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Research Article

Common Coupled Fixed Point Theorems on $C^\star$-Algebra-Valued Partial Metric Spaces

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In this paper, we prove common coupled fixed point theorems on complete $C^\star$-algebra-valued partial metric spaces. An example and application to support our result are presented.

1. Introduction

In 1987, Guo and Lakshmikantham [1] introduced the concept of a coupled fixed point. In 2006, Bhaskar and Lakshmikantham [2] introduced the concept of a mixed monotone property for the first time and investigated some coupled fixed point theorems for mappings. As a result, many authors obtained many coupled fixed point and coupled coincidence theorems (see [3–21] and references therein).

In 2014, Ma et al. [22] introduced the notion of a $\mathscr{C}^\star$-algebra-valued metric space and proved fixed point theorem. In 2015, Batul and Kamran [23] proved fixed theorems on $\mathscr{C}^\star$-algebra-valued metric space. In 2016, Alsulami et al. [24] proved fixed point theorems on $C^\star$-algebra-valued metric space. In 2016, Cao and Xin [25] proved common coupled fixed point theorems in $C^\star$-algebra-valued metric spaces. The details on $\mathscr{C}^\star$-algebra are available in [26–29]. In 2011, Aydi et al. [30] proved coupled fixed point theorems on ordered partial metric space. The details on partial metric space are available in [31–43]. In 2019, Chandok et al. [44] proved fixed point theorems on C^*-algebra-valued partial metric space. In this paper, we prove common coupled fixed point theorems on C^*-algebra-valued partial metric space.

2. Preliminaries

First of all, we recall some basic definitions, notations, and results of $C^\star$-algebra that can be found in [27]. An algebra $\mathbb{A}$, together with a conjugate linear involution map $\mathfrak{a} \mapsto \mathfrak{a}^\star$, is called a $\star$-algebra if $(ab)^\star = \mathfrak{b}^\star \mathfrak{a}^\star$ and $(a^\star)^\star = \mathfrak{a}$ for all $\mathfrak{a}$, $\mathfrak{b} \in \mathbb{A}$. Moreover, the pair $(\mathbb{A}, \star)$ is called a unital $\star$-algebra if $\mathbb{A}$ contains the identity element $1_{\mathbb{A}}$. By a Banach $\star$-algebra, we mean a complete normed unital å-algebra $(\mathbb{A}, \star)$ such that the norm on $\mathbb{A}$ is submultiplicative and satisfies

$\|\mathfrak{a}^\star\| = \|\mathfrak{a}\|$ for all $\mathfrak{a} \in \mathbb{A}$. Further, if for all $\mathfrak{a} \in \mathbb{A}$, we have $\|\mathfrak{a}^\star \mathfrak{a}\| = \|\mathfrak{a}\|^2$ in a Banach $\star$-algebra $(\mathbb{A}, \star)$, then $\mathbb{A}$ is known as a $\mathscr{C}^\star$-algebra. A positive element of $\mathbb{A}$ is an element $\mathfrak{a} \in \mathbb{A}$ such that $\mathfrak{a} = \mathfrak{a}^\star$ and its spectrum $\sigma(\mathfrak{a}) \subset \mathbb{R}_+$, where $\sigma(\mathfrak{a}) = \{v \in \mathbb{R} : v1_{\mathbb{A}} - \mathfrak{a} \text{ is noninvertible}\}$. The set of all positive elements will be denoted by $\mathbb{A}_+$. Such elements allow us to define a parial ordering $\succeq$ on the elements of $\mathbb{A}$. That is,

$$\mathfrak{b} \succeq \mathfrak{a} \text{ if and only if } \mathfrak{b} - \mathfrak{a} \in \mathbb{A}_+. \tag{1}$$

If $\mathfrak{a} \in \mathbb{A}$ is positive, then we write $\mathfrak{a} \succeq 0_{\mathbb{A}}$, where $0_{\mathbb{A}}$ is the zero element of $\mathbb{A}$. Each positive element $\mathfrak{a}$ of a $\mathscr{C}^\star$-algebra $\mathbb{A}$ has a unique positive square root. From now on, by $\mathbb{A}$, we mean a unital $\mathscr{C}^\star$-algebra with identity element $1_{\mathbb{A}}$. Further, $\mathbb{A}_+ = \{\mathfrak{a} \in \mathbb{A} : \mathfrak{a} \pm 0_{\mathbb{A}}\}$ and $(\mathfrak{a}^\star \mathfrak{a})^{1/2} = |\mathfrak{a}|$.

Now, we recall the definition of $C^\star$-algebra-valued partial metric space introduced by Chandok et al. [44].

Definition 1. Let Γ be a nonvoid set and the mapping $\rho : \Gamma \times \Gamma \longrightarrow \mathbb{A}$ are defined, with the following properties:

(A1) $0_{\mathbb{A}} \preceq \rho(\aleph, \varpi)$ for all $\aleph, \varpi \in \Gamma$ and $\rho(\aleph, \aleph) = \rho(\varpi, \varpi) = \rho(\aleph, \varpi)$ if and only if $\aleph = \varpi$

(A2) $\rho(\aleph, \aleph) \preceq \rho(\aleph, \varpi)$

(A3) $\rho(\aleph, \varpi) = \rho(\varpi, \aleph)$ for all $\aleph, \varpi \in \Gamma$

(A4) $\rho(\aleph, \varpi) \preceq \rho(\aleph, \gamma) + \rho(\gamma, \varpi) - \rho(\gamma, \gamma)$ for all $\aleph, \varpi, \gamma \in \Gamma$

Then, ρ is said to be a $C^\star$-algebra-valued partial metric on Γ, and $(\Gamma, \mathbb{A}, \rho)$ is said to be a $C^\star$-algebra-valued partial metric space.

Definition 2. A sequence $\{\aleph_\alpha\}$ in $(\Gamma, \mathbb{A}, \rho)$ is called convergent (with respect to $\mathbb{A}$) to a point $\aleph \in \Gamma$, if for given $\varepsilon > 0$, $\exists\, \mathfrak{k} \in \mathbb{N}$ such that $\|\rho(\aleph_\alpha, \aleph) - \rho(\aleph, \aleph\| < \varepsilon, \forall \alpha > \mathfrak{k}$.

Definition 3. A sequence $\{\aleph_\alpha\}$ in $(\Gamma, \mathbb{A}, \rho)$ is called Cauchy (with respect to $\mathbb{A}$), if $\lim_{\alpha \longrightarrow \infty} \rho(\aleph_\alpha, \aleph_{\mathfrak{m}})$ exists, and it is finite.

Definition 4. The triplet $(\Gamma, \mathbb{A}, \rho)$ is called complete $C^\star$ -algebra-valued partial metric space if every Cauchy sequence in Γ is convergent to some point $\aleph$ in Γ such that

$$\lim_{\alpha \longrightarrow \infty} \rho(\aleph_\alpha, \aleph_{\mathfrak{m}}) = \lim_{\alpha \longrightarrow \infty} \rho(\aleph_\alpha, \aleph) = \rho(\aleph, \aleph). \tag{2}$$

Definition 5 (see [18]). Let Γ be a nonvoid set. An element $(\aleph, \varpi) \in \Gamma \times \Gamma$ is said to be

(1) A couple fixed point of the mapping $\varphi : \Gamma \times \Gamma \longrightarrow \Gamma$ if $\varphi(\aleph, \varpi) = \aleph$ and $\varphi(\varpi, \aleph) = \varpi$

(2) A coupled coincidence point of the mapping $\varphi : \Gamma \times \Gamma \longrightarrow \Gamma$ and $g : \Gamma \longrightarrow \Gamma$ if $\varphi(\aleph, \varpi) = g\aleph$ and $\varphi(\varpi, \aleph) = g\varpi$. In this case, $(g\aleph, g\varpi)$ is said to be coupled point of coincidence

(3) A common coupled fixed point of the mapping $\varphi : \Gamma \times \Gamma \longrightarrow \Gamma$ and $g : \Gamma \longrightarrow \Gamma$ if $\varphi(\aleph, \varpi) = g\aleph = \aleph$ and $\varphi(\varpi, \aleph) = g\varpi = \varpi$

Note that Definition 5 (3) reduces to Definition 5 (1) if the mapping g is the identity mapping.

Definition 6 (see [18]). The mappings $\varphi : \Gamma \times \Gamma \longrightarrow \Gamma$ and $g : \Gamma \longrightarrow \Gamma$ is said to be ω-compatible if $g(\varphi(\aleph, \varpi)) = \varphi(g\aleph, g\varpi)$ whenever $g\aleph = \varphi(\aleph, \varpi)$ and $g\varpi = \varphi(\varpi, \aleph)$.

3. Main Results

Now, we give our main results.

Theorem 7. *Let $(\Gamma, \mathbb{A}, \rho)$ be a complete $C^\star$ -algebra-valued partial metric space. Suppose that the mappings $\varphi : \Gamma \times \Gamma \longrightarrow \Gamma$ and $g : \Gamma \longrightarrow \Gamma$ such that*

$$\rho(\varphi(\aleph, \varpi), \varphi(i, v)) \preceq \rho(g\aleph, gi)r + r^\star \rho(g\varpi, gv)r, \text{for any } \aleph, \varpi, i, v \in \Gamma, \tag{3}$$

where $r \in \mathbb{A}$ with $\|r\| < (1/\sqrt{2})$. If $\varphi(\Gamma \times \Gamma) \subseteq g(\Gamma)$ and $g(\Gamma)$ is complete in Γ, then φ and g have a coupled coincidence point and $\rho(g\aleph, g\aleph) = 0_{\mathbb{A}}$, $\rho(g\varpi, g\varpi) = 0_{\mathbb{A}}$. Moreover, if φ and g are ω-compatible, then they have unique common coupled fixed point in Γ.

Proof. Let $\aleph_0, \varpi_0 \in \Gamma$, then $g(\aleph_1) = \varphi(\aleph_0, \varpi_0)$, and $g(\varpi_1) = \varphi(\varpi_0, \aleph_0)$. One can obtain two sequences $\{\aleph_\alpha\}$ and $\{\varpi_\alpha\}$ by continuing this process such that $g(\aleph_{\alpha+1}) = \varphi(\aleph_\alpha, \varpi_\alpha)$, and $g(\varpi_{\alpha+1}) = \varphi(\varpi_\alpha, \aleph_\alpha)$. Then,

$$\begin{aligned} \rho(g\aleph_\alpha, g\aleph_{\alpha+1}) &= \rho(\varphi(\aleph_{\alpha-1}, \varpi_{\alpha-1}), \varphi(\aleph_\alpha, \varpi_\alpha)) \\ &\preceq r^\star(\rho(g\aleph_{\alpha-1}, g\aleph_\alpha))r + r^\star(\rho(g\varpi_{\alpha-1}, g\varpi_\alpha))r \\ &\preceq r^\star(\rho(g\aleph_{\alpha-1}, g\aleph_\alpha)) + (\rho(g\varpi_{\alpha-1}, g\varpi_\alpha))r. \end{aligned} \tag{4}$$

Similarly,

$$\begin{aligned} \rho(g\varpi_\alpha, g\varpi_{\alpha+1}) &= \rho(\varphi(\varpi_{\alpha-1}, \aleph_{\alpha-1}), \varphi(\varpi_\alpha, \aleph_\alpha)) \\ &\preceq r^\star(\rho(g\varpi_{\alpha-1}, g\varpi_\alpha))r + r^\star(\rho(g\aleph_{\alpha-1}, g\aleph_\alpha))r \\ &\preceq r^\star(\rho(g\varpi_{\alpha-1}, g\varpi_\alpha)) + (\rho(g\aleph_{\alpha-1}, g\aleph_\alpha))r. \end{aligned} \tag{5}$$

Let

$$\mathfrak{I}_\alpha = \rho(g\aleph_\alpha, g\aleph_{\alpha+1}) + \rho(g\varpi_\alpha, g\varpi_{\alpha+1}). \tag{6}$$

Using (4) and (5), we have

$$\begin{aligned} \mathfrak{I}_\alpha &= \rho(g\aleph_\alpha, g\aleph_{\alpha+1}) + \rho(g\varpi_\alpha, g\varpi_{\alpha+1}) \\ &\preceq r^\star(\rho(g\aleph_{\alpha-1}, g\aleph_\alpha) + \rho(g\varpi_{\alpha-1}, g\varpi_\alpha))r \\ &\quad + r^\star(\rho(g\varpi_{\alpha-1}, g\varpi_\alpha) + \rho(g\aleph_{\alpha-1}, g\aleph_\alpha))r \\ &\preceq \left(\sqrt{2}r\right)^\star(\rho(g\aleph_{\alpha-1}, g\aleph_\alpha) + \rho(g\varpi_{\alpha-1}, g\varpi_\alpha))\left(\sqrt{2}r\right) \\ &\preceq \left(\sqrt{2}r\right)^\star \mathfrak{I}_{\alpha-1}\left(\sqrt{2}r\right). \end{aligned} \tag{7}$$

Let $s, \mathfrak{t} \in \mathbb{A}_{\boxtimes}$, then $s \preceq \mathfrak{t}$ implies $r^\star s r \preceq r^\star \mathfrak{t} r$ (Theorem 2.2.5 in [27]). Therefore, for each $\alpha \in \mathbb{N}$,

$$0_{\mathbb{A}} \preceq \mathfrak{I}_{\alpha} \preceq \left(\sqrt{2}r\right)^{*} \mathfrak{I}_{\alpha-1}\left(\sqrt{2}r\right) \preceq \cdots \preceq \left[\left(\sqrt{2}r\right)^{*}\right]^{\alpha} \mathfrak{I}_{0}\left(\sqrt{2}r\right)^{\alpha}. \quad (8)$$

If $\mathfrak{I}_0 = 0_{\mathbb{A}}$, then φ and g have a coupled coincidence point $(\aleph_0, \varpi_0)$. Now, letting $0_{\mathbb{A}} \preceq \mathfrak{I}_0$, then for each $\alpha, \wp \in \mathbb{N}$,

$$\begin{aligned}
\rho\left(g\aleph_{\alpha+\wp}, g\aleph_{\alpha}\right) \preceq & \rho\left(g\aleph_{\alpha+\wp}, g\aleph_{\alpha+\wp-1}\right) + \rho\left(g\aleph_{\alpha+\wp-1}, g\aleph_{\alpha+\wp-2}\right) \\
& - \rho\left(g\aleph_{\alpha+\wp-1}, g\aleph_{\alpha+\wp-1}\right) + \cdots \\
& + \rho\left(g\aleph_{\alpha+2}, g\aleph_{\alpha+1}\right) + \rho\left(g\aleph_{\alpha+1}, g\aleph_{\alpha}\right) \\
& - \rho\left(g\aleph_{\alpha+1}, g\aleph_{\alpha+1}\right) \\
\preceq & \rho\left(g\aleph_{\alpha+\wp}, g\aleph_{\alpha+\wp-1}\right) + \rho\left(g\aleph_{\alpha+\wp-1}, g\aleph_{\alpha+\wp-2}\right) \\
& + \cdots + \rho\left(g\aleph_{\alpha+2}, g\aleph_{\alpha+1}\right) + \rho\left(g\aleph_{\alpha+1}, g\aleph_{\alpha}\right), \\
\rho\left(g\varpi_{\alpha+\wp}, g\varpi_{\alpha}\right) \preceq & \rho\left(g\varpi_{\alpha+\wp}, g\varpi_{\alpha+\wp-1}\right) + \rho\left(g\varpi_{\alpha+\wp-1}, g\varpi_{\alpha+\wp-2}\right) \\
& - \rho\left(g\varpi_{\alpha+\wp-1}, g\varpi_{\alpha+\wp-1}\right) + \cdots \\
& + \rho\left(g\varpi_{\alpha+2}, g\varpi_{\alpha+1}\right) + \rho\left(g\varpi_{\alpha+1}, g\varpi_{\alpha}\right) \\
& - \rho\left(g\varpi_{\alpha+1}, g\varpi_{\alpha+1}\right) \preceq \rho\left(g\varpi_{\alpha+\wp}, g\varpi_{\alpha+\wp-1}\right) \\
& + \rho\left(g\varpi_{\alpha+\wp-1}, g\varpi_{\alpha+\wp-2}\right) + \cdots \\
& + \rho\left(g\varpi_{\alpha+2}, g\varpi_{\alpha+1}\right) + \rho\left(g\varpi_{\alpha+1}, g\varpi_{\alpha}\right).
\end{aligned} \quad (9)$$

Consequently,

$$\begin{aligned}
\rho\left(g\aleph_{\alpha+\wp}, g\aleph_{\alpha}\right) + \rho\left(g\varpi_{\alpha+\wp}, g\varpi_{\alpha}\right) & \preceq \mathfrak{I}_{\alpha+\wp-1} + \mathfrak{I}_{\alpha+wp-2} + \cdots + \mathfrak{I}_{\alpha} \\
& \preceq \sum_{\Bbbk=\alpha}^{\alpha+\wp-1} \left[\left(\sqrt{2}r\right)^{*}\right]^{\Bbbk} \mathfrak{I}_0 \left[\sqrt{2}r\right]^{\Bbbk},
\end{aligned} \quad (10)$$

which implies that

$$\begin{aligned}
\left\|\rho\left(g\aleph_{\alpha+\wp}, g\aleph_{\alpha}\right) + \rho\left(g\varpi_{\alpha+\wp}, g\varpi_{\alpha}\right)\right\| & \preceq \sum_{\Bbbk=\alpha}^{\alpha+\wp-1} \left\|\sqrt{2}r\right\|^{2\Bbbk} \mathfrak{I}_0 \\
& \preceq \sum_{\Bbbk=\alpha}^{\infty} \left\|\sqrt{2}r\right\|^{2\Bbbk} \mathfrak{I}_0 \\
& = \frac{\left\|\sqrt{2}r\right\|^{2\alpha}}{1 - \left\|\sqrt{2}r\right\|^{2}} \mathfrak{I}_0.
\end{aligned} \quad (11)$$

Since $\|r\| < (1/\sqrt{2})$, we have

$$\left\|\rho\left(g\aleph_{\alpha+\wp}, g\aleph_{\alpha}\right) + \rho\left(g\varpi_{\alpha+\wp}, g\varpi_{\alpha}\right)\right\| \preceq \frac{\left\|\sqrt{2}r\right\|^{2\alpha}}{1 - \left\|\sqrt{2}r\right\|^{2}} \mathfrak{I}_0 \longrightarrow 0, \quad (12)$$

which is together with

$$\rho\left(g\aleph_{\alpha+\wp}, g\aleph_{\alpha}\right) \preceq \rho\left(g\aleph_{\alpha+\wp}, g\aleph_{\alpha}\right) + \rho\left(g\varpi_{\alpha+\wp}, g\varpi_{\alpha}\right), \quad (13)$$

and

$$\rho\left(g\varpi_{\alpha+\wp}, g\varpi_{\alpha}\right) \preceq \rho\left(g\aleph_{\alpha+\wp}, g\aleph_{\alpha}\right) + \rho\left(g\varpi_{\alpha+\wp}, g\varpi_{\alpha}\right). \quad (14)$$

Therefore, $\{g\aleph_{\alpha}\}$ and $\{g\varpi_{\alpha}\}$ are Cauchy sequences in $g(\Gamma)$. Since $\{g\varpi_{\alpha}\}$ is complete, $\exists \aleph, \varpi \in \Gamma$ such that $\lim_{\alpha \leq \infty} g\aleph_{\alpha} = g\aleph$ and

$$\rho(g\aleph, g\aleph) = \lim_{n \longrightarrow \infty} \rho(g\aleph_{\alpha}, g\aleph) = \lim_{n \longrightarrow \infty} \rho(g\aleph_{\alpha}, g\aleph_{\alpha}) = 0_{\mathbb{A}}, \quad (15)$$

$\lim_{\alpha \leq \infty} g\varpi_{\alpha} = g\varpi$, and

$$\rho(g\varpi, g\varpi) = \lim_{n \longrightarrow \infty} \rho(g\varpi_{\alpha}, g\varpi) = \lim_{n \longrightarrow \infty} \rho(g\varpi_{\alpha}, g\varpi_{\alpha}) = 0_{\mathbb{A}}. \quad (16)$$

Now, we show that $\varphi(\aleph, \varpi) = g\aleph$ and $\varphi(\varpi, \aleph) = g\varpi$. For this,

$$\begin{aligned}
\rho(\varphi(\aleph, \varpi), g\aleph) & \preceq \rho(\varphi(\aleph, \varpi), g\aleph_{\alpha+1}) + \rho(g\aleph_{\alpha+1}, g\aleph) \\
& \preceq \rho(\varphi(\aleph, \varpi), \varphi(\aleph_{\alpha}, \varpi_{\alpha})) + \rho(g\aleph_{\alpha+1}, g\aleph) \\
& \preceq r^{*} \rho(g\aleph_{\alpha}, g\aleph) r + r^{*} \rho(g\varpi_{\alpha}, g\varpi) r + \rho(g\aleph_{\alpha+1}, g\aleph).
\end{aligned} \quad (17)$$

As $\alpha \longrightarrow \infty$, we get $\rho(\varphi(\aleph, \varpi), g\aleph) = 0_{\mathbb{A}}$, and hence, $\varphi(\aleph, \varpi) = g\aleph$. Similarly, $\varphi(\varpi, \aleph) = g\varpi$. Therefore, φ and g have a coupled coincidence point $(\aleph, \varpi)$.

Let $(\aleph', \varpi')$ be another coupled coincidence point of φ and g. Then,

$$\begin{aligned}
\rho\left(g\aleph, g\aleph'\right) & = \rho\left(\varphi(\aleph, \varpi), \varphi\left(\aleph', \varpi'\right)\right) \\
& \preceq r^{*} \rho\left(g\aleph, g\aleph'\right) r + r^{*} \rho\left(g\varpi, g\varpi'\right) r, \\
\rho\left(g\varpi, g\varpi'\right) & = \rho\left(\varphi(\varpi, \aleph), \varphi\left(\varpi', \aleph'\right)\right) \\
& \preceq r^{*} \rho\left(g\varpi, g\varpi'\right) r + r^{*} \rho\left(g\aleph, g\aleph'\right) r.
\end{aligned} \quad (18)$$

Consequently,

$$\begin{aligned}
& \rho\left(g\aleph, g\aleph'\right) + \rho\left(g\varpi, g\varpi'\right) \\
& \preceq \left(\sqrt{2}r\right)^{*} \left(\rho\left(g\aleph, g\aleph'\right) + \rho\left(g\varpi, g\varpi'\right)\right) \left(\sqrt{2}r\right),
\end{aligned} \quad (19)$$

which implies that

$$\begin{aligned}
& \left\|\rho\left(g\aleph, g\aleph'\right) + \rho\left(g\varpi, g\varpi'\right)\right\| \\
& \preceq \left\|\left(\sqrt{2}r\right)\right\|^{2} \left\|\rho\left(g\aleph, g\aleph'\right) + \rho\left(g\varpi, g\varpi'\right)\right\|.
\end{aligned} \quad (20)$$

Since $\|(\sqrt{2}r)\| < 1$, then $\|\rho(g\aleph, g\aleph') + \rho(g\varpi, g\varpi')\| = 0$. Hence, we get $g\aleph = g\aleph'$ and $g\varpi = g\varpi'$. Similarly, we can prove that $g\aleph = g\varpi'$ and $g\varpi = g\aleph'$. Then, φ and g have a unique coupled point of coincidence $(g\aleph, g\aleph)$. Moreover, set $v = g\aleph$, then $v = g\aleph = \varphi(\aleph, \aleph)$. Since φ and g are ω-compatible,

$$gv = g(g\aleph) = g(\varphi(\aleph, \aleph)) = \varphi(g\aleph, g\aleph) = \varphi(v, v). \tag{21}$$

Therefore, φ and g have a coupled point of coincidence (gv, gv). We know $gv = g\aleph$, then $v = gv = \varphi(v, v)$. Therefore, φ and g have a unique common coupled fixed point (v, v). □

Example 1. Let $\Gamma = \mathscr{R}$ and $\mathbb{A} = \mathscr{M}_2(\mathbb{C})$, and the map $\rho : \Gamma \times \Gamma \longrightarrow \mathbb{A}$ is defined by

$$\rho(\aleph, \varpi) = \begin{bmatrix} |\aleph - \varpi| & 0 \\ 0 & \Bbbk|\aleph - \varpi| \end{bmatrix} + \begin{bmatrix} \max\{\aleph, \varpi\} & 0 \\ 0 & \Bbbk \max\{\aleph, \varpi\} \end{bmatrix}, \tag{22}$$

where $\Bbbk > 0$ is a constant. Then, $(\Gamma, \mathbb{A}, \rho)$ is a complete $C^\star$-algebra-valued partial metric space. Consider the mappings $\varphi : \Gamma \times \Gamma \longrightarrow \Gamma$ with $\varphi(\aleph, \varpi) = (\aleph + \varpi)/2$ and $g : \Gamma \longrightarrow \Gamma$ with $g(\aleph) = 2\aleph$. Set $\lambda \in \mathbb{C}$ with $|\lambda| < (1/\sqrt{2})$, and $r = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, then $r \in \mathbb{A}$ and $\|r\|_\infty = |\lambda|$. Clearly, φ and g are ω-compatible. Moreover, one can verify that φ satisfies the contractive condition

$$\rho(\varphi(\aleph, \varpi), \varphi(u, v)) \preceq r^\star \varphi(\aleph, u) r + r^\star \varphi(\varpi, v) r, \text{ for any } \aleph, \varpi, u, v \in \Gamma. \tag{23}$$

In this case, $(0, 0)$ is coupled coincidence point of φ and g. Moreover, $(0, 0)$ is a unique common coupled fixed point of φ and g.

Corollary 8. *Let $(\Gamma, \mathbb{A}, \rho)$ be a complete $C^\star$-algebra-valued partial metric space. Suppose that mapping $\varphi : \Gamma \times \Gamma \longrightarrow \Gamma$ such that*

$$\rho(\varphi(\aleph, \varpi), \varphi(u, v)) \preceq r^\star \rho(\aleph, u) r + r^\star \rho(\varpi, v) r, \textit{ for any } \aleph, \varpi, u, v \in \Gamma, \tag{24}$$

where $r \in \mathbb{A}$ with $\|r\| < (1/\sqrt{2})$. Then, φ has a unique coupled fixed point.

We recall the following lemma of [27].

Lemma 9. *Suppose that $\mathbb{A}$ is a unital $C^\star$-algebra with a unit $1_{\mathbb{A}}$.*

(1) If $r \in \mathbb{A}_+$ with $\|r\| < (1/2)$, then $1_{\mathbb{A}} - r$ is invertible

(2) If $r, s \in \mathbb{A}_+$ and $rs = sr$, then $0_{\mathbb{A}} \preceq rs$

(3) If $r, s \in \mathbb{A}_{\boxtimes}$ and $\mathfrak{t} \in \mathbb{A}'_+$ then $r \preceq s$ deduces $\mathfrak{t}r \preceq \mathfrak{t}s$, where $\mathbb{A}'_+ = \mathbb{A}_+ \cap \mathbb{A}'$

Theorem 10. *Let $(\Gamma, \mathbb{A}, \rho)$ is a complete $C^\star$-algebra-valued partial metric space. Suppose that the mappings $\varphi : \Gamma \times \Gamma \longrightarrow \Gamma$ and $g : \Gamma \longrightarrow \Gamma$ such that*

$$\begin{aligned} &\rho(\varphi(\aleph, \varpi), \varphi(u, v)) \\ &\quad \preceq r\rho(\varphi(\aleph, \varpi), g\aleph) + s\rho(\varphi(u, v), gu), \textit{ for any } \aleph, \varpi, u, v \in \Gamma, \end{aligned} \tag{25}$$

where $r, s \in \mathbb{A}'_+$ with $\|r\| + \|s\| < 1$. If $\varphi(\Gamma \times \Gamma) \subseteq g(\Gamma)$ and $g(\Gamma)$ is complete in Γ, then φ and g have a coupled coincidence point and $\rho(g\aleph, g\aleph) = 0_{\mathbb{A}}$, $\rho(g\varpi, g\varpi) = 0_{\mathbb{A}}$. Moreover, if φ and g are ω-compatible, then they have unique common coupled fixed point in Γ.

Proof. Similar to Theorem 7, construct two sequences $\{\aleph_\alpha\}$ and $\{\varpi_\alpha\}$ in Γ such that $g\aleph_{\alpha+1} = \varphi(\aleph_\alpha, \varpi_\alpha)$ and $g\varpi_{\alpha+1} = \varphi(\varpi_\alpha, \aleph_\alpha)$. Then, by applying (25), we have

$$\begin{aligned} &(1_{\mathbb{A}} - s)\rho(g\aleph_\alpha, g\aleph_{\alpha+1}) \preceq r\rho(g\aleph_\alpha, g\aleph_{\alpha-1}), \\ &(1_{\mathbb{A}} - s)\rho(g\varpi_\alpha, g\varpi_{\alpha+1}) \preceq r\rho(g\varpi_\alpha, g\varpi_{\alpha-1}). \end{aligned} \tag{26}$$

Since $r, s \in \mathbb{A}'_+$ with $\|r\| + \|s\| < 1$, we have $1_{\mathbb{A}} - s$ is invertible and $(1_{\mathbb{A}} - s)^{-1} r \in \mathbb{A}'_+$. Therefore,

$$\begin{aligned} &\rho(g\aleph_\alpha, g\aleph_{\alpha+1}) \preceq (1_{\mathbb{A}} - s)^{-1} r\rho(g\aleph_\alpha, g\aleph_{\alpha-1}), \\ &\rho(g\varpi_\alpha, g\varpi_{\alpha+1}) \preceq (1_{\mathbb{A}} - s)^{-1} r\rho(g\varpi_\alpha, g\varpi_{\alpha-1}). \end{aligned} \tag{27}$$

Then,

$$\begin{aligned} &\|\rho(g\aleph_\alpha, g\aleph_{\alpha+1})\| \preceq \left\|(1_{\mathbb{A}} - s)^{-1} r\right\| \|\rho(g\aleph_\alpha, g\aleph_{\alpha-1})\|, \\ &\|\rho(g\varpi_\alpha, g\varpi_{\alpha+1})\| \preceq \left\|(1_{\mathbb{A}} - s)^{-1} r\right\| \|\rho(g\varpi_\alpha, g\varpi_{\alpha-1})\|. \end{aligned} \tag{28}$$

Since,

$$\left\|(1_{\mathbb{A}} - s)^{-1} r\right\| \preceq \left\|(1_{\mathbb{A}} - s)^{-1}\right\| \|\mathrm{r}\| \preceq \sum_{\Bbbk=0}^{\infty} \|s\|^{\Bbbk} \|\mathrm{r}\| = \frac{\|\mathrm{r}\|}{1 - \|s\|}. < 1. \tag{29}$$

Therefore, $\{g\aleph_\alpha\}$ and $\{g\varpi_\alpha\}$ are Cauchy sequences in $g(\Gamma)$. By the completeness of $g(\Gamma)$, $\exists \aleph, \varpi \in \Gamma$ such that $\lim_{\alpha \circ \infty} g\aleph_\alpha = g\aleph$ and

$$\rho(g\aleph, g\aleph) = \lim_{n \longrightarrow \infty} \rho(g\aleph_\alpha, g\aleph) = \lim_{n \longrightarrow \infty} \rho(g\aleph_\alpha, g\aleph_\alpha) = 0_{\mathbb{A}}, \tag{30}$$

$\lim_{\alpha\longrightarrow\infty} g\varpi_\alpha = g\varpi$, and

$$\rho(g\varpi, g\varpi) = \lim_{n\longrightarrow\infty}\rho(g\varpi_\alpha, g\varpi) = \lim_{n\longrightarrow\infty}\rho(g\varpi_\alpha, g\varpi_\alpha) = 0_{\mathbb{A}}. \tag{31}$$

Since,

$$\begin{aligned}\rho(\varphi(\aleph,\varpi), g\aleph) &\preceq \rho(g\aleph_{\alpha+1}, \varphi(\aleph,\varpi)) + \rho(g\aleph_{\alpha+1}, g\aleph)\\ &= \rho(\varphi(\aleph_\alpha,\varpi_\alpha), \varphi(\aleph,\varpi)) + \rho(g\aleph_{\alpha+1}, g\aleph)\\ &\preceq r\rho(\varphi(\aleph_\alpha,\varpi_\alpha), g\aleph_\alpha) + s\rho(\varphi(\aleph,\varpi), g\aleph)\\ &\quad + \rho(g\aleph_{\alpha+1}, g\aleph)\\ &\preceq r\rho(g\aleph_{\alpha+1}, g\aleph_\alpha) + s\rho(\varphi(\aleph,\varpi), g\aleph)\\ &\quad + \rho(g\aleph_{\alpha+1}, g\aleph),\end{aligned} \tag{32}$$

which implies that

$$\rho(\varphi(\aleph,\varpi), g\aleph) \preceq (1-s)^{-1} r\rho(g\aleph_{\alpha+1}, g\aleph_\alpha) + (1-s)^{-1}\rho(g\aleph_{\alpha+1}, g\aleph_\alpha). \tag{33}$$

Then, $\rho(\varphi(\aleph,\varpi), g\aleph) = 0_{\mathbb{A}}$ or equivalently $\varphi(\aleph,\varpi) = g\aleph$. Similarly, one can obtain $\varphi(\varpi,\aleph) = g\varpi$. Let $(\aleph',\varpi')$ be another coupled coincidence point of φ and g, then

$$\begin{aligned}\rho\left(g\aleph', g\aleph\right) &\preceq \rho\left(\varphi\left(\aleph',\varpi'\right), \varphi(\aleph,\varpi)\right)\\ &\preceq r\rho\left(\varphi\left(\aleph',\varpi'\right), g\aleph'\right) + s\rho(\varphi(\aleph,\varpi), g\aleph)\\ &= r\rho\left(g\aleph', g\aleph'\right) + s\rho(g\aleph, g\aleph) = 0_{\mathbb{A}},\end{aligned} \tag{34}$$

and

$$\begin{aligned}\rho\left(g\varpi', g\varpi\right) &\preceq \rho\left(\varphi\left(\varpi',\aleph'\right), \varphi(\varpi,\aleph)\right)\\ &\preceq r\rho\left(\varphi\left(\varpi',\aleph'\right), g\varpi'\right) + s\rho(\varphi(\varpi,\aleph), g\varpi)\\ &= r\rho\left(g\varpi', g\varpi'\right) + s\rho(g\varpi, g\varpi) = 0_{\mathbb{A}},\end{aligned} \tag{35}$$

which implies that $g\aleph' = g\aleph$ and $g\varpi' = g\varpi$. Similarly, we have $g\aleph' = g\varpi$ and $g\varpi' = g\aleph$. Hence, φ and g have a unique coupled point of coincidence $(g\aleph, g\aleph)$. Moreover, we can show that φ and g have a unique common coupled fixed point. □

Theorem 11. *Let $(\Gamma, \mathbb{A}, \rho)$ be a complete C^* -algebra-valued partial metric space. Suppose that mappings $\varphi : \Gamma\times\Gamma \longrightarrow \Gamma$ and $g : \Gamma \longrightarrow \Gamma$ such that*

$$\begin{aligned}&\rho(\varphi(\aleph,\varpi), \varphi(u,v))\\ &\quad\preceq r\rho(\varphi(\aleph,\varpi), gu) + s\rho(\varphi(u,v), g\aleph), \text{for any } \aleph,\varpi,u,v\in\Gamma,\end{aligned} \tag{36}$$

where $r, s \in {\mathbb{A}_+}'$ with $\|r\|+\|s\|<1$. If $\varphi(\Gamma\times\Gamma)\subseteq g(\Gamma)$ and $g(\Gamma)$ is complete in Γ, then φ and g have a coupled coincidence point and $\rho(g\aleph, g\aleph) = 0_{\mathbb{A}}$, $\rho(g\varpi, g\varpi) = 0_{\mathbb{A}}$. Moreover, if φ and g are ω-compatible, then they have unique common coupled fixed point in Γ.

Proof. Following similar process given in Theorem 7, we construct two sequences $\{\aleph_\alpha\}$ and $\{\varpi_\alpha\}$ in Γ such that $g(\aleph_{\alpha+1}) = \varphi(\aleph_\alpha, \varpi_\alpha)$ and $g(\varpi_{\alpha+1}) = \varphi(\varpi_\alpha, \aleph_\alpha)$. From (36), we have

$$\begin{aligned}\rho(g\aleph_\alpha, g\aleph_{\alpha+1}) &= \rho(\varphi(\aleph_{\alpha-1},\varpi_{\alpha-1}), \varphi(\aleph_\alpha,\varpi_\alpha))\\ &\preceq r\rho(\varphi(\aleph_{\alpha-1},\varpi_{\alpha-1}), g\aleph_\alpha)\\ &\quad + s\rho(\varphi(\aleph_\alpha,\varpi_\alpha), g\aleph_{\alpha-1})\\ &\preceq r\rho(g\aleph_\alpha, g\aleph_\alpha) + s\rho(g\aleph_{\alpha+1}, g\aleph_{\alpha-1})\\ &\preceq r\rho(g\aleph_{\alpha+1}, g\aleph_\alpha) + s\rho(g\aleph_{\alpha+1}, g\aleph_\alpha)\\ &\quad + s\rho(g\aleph_\alpha, g\aleph_{\alpha-1}) - s\rho(\rho(g\aleph_\alpha, g\aleph_\alpha)\\ &\preceq r\rho(g\aleph_{\alpha+1}, g\aleph_\alpha) + s\rho(g\aleph_{\alpha+1}, g\aleph_\alpha)\\ &\quad + s\rho(g\aleph_\alpha, g\aleph_{\alpha-1}),\end{aligned} \tag{37}$$

which implies that

$$(1_{\mathbb{A}} - (r+s))\rho(g\aleph_\alpha, g\aleph_{\alpha+1}) \preceq s\rho(g\aleph_\alpha, g\aleph_{\alpha-1}). \tag{38}$$

Because of the symmetry in (36),

$$\begin{aligned}\rho(g\aleph_{\alpha+1}, g\aleph_\alpha) &= \rho(\varphi(\aleph_\alpha,\varpi_\alpha), \varphi(\aleph_{\alpha-1},\varpi_{\alpha-1}))\\ &\preceq r\rho(\varphi(\aleph_\alpha,\varpi_\alpha), g\aleph_{\alpha-1})\\ &\quad + s\rho(\varphi(\aleph_{\alpha-1},\varpi_{\alpha-1}), g\aleph_\alpha)\\ &\preceq r\rho(g\aleph_{\alpha+1}, g\aleph_{\alpha-1}) + s\rho(g\aleph_\alpha, g\aleph_\alpha)\\ &\preceq r\rho(g\aleph_{\alpha+1}, g\aleph_\alpha) + r\rho(g\aleph_\alpha, g\aleph_{\alpha-1})\\ &\quad - r\rho(g\aleph_\alpha, g\aleph_\alpha) + s\rho(g\aleph_{\alpha+1}, g\aleph_\alpha)\\ &\preceq r\rho(g\aleph_{\alpha+1}, g\aleph_\alpha) + r\rho(g\aleph_\alpha, g\aleph_{\alpha-1})\\ &\quad + s\rho(g\aleph_{\alpha+1}, g\aleph_\alpha),\end{aligned} \tag{39}$$

which implies that

$$(1_{\mathbb{A}} - (r+s))\rho(g\aleph_\alpha, g\aleph_{\alpha+1}) \preceq r\rho(g\aleph_\alpha, g\aleph_{\alpha-1}). \tag{40}$$

From (38) and (40), we obtain

$$(1_{\mathbb{A}} - (r+s))\rho(g\aleph_\alpha, g\aleph_{\alpha+1}) \preceq \frac{r+s}{2}\rho(g\aleph_\alpha, g\aleph_{\alpha-1}). \tag{41}$$

Since $r, s \in {\mathbb{A}_+}'$ with $\|r+s\| \preceq \|r\|+\|s\|<1$, then $(1_{\mathbb{A}} - (r+s))^{-1} \in {\mathbb{A}_+}'$, which together with Lemma 9 (3), we obtain

$$\rho(g\aleph_\alpha, g\aleph_{\alpha+1}) \preceq (1_{\mathbb{A}} - (r+s))^{-1}\frac{r+s}{2}\rho(g\aleph_\alpha, g\aleph_{\alpha-1}). \tag{42}$$

Let $\mathfrak{e} = (1_{\mathbb{A}} - (r+s))^{-1}((r+s)/2)$, then $\|\mathfrak{e}\| = \|(1_{\mathbb{A}} - (r+s))^{-1}((r+s)/2)\|<1$. The same argument in

Theorem 10 tells that $\{g\aleph_\alpha\}$ is a Cauchy sequence in $g(\Gamma)$. Similarly, we can derive that $\{g\varpi_\alpha\}$ is also a Cauchy sequence in $g(\Gamma)$. By the completeness of $g(\Gamma)$, $\exists\aleph, \varpi \in \Gamma$ such that $\lim_{\alpha\leq\infty} g\aleph_\alpha = g\aleph$ and

$$\rho(g\aleph, g\aleph) = \lim_{n\longrightarrow\infty} \rho(g\aleph_\alpha, g\aleph) = \lim_{n\longrightarrow\infty} \rho(g\aleph_\alpha, g\aleph_\alpha) = 0_{\mathbb{A}}, \tag{43}$$

$\lim_{\alpha\leq\infty} g\varpi_\alpha = g\varpi$, and

$$\rho(g\varpi, g\varpi) = \lim_{n\longrightarrow\infty} \rho(g\varpi_\alpha, g\varpi) = \lim_{n\longrightarrow\infty} \rho(g\varpi_\alpha, g\varpi_\alpha) = 0_{\mathbb{A}}. \tag{44}$$

Now, we show that $\varphi(\aleph, \varpi) = g\aleph$ and $\varphi(\varpi, \aleph) = g\varpi$. For this,

$$\begin{aligned}\rho(\varphi(\aleph, \varpi), g\aleph) &\preceq \rho(g\aleph_{\alpha+1}, \varphi(\aleph, \varpi)) + \rho(g\aleph_{\alpha+1}, g\aleph)\\ &= \rho(\varphi(\aleph_\alpha, \varpi_\alpha), \varphi(\aleph, \varpi)) + \rho(g\aleph_{\alpha+1}, g\aleph)\\ &\preceq r\rho(\varphi(\aleph_\alpha, \varpi_\alpha), g\aleph) + s\rho(\varphi(\aleph, \varpi), g\aleph_\alpha)\\ &\quad + \rho(g\aleph_{\alpha+1}, g\aleph)\\ &\preceq r\rho(g\aleph_{\alpha+1}, g\aleph) + s\rho(\varphi(\aleph, \varpi), g\aleph_\alpha)\\ &\quad + \rho(g\aleph_{\alpha+1}, g\aleph),\end{aligned} \tag{45}$$

which implies that

$$\begin{aligned}\|\rho(\varphi(\aleph, \varpi), g\aleph)\| &\leq \|r\| \|\rho(g\aleph_{\alpha+1}, g\aleph)\| + \|s\| \|\rho(\varphi(\aleph, \varpi), g\aleph_\alpha)\|\\ &\quad + \|\rho(g\aleph_{\alpha+1}, g\aleph)\|.\end{aligned} \tag{46}$$

By the continuity of the metric and the norm, we obtain

$$\|\rho(\varphi(\aleph, \varpi), g\aleph)\| \leq \|s\| \|\rho(\varphi(\aleph, \varpi), g\aleph)\|. \tag{47}$$

Since $\|s\|<1$; therefore, $\|\rho(\varphi(\aleph, \varpi), g\aleph)\| = 0$. Thus, $\varphi(\aleph, \varpi) = g\aleph$. Similarly, $\varphi(\varpi, \aleph) = g\varpi$. Hence, $(\aleph, \varpi)$ is a coupled coincidence point of φ and g. The same reasoning that Theorem 10 tells us that φ and g have unique common coupled fixed point in Γ. □

In 2015, Ma and Jiang [45] proved fixed point theorems in C^*-algebra-valued b-metric spaces with an application of Fredholm integral equations. In 2016, Xin et al. [46] proved common fixed point theorems in C^*-algebra-valued metric spaces with an application of Fredholm integral equations. In 2020, Mlaiki et al. [47] proved fixed point results on C^*-algebra valued partial b-metric spaces with an application of Fredholm integral equations. In 2021, Tomar et al. [48] proved fixed point theorems in C^*-algebra valued partial metric space with an application of Fredholm integral equations.

4. Application

As an application of Corollary 8, we find an existence and uniqueness result for a type of following system of Fredholm integral equations:

$$\aleph(\mu) = \int_{\mathscr{E}} \mathscr{G}(\mu, p, \aleph(p), \varpi(p))dp + \delta(\mu), \mu, p \in \mathscr{E}, \tag{48}$$

$$\varpi(\mu) = \int_{\mathscr{E}} \mathscr{G}(\mu, p, \varpi(p), \aleph(p))dp + \delta(\mu), \ \mu, p \in \mathscr{E}, \tag{49}$$

where $\mathscr{E}$ is a measurable, $\mathscr{G} : \mathscr{E} \times \mathscr{E} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\delta \in \mathscr{L}^\infty(\mathscr{E})$. Let $\Gamma = \mathscr{L}^\infty(\mathscr{E})$, $\mathscr{K} = \mathscr{L}^2(\mathscr{E})$, and $\mathscr{L}(\mathscr{K}) = \mathbb{A}$. Define $\rho : \Gamma \times \Gamma \longrightarrow \mathbb{A}$ by (for all $\delta, \theta, I \in \Gamma$ and $\|\tau\| = \theta < 1$):

$$\rho(\delta, \theta) = \pi_{|\delta-\theta|} + I, \tag{50}$$

where $\pi_{\mathfrak{q}} : \mathscr{K} \longrightarrow \mathscr{K}$ is the multiplicative operator, which is defined by:

$$\pi_{\mathfrak{q}}(\psi) = \mathfrak{q}.\psi. \tag{51}$$

Now, we state and prove our result, as follows:

Theorem 12. *Suppose that (for all $\aleph, \varpi \in \Gamma$)*

(S1) There exists a continuous function $\kappa : \mathscr{E} \times \mathscr{E} \longrightarrow \mathbb{R}$ and $\theta \in (0, 1)$, such that

$$\begin{aligned}&|\mathscr{G}(\mu, p, \aleph(p), \varpi(p)) - \mathscr{G}(\mu, p, u(p), u(p))|\\ &\quad \leq \theta|\kappa(\mu, p)|\left(|\aleph(p) - u(p)| + |\varpi(p) - v(p)| + I - \theta^{-1}I\right),\end{aligned} \tag{52}$$

for all $\mu, p \in \mathscr{E}$.

(S2) $\sup_{\mu\in\mathscr{E}} \int_{\mathscr{E}} |\kappa(\mu, p)| dp \leq 1$.

Subsequently, the integral Equation (49) has a unique solution in Γ.

Proof. Define $\varphi : \Gamma \times \Gamma \longrightarrow \Gamma$ by:

$$\varphi(\aleph, \varpi)(\mu) = \int_{\mathscr{E}} \mathscr{G}(\mu, p, \aleph(p), \varpi(p))dp + \delta(\mu), \forall \mu, p \in \mathscr{E}, \tag{53}$$

Set $\tau = \theta I$, then $\tau \in \mathbb{A}$. For any $z \in \mathscr{K}$, we have

$$\begin{aligned}
\|\rho(\varphi(\aleph,\varpi),\varphi(u,v))\| &= \sup_{\|z\|=1}\left(\pi_{|\varphi(\aleph,\varpi)-\varphi(u,v)|+I}z, z\right)\\
&= \sup_{\|z\|=1}\int_{\mathscr{E}}(|\varphi(\aleph,\varpi)-\varphi(u,v)|+I)z(\mu)\bar{z(\mu)}d\mu\\
&\leq \sup_{\|z\|=1}\int_{\mathscr{E}}\int_{\mathscr{E}}|\mathscr{G}(\mu,p,\aleph(p),\varpi(p))\\
&\quad - \mathscr{G}(\mu,p,\aleph(p),\varpi(p))|dp|z(\mu)|^2d\mu\\
&\quad + \sup_{\|z\|=1}\int_{\mathscr{E}}\int_{\mathscr{E}}dp|z(\mu)|^2d\mu I\\
&\leq \sup_{\|z\|=1}\int_{\mathscr{E}}\left[\int_{\mathscr{E}}\theta\mid\kappa(\mu,p)\mid(|\aleph(p)-u(p)|+|\varpi(p)\right.\\
&\quad \left.-v(p)|+I-\theta^{-1}I)dp\right]|z(\mu)|^2d\mu + I\\
&\leq \theta\sup_{\|z\|=1}\int_{\mathscr{E}}\left[\int_{\mathscr{E}}\mid\kappa(\mu,p)\mid dp\right]|z(\mu)|^2d\mu\\
&\quad \cdot\left(\|\aleph-u\|_\infty + \|\varpi-v\|_\infty\right)\\
&\leq \theta\sup_{\mu\in\mathscr{E}}\int_{\mathscr{E}}|\kappa(\mu,p)|dp\sup_{\|z\|=1}\int_{\mathscr{E}}|z(\mu)|^2d\mu\\
&\quad \cdot\left(\|\aleph-u\|_\infty + \|\varpi-v\|_\infty\right)\\
&\leq \theta\left[\|\aleph-u\|_\infty + \|\varpi-v\|_\infty\right]\\
&= \|\tau\|[\|\rho(\aleph,u)\| + \|\rho(\varpi,v)\|].
\end{aligned} \tag{54}$$

Hence, all the hypotheses of Corollary 8 are verified, and consequently, the integral Equation (49) has a unique solution. □

5. Conclusion

In this paper, we proved common coupled fixed point theorems on C^*-algebra-valued partial metric space using ω-compatible mappings. An illustrative example is provided that shows the validity of the hypothesis and the degree of usefulness of our findings. Moreover, we introduced an application to show that the useful of C^*-algebra-valued metric space to study the existence and uniqueness of system of Fredholm integral equations. Recently, Mutlu et al. [49] proved coupled fixed point theorems on bipolar metric spaces. It is an interesting open problem to study the C^*-algebra-valued bipolar metric space instead of C^*-algebra-valued metric space and obtain common coupled fixed point results on C^*-algebra-valued bipolar metric spaces.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that there is not any competing interest regarding the publication of this manuscript.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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