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Research Article

A Numerical Approach for the Analytical Solution of the Fourth-Order Parabolic Partial Differential Equations

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In this study, we propose a new iterative scheme (NIS) to investigate the approximate solution of the fourth-order parabolic partial differential equations (PDEs) that arises in transverse vibration problems. We introduce the Mohand transform as a new operator that is very easy to implement coupled with the homotopy perturbation method. This NIS is capable of reducing the linearization, perturbation, and restrictive assumptions that ruin the nature of the numerical problems. Some numerical examples are demonstrated to legitimate the accuracy and authenticity of this NIS. The computational results are obtained in the shape of a series that converges only after a few iterations. The comparison of the graphical representations shows that NIS is a very simple but also an effective approach for other numerical problems involving complex variables.

1. Introduction

Many physical phenomena of differential equations in complex variables play an important role in science and engineering such as physics, chemical energy, biology, medicine, and engineering [1–3]. These physical phenomena are of great interest in this modern era and are introduced by parabolic PDEs. It is still very difficult to investigate the exact solution of the PDEs in most numerical problems. Therefore, most of the researchers introduced numerous analytical and numerical approaches to provide the approximate solution for these PDEs such as the quintic B-spline collocation method [4], q-HATM [5], quintic B-spline [6], Legendre wavelet method [7], homotopy perturbation transform method [8], and so on [9–11].

Consider the fourth-order parabolic PDEs with variable coefficients [12, 13]

$$\frac{\partial^2\Psi}{\partial\eta^2}+\alpha(\xi,\S,\theta)\frac{\partial^4\Psi}{\partial\xi^4}+\frac{1}{\S}\beta(\xi,\S,\theta)\frac{\partial^4\Psi}{\partial\S^4}+\frac{1}{\theta}\gamma(\xi,\S,\theta)\frac{\partial^4\Psi}{\partial\theta^4}=g(\xi,\S,\theta,\eta), \tag{1}$$

where $\alpha, \beta, \gamma > 0$, subjected to the following initial conditions

$$\begin{aligned}\Psi(\xi,\S,\theta,0)&=f_1(\xi,\S,\theta),\\ \frac{\partial\Psi}{\partial\eta}(\xi,\S,\theta,0)&=f_2(\xi,\S,\theta),\end{aligned} \tag{2}$$

and boundary conditions

$$\begin{aligned}\Psi(a,\S,\theta,\eta)&=g_0(\S,\theta,\eta),\\ \Psi(b,\S,\theta,\eta)&=g_1(\S,\theta,\eta),\\ \Psi(\xi,a,\theta,\eta)&=k_0(\xi,\theta,\eta),\\ \Psi(\xi,b,\theta,\eta)&=k_1(\xi,\theta,\eta),\\ \Psi(\xi,\S,a,\eta)&=h_0(\xi,\S,\eta),\\ \Psi(\xi,\S,b,\eta)&=h_1(\xi,\S,\eta),\end{aligned}$$

$$\frac{\partial^2\Psi}{\partial\xi^2}(a,\S,\theta,\eta)=\bar{g}_0(\S,\theta,\eta),$$
$$\frac{\partial^2\Psi}{\partial\xi^2}(b,\S,\theta,\eta)=\bar{g}_1(\S,\theta,\eta),$$
$$\frac{\partial^2\Psi}{\partial\S^2}(\xi,a,\theta,\eta)=\bar{k}_0(\xi,\theta,\eta),$$
$$\frac{\partial^2\Psi}{\partial\S^2}(\xi,b,\theta,\eta)=\bar{k}_1(\xi,\theta,\eta),$$
$$\frac{\partial^2\Psi}{\partial\theta^2}(\xi,\S,a,\eta)=\bar{h}_0(\xi,\S,\eta),$$
$$\frac{\partial^2\Psi}{\partial\theta^2}(\xi,\S,b,\eta)=\bar{h}_1(\xi,\S,\eta), \quad (3)$$

where f_j, g_j, h_j, k_j, $\bar{g}_j$, $\bar{h}_j$, and $\bar{k}_j$ are continuous functions and j varies from 0 to 1.

Wazwaz [14] used the Adomian decomposition method to examine the analytical solution of transverse vibrations of a uniform flexible beam. Aziz et al. [15] studied the fourth-order nonhomogeneous parabolic partial differential equations that govern the behavior of a vibrating beam by using a new three-level method based on the parametric quintic spline in space and finite difference discretization in time. Biazar and Ghazvini [16] used the variational iteration method for the analytical solution of the fourth-order parabolic equations. Dehghan and Manafian [17] applied HPM for the solution of the fourth-order parabolic PDEs. El-Gamel [18] used the sinc-Galerkin method to examine the fourth-order PDEs in one space variable coefficient. Rashidinia and Mohammadi [19] reported new three-level implicit methods for the numerical solution of the fourth-order nonhomogeneous parabolic PDEs with variable coefficients. Mittal and Jain [20] applied the quintic B-spline method, and Birol [21] used the reduced differential transformation method for the fourth-order nonhomogeneous parabolic partial differential equation. Khan and Sultana [22] used the parametric septic spline for the numerical solution of the fourth-order parabolic PDEs.

The homotopy perturbation method (HPM) was developed by He [23, 24]. HPM gives the solution in the form of a rapid and consecutive series toward the exact solution. Dehghan and Manafian [17] used HPM to obtain the numerical results for the linear and nonlinear boundary value problems. The convergence rate of HPM can be studied through [25]. Nadeem et al. [13] applied the Laplace transform coupled with the homotopy perturbation method to solve the fourth-order parabolic PDEs with variable coefficients. Luo et al. [26] introduced a combined form of the Mohand transform and the homotopy perturbation method to provide the analytical solution of the delay differential equations. Recently, many integral transformations have been introduced to find the approximate solution of ordinary and partial differential equations such as the Elzaki transform [27, 28], Sumudu transform [29], Aboodh transformation [30], Mohand transform [31], and homotopy perturbation method [24].

In this paper, we construct the idea of NIS with the help of the Mohand transform and the homotopy perturbation method for obtaining the approximate solution of partial differential equations. This NIS provides the results in the form of a series that converges to the exact solution very rapidly. This scheme does not require any linearization, variation, and limiting expectations. In particular, this study is organized as follows. In Section (2), we recall some basic definitions of the Mohand transform. In Sections (3) and (4), first, we present the basic idea of HPM and then formulate the idea of NIS for finding the approximate solution of PDEs. We illustrate three examples to present the accuracy and validity of NIS in Section (5). We give a brief discussion of the obtained results in Section (6), and finally, the conclusion is presented in Section (7).

2. Fundamental Concepts of the Mohand Transform

In this section, we introduce some basic definitions and preliminary concepts of the Mohand transform, which reveals the idea of its implementations to functions.

Definition 1. Mohand and Mahgoub [31] presented a new scheme Mohand transform $M(.)$ in order to gain the results of ordinary differential equations, which is defined as

$$\mathbf{M}\{\Psi(\eta)\}=R(w)=w^2\int_0^\infty \Psi(\eta)e^{-w\eta}d\eta, \quad k_1\le w\le k_2. \quad (4)$$

On the other hand, if $R(w)$ is the Mohand transform of a function $\Psi(\eta)$, then $\Psi(\eta)$ is the inverse of $R(w)$such that

$$\mathbf{M}^{-1}\{R(w)\}=\Psi(\eta), \quad \mathbf{M}^{-1} \text{ is the inverse Mohand operator}. \quad (5)$$

Definition 2. If $\Psi(\eta)=\eta^n$,

$$R(w)=\frac{n!}{w^{n-1}}. \quad (6)$$

Definition 3. If $\mathbf{M}\{\Psi(\eta)\}=R(w)$, then it has the following differential properties:

(i) $\mathbf{M}\{\Psi'(\eta)\}=wR(w)-w^2F(0)$

(ii) $\mathbf{M}\{\Psi''(\eta)\}=w^2R(w)-w^3F(0)-w^2F'(0)$

(iii) $\mathbf{M}\{Fu^n(\eta)\}=w^nR(w)-w^{n+1}F(0)-w^nF'(0)-\cdots-w^nF^{n-1}(0)$

3. Basic Idea of HPM

In this segment, we illustrate a nonlinear functional equation to explain the basic view of HPM [32, 33]. Consider

$$T(\Psi) - g(h) = 0, \quad h \in \Omega, \tag{7}$$

with conditions

$$S\left(\Psi, \frac{\partial \Psi}{\partial n}\right) = 0, \quad h \in \Gamma, \tag{8}$$

where T and S are known as the general functional operator and boundary operator, respectively, and $g(h)$ is a known function with Γ as an interval of the domain Ω. We now divide T into two units such thatT_1 represents a linear and T_2 a nonlinear operator. As a result, we can express Equation (8) such that

$$T_1(\Psi) + T_2(\Psi) - g(h) = 0. \tag{9}$$

Assume a homotopy $\vartheta(h, p)$: $\Omega \times [0, 1] \longrightarrow \mathbb{H}$ in such a way that it is appropriate for

$$H(\vartheta, p) = (1 - p)[T_1(\vartheta) - T_1(\Psi_0)] + p[T_1(\vartheta) - T_2(\vartheta) - g(h)], \tag{10}$$

or

$$H(\vartheta, p) = T_1(\vartheta) - T_1(\Psi_0) + qL(\Psi_0) + p[T_2(\vartheta) - g(h)] = 0, \tag{11}$$

where $p \in [0, 1]$ is the embedding parameter and Ψ_0 is an initial guess of Equation (7), which is suitable for the boundary conditions. The theory of HPM states that p is considered a slight variable, and the solution of Equation (7) in the resulting form of $p$is

$$\vartheta = \vartheta_0 + p\vartheta_1 + p^2\vartheta_2 + p^3\vartheta_3 + \cdots = \sum_{i=0}^{\infty} p^i \vartheta_i. \tag{12}$$

Let $p = 1$, and then the particular solution of Equation (8) is written as

$$\Psi = \lim_{p \longrightarrow 1} \vartheta = \vartheta_0 + \vartheta_1 + \vartheta_2 + \vartheta_3 + \cdots = \sum_{i=0}^{\infty} \vartheta_i. \tag{13}$$

The nonlinear terms can be calculated as

$$T_2\Psi(x, t) = \sum_{n=0}^{\infty} p^n H_n(\Psi). \tag{14}$$

Then, He's polynomials $H_n(\Psi)$ can be obtained using the following expression:

$$H_n(\Psi_0 + \Psi_1 + \cdots + \Psi_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left(T_2 \left(\sum_{i=0}^{\infty} p^i \Psi_i \right) \right)_{p=0}, \quad n = 0, 1, 2, \cdots. \tag{15}$$

The series solution in Equation (14) is mostly convergent due to the convergence rate of the series depending on the nonlinear operator T_2.

4. Formulation of NIS

This segment presents the formulation of a new iterative scheme (NIS) for obtaining the approximate solution of the fourth-order parabolic PDEs. Let us consider a second-order differential equation of the form

$$\Psi''(\xi, \eta) + \Psi(\xi, \eta) + g(\Psi) = g(\xi, \eta), \tag{16}$$

with the following conditions:

$$\begin{aligned} \Psi(\xi, 0) &= a, \\ \Psi'(\xi, 0) &= b, \end{aligned} \tag{17}$$

where Ψ is a function in time domain η, $g(\Psi)$ represents a nonlinear term, and $g(\eta)$ is a source term, whereas a and b are constants. Rewrite Equation (16) again as

$$\Psi''(\xi, \eta) = -\Psi(\xi, \eta) - g(\Psi) + g(\xi, \eta). \tag{18}$$

Now, taking MT on both sides of Equation (18), we obtain

$$\mathbf{M}\left[\Psi''(\xi, \eta)\right] = \mathbf{M}[-\Psi(\xi, \eta) - g(\Psi) + g(\xi, \eta)]. \tag{19}$$

Applying the differential properties of MT, we get

$$w^2 R[w] - w^3 \Psi(\xi, 0) - w^2 \Psi'(\xi, 0) = \mathbf{M}[-\Psi(\xi, \eta) - g(\Psi) + g(\xi, \eta)]. \tag{20}$$

Thus, $R(w)$ can be obtained from Equation (20) such that

$$R[w] = wu(\xi, 0) + \Psi'(\xi, 0) - \frac{1}{w^2}\mathbf{M}[\Psi(\xi, \eta) + g(\Psi) - g(\xi, \eta)]. \tag{21}$$

Operating the inverse Mohand transform on Equation (21), we get

$$\Psi(\xi, \eta) = G(\xi, \eta) - \mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}[\Psi(\xi, \eta) + g(\Psi)]\right], \tag{22}$$

where Equation (22) is called the NIS and

$$G(\xi, \eta) = \mathbf{M}^{-1}\left[wu(0) + \Psi'(0) + \frac{1}{w^2} g(\xi, \eta)\right]. \tag{23}$$

Now, we apply HPM on Equation (22). Let

$$\Psi(\eta) = \sum_{i=0}^{\infty} p^i \Psi_i(n) = \Psi_0 + p^1\Psi_1 + p^2\Psi_2 + \cdots, \quad (24)$$

and nonlinear terms $g(\Psi)$ can be calculated by using the following formula:

$$g(\Psi) = \sum_{i=0}^{\infty} p^i H_i(\Psi) = H_0 + p^1 H_1 + p^2 H_2 + \cdots, \quad (25)$$

where $H_n{}'s$ is He's polynomial, which may be computed using the following procedure:

$$H_n(\Psi_0 + \Psi_1 + \cdots + \Psi_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(g\left(\sum_{i=0}^{\infty} p^i \Psi_i\right)\right)_{p=0}, \quad n = 0, 1, 2, \cdots. \quad (26)$$

Put Equations (24)–(26) in Equation (22), and comparing the similar factors of p, we get the following consecutive elements:

$$\begin{aligned} p^0 &: \Psi_0(\xi,\eta) = G(\xi,\eta), \\ p^1 &: \Psi_1(\xi,\eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\{\Psi_0(\xi,\eta) + H_0(\Psi)\}\right], \\ p^2 &: \Psi_2(\xi,\eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\{\Psi_1(\xi,\eta) + H_1(\Psi)\}\right], \\ p^3 &: \Psi_3(\xi,\eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\{\Psi_2(\xi,\eta) + H_2(\Psi)\}\right], \\ &\vdots . \end{aligned} \quad (27)$$

In continuing the similar process, we can summarize this series to get the approximate solution such that

$$\Psi(\xi,\eta) = \Psi_0 + \Psi_1 + \Psi_2 + \cdots = \sum_{i=0}^{\infty} \Psi_i. \quad (28)$$

Thus, Equation (28) is to be considered an approximate solution of differential equations of Equation (16).

5. Numerical Examples

In this part, we consider three numerical problems to check the authenticity and validity of NIS. We also demonstrate the solution surface of the illustrated problems for the behavior and a better understanding of this strategy where we see that the solution graphs of the approximate solution and the particular solution coincide with each other only after a few iterations.

5.1. Example 1. Consider the one-dimensional fourth-order parabolic PDEs

$$\frac{\partial^2 \Psi}{\partial \eta^2} + \left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\frac{\partial^4 \Psi}{\partial \xi^4} = 0, \quad (29)$$

with the initial conditions

$$\begin{aligned} \Psi(\xi,0) &= 0, \\ \Psi_\eta(\xi,0) &= 1 + \frac{\xi^5}{120}. \end{aligned} \quad (30)$$

Applying MT on Equation (29) together with the differential property as defined in Equation (6), we get

$$w^2 R(w) - w^3 \Psi(\xi,0) - w^2 \Psi_\eta(\xi,0) = -\mathbf{M}\left[\left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\frac{\partial^4 \Psi}{\partial \xi^4}\right]. \quad (31)$$

Thus, $R(w)$ yields

$$R(w) = w\Psi(\xi,0) - \Psi_\eta(\xi,0) - \frac{1}{w^2}\mathbf{M}\left[\left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\frac{\partial^4 \Psi}{\partial \xi^4}\right]. \quad (32)$$

Using the inverse Mohand transform, we get

$$\Psi(\xi,\eta) = \Psi(\xi,0) - \eta\Psi_\eta(\xi,0) - \mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{\left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\frac{\partial^4 \Psi}{\partial \xi^4}\right\}\right]. \quad (33)$$

Applying MHPTM to get He's polynomials, we get

$$\sum_{i=0}^{\infty} p^i \Psi_i(n) = \Psi(\xi,0) - \eta\Psi_\eta(\xi,0) - \mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{\left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\sum_{i=0}^{\infty} p^i \frac{\partial^4 \Psi_i}{\partial \xi^4}\right\}\right]. \quad (34)$$

Observing the similar powers of p, we get

$$\begin{aligned} p^0 &: \Psi_0(\xi,\eta) = \left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\eta, \\ p^1 &: \Psi_1(\xi,\eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{\left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\frac{\partial^4 \Psi_0}{\partial \xi^4}\right\}\right] = -\left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\frac{\eta^3}{3!}, \\ p^2 &: \Psi_2(\xi,\eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{\left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\frac{\partial^4 \Psi_1}{\partial \xi^4}\right\}\right] = \left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\frac{\eta^5}{5!}, \\ p^3 &: \Psi_3(\xi,\eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{\left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\frac{\partial^4 \Psi_2}{\partial \xi^4}\right\}\right] = -\left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\frac{\eta^7}{7!}, \\ p^4 &: \Psi_4(\xi,\eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{\left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\frac{\partial^4 \Psi_3}{\partial \xi^4}\right\}\right] = \left(\frac{1}{\xi} + \frac{\xi^4}{120}\right)\frac{\eta^9}{9!}, \\ &\vdots . \end{aligned} \quad (35)$$

In continuing this process, the approximate solution results can be summarized as

$$\begin{aligned}\Psi(\xi,\eta) &= \Psi_0(\xi,\eta)+\Psi_1(\xi,\eta)+\Psi_2(\xi,\eta)+\Psi_3(\xi,\eta)+\Psi_4(\xi,\eta)+\cdots \\ &= \left(1+\frac{\xi^5}{120}\right)\left(\eta-\frac{\eta^3}{3!}+\frac{\eta^5}{5!}-\frac{\eta^7}{7!}+\frac{\eta^9}{9!}\right)+\cdots.\end{aligned} \tag{36}$$

This series converges to the particular solution

$$\Psi(\xi,\eta) = \left(1+\frac{\xi^5}{120}\right)\sin\eta. \tag{37}$$

5.2. Example 2. Consider the two-dimensional fourth-order parabolic PDEs

$$\frac{\partial^2\Psi}{\partial\eta^2}+2\left(\frac{1}{\xi^2}+\frac{\xi^4}{6!}\right)\frac{\partial^4\Psi}{\partial\xi^4}+2\left(\frac{1}{\S^2}+\frac{\S^4}{6!}\right)\frac{\partial^4\Psi}{\partial\S^4}=0, \tag{38}$$

with the initial conditions

$$\begin{aligned}\Psi(\xi,\S,0) &= 0,\\ \Psi_\eta(\xi,\S,0) &= 2+\frac{\xi^6}{6!}+\frac{\S^6}{6!}.\end{aligned} \tag{39}$$

Applying NIM, we get

$$\begin{aligned}\Psi(\xi,\S,\eta) &= \Psi(\xi,\S,0)-\eta\Psi_\eta(\xi,\S,0)-\mathbf{M}^{-1}\\ &\cdot\left[\frac{1}{w^2}\mathbf{M}\left\{2\left(\frac{1}{\xi^2}+\frac{\xi^4}{6!}\right)\frac{\partial^4\Psi}{\partial\xi^4}+2\left(\frac{1}{\S^2}+\frac{\S^4}{6!}\right)\frac{\partial^4\Psi}{\partial\S^4}\right\}\right].\end{aligned} \tag{40}$$

This equation provides He's polynomials

$$\begin{aligned}\sum_{i=0}^{\infty}p^i\Psi_i(n) &= \Psi(\xi,\S,0)-\eta\Psi_\eta(\xi,\S,0)-\mathbf{M}^{-1}\\ &\cdot\left[\frac{1}{w^2}\mathbf{M}\left\{2\left(\frac{1}{\xi^2}+\frac{\xi^4}{6!}\right)\sum_{i=0}^{\infty}p^i\frac{\partial^4\Psi_i}{\partial\xi^4}+2\left(\frac{1}{\S^2}+\frac{\S^4}{6!}\right)\sum_{i=0}^{\infty}p^i\frac{\partial^4\Psi_i}{\partial\S^4}\right\}\right].\end{aligned} \tag{41}$$

Observing the similar powers of p, we get

$$\begin{aligned}p^0 &: \Psi_0(\xi,\S,\eta) = \left(2+\frac{\xi^6}{6!}+\frac{\S^6}{6!}\right)\eta,\\ p^1 &: \Psi_1(\xi,\S,\eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{2\left(\frac{1}{\xi^2}+\frac{\xi^4}{6!}\right)\frac{\partial^4\Psi}{\partial\xi^4}+2\left(\frac{1}{\S^2}+\frac{\S^4}{6!}\right)\frac{\partial^4\Psi}{\partial\S^4}\right\}\right]\\ &= -\left(2+\frac{\xi^6}{6!}+\frac{\S^6}{6!}\right)\frac{\eta^3}{3!},\\ p^2 &: \Psi_2(\xi,\S,\eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{2\left(\frac{1}{\xi^2}+\frac{\xi^4}{6!}\right)\frac{\partial^4\Psi}{\partial\xi^4}+2\left(\frac{1}{\S^2}+\frac{\S^4}{6!}\right)\frac{\partial^4\Psi}{\partial\S^4}\right\}\right]\\ &= \left(2+\frac{\xi^6}{6!}+\frac{\S^6}{6!}\right)\frac{\eta^5}{5!},\\ p^3 &: \Psi_3(\xi,\S,\eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{2\left(\frac{1}{\xi^2}+\frac{\xi^4}{6!}\right)\frac{\partial^4\Psi}{\partial\xi^4}+2\left(\frac{1}{\S^2}+\frac{\S^4}{6!}\right)\frac{\partial^4\Psi}{\partial\S^4}\right\}\right]\\ &= -\left(2+\frac{\xi^6}{6!}+\frac{\S^6}{6!}\right)\frac{\eta^7}{7!},\\ p^4 &: \Psi_4(\xi,\S,\eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{2\left(\frac{1}{\xi^2}+\frac{\xi^4}{6!}\right)\frac{\partial^4\Psi}{\partial\xi^4}+2\left(\frac{1}{\S^2}+\frac{\S^4}{6!}\right)\frac{\partial^4\Psi}{\partial\S^4}\right\}\right]\\ &= \left(2+\frac{\xi^6}{6!}+\frac{\S^6}{6!}\right)\frac{\eta^9}{9!},\\ &\vdots.\end{aligned} \tag{42}$$

In continuing this process, the approximate solution results can be summarized as

$$\begin{aligned}\Psi(\xi,\S,\eta) &= \Psi_0(\xi,\S,\eta)+\Psi_1(\xi,\S,\eta)+\Psi_2(\xi,\S,\eta)\\ &\quad+\Psi_3(\xi,\S,\eta)+\Psi_4(\xi,\S,\eta)+\cdots\\ &= \left(2+\frac{\xi^6}{6!}+\frac{\S^6}{6!}\right)\left(\eta-\frac{\eta^3}{3!}+\frac{\eta^5}{5!}-\frac{\eta^7}{7!}+\frac{\eta^9}{9!}\right)+\cdots.\end{aligned} \tag{43}$$

This series converges to the particular solution

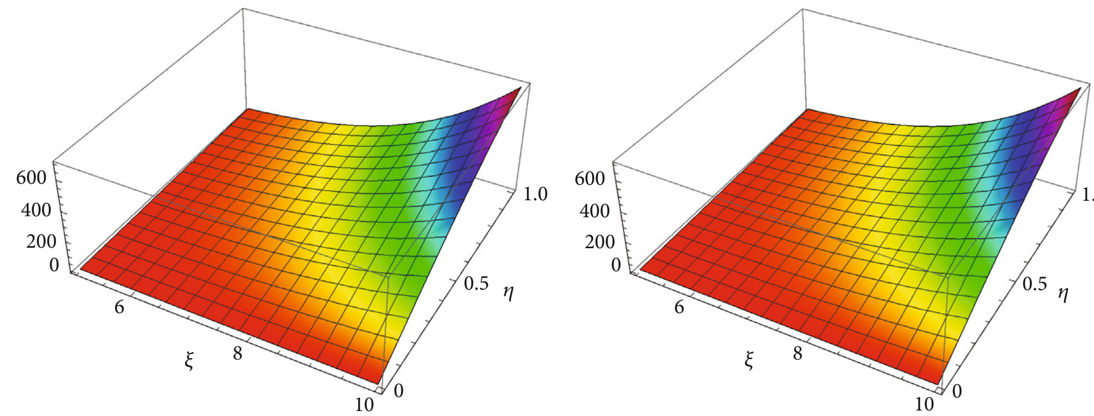
$$\Psi(\xi,\S,\eta) = \left(2+\frac{\xi^6}{6!}+\frac{\S^6}{6!}\right)\sin\eta. \tag{44}$$

5.3. Example 3. Consider the three-dimensional fourth-order parabolic PDEs

$$\begin{aligned}&\frac{\partial^2\Psi}{\partial\eta^2}+\left(2\frac{\S+\theta}{\cos(\xi)}-1\right)\frac{\partial^4\Psi}{\partial\xi^4}+\left(\frac{\xi+\theta}{2\cos(\S)}-1\right)\frac{\partial^4\Psi}{\partial\S^4}\\ &+\left(\frac{\S+\xi}{2\cos(\theta)}-1\right)\frac{\partial^4\Psi}{\partial\theta^4}=0,\end{aligned} \tag{45}$$

with the initial conditions

$$\begin{aligned}\Psi(\xi,\S,\theta,0) &= \xi+\S+\theta-(\cos(\xi)+\cos(\S)+\cos(\theta)),\\ \Psi(\xi,\S,\theta,0) &= (\cos(\xi)+\cos(\S)+\cos(\theta))-(\xi+\S+\theta).\end{aligned} \tag{46}$$



(a) Approximate solution of $\Psi(\xi, \eta)$ for Equation (29)

(b) Particular solution of $\Psi(\xi, \eta)$ for Equation (29)

FIGURE 1: Surface solutions for the one-dimensional parabolic differential equation.

Applying NIM, we get

$$\Psi(\xi, \S, \theta, \eta) = \Psi(\xi, \S, 0) - \eta\Psi_\eta(\xi, \S, 0) - \mathbf{M}^{-1} \cdot \left[\frac{1}{w^2}\mathbf{M}\left\{\left(2\frac{\S+\theta}{\cos(\xi)} - 1\right)\frac{\partial^4\Psi}{\partial\xi^4} + \left(\frac{\xi+\theta}{2\cos(\S)} - 1\right)\frac{\partial^4\Psi}{\partial\S^4} + \left(\frac{\S+\xi}{2\cos(\theta)} - 1\right)\frac{\partial^4\Psi}{\partial\theta^4}\right\}\right]. \tag{47}$$

This equation provides He's polynomials

$$p^0 : \Psi_0(\xi, \S, \theta, \eta) = w\Psi(\xi, \S, 0) - \Psi_\eta(\xi, \S, 0)(1-\eta),$$

$$p^1 : \Psi_1(\xi, \S, \theta, \eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{\left(2\frac{\S+\theta}{\cos(\xi)} - 1\right)\frac{\partial^4\Psi}{\partial\xi^4} + \left(\frac{\xi+\theta}{2\cos(\S)} - 1\right)\frac{\partial^4\Psi}{\partial\S^4} + \left(\frac{\S+\xi}{2\cos(\theta)} - 1\right)\frac{\partial^4\Psi}{\partial\theta^4}\right\}\right],$$

$$p^2 : \Psi_2(\xi, \S, \theta, \eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{\left(2\frac{\S+\theta}{\cos(\xi)} - 1\right)\frac{\partial^4\Psi}{\partial\xi^4} + \left(\frac{\xi+\theta}{2\cos(\S)} - 1\right)\frac{\partial^4\Psi}{\partial\S^4} + \left(\frac{\S+\xi}{2\cos(\theta)} - 1\right)\frac{\partial^4\Psi}{\partial\theta^4}\right\}\right],$$

$$p^3 : \Psi_3(\xi, \S, \theta, \eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{\left(2\frac{\S+\theta}{\cos(\xi)} - 1\right)\frac{\partial^4\Psi}{\partial\xi^4} + \left(\frac{\xi+\theta}{2\cos(\S)} - 1\right)\frac{\partial^4\Psi}{\partial\S^4} + \left(\frac{\S+\xi}{2\cos(\theta)} - 1\right)\frac{\partial^4\Psi}{\partial\theta^4}\right\}\right],$$

$$p^4 : \Psi_4(\xi, \S, \theta, \eta) = -\mathbf{M}^{-1}\left[\frac{1}{w^2}\mathbf{M}\left\{\left(2\frac{\S+\theta}{\cos(\xi)} - 1\right)\frac{\partial^4\Psi}{\partial\xi^4} + \left(\frac{\xi+\theta}{2\cos(\S)} - 1\right)\frac{\partial^4\Psi}{\partial\S^4} + \left(\frac{\S+\xi}{2\cos(\theta)} - 1\right)\frac{\partial^4\Psi}{\partial\theta^4}\right\}\right], \tag{48}$$

which gives

$$\begin{aligned}
\Psi_0(\xi, \S, \eta) &= (\xi + \S + \theta - \cos(\xi) - \cos(\S) - \cos(\theta))(1-\eta),\\
\Psi_1(\xi, \S, \eta) &= (\xi + \S + \theta - \cos(\xi) - \cos(\S) - \cos(\theta))\left(\frac{\eta^2}{2!} - \frac{\eta^3}{3!}\right),\\
\Psi_2(\xi, \S, \eta) &= \left(\xi + \S + \theta - (\cos(\xi) - \cos(\S) - \cos(\theta))\left(\frac{\eta^4}{4!} - \frac{\eta^5}{5!}\right),\right.\\
\Psi_3(\xi, \S, \eta) &= (\xi + \S + \theta - \cos(\xi) - \cos(\S) - \cos(\theta))\left(\frac{\eta^6}{6!} - \frac{\eta^7}{7!}\right),\\
\Psi_4(\xi, \S, \eta) &= (\xi + \S + \theta - \cos(\xi) - \cos(\S) - \cos(\theta))\left(\frac{\eta^8}{8!} - \frac{\eta^9}{9!}\right),\\
&\vdots .
\end{aligned} \tag{49}$$

In continuing this process, the approximate solution results can be summarized as

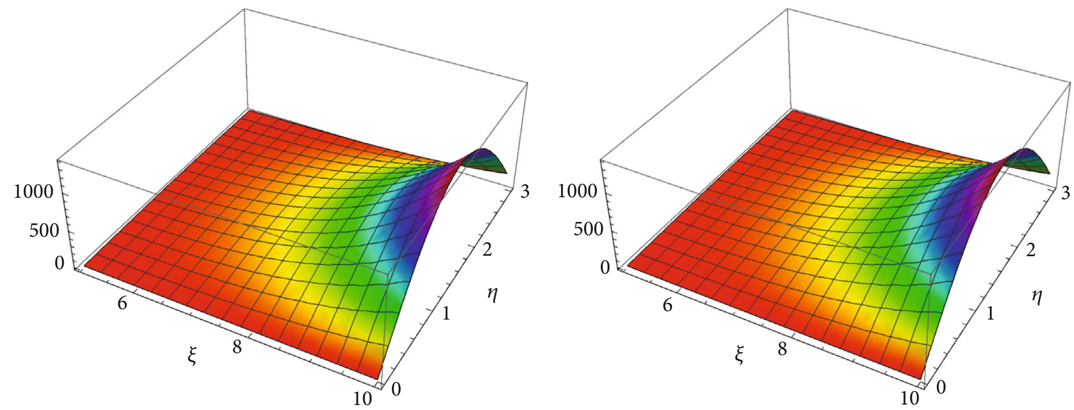
$$\begin{aligned}
\Psi(\xi, \eta) &= \Psi_0(\xi, \eta) + \Psi_1(\xi, \eta) + \Psi_2(\xi, \eta) + \Psi_3(\xi, \eta) + \Psi_4(\xi, \eta) + \cdots\\
&= (\xi + \S + \theta - \cos(\xi) - \cos(\S) - \cos(\theta))\\
&\quad \cdot \left(1 - \eta + \frac{\eta^2}{2!} - \frac{\eta^3}{3!} + \frac{\eta^4}{4!} - \frac{\eta^5}{5!} + \frac{\eta^6}{6!} - \frac{\eta^7}{7!} + \frac{\eta^8}{8!} - \frac{\eta^9}{9!} + \cdots\right).
\end{aligned} \tag{50}$$

This series converges to the particular solution

$$\Psi(\xi, \eta) = (\xi + \S + \theta - \cos(\xi) - \cos(\S) - \cos(\theta))e^{-\eta}. \tag{51}$$

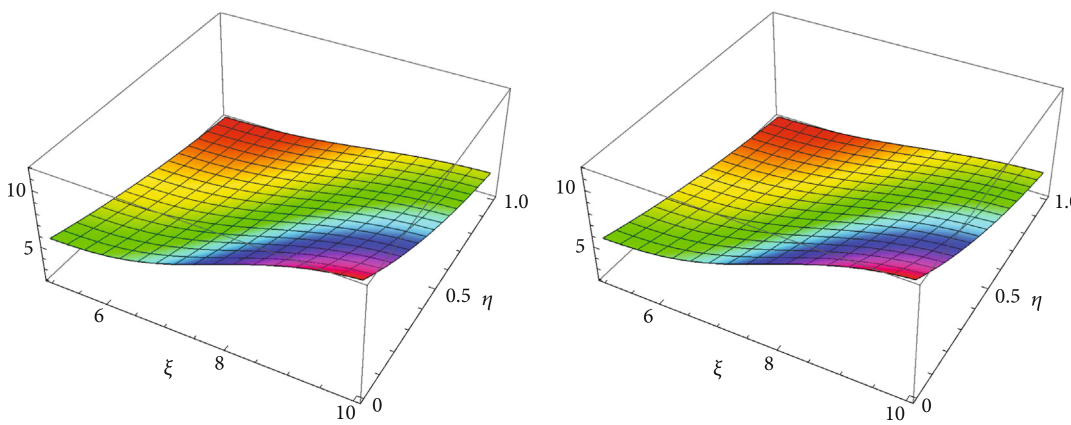
6. Results and Discussion

In this segment, we present the discussion of some graphical representations in Figures 1–3. It can be seen that the formulated series converges to the particular solution only after a few iterations very rapidly. Figures 1(a) and 1(b) represent the comparison between the approximate solution and the exact solution of Equations (36) and (37) at $0 \le \eta \le 1$ and $0 \le \xi \le 10$, respectively. Figures 2(a) and 2(b) show the comparison between the approximate solution and the particular



(a) Approximate solution of $\Psi(\xi, \S, \eta)$ for Equation (38) (b) Particular solution of $\Psi(\xi, \S, \eta)$ for Equation (38)

Figure 2: Surface solutions for the two-dimensional parabolic differential equation.



(a) Approximate solution of $\Psi(\xi, \S, \theta, \eta)$ for Equation (45) (b) Particular solution of $\Psi(\xi, \S, \theta, \eta)$ for Equation (45)

Figure 3: Surface solutions for the three-dimensional parabolic differential equation.

solution of Equations (43) and (44) at $0 \le \eta \le 3$ and $0 \le \xi \le 10$, respectively, and similarly, Figures 3(a) and 3(b) represent the comparison between the approximate solution and the particular solution of Equations (50) and (51) at $0 \le \eta \le 1$ and $0 \le \xi \le 10$, respectively. This comparison shows that NIS is easy to implement and does not require any heavy calculation for the computation of the approximate solution of the fourth-order parabolic PDEs with variable coefficients.

7. Conclusion and Future Work

In this analysis, we successfully employed the NIS to examine the approximate solution of the fourth-order parabolic partial differential equations with variable coefficients. The Mohand transform coupled with HPM has been used to construct the idea of this scheme. This NIS approach is applicable for both the linear and nonlinear partial differential equations. This approach does not require the recurrence relation for the assumption of a variable. This NIS formulates the obtained results of the illustrated problems in the form of a series that converges to the particular solution very rapidly. This approach has an advantage of direct implementation to the numerical problems and confirms the accuracy with full agreement. This NIS is also applicable for the other partial differential equations with fractional derivatives in science and engineering.

Data Availability

All the data are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Fenglian Liu worked on the investigation, methodology, and writing (original draft) of the manuscript. Muhammad Nadeem did work in validation, editing, and improvement of the English language during the revision of the manuscript. Ibrahim Mahariq implemented the software programming to provide the graphical results, whereas Suliman Dawood supervised and approved the manuscript for submission.

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