alpha_1}\lambda_1\left(\frac{x_1(t)}{f_1(t,x_1(t),x_2(t))}\right)+a_2, \end{aligned} \tag{18}$$

where $a_0, a_1, a_2 \in \mathbb{R}$. □

According to the condition ${}^cD^{2\alpha_1}(x_1(0)/f_1(0,x_1(0),x_2(0))) + \mu_1{}^cD^{2\alpha_1}(x_1(1)/f_1(1,x(1),x_2(1))) = \sigma_{31}\int_0^1 k_1(s,x_1(s))ds$, we find that

$$\begin{aligned} a_1 &= \Gamma(2-\alpha_1)\left(\frac{\sigma_{31}}{\mu_1}\int_0^1 k_1(s,x_1(s))\right)ds + \frac{\lambda_1\sigma_{21}}{\mu_1}\int_0^1 h_1(s,x_1(s))ds \\ &\quad - \frac{1}{\Gamma(\beta_1-\alpha_1)}\int_0^1 (1-s)^{\beta_1-\alpha_1}y_1(s)ds\Bigg). \end{aligned} \tag{19}$$

Using the facts that ${}^cD^{\alpha_1}(x_1(0)/f_1(0,x_1(0),x_2(0))) + \mu_1{}^cD^{\alpha_1}(x_1(1)/f_1(1,x_1(1),x_2(1))) = \sigma_{21}\int_0^1 h_1(s,x_1(s))ds$ and $(x_1(0)/f_1(0,x_1(0),x_2(0))) + \mu_1(x_1(1)/f_1(1,x_1(1),x_2(1))) = \sigma_{11}\int_0^1 g_1(s,x_1(s))ds$, we have

$$\begin{aligned} a_0 &= \frac{-\Gamma(2-\alpha_1)\sigma_{31}}{1+\mu_1}\int_0^1 k_1(s,x_1(s))ds + \frac{(1-\lambda_1\Gamma(2-\alpha_1))\sigma_{21}}{1+\mu_1}\int_0^1 h_1(s,x_1(s))ds \\ &\quad + \frac{\lambda_1\sigma_{11}}{1+\mu_1}\int_0^1 g_1(s,x_1(s))ds + \frac{\Gamma(2-\alpha_1)\mu_1}{(1+\mu_1)\Gamma(\beta_1-\alpha_1)}\int_0^1 (1-s)^{\beta_1-\alpha_1-1}y_1(s)ds \\ &\quad - \frac{\mu_1}{(1+\mu_1)\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1}y_1(s)ds, \\ a_2 &= \frac{\mu_1\lambda_1}{(1+\mu_1)\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1}x_1(s)ds \\ &\quad + \left(\frac{\sigma_{11}}{1+\mu_1} - \frac{\mu_1\lambda_1\sigma_{11}}{\Gamma(\alpha_1+1)(1+\mu_1)^2}\right)\int_0^1 g_1(s,x_1(s))ds \\ &\quad - \frac{\mu_1}{(1+\mu_1)\Gamma(\alpha_1+\beta_1)}\int_0^1 (1-s)^{\alpha_1+\beta_1-1}y_1(s)ds \\ &\quad + \frac{\mu_1^2}{(1+\mu_1)^2\Gamma(\beta_1)\Gamma(\alpha_1+1)}\int_0^1 (1-s)^{\beta_1-1}y_1(s)ds + \frac{\Gamma(2-\alpha_1)}{\Gamma(\beta_1-\alpha_1)} \\ &\quad \cdot \left(\frac{\mu_1}{(1+\mu_1)\Gamma(\alpha_1+2)} - \frac{\mu_1^2}{(1+\mu_1)^2\Gamma(\alpha_1+1)}\right)\int_0^1 (1-s)^{\beta_1-\alpha_1-1}y_1(s)ds \\ &\quad + \left[\left(\frac{\mu_1^2}{(1+\mu_1)^2\Gamma(\alpha_1+1)} - \frac{\mu_1}{(1+\mu_1)\Gamma(\alpha_1+2)}\right)\frac{\Gamma(2-\alpha_1)\lambda_1\sigma_{21}}{\mu_1}\right. \\ &\quad \left. - \frac{\mu_1\sigma_{21}}{(1+\mu_1)^2\Gamma(\alpha_1+1)}\right] \times \int_0^1 h_1(s,x_1(s))ds. \end{aligned} \tag{20}$$

Substituting the values of a_0,a_1, and a_2, we obtain

$$\begin{aligned} x_1(t) &= f_1(t,x_1(t),x_2(t))\left[\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t (t-s)^{\alpha_1+\beta_1-1}y_1(s)ds\right. \\ &\quad + A_{11}(t)\int_0^1 h_1(s,x_1(s))ds + A_{21}(t)\int_0^1 g_1(s,x_1(s))ds \\ &\quad + A_{32}(t)\int_0^1 k_1(s,x_1(s))ds + \frac{A_{41}(t)}{\Gamma(\beta_1-\alpha_1)}\int_0^1 (1-s)^{\beta-\alpha-1}y_1(s)ds \\ &\quad + \frac{A_{51}(t)}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1}y_1(s)ds + \frac{\mu_1\lambda_1}{(1+\mu_1)\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1}x_1(s)ds \\ &\quad \left. - \frac{\mu_1}{(1+\mu_1)\Gamma(\alpha_1+\beta_1)}\int_0^1 (1-s)^{\alpha_1+\beta_1-1}y_1(s)ds\right] \\ &\quad - \frac{\lambda_1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha-1}x_1(s)ds. \end{aligned} \tag{21}$$

Analogously, we can deduce that

$$\begin{aligned} x_2(t) &= f_2(t,x_1(t),x_2(t))\left[\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t (t-s)^{\alpha_2+\beta_2-1}y_2(s)ds\right. \\ &\quad + A_{12}(t)\int_0^1 h_2(s,x_2(s))ds + A_{22}(t)\int_0^1 g_2(s,x_2(s))ds \\ &\quad + A_{32}(t)\int_0^1 k_2(s,x_2(s))ds + \frac{A_{42}(t)}{\Gamma(\beta_2-1-\alpha_2-1)}\int_0^1 (1-s)^{\beta_2-\alpha_2-1}y_2(s)ds \\ &\quad + \frac{A_{51}(t)}{\Gamma(\beta_2)}\int_0^1 (1-s)^{\beta_2-1}y_2(s)ds + \frac{\mu_2\lambda_2}{(1+\mu_2)\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-1}x_2(s)ds \\ &\quad \left. - \frac{\mu_2}{(1+\mu_2)\Gamma(\alpha_2+\beta_2)}\int_0^1 (1-s)^{\alpha_2+\beta_2}y_2(s)ds\right] - \frac{\lambda_2}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1}x_2(s)ds. \end{aligned} \tag{22}$$

By a direct computation, the converse of the lemma can be easily verified.

3. Main Results

In view of Lemma 7, we define the operator $U : X \times X \longrightarrow X \times X$ by $U(x_1, x_2) = (U_1(x_1, x_2), U_2(x_1, x_2))$.

Where,

$$\begin{aligned} U_1(x_1,x_2)(t) = f_1(t,x_1(t),x_2(t)) \Bigg[& \frac{1}{\Gamma(\alpha_1+\beta_1)} \int_0^t (t-s)^{\alpha_1+\beta_1-1} w_1(s,x_1(s),x_2(s))ds \\ & + A_{11}(t)\int_0^1 h_1(s,x_1(s))ds + A_{21}(t)\int_0^1 g_1(s,x_1(s))ds \\ & + A_{32}(t)\int_0^1 k_1(s,x_1(s))ds \\ & + \frac{A_{41}(t)}{\Gamma(\beta_1-\alpha_1)}\int_0^1 (1-s)^{\beta_1-\alpha_1-1} w_1(s,x_1(s),x_2(s))ds \\ & + \frac{A_{51}(t)}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1} w_1(s,x_1(s),x_2(s))ds \\ & + \frac{\mu_1\lambda_1}{(1+\mu_1)\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1} x_1(s)ds \\ & - \frac{\mu_1}{(1+\mu_1)\Gamma(\alpha_1+\beta_1)}\int_0^1 (1-s)^{\alpha_1+\beta_1-1} w_1(s,x_1(s),x_2(s))ds \Bigg] \\ & - \frac{\lambda_1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha-1} x_1(s)ds, \end{aligned} \tag{23}$$

$$\begin{aligned} U_2(x_1,x_2)(t) = {} & f_2(t,x_1(t),x_2(t)) \\ & \cdot \Bigg[\frac{1}{\Gamma(\alpha_2+\beta_2)} \int_0^t (t-s)^{\alpha_2+\beta_2-1} w_2(s,x_1(s),x_2(s))ds \\ & + A_{12}(t)\int_0^1 h_2(s,x_2(s))ds + A_{22}(t)\int_0^1 g_2(s,x_2(s))ds \\ & + A_{32}(t)\int_0^1 k_2(s,x_2(s))ds \\ & + \frac{A_{42}(t)}{\Gamma(\beta_2-1-\alpha_2-1)}\int_0^1 (1-s)^{\beta_2-\alpha_2-1} w_2(s,x_1(s),x_2(s))ds \\ & + \frac{A_{51}(t)}{\Gamma(\beta_2)}\int_0^1 (1-s)^{\beta_2-1} w_2(s,x_1(s),x_2(s))ds \\ & + \frac{\mu_2\lambda_2}{(1+\mu_2)\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-1} x_2(s)ds \\ & - \frac{\mu_2}{(1+\mu_2)\Gamma(\alpha_2+\beta_2)}\int_0^1 (1-s)^{\alpha_2+\beta_2} w_2(s,x_1(s),x_2(s))ds \Bigg] \\ & - \frac{\lambda_2}{\Gamma(\alpha_2)}\int_0^t (t-s)^{\alpha_2-1} x_2(s)ds. \end{aligned} \tag{24}$$

For computational convenience, we set

$$\begin{aligned} r_{11} = \max \Bigg\{ \Bigg[& L_1 q_{11} \left(\frac{1}{\Gamma(\alpha_1+\beta_1+1)} + \frac{A_{41}}{\Gamma(\beta_1-\alpha_1+1)} + \frac{A_{51}}{\Gamma(\beta_1+1)} + \left| \frac{\mu_1}{(1+\mu_1)\Gamma(\alpha_1+\beta_1+1)} \right| \right) \\ & + \left| \frac{\mu_1\lambda_1}{(1+\mu_1)\Gamma(\alpha_1+1)} \right| + A_{11}q_{51} + A_{21}q_{41} + A_{31}q_{61} \Bigg] + \frac{\lambda_1}{\Gamma(\alpha_1+1)}, L_1 q_{21} \\ & \times \left(\frac{1}{\Gamma(\alpha_1+\beta_1+1)} + \frac{A_{41}}{\Gamma(\beta_1-\alpha_1+1)} + \frac{A_{51}}{\Gamma(\beta_1+1)} + \left| \frac{\mu_1}{(1+\mu_1)\Gamma(\alpha_1+\beta_1+1)} \right| \right) \Bigg\}, \\ r_{12} = \max \Bigg\{ \Bigg[& L_2 q_{12} \left(\frac{1}{\Gamma(\alpha_2+\beta_2+1)} + \frac{A_{42}}{\Gamma(\beta_2-\alpha_2+1)} + \frac{A_{52}}{\Gamma(\beta_2+1)} + \left| \frac{\mu_2}{(1+\mu_2)\Gamma(\alpha_2+\beta_2+1)} \right| \right) \\ & + \left| \frac{\mu_2\lambda_2}{(1+\mu_2)\Gamma(\alpha_2+1)} \right| + A_{12}q_{52} + A_{22}q_{42} + A_{32}q_{62} \Bigg] + \frac{\lambda_2}{\Gamma(\alpha_2+1)}, q_{22} L_2 \\ & \times \left(\frac{1}{\Gamma(\alpha_2+\beta_2+1)} + \frac{A_{42}}{\Gamma(\beta_2-\alpha_2+1)} + \frac{A_{52}}{\Gamma(\beta_2+1)} + \left| \frac{\mu_2}{(1+\mu_2)\Gamma(\alpha_2+\beta_2+1)} \right| \right) \Bigg\}, \\ & A_{ij} = \max_{t\in[0,1]} |A_{ij}(t)| \quad \text{for } i = 1, 2, .., 5, j = 1, 2. \end{aligned} \tag{25}$$

Before giving the main results, we impose the following assumptions:

(H_1) The functions f_i are continuous and bounded; that is, there exist positive numbers $L_i > 0$ such that $|f_i(t, u, v)| \le L_i$ for all $(t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R} (i = 1, 2)$.

$(H_2) w_1, w_2 : [0, 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $h_1, g_1, k_1, h_2, g_2, k_2 : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions.

(H_3) There exist positive constants $q_{11}, q_{21}, q_{31}, q_{12}, q_{22}, q_{32}$ such that for all $t \in [0, 1]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$\begin{aligned} |w_1(t, x_1, y_1) - w_1(t, x_2, y_2)| &\le q_{11}|x_1 - x_2| + q_{21}|y_1 - y_2|, \\ |w_2(t, x_1, y_1) - w_2(t, x_2, y_2)| &\le q_{12}|x_1 - x_2| + q_{22}|y_1 - y_2|. \end{aligned} \tag{26}$$

(H_4) There exist positive constants $q_{41}, q_{51}, q_{61}, q_{42}, q_{52}, q_{62}$ such that

$$\begin{aligned} & |g_1(t, x_1) - g_1(t, x_2)| \\ & \quad \le q_{41}|x_1 - x_2|, |g_2(t, x_1) - g_2(t, x_2)| \\ & \quad \le q_{42}|x_1 - x_2|, |h_1(t, x_1) - h_1(t, x_2)| \\ & \quad \le q_{51}|x_1 - x_2|, |h_2(t, x_1) - h_2(t, x_2)| \\ & \quad \le q_{52}|x_1 - x_2|, \\ & |k_1(t, x_1) - k_1(t, x_2)| \\ & \quad \le q_{61}|x_1 - x_2|, |k_2(t, x_1) - k_2(t, x_2)| \\ & \quad \le q_{62}|x_1 - x_2|, \forall x_1, x_2 \in \mathbb{R}, t \in [0, 1]. \end{aligned} \tag{27}$$

$(H_5) |f_i(t, x, y)| \le m_i(t)$; $|h_i(t, x)| \le \rho_i(t)$; $|k_i(t, x)| \le \psi_i(t)$; $|g_i(t, x)| \le \phi_i(t)$, $\forall (t, x, y) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$ with $m_i, \rho_i, \phi_i, \psi_i \in C([0, 1] ; \mathbb{R}^+)$, for $i = 1, 2$.

3.1. First Result. In our first result, we discuss the existence of solutions for system (7)–(8) by means of the Banach fixed point theorem.

Theorem 8. *Suppose that (H_1)–(H_3) are satisfied.*

Then, there exist a unique solution for systems (7)–(8) provided that $r_{11} + r_{12} < 1$.

Proof. We put $\sup_{0\le t\le 1} |f_i(t, 0, 0)| = M_{0i}$, $\sup_{0\le t\le 1} |g_i(t, 0)| = M_{1i}$, $\sup_{0\le t\le 1} |h_i(t, 0)| = M_{2i}$, $\sup_{0\le t\le 1} |k_i(t, 0)| = M_{3i}$, for $i = 1, 2$.

Let $B_r = \{(x_1, x_2) \in X \times X : \|(x_1, x_2)\| \le r\}$ with $r \ge (r_{21} + r_{22})/(1 - (r_{11} + r_{12}))$, where

$$\begin{aligned} r_{21} = {} & \frac{M_{01}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{M_{01}A_{41}}{\Gamma(\beta_1-\alpha_1+1)} \\ & + A_{11}M_{21} + A_{21}M_{11} + A_{31}M_{31} \\ & + \frac{A_{51}M_{01}}{\Gamma(\beta_1+1)} + \left| \frac{\mu_1 M_{01}}{(1+\mu_1)\Gamma(\alpha_1+\beta_1+1)} \right|, \end{aligned}$$

$$
\begin{aligned}
r_{22} = {} & \frac{M_{02}}{\Gamma(\alpha_2 + \beta_2 + 1)} + \frac{M_{02} A_{42}}{\Gamma(\beta_2 - \alpha_2 + 1)} \\
& + A_{12} M_{22} + A_{22} M_{12} + A_{32} M_{32} \\
& + \frac{A_{52} M_{02}}{\Gamma(\beta_2 + 1)} + \left| \frac{\mu_2 M_{02}}{(1 + \mu_2)\Gamma(\alpha_2 + \beta_2 + 1)} \right|.
\end{aligned} \tag{28}
$$

□

We prove that $U(B_r) \subseteq B_r$.
For $(x_1, x_2) \in B_r$, $t \in [0, 1]$, we have

$$
\begin{aligned}
& |U_1(x_1, x_2)(t)| \\
& \le |f_1(s, x_1(s), x_2(s))| \Bigg[\frac{1}{\Gamma(\alpha_1 + \beta_1)} \int_0^t (t - s)^{\alpha_1 + \beta_1 - 1} (|w_1(s, x_1(s), x_2(s)) \\
& \quad - w_1(s, 0, 0)| + |w_1(s, 0, 0)|) ds + |A_{11}(t)| \left| \int_0^1 \right| h_1(s ; x_1(s)) \\
& \quad - h_1(s ; 0)| + |h_1(s ; 0)| ds + |A_{21}(t)| \left| \int_0^1 \right| g_1(s ; x_1(s)) \\
& \quad - g_1(s ; 0)| + |g_1(s ; 0)| ds + |A_{31}(t)| \left| \int_0^1 \right| k_1(s ; x_1(s)) \\
& \quad - k_1(s ; 0)| + |k_1(s ; 0)| ds + \frac{|A_{41}(t)|}{\Gamma(\beta_1 - \alpha_1)} \\
& \quad \times \int_0^1 (1 - s)^{\beta_1 - \alpha_1 - 1} (|w_1(s, x_1(s), x_2(s)) - w_1(s, 0, 0)| + |w_1(s, 0, 0)|) ds \\
& \quad + \frac{|A_{51}(t)|}{\Gamma(\beta_1)} \int_0^1 (1 - s)^{\beta_1 - 1} (|w_1(s, x_1(s), x_2(s)) - w_1(s, 0, 0)| + |w_1(s, 0, 0)|) ds \\
& \quad + \left| \frac{\mu_1 \lambda_1}{(1 + \mu_1)\Gamma(\alpha_1)} \right| \left| \int_0^1 (1 - s)^{\alpha_1 - 1} |x_1(s)| ds + \left| \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1)} \right| \right. \\
& \quad \times \int_0^1 (1 - s)^{\alpha_1 + \beta_1 - 1} (|w_1(s, x_1(s), x_2(s)) - w_1(s, 0, 0)| + |w_1(s, 0, 0)|) ds \Bigg] \\
& \quad + \frac{|\lambda_1|}{\Gamma(\alpha_1)} \int_0^t (t - s)^{\alpha_1 - 1} |x_1(s)| ds.
\end{aligned} \tag{29}
$$

Consequently,

$$
\begin{aligned}
\| U_1(x_1, x_2) \| \le {} & L_1 \Bigg[\frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} (q_{11} \|x_1\| + q_{21} \|x_2\| + \|x_2\| + M_{01}) \\
& + A_{11} [q_{51} \|x_1\| + M_{21}] + A_{21} [q_{41} \|x_1\| + M_{11}] + A_{31} [q_{61} \|x_1\| + M_{31}] \\
& + \frac{A_{41}}{\Gamma(\beta_1 - \alpha_1 + 1)} (q_{11} \|x_1\| + q_{21} \|x_2\| + \|x_2\| + M_{01}) \\
& + \frac{A_{51}}{\Gamma(\beta_1 + 1)} (q_{11} \|x_1\| + q_{21} \|x_2\| + \|x_2\| + M_{01}) + \left| \frac{\mu_1 \lambda_1}{(1 + \mu_1)\Gamma(\alpha_1 + 1)} \right| \|x_1\| \\
& + \left| \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1 + 1)} \right| (q_{11} \|x_1\| + q_{21} \|x_2\| + \|x_2\| + M_{01}) \Bigg] \\
& + \frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)} \|x_1\| \\
\le {} & \Bigg[L_1 \Bigg(q_{11} \Bigg(\frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{A_{41}}{\Gamma(\beta_1 - \alpha_1 + 1)} + \frac{A_{51}}{\Gamma(\beta_1 + 1)} \\
& + \left| \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1 + 1)} \right| \Bigg) + \left| \frac{\mu_1 \lambda_1}{(1 + \mu_1)\Gamma(\alpha_1 + 1)} \right| + A_{11} q_{51} + A_{21} q_{41} \\
& + A_{31} q_{61} \Bigg) + \frac{\lambda_1}{\Gamma(\alpha_1 + 1)} \Bigg] \|x_1\| L_1 \Bigg[\Bigg(q_{21} \Bigg(\frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{A_{41}}{\Gamma(\beta_1 - \alpha_1 + 1)} \\
& + \frac{A_{51}}{\Gamma(\beta_1 + 1)} + \left| \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1 + 1)} \right| \Bigg) \Bigg) \|x_2\| + \frac{M_{01}}{\Gamma(\alpha_1 + \beta_1 + 1)} \\
& + \frac{M_{01} A_{41}}{\Gamma(\beta_1 - \alpha_1 + 1)} + A_{11} M_{21} + A_{21} M_{11} + A_{31} M_{31} + \frac{A_{51} M_{01}}{\Gamma(\beta_1 + 1)} \\
& + \left| \frac{\mu_1 M_{01}}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1 + 1)} \right| \Bigg] \le r_{11} r + r_{21}.
\end{aligned} \tag{30}
$$

In the same way, we obtain that

$$
\| U_2(x_1, x_2) \| \le r_{12} r + r_{22}. \tag{31}
$$

Therefore, we have

$$
\begin{aligned}
\| \mathrm{U}(x_1, x_2) \| & = \| U_1(x_1, x_2) \| + \| U_2(x_1, x_2) \| \\
& \le (r_{11} + r_{12}) r + r_{21} + r_{22} \le r.
\end{aligned} \tag{32}
$$

Now, for $(x_1, x_2), (x_1', x_2') \in X \times X$ and $t \in [0 ; 1]$, we get

$$
\begin{aligned}
& \left| U_1(x_1, x_2)(t) - U_1\left(x_1', x_2'\right)(t) \right| \\
& \le \left| f_1(s, x_1(s), x_2(s)) - f_1\left(s, x_1'(s), x_2'(s)\right) \right| \Bigg[\frac{1}{\Gamma(\alpha_1 + \beta_1)} \int_0^t (t - s)^{\alpha_1 + \beta_1 - 1} \Big| w_1(s, x_1(s), x_2(s)) \\
& \quad - w_1\left(s, x_1'(s), x_2'(s)\right) \Big| ds + |A_{11}(t)| \int_0^1 \left| h_1(s ; x_1(s)) - h_1\left(s ; x_1'(s)\right) \right| ds \\
& \quad + |A_{21}(t)| \int_0^1 \left| g_1(s ; x_1(s)) - g_1\left(s ; x_1'(s)\right) \right| ds + |A_{31}(t)| \times \int_0^1 \Big| k_1(s ; x_1(s)) \\
& \quad - k_1\left(s ; x_1'(s)\right) \Big| ds + \frac{|A_{41}(t)|}{\Gamma(\beta_1 - \alpha_1)} \int_0^1 (1 - s)^{\beta_1 - \alpha_1 - 1} \times \Big| w_1(s, x_1(s), x_2(s)) \\
& \quad - w_1\left(s, x_1'(s), x_2'(s)\right) \Big| ds + \frac{|A_{51}(t)|}{\Gamma(\beta_1)} \int_0^1 (1 - s)^{\beta_1 - 1} \Big| w_1(s, x_1(s), x_2(s)) \\
& \quad - w_1\left(s, x_1'(s), x_2'(s)\right) \Big| ds + \left| \frac{\mu_1 \lambda_1}{(1 + \mu_1)\Gamma(\alpha_1)} \right| \times \int_0^1 (1 - s)^{\alpha_1 - 1} \left| x_1(s) - x_1'(s) \right| ds \\
& \quad + \frac{|\mu_1|}{|1 + \mu_1| \Gamma(\alpha_1 + \beta_1)} \int_0^1 (1 - s)^{\alpha_1 + \beta_1 - 1} \left| w_1(s, x_1(s), x_2(s)) - w_1\left(s, x_1'(s), x_2'(s)\right) \right| ds \Bigg] \\
& \quad + \frac{|\lambda_1|}{\Gamma(\alpha_1)} \int_0^t (t - s)^{\alpha_1 - 1} \left| x_1(s) - x_1'(s) \right| ds \\
& \le L_1 \Bigg[\frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} \left(q_{11} \left\| x_1 - x_1' \right\| + \left(q_{21} + \frac{q_{31}}{\Gamma(p_2 + 1)} \right) \left\| x_2 - x_2' \right\| \right) \\
& \quad + \frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)} \left\| x_1 - x_1' \right\| + A_{11} q_{51} \left\| x_1 - x_1' \right\| + A_{21} q_{41} \left\| x_1 - x_1' \right\| + A_{31} q_{61} \left\| x_1 - x_1' \right\| \\
& \quad + \frac{A_{41}}{\Gamma(\beta_1 - \alpha_1 + 1)} \left(q_{11} \left\| x_1 - x_1' \right\| + \left(q_{21} + \frac{q_{31}}{\Gamma(p_2 + 1)} \right) \left\| x_2 - x_2' \right\| \right) \\
& \quad + \frac{A_{51}}{\Gamma(\beta_1 + 1)} \left(q_{11} \left\| x_1 - x_1' \right\| + \left(q_{21} + \frac{q_{31}}{\Gamma(p_2 + 1)} \right) \left\| x_2 - x_2' \right\| \right) \\
& \quad + \left| \frac{\mu_1 \lambda_1}{(1 + \mu_1)\Gamma(\alpha_1 + 1)} \right| \left\| x_1 - x_1' \right\| \\
& \quad + \left| \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1 + 1)} \right| \left(q_{11} \left\| x_1 - x_1' \right\| + q_{21} \left\| x_2 - x_2' \right\| \right) \Bigg] + \frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)} \left\| x_1 - x_1' \right\| \\
& \le L_1 \Bigg[\Bigg[q_{11} \left(\frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{A_{41}}{\Gamma(\beta_1 - \alpha_1 + 1)} + \frac{A_{51}}{\Gamma(\beta_1 + 1)} + \left| \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1 + 1)} \right| \right) \\
& \quad + \left| \frac{\mu_1 \lambda_1}{(1 + \mu_1)\Gamma(\alpha_1 + 1)} \right| + A_{11} q_{51} + A_{21} q_{41} + A_{31} q_{61} \Bigg] \left\| x_1 - x_1' \right\| + q_{21} \\
& \quad \times \left(\frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{A_{41}}{\Gamma(\beta_1 - \alpha_1 + 1)} + \frac{A_{51}}{\Gamma(\beta_1 + 1)} + \left| \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1 + 1)} \right| \right) \\
& \quad \times \left\| x_2 - x_2' \right\| + \frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)} \left\| x_1 - x_1' \right\| \Bigg] \le r_{11} \left(\left\| x_1 - x_1' \right\| + \left\| x_2 - x_2' \right\| \right).
\end{aligned} \tag{33}
$$

Analogously, one has

$$
\left| U_2(x_1, x_2)(t) - U_2\left(x_1', x_2'\right)(t) \right| \le r_{12} \left(\left\| x_1 - x_1' \right\| + \left\| x_2 - x_2' \right\| \right), \tag{34}
$$

and thus

$$
\left\| U(x_1, x_2) - U\left(x'_1, x'_2\right) \right\| \le (r_{11} + r_{12}) \left(\left\| x_1 - x_1' \right\| + \left\| x_2 - x_2' \right\| \right). \tag{35}
$$

Since $r_{11} + r_{12} < 1$,by thene the operator U is a contraction mapping. Hence, we deduce that systems (7)–(8) have a unique solution.

3.2. Second Result. In our second result, we discuss the existence of solutions for system (7)–(8) by means of the so-called Leray-Schauder alternative.

Theorem 9. *Assume that conditions* (H_1)–(H_3) *and* (H_5) *hold. Furthermore, it is assumed that*

$$\pi_1 + \pi_2(\eta_0 + \eta_1\|x_1\| + \eta_2\|x_2\|) + \pi_3\|x_1\| + \pi_4\|x_2\| < 1. \quad (36)$$

Then, system (7)–(8) have at least one solution.

Proof. We will show that the operator $Y : X \times Y \longrightarrow X \times Y$ satisfes all the assumptions of Lemma 6.

In the first step, we prove that the operator Y is completely continuous.

Clearly, it follows by the continuity of functions f_1, f_2, g_1, and g_2 that the operator Y is continuous.

Let $S \subset X \times Y$ be bounded. Then, we can find positive constants H_1 and H_2 such that

$$|w_1(t, x_1, x_2)| \le H_1, |w_2(t, x_1, x_2)| \le H_2, \quad \forall (x_1, x_2) \in S. \quad (37)$$

Thus, for any $x_1, x_2 \in S$, we get

$$\begin{aligned}
|Y_1(x_1,x_2)(t)| \le{} & L_1\Bigg[\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t (t-s)^{\alpha_1+\beta_1-1}|w_1(s,x_1(s),x_2(s))|ds \\
& + |A_{11}(t)| \times \int_0^1 |h_1(s;x_1(s))|ds + |A_{21}(t)|\left|\int_0^1 |g_1(s;x_1(s))|ds\right. \\
& + |A_{31}(t)|\left|\int_0^1 |k_1(s;x_1(s))|ds\right. \\
& + \frac{|A_{41}(t)|}{\Gamma(\beta_1-\alpha_1)}\int_0^1 (1-s)^{\beta_1-\alpha_1-1}|w_1(s,x_1(s),x_2(s))|ds \\
& + \frac{|A_{51}(t)|}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1}|f_1(s,x_1(s),x_2(s),I^{p_2}x_2(s))|ds \\
& + \frac{|\mu_1|}{|1+\mu_1|\Gamma(\alpha_1+\beta_1)}\int_0^1 (1-s)^{\alpha_1+\beta_1-1}|w_1(s,x_1(s),x_2(s))|ds \\
& + \left|\frac{\mu_1\lambda_1}{(1+\mu_1)\Gamma(\alpha_1)}\right|\left|\int_0^1 (1-s)^{\alpha_1-1}|x_1(s)|ds\right] + \frac{|\lambda_1|}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}|x_1(s)|ds \\
\le{} & L_1\Bigg[\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t (t-s)^{\alpha_1+\beta_1-1}H_1 ds + A_{11}\int_0^1 \rho_1 ds + A_{21}\int_0^1 \phi_1 ds \\
& + A_{31}\int_0^1 \psi_1 ds + \frac{A_{41}}{\Gamma(\beta_1-\alpha_1)}\int_0^1 (1-s)^{\beta_1-\alpha_1-1}H_1 ds \\
& + \frac{A_{51}}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1}H_1 ds + \frac{|\mu_1|}{(|1+\mu_1|)\Gamma(\alpha_1+\beta_1)}\int_0^1 (1-s)^{\alpha_1+\beta_1-1} H_1 ds \\
& + \frac{|\mu_1\lambda_1|}{(|1+\mu_1|)\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1}|x_1(s)|ds\Bigg] + \left|\frac{\lambda_1}{\Gamma(\alpha_1)}\right|\left|\int_0^t (t-s)^{\alpha_1-1}|x_1(s)|ds\right. \\
\le{} & L_1\Bigg[\frac{H_1}{\Gamma(\alpha_1+\beta_1+1)} + A_{11}\rho_1 + A_{21}\phi_1 + A_{31}\psi_1 + \frac{A_{41}H_1}{\Gamma(\beta_1-\alpha_1+1)} \\
& + \frac{A_{51}H_1}{\Gamma(\beta_1+1)} + \frac{|\mu_1|\,H_1}{(|1+\mu_1|)\Gamma(\alpha_1+\beta_1+1)} + \frac{|\mu_1\lambda_1|}{(|1+\mu_1|)\Gamma(\alpha_1+1)}\|x_1\|\Bigg] \\
& + \left|\frac{\lambda_1}{\Gamma(\alpha_1+1)}\right|\|x_1\|.
\end{aligned} \quad (38)$$

In a similar manner, we have

$$\begin{aligned}
|Y_2(x_1,x_2)(t)| \le{} & L_2\left[\frac{H_2}{\Gamma(\alpha_2+\beta_2+1)} + A_{12}\rho_2 + A_{22}\phi_2\| + A_{32}\psi_2\right. \\
& + \frac{A_{42}H_2}{\Gamma(\beta_2-\alpha_2+1)} + \frac{A_{52}H_2}{\Gamma(\beta_2+1)} + \frac{|\mu_2|H_2}{(|1+\mu_2|)\Gamma(\alpha_2+\beta_2+1)} \\
& + \frac{|\mu_2\lambda_2|}{(|1+\mu_2|)\Gamma(\alpha_2+1)}\|x_2\|] + \left|\frac{\lambda_2}{\Gamma(\alpha_2+1)}\right|\|x_2\|.
\end{aligned} \quad (39)$$

From the inequalities above, we deduce that the operator Y is uniformly bounded.

Next, we prove that Y is equicontinuous.

The continuity of $f_1, f_2, h_1, h_2, g_1, g_2, k_1, k_2$ implies that the operator Y_1 is continuous. Moreover, Y_1 is uniformly bounded on $B_{r'}$.

Suppose that $0 \le t_1 < t_2 \le 1$. Then we have

$$\begin{aligned}
& |Y_1(x_1,x_2)(t_2) - Y_1(x_1,x_2)(t_1)| \\
& \le L_1\Bigg[\frac{1}{\Gamma(\alpha_1+\beta_1)}\Bigg|\int_0^{t_2}(t_2-s)^{\alpha_1+\beta_1-1}w_1(s,x_1(s),x_2(s))ds \\
& \quad - \int_0^{t_1}(t_1-s)^{\alpha_1+\beta_1-1}w_1(s,x_1(s),x_2(s))ds\Bigg| \\
& \quad + |A_{11}(t_2) - A_{11}(t_1)| \times \left|\int_0^1 h_1(s;x_1(s))ds\right| + |A_{21}(t_2) \\
& \quad - A_{21}(t_1)|\left|\int_0^1 g_1(s;x_1(s))ds\right| + |A_{31}(t_2) \\
& \quad - A_{31}(t_1)|\left|\int_0^1 k_1(s;x_1(s))ds\right| \\
& \quad + \frac{|A_{41}(t_2) - A_{41}(t_1)|}{\Gamma(\beta_1-\alpha_1)}\left|\int_0^1 (1-s)^{\beta_1-\alpha_1-1}w_1(s,x_1(s),x_2(s))ds\right| \\
& \quad + \frac{|A_{51}(t_2) - A_{51}(t_1)|}{\Gamma(\beta_1)}\left|\int_0^1 (1-s)^{\beta_1-1}w_1(s,x_1(s),x_2(s))\right|ds \\
& \quad + \frac{\lambda_1}{\Gamma(\alpha_1)}\left|\int_0^{t_2}(t_2-s)^{\alpha_1-1}x_1(s)ds - \int_0^{t_1}(t_1-s)^{\alpha_1-1}x_1(s)ds\right|\Bigg] \\
& \le L_1\Bigg[\frac{m_1}{\Gamma(\alpha_1+\beta_1+1)}\left(t_2^{\alpha_1+\beta_1} - t_1^{\alpha_1+\beta_1}\right) + \rho_1|A_{11}(t_2) - A_{11}(t_1)| \\
& \quad + \phi_1|A_{21}(t_2) - A_{21}(t_1)| + \psi_1|A_{31}(t_2) \\
& \quad - A_{31}(t_1)|\frac{m_1}{\Gamma(\beta_1-\alpha_1+1)}(|A_{41}(t_2) - A_{41}(t_1)|) \\
& \quad + \frac{m_1}{\Gamma(\beta_1+1)}(|A_{51}(t_2) - A_{51}(t_1)|) + \frac{\lambda_1}{\Gamma(\alpha_1)}\Bigg|\int_0^{t_1}((t_1-s)^{\alpha_1-1} \\
& \quad - (t_2-s)^{\alpha_1-1}x_1(s)ds - \int_{t_1}^{t_2}((t_1-s)^{\alpha_1-1}x_1(s)ds\Bigg|\Bigg].
\end{aligned} \quad (40)$$

Similarly, one has

$$\begin{aligned}
&|Y_2(x_1,x_2)(t_2)-Y_2(x_1,x_2)(t_1)| \\
&\le L_2\left[\frac{m_2}{\Gamma(\alpha_2+\beta_2+1)}\left(t_2^{\alpha_2+\beta_2}-t_1^{\alpha_2+\beta_2}\right)+\rho_2|A_{12}(t_2)-A_{12}(t_1)|\right. \\
&\quad+\phi_2|A_{22}(t_2)-A_{22}(t_1)|+\psi_2|A_{32}(t_2) \\
&\quad-A_{32}(t_1)|\frac{m_2}{\Gamma(\beta_2-\alpha_2+1)}(|A_{42}(t_2)-A_{42}(t_1)|) \\
&\quad+\frac{m_2}{\Gamma(\beta_2+1)}(|A_{52}(t_2)-A_{52}(t_1)|)+\frac{\lambda_1}{\Gamma(\alpha_1)}\left|\int_0^{t_1}((t_1-s)^{\alpha_2-1}\right. \\
&\quad\left.\left.-(t_2-s)^{\alpha_1-1}x_1(s)ds-\int_{t_1}^{t_2}((t_1-s)^{\alpha_2-1}x_1(s)ds\right|\right],
\end{aligned} \tag{41}$$

which tend to 0 independently of (x_1,x_2). This implies that the operator $Y(x_1,x_2)$ is equicontinuous. Thus, by the above fndings, the operator $Y(x_1,x_2)$ is completely continuous.

In the next step, we will prove that the set $P=\{(x_1,x_2)\in X\times Y/x_1,x_2=\lambda Y(x_1,x_2), 0\le\lambda\le 1\}$ is bounded.

Let $(x_1,x_2)\in P$. Then, we have $(x_1,x_2)=\lambda Y(x_1,x_2)$. Thus, for any $t\in[0,1]$, we can write

$$\begin{aligned}
x_1(t)&=\lambda Y_1(x_1,x_2)(t),\\
x_2(t)&=\lambda Y_2(x_1,x_2)(t).
\end{aligned} \tag{42}$$

Then,

$$\begin{aligned}
\|x_1\|&\le L_1\left[\frac{1}{\Gamma(\alpha_1+\beta_1+1)}(\eta_0+\eta_1\|x_1\|+\eta_2\|x_2\|)+A_{11}\rho_1\right.\\
&\quad+A_{21}\phi_1+A_{31}\psi_1+\frac{A_{41}}{\Gamma(\beta_1-\alpha_1+1)}(\eta_0+\eta_1\|x_1\|+\eta_2\|x_2\|)\\
&\quad+\frac{A_{51}}{\Gamma(\beta_1+1)}(\eta_0+\eta_1\|x_1\|+\eta_2\|x_2\|)+\frac{|\mu_1||\lambda_1|}{(1+\mu_1)\Gamma(\alpha_1)}\|x_1\|\\
&\quad\left.+\frac{|\mu_1|}{(|1+\mu_1|)\Gamma(\alpha_1+\beta_1+1)}(\eta_0+\eta_1\|x_1\|+\eta_2\|x_2\|)\right]+\frac{|\lambda_1|}{\Gamma(\alpha_1+1)}\|x_1\|\\
&\le L_1\left[A_{11}\rho_1+A_{21}\phi_1+A_{32}\psi_1+\left(\frac{1}{\Gamma(\alpha_1+\beta_1+1)}+\frac{A_{41}}{\Gamma(\beta_1-\alpha_1+1)}\right.\right.\\
&\quad\left.\left.+\frac{|\mu_1|}{(|1+\mu_1|)\Gamma(\alpha_1+\beta_1+1)}+\frac{A_{51}}{\Gamma(\beta_1+1)}\right)(\eta_0+\eta_1\|x_1\|+\eta_2\|x_2\|)\right]\\
&\quad+\left(L_1\frac{|\mu_1||\lambda_1|}{(1+\mu_1)\Gamma(\alpha_1)}+\frac{|\lambda_1|}{\Gamma(\alpha_1+1)}\right)\|x_1\|,\\
\|x_2\|&\le L_2\left[\frac{1}{\Gamma(\alpha_2+\beta_2+1)}(\eta_0+\eta_1\|x_1\|+\eta_2\|x_2\|)+A_{12}\rho_2+A_{22}\phi_2+A_{32}\psi_2\right.\\
&\quad+\frac{A_{42}}{\Gamma(\beta_2-\alpha_2+1)}(\eta_0+\eta_1\|x_1\|+\eta_2\|x_2\|)\\
&\quad+\frac{A_{52}}{\Gamma(\beta_2+1)}(\eta_0+\eta_1\|x_1\|+\eta_2\|x_2\|)+\frac{|\mu_2||\lambda_2|}{(1+\mu_2)\Gamma(\alpha_2)}\|x_2\|\\
&\quad\left.+\frac{|\mu_2|}{(|1+\mu_2|)\Gamma(\alpha_2+\beta_2+1)}(\eta_0+\eta_1\|x_1\|+\eta_2\|x_2\|)\right]+\frac{|\lambda_2|}{\Gamma(\alpha_2+1)}\|x_2\|\\
&\le L_2\left[A_{12}\rho_2+A_{22}\phi_2+A_{32}\psi_2+\left(\frac{1}{\Gamma(\alpha_2+\beta_2+1)}+\frac{A_{42}}{\Gamma(\beta_2-\alpha_2+1)}\right.\right.\\
&\quad\left.\left.+\frac{|\mu_2|}{(|1+\mu_2|)\Gamma(\alpha_2+\beta_2+1)}+\frac{A_{52}}{\Gamma(\beta_2+1)}\right)(\eta_0+\eta_1\|x_1\|+\eta_2\|x_2\|)\right]\\
&\quad+\left(L_2\frac{|\mu_2||\lambda_2|}{(1+\mu_2)\Gamma(\alpha_2)}+\frac{|\lambda_2|}{\Gamma(\alpha_2+1)}\right)\|x_2\|.
\end{aligned} \tag{43}$$

In consequence, we have

$$\begin{aligned}
\|x_2+x_1\|&\le L_1A_{11}\rho_1+L_1A_{21}\phi_1+L_1A_{32}\psi_1+L_1A_{12}\rho_2\\
&\quad+L_1A_{22}\phi_2+L_1A_{32}\psi_2\\
&\quad+\left(\frac{L_1}{\Gamma(\alpha_1+\beta_1+1)}+\frac{L_1A_{41}}{\Gamma(\beta_1-\alpha_1+1)}\right.\\
&\quad+\frac{L_1|\mu_1|}{(|1+\mu_1|)\Gamma(\alpha_1+\beta_1+1)}+\frac{L_1A_{51}}{\Gamma(\beta_1+1)}\\
&\quad+\frac{L_2}{\Gamma(\alpha_2+\beta_2+1)}+\frac{L_2A_{42}}{\Gamma(\beta_2-\alpha_2+1)}\\
&\quad\left.+\frac{L_2|\mu_2|}{(|1+\mu_2|)\Gamma(\alpha_2+\beta_2+1)}+\frac{L_2A_{52}}{\Gamma(\beta_2+1)}\right)\\
&\quad\cdot(\eta_0+\eta_1\|x_1\|+\eta_2\|x_2\|)\\
&\quad+\left(L_2\frac{|\mu_1||\lambda_1|}{(1+\mu_1)\Gamma(\alpha_1)}+\frac{|\lambda_1|}{\Gamma(\alpha_1+1)}\right)\|x_1\|\\
&\quad+\left(L_2\frac{|\mu_2||\lambda_2|}{(1+\mu_2)\Gamma(\alpha_2)}+\frac{|\lambda_2|}{\Gamma(\alpha_2+1)}\right)\|x_2\|,
\end{aligned} \tag{44}$$

then $\|(x_2,x_1)\|\le\pi_1+\pi_2(\eta_0+\eta_1\|x_1\|+\eta_2\|x_2\|)+\pi_3\|x_1\|+\pi_4\|x_2\|$ with

$$\begin{aligned}
\pi_1&=L_1A_{11}\rho_1+L_1A_{21}\phi_1+L_1A_{32}\psi_1+L_1A_{12}\rho_2+L_1A_{22}\phi_2\\
&\quad+L_1A_{32}\psi_2,\\
\pi_2&=\left(\frac{L_1}{\Gamma(\alpha_1+\beta_1+1)}+\frac{L_1A_{41}}{\Gamma(\beta_1-\alpha_1+1)}\right.\\
&\quad+\frac{L_1|\mu_1|}{(|1+\mu_1|)\Gamma(\alpha_1+\beta_1+1)}+\frac{L_1A_{51}}{\Gamma(\beta_1+1)}\\
&\quad+\frac{L_2}{\Gamma(\alpha_2+\beta_2+1)}+\frac{L_2A_{42}}{\Gamma(\beta_2-\alpha_2+1)}\\
&\quad\left.+\frac{L_2|\mu_2|}{(|1+\mu_2|)\Gamma(\alpha_2+\beta_2+1)}+\frac{L_2A_{52}}{\Gamma(\beta_2+1)}\right),\\
\pi_3&=\left(L_1\frac{|\mu_1||\lambda_1|}{(1+\mu_1)\Gamma(\alpha_1)}+\frac{|\lambda_1|}{\Gamma(\alpha_1+1)}\right),\\
\pi_4&=\left(L_2\frac{|\mu_2||\lambda_2|}{(1+\mu_2)\Gamma(\alpha_2)}+\frac{|\lambda_2|}{\Gamma(\alpha_2+1)}\right).
\end{aligned} \tag{45}$$

This shows that the set P is bounded. Hence, all the conditions of Lemma 6 are satisfied, and consequently, the operator Y has at least one fxed point, which corresponds to a solution of system (7)–(8). This completes the proof.

4. Examples

4.1. Example 1. Consider the following system of fractional hybrid differentiel equation:

$$\begin{aligned}
&cD^{4/3}\left({}^cD^{1/3}+\frac{1}{700}\right)\frac{x_1(t)}{f_1(t,x_1(t),x_2(t))}\\
&\quad=\frac{1}{1+t^4}\left(\frac{t^4|x_1(t)|}{(4|x_2(t)|+10)}+\frac{|x_2(t)|}{(4|x_1(t)|+6)}+\frac{1}{\Gamma(4/3)}\int_0^t(t-s)^{1/3}\frac{ds}{1+x_1^2(s)}\right),\\
&{}^cD^{4/3}\left({}^cD^{1/3}+\frac{1}{300}\right)\frac{x_2(t)}{f_2(x_1(t)/f_1(t,x_1(t),x_2(t)))}\\
&\quad=\frac{1}{500+t^2}\left(\sin(x_2(t))+\frac{|x_1(t)|}{1+|x_2(t)|}+\frac{1}{\Gamma(15/2)}\int_0^t(t-s)^{13/2}x_2(s)ds\right)\Bigg),\\
&\frac{x_1(0)}{f_1(0,x_1(0),x_2(0))}+\frac{x_1(1)}{f_1(1,x_1(1),x_2(1))}=\frac{1}{300}\int_0^1\frac{1}{s+100}\frac{|x_1(s)|}{300+|x_1(s)|}ds,\\
&{}^cD^{1/3}\left(\frac{x_1(0)}{f_1(0,x_1(0),x_2(0))}\right)+{}^cD^{1/3}\left(\frac{x_1(1)}{f_1(1,x_1(1),x_2(1))}\right)=\frac{1}{300}\int_0^1\left(\frac{1}{s+8000}\right)\frac{|x_1(s)|}{30+|x_1(s)|}ds,\\
&{}^cD^{2/3}\left(\frac{x_1(0)}{f_1(0,x_1(0),x_2(0))}\right)+{}^cD^{2/3}\left(\frac{x_1(1)}{f_1(1,x_1(1),x_2(1))}\right)=\frac{1}{300}\int_0^1\left(\frac{1}{s+1600}\right)\frac{|x_1(s)|}{30+|x_1(s)|}ds.\\
&\frac{x_2(0)}{f_2(0,x_1(0),x_2(0))}+\frac{x_2(1)}{f_2(1,x_1(1),x_2(1))}=\frac{1}{200}\int_0^1\frac{|x_2(s)|}{300+|x_2(s)|}ds,\\
&cD^{1/3}\left(\frac{x_2(0)}{f_2(0,x_1(0),x_2(0))}\right)+{}^cD^{1/3}\left(\frac{x_2(1)}{f_2(1,x_1(1),x_2(1))}\right)=\frac{1}{200}\int_0^1\left(\frac{1}{s+2}\right)^3\frac{|x_2(s)|}{30+|x_2(s)|}ds,\\
&{}^cD^{2/3}\left(\frac{x_2(0)}{f_2(0,x_1(0),x_2(0))}\right)+{}^cD^{2/3}\left(\frac{x_2(1)}{f_2(1,x_1(1),x_2(1))}\right)=\frac{1}{200}\int_0^1\left(\frac{1}{s+4}\right)^2\frac{|x_2(s)|}{30+|x_2(s)|}ds,
\end{aligned}\tag{46}$$

$\lambda_2=1/300, \mu_1=\mu_2=1, \sigma_{i1}=1/300, \sigma_{i2}=1/200, i=1,2,3$ and

$$\begin{aligned}
w_1(t,x_1(t),x_2(t)))&=\frac{1}{1+t^4}\left(\frac{t^4|x_1(t)|}{(4|x_2(t)|+10)}+\frac{|x_2(t)|}{(4|x_1(t)|+6)}+\frac{1}{\Gamma(4/3)}\int_0^t(t-s)^{1/3}\frac{ds}{1+x_1^2(s)}\right),\\
f_1(t,x_1(t),x_2(t)))&=\frac{(t+1)^2}{100}\left(\sin x_1(t)+\frac{|x_2(t)|}{1+|x_1(t)|}+3\right),\\
g_1(t,x_1t))&=\frac{|x_1(t)|}{300+|x_1(t)|}, h_1t,x_1(t))=\left(\frac{1}{t+2}\right)^3\frac{|x_1(t)|}{30+|x_1(t)|},\\
k_1(t,x_1(t))&=\left(\frac{1}{t+4}\right)^2\frac{|x_1(t)|}{30+|x_1(t)|},\\
w_2(t,x_1(t),x_2(t)))&=\frac{1}{500+t^2}\left(\sin(x_2(t))+\frac{|x_1(t)|}{1+|x_2(t)|}+\frac{1}{\Gamma(15/2)}\int_0^t(t-s)^{13/2}x_2(s)ds\right)\Bigg),\\
f_2(t,x_1(t),x_2(t)))&=\frac{1}{2(35+t)}\left(\frac{x_2^2(t)+|x_1(t)|}{1++|x_2(t)|}\right)+t^{1/3}+1,\\
g_2(t,x_2(t))&=\frac{1}{t+100}\frac{|x_2(t)|}{300+|x_2(t)|},\\
h_2(t,x_2(t))&=\left(\frac{1}{t+8000}\right)\frac{|x_2(t)|}{30+|x_2(t)|},\\
k_2(t,x_2(t))&=\left(\frac{1}{t+1600}\right)\frac{|x_2(t)|}{30+|x_2(t)|}.
\end{aligned}\tag{47}$$

Clearly, $q_{11}=q_{21}=1/400, q_{12}=q_{22}=1/500$, $q_{41}=q_{51}=q_{61}=1/300$, $q_{42}=q_{52}=q_{62}=1/200$, and $q_{61}=q_{62}=1/400$; furthermore, we have

$$\begin{aligned}
r_{11}+r_{12}&\approx\max(0.007421,0.0128)+\max(0.012255,0.007583)\\
&\approx 0.026<1.
\end{aligned}\tag{48}$$

Thus, by Theorem 8, system (46) has an unique solution.

4.2. Example 2. Consider the following system:

$$\begin{aligned}
&{}^cD^{2/3}\left({}^cD^{1/2}+\frac{1}{600}\right)\frac{x_1(t)}{f_1(t,x_1(t),x_2(t))}\\
&\quad=\frac{1}{t^2+4}\left(\frac{t^2|x_1(t)|}{(3|x_1(t)|+1)}+\frac{|x_2(t)|}{(6|x_2(t)|+10)}+\frac{1}{\Gamma(4/3)}\int_0^t(t-s)^{1/3}\frac{ds}{1+x_2^2(s)}\right),\\
&cD^{4/3}\left({}^cD^{1/3}+\frac{1}{700}\right)\frac{x_2(t)}{f_2(t,x_1(t),x_2(t))}\\
&\quad=\frac{1}{1+t^4}\left(\frac{t^4|x_1(t)|}{(4|x_1(t)|+10)}+\frac{|x_2(t)|}{(4|x_2(t)|+6)}+\frac{1}{\Gamma(4/3)}\int_0^t(t-s)^{1/3}\frac{ds}{1+x_1^2(s)}\right),\\
&\frac{x_1(0)}{f_1(0,x_1(0),x_2(0))}+\frac{x_1(1)}{f_1(1,x_1(1),x_2(1))}=\frac{1}{300}\int_0^1\frac{1}{s+100}\frac{|x_1(s)|}{300+|x_1(s)|}ds,\\
&{}^cD^{1/2}\left(\frac{x_1(0)}{f_1(0,x_1(0),x_2(0))}\right)+{}^cD^{1/2}\left(\frac{x_1(1)}{f_1(1,x_1(1),x_2(1))}\right)=\frac{1}{300}\int_0^1\left(\frac{1}{s+20}\right)^3\frac{|x_1(s)|}{30+|x_1(s)|}ds,\\
&{}^cD^{1}\left(\frac{x_1(0)}{f_1(0,x_1(0),x_2(0))}\right)+{}^cD^{1}\left(\frac{x_1(1)}{f_1(1,x_1(1),x_2(1))}\right)=\frac{1}{300}\int_0^1\left(\frac{1}{s+40}\right)^2\frac{|x_1(s)|}{30+|x_1(s)|}ds,\\
&\frac{x_2(0)}{f_2(0,x_1(0),x_2(0))}+\frac{x_2(1)}{f_2(1,x_1(1),x_2(1))}\frac{1}{300}\int_0^1\frac{1}{s+200}\frac{|x_2(s)|}{300+|x_2(s)|}ds,\\
&{}^cD^{1/3}\frac{x_2(0)}{f_2(0,x_1(0),x_2(0))}+{}^cD^{1/3}\frac{x_2(1)}{f_2(1,x_1(1),x_2(1))}=\frac{1}{300}\int_0^1\left(\frac{1}{s+8000}\right)\frac{|x_2(s)|}{30+|x_2(s)|}ds,\\
&{}^cD^{2/3}\frac{x_2(0)}{f_2(0,x_1(0),x_2(0))}+{}^cD^{2/3}\frac{x_2(1)}{f_2(1,x_1(1),x_2(1))}=\frac{1}{300}\int_0^1\left(\frac{1}{s+1600}\right)\frac{|x_2(s)|}{30+|x_2(s)|}ds,
\end{aligned}\tag{49}$$

where $t\in[0,1], \beta_1=3/2$, $\alpha_1=1/2$, $\beta_2=4/3$, $\alpha_2=1/3$, $\lambda_1=1/600$, $\lambda_2=1/700$, $\mu_1=1$, $\mu_2=1$, $\sigma_{11}=\sigma_{21}=\sigma_{31}=\sigma_{12}$

$= \sigma_{22} = \sigma_{32} = 1/300$, and

$$\begin{aligned}
w_1(t, x_1(t), x_2(t))) &= \frac{1}{t^2+4}\left(\frac{t^2|x_1(t)|}{(3|x_1(t)|+1)} + \frac{|x_2(t)|}{(6|x_2(t)|+10)}\right),\\
f_1(t, x_1(t), x_2(t))) &= \frac{t^2}{200}\left(\cos x_1(t) + \frac{|x_2(t)|}{1+|x_1(t)|} + 3\right),\\
g_1(t, x_1 t)) = \frac{1}{t+100}\frac{|x_1(t)|}{300+|x_1(t)|}, & \; h_1(t, x_1(t)) = \left(\frac{1}{t+20}\right)^3 \frac{|x_1(t)|}{30+|x_1(t)|},\\
k_1(t, x_1(t)) &= \left(\frac{1}{t+40}\right)^2 \frac{|x_1(t)|}{30+|x_1(t)|},\\
w_2(t, x_1(t), x_2(t))) &= \frac{1}{1+t^4}\left(\frac{t^4|x_1(t)|}{(4|x_1(t)|+10)} + \frac{|x_2(t)|}{(4|x_2(t)|+6)}\right),\\
f_2(t, x_1(t), x_2(t))) &= \frac{1}{2(35+t)}\left(\frac{x_2^2(t)+|x_1(t)|}{1++|x_2(t)|}\right) + t^{1/3} + 1\\
g_2(t, x_2(t)) &= \frac{1}{t+200}\frac{|x_2(t)|}{300+|x_2(t)|},\\
h_2(t, x_2(t)) &= \left(\frac{1}{t+8000}\right)\frac{|x_2(t)|}{30+|x_2(t)|},\\
k_2(t, x_2(t)) &= \left(\frac{1}{t+1600}\right)\frac{|x_2(t)|}{30+|x_2(t)|}.
\end{aligned} \tag{50}$$

In this conrete application, we have

$$\pi_1 + \pi_2(\eta_0 + \eta_1\|x_1\| + \eta_2\|x_2\|) + \pi_3\|x_1\| + \pi_4\|x_2\| = 0.0029 < 1. \tag{51}$$

The riview of Theorem 9, problem (49) has a least one solution.

5. Conclusion

It is known that most natural phenomena are modeled by different types of fractional differential equations. This diversity in investigating complicated fractional differential equations increases our ability for exact modeling of different phenomena. This is useful in making modern software which helps us to allow for more cost-free testing and less material consumption. For our contribution in this present work, we investigate a fractional hybrid differential system with mixed integral hybrid and boundary hybrid conditions. We investigatedtwo numerical examples to illustrate our main results

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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