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Research Article

The Existence of Periodic Solutions of Delay Differential Equations by E^+-Conley Index Theory

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In this paper, the E^+-Conley index theory has been used to study the existence of periodic solutions of nonautonomous delay differential equations (in short, DDEs). The variational structure for DDEs is built, and the existence of periodic solutions of DDEs is transferred to that of critical points of the associated function. When DDEs are 2π-nonresonant, some sufficient conditions are obtained to guarantee the existence of periodic solutions. When the system is 2π-resonant at infinity, by making use a second disturbing of the original functional, some sufficient conditions are obtained to guarantee the existence of periodic solutions to DDEs.

1. Introduction

In the last several decades, there has been increasing interest in the dynamical properties of delay differential equations due to their important applications in many fields such as biological, physical, and social sciences. Many rich mathematical investigations and interesting results on DDEs have been available in the literature. In particular, the existence of periodic solutions is of great interest. As far as the author's knowledge, it was Jones who first studied the existence of periodic solutions of DDEs. Since then, some methods had been developed to search for periodic solutions of DDEs, such as fixed point theorems (cf. [1–4]), cone mapping method (cf. [5]), coincidence degree theory (cf. [6, 7]), Poincaré-Bendixson theorem (cf. [8–10]), and Hopf bifurcation theorem (cf. [11, 12]).

In 1973, Rabinowitz built the variational approach. It had been developed quickly. The variational approach had been used in two different ways to study the existence of periodic solutions of DDEs. Firstly, it can be used indirectly. More specially, in 1974, Kaplan and Yorke studied the following equation:

$$x'(t) = -f(x(t-1)), x \in R. \tag{1}$$

They (cf. [13]) translated the existence of periodic solutions of (1) to that of an associated plane ordinary differential equations. Li and his cooperators developed Kaplan and Yorke's technique and applied to the following equation:

$$x'(t) = -f(x(t-1)) - f(x(t-2)) - \cdots - f(x(t-n)), x \in R. \tag{2}$$

They showed that the associated equation of (2) is a Hamiltonian system (resp., general Hamiltonian system) (cf. [14–16]). By noting the symmetric property, Fei (cf. [17, 18]) made use of variational methods to study the existence of periodic solutions of the associated Hamiltonian system and obtain the existence of multiple periodic solutions of (2). In 2019, Li et al. (cf. [19]) proved the multiplicity of periodic solutions to the following equation:

$$\begin{cases} x^{(2n+1)}(t) = -\sum_{i=1}^{2k} f(x(t-i)), a.e.t \in [0, 2(2k+1)], \\ x(t) - x(t-2(2k+1)) = 0, x \in R. \end{cases} \tag{3}$$

For more results in this direction, we refer to [20] and the reference therein.

In 2005, Guo and Yu firstly established a variational structure for DDEs below:

$$x'(t) = -f(x(t-1)), x \in R^n. \tag{4}$$

They converted the existence of periodic solutions of (4) to that of critical points of the associated variational functional. By using of pseudo-index theory, they (cf. [21]) proved multiple periodic solutions for (4). Later, Guo and Yu and Zheng and Guo studied the following systems:

$$x'(t) = -f(x(t-1)) - f(x(t-2)) - \cdots - f(x(t-n)), x \in R^n. \tag{5}$$

They (cf. [22, 23]) proved the existence of multiple periodic solutions of (5). In 2009, C.J. Guo and Z.M. Guo studied second order system as follows:

$$x''(t) = -f(x(t-1)), x \in R^n. \tag{6}$$

They (cf. [24]) obtained some sufficient conditions to guarantee the existence of multiple periodic solutions of (6). For more results on this direction, we refer to [25] and the reference therein.

As for nonautonomous DDEs, Yu and Xiao studied the following system:

$$x'(t) = -f(t, x(t-1)), x \in R^n. \tag{7}$$

By making use of the Maslov-type index theory, they (cf. [26]) proved the existence of multiple periodic solutions. For more reference on this direction, we refer to [27].

It is well-known that there are many index theories in variational approach. However, only a few had been used to study the existence and multiplicity of periodic solutions to DDEs. A nature question is that whether those index theories can be used to study the existence of periodic solutions to DDEs. It is well known that E^+-Conley index, which is a relative Morse index, was introduced by Abbondandolo [28, 29]. It has been used to study periodic solutions of Hamiltonian systems by proving the Morse-Conley relations. Motivated by [21, 28, 29], we use E^+-Conley index to study the existence of periodic solutions of the following equations:

$$x'(t) = -f\left(t, x\left(t - \frac{\pi}{2}\right)\right), \tag{8}$$

where $x \in R^n, f \in C(R \times R^n, R^n)$. Assume that

$(F1)$ f is odd with respect to variable x and π-periodic with respect to variable t, i.e., for any $x \in R^n$, $f(t,-x) = -f(t, x)$ uniformly for $t \in [0, \pi]$ and $f(t+\pi, x) = f(t, x)$ uniformly for $x \in R^n$

$(F2)$ there exists a twice continuously differentiable function F, such that the gradient of F is f, i.e., $\nabla_x F(t, x) = f(t, x)$

$(F3)$ there exists a π-periodic continuous loop of symmetric matrices $B_\infty(t)$ and a function $G \in C^2(\mathbb{R} \times \mathbb{R}^n)$ such that

$$f(t, x) = B_\infty(t)x + \nabla G(t, x), \tag{9}$$

where $\nabla G(t, x) = o(|x|)$ as $|x| \longrightarrow \infty$, uniformly with respect to $t \in [0, \pi]$

$(F4)$ D^2G is bounded

Theorem 1. *Assume that F satisfies $(F1)$ - $(F4)$. Assume that (8) is 2π-nonresonant at infinity (see Definition 31). If all the 2π-periodic solutions of (8) are 2π-nonresonant, then there is an odd number of them and the following relation holds:*

$$\sum \lambda^{E^+-\dim\ W^-} = \lambda^{E^+-\dim\ V_\infty^-} + (1+\lambda)Q(\lambda), \tag{10}$$

where the sum is taken over all the 2π-periodic solutions and Q is a Laurent polynomial with positive coefficients.

Assume the following:

$(F5)$ $|D^2G(t, x)| \longrightarrow 0$ as $|x| \longrightarrow \infty$, uniformly with respect to t

$(F6)$ $\nabla G(t, x)$ is bounded

Theorem 2. *Assume that (F1)-(F3), (F5), and (F6) hold. Assume that x_0 is a 2π-nonresonant periodic solution of the system (8). If*

$$E^+ - \dim\ W_{x_0} \notin \left[E^+ - \dim\ V_\infty^- - 1, E^+ - \dim\ V_\infty^- + E^+ - \dim\ V_\infty^0 + 1\right], \tag{11}$$

then (8) has at least another 2π-periodic solution.

The paper is organized as follows: in Section 2, we present the space which the variational functional is built; in Section 3, we summarize some basic knowledge on the E^+-Conley index; in Section 4, we will study system which is 2 π-nonresonant system at infinity; in Section 5, we will study system which is 2π-resonant system at infinity.

2. Preparation

Denote by $S^1 = R/(2\pi Z)$. A similar discussion as reference [21], we can build the Hilbert space $H = H^{1/2}(S^1, R^n)$. If $x \in H$, it has Fourier expansion

$$x(t) = \frac{a_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{j=1}^{\infty} \left(a_j \cos jt + b_j \sin jt\right), \tag{12}$$

where $a_0, a_j, b_j \in R^n, j \in N^*$. The space H equips with the following norm and inner product:

$$\begin{aligned} \|x\|^2 &= |a_0|^2 + \sum_{j=1}^{\infty} j\left(\left|a_j\right|^2 + \left|b_j\right|^2\right), \\ \langle x, y \rangle &= (a_0, c_0) + \sum_{j=1}^{\infty} j\left[\left(a_j, c_j\right) + \left(b_j, d_j\right)\right], \end{aligned} \tag{13}$$

where $y = c_0/\sqrt{2\pi} + 1/\sqrt{\pi}\sum_{j=1}^{\infty}(c_j \cos jt + d_j \sin jt)$, $c_0, c_j, d_j \in R^n$, $j \in N^*$.

Proposition 3 (see [30]). *For every $p \in [1,+\infty)$, H is compactly embedded into the Banach space $L^p(R/(2\pi Z); R^n)$.*

Let $x \in L^2(S^1, R^n)$. If for every $z \in C^\infty(S^1, R^n)$,

$$\int_0^{2\pi}\left(x(t), z'(t)\right)dt = -\int_0^{2\pi}(y(t), z(t))dt, \tag{14}$$

then y is called a weak derivative of x, denoted by $\dot{x}$.

Now, the variational function defined on H is

$$J(x) = \int_0^{2\pi}\left[\frac{1}{2}\left(\dot{x}\left(t+\frac{\pi}{2}\right), x(t)\right) + F(t, x(t))\right]dt. \tag{15}$$

Defined an operator by extending the bilinear forms

$$\langle Ax, y\rangle = \int_0^{2\pi}\left(\dot{x}\left(t+\frac{\pi}{2}\right), y(t)\right)dt, x, y \in H. \tag{16}$$

It is easily to check that A is a linear bounded operator on H.

Define

$$\varphi(x) = \int_0^{2\pi} F(t, x(t))dt, \forall x \in H. \tag{17}$$

Then, J can be rewritten as

$$J(x) = \frac{1}{2}\langle Ax, x\rangle + \varphi(x), \forall x \in H. \tag{18}$$

Now we state a useful lemma, which can be proved as Proposition B.37 in the book of Rabinowitz [30] and Appendix A3 in the book of Hofer and Zehnder [31].

Lemma 4. *Assume that (F1)-(F3) are satisfied; then, $\varphi \in C^2(H)$ and*

$$\left\langle \varphi'(x), y\right\rangle = \int_0^{2\pi}(f(t, x(t)), y(t))dt,$$

$$d^2b(x)[y, z] = \left\langle D^2\varphi(x)y, z\right\rangle = \int_0^{2\pi}\left(D^2F(t, x(t))y(t), z(t)\right)dt. \tag{19}$$

Moreover, the map $\varphi' : H \longrightarrow H$ is completely continuous.

Define an operator $\Gamma : H \longrightarrow H$

$$\Gamma x(t) = x\left(t + \frac{\pi}{2}\right), \text{for all } x \in H. \tag{20}$$

Clearly, Γ is a bounded linear operator and satisfies $\|\Gamma(x)\| = \|x\|$.

Next, we set

$$E = \left\{x \in H \mid \Gamma^2(x) = -x\right\}. \tag{21}$$

Then, E is a closed space of H. It is easily to check that A is a linear bounded self-adjoint operator on E. If $x \in E$, it has the following Fourier expansion:

$$x(t) = \frac{1}{\sqrt{\pi}}\sum_{j=1}^{\infty}\left[a_j \cos(2j-1)t + b_j \sin(2j-1)t\right]. \tag{22}$$

Lemma 5. *Assume that (F1)-(F4) are satisfied. Then, J is twice continuously differentiable. The existence of 2π-periodic solutions $x(t)$ for (1) is equivalent to the existence of critical points of functional J restricted to E. Moreover, if x is a critical point of J, the linear operator $D^2\varphi(x)$ is compact.*

Proof. Based on the definition of J and Lemma 4, J is twice continuously differentiable on E. □

By a standard argument as in [32] (cf. Proposition 4.1.3) and [21] (cf. Lemma 4), we can prove the second and third assertions.

Remark 6. Since $(F5)$ and $(F6)$ imply that of $(F4)$, Lemma 5 holds when F satisfies $(F1)$-$(F3)$, $(F5)$, and $(F6)$.

Now, we restrict our discussion on the space E. Then, A can be computed explicitly in terms of the Fourier expansion of x:

$$Ax = \frac{1}{\sqrt{\pi}}\sum_{j=1}^{\infty}\left[(-1)^j\left(a_j \cos(2j-1)t + b_j \sin(2j-1)\right)\right]. \tag{23}$$

Define the following family of subspaces of E.

$$E(j) = a_j \cos(2j-1)t + b_j \sin(2j-1)t. \tag{24}$$

Then, $Ax = x$ on $E(2j)$ and $Ax = -x$ on $E(2j-1)$ for all $j \in N^*$.

Set

$$E^- = \overset{\infty}{\underset{j=1}{\oplus}} E(2j-1), E^+ = \overset{\infty}{\underset{j=1}{\oplus}} E(2j). \tag{25}$$

Then, $E = E^- \oplus E^+$. Notice that the null space of A is $\{0\}$. Then, A is a self-adjoint invertible Fredholm operator.

3. E^+-Conley Index Theory

3.1. The E^+-Dimension and the E^+-Morse Index. Let E be a real Hilbert space. We denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the Hilbert norm and scalar product on E, respectively. Assume that E has an orthogonal splitting

$$E = E^- \oplus E^+, \tag{26}$$

where both E^+ and E^- are infinite dimensional.

Definition 7. Two closed subspaces V and W of E are called commensurable if the quotient projections $W \longrightarrow E/V$ and $V \longrightarrow E/W$ are compact.

Let P_V denote the orthogonal projection onto V and $V^\perp$ denote the orthogonal complement of V. Two closed subspaces V and W of E are commensurable if and only if both $P_{V^\perp}$ restricted to W and $P_{W^\perp}$ restricted to V are compact. Subsequently, both dim $(V \cap W^\perp)$ and dim $(V^\perp \cap W)$ are finite.

Definition 8. Let V be a closed subspace of E commensurable with E^-. The E^+-dimension of V is defined as

$$\begin{aligned} E^+ - \dim\ V &= \dim\ V \cap E^+ - co \dim \left(V + E^+\right) \\ &= \dim\ V \cap E^+ - \dim\ V^\perp \cap E^-. \end{aligned} \tag{27}$$

Let J be a twice differentiable functional on E and $d^2 J(x)$ be its second order differential at x. Denote by $D^2 J(x)$ the associated self-adjoint operator:

$$\left\langle D^2 J(x) y, z \right\rangle = d^2 J(x)[y, z]. \tag{28}$$

Definition 9. A critical point x of J is called nondegenerate if $D^2 J(x)$ is invertible. A critical point x of J is called weakly nondegenerate if $D^2 J(x)$ is a Fredholm operator.

If x is a weakly nondegenerate critical point of J, E splits into three closed subspaces.

$$E = V^+ \oplus V^0 \oplus V^-, \tag{29}$$

where V^+, V^0, and V^- are positive, null, and negative eigenspaces of $D^2 J(x)$, respectively. Obviously, V^0 is finite dimensional. And there exists $\alpha > 0, \beta > 0$ such that

$$\begin{aligned} D^2 J(x)[x, x] \geq \alpha \|x\|^2, \forall x \in V^+, \\ D^2 J(x)[x, x] \leq -\beta \|x\|^2, \forall x \in V^-. \end{aligned} \tag{30}$$

Definition 10. Let x be a weakly nondegenerate critical point of J. Assume moreover that the negative eigenspace V^- of $D^2 J(x)$ is commensurable with E^-. The E^+-Morse index is the finite relative integer

$$E^+ - m(x) = E^+ - m(x, J) = E^+ - \dim\ V^-. \tag{31}$$

The large E^+-Morse index is the finite relative integer

$$E^+ - m^*(x) = E^+ - m^*(x, J) = E^+ - \dim\ \left(V^- \oplus V^0\right). \tag{32}$$

Proposition 11 (see [29]). *Let L be a Fredholm operator whose negative eigenspace V^- is commensurable with E^-. Then,*

$$E^+ - \dim\ V^- = \max E^+ - \dim\ W, \tag{33}$$

where the maximum is taken over the family of all closed subspaces of W of E which are commensurable with E^- and such that L is strictly negative on W.

Lemma 12 (see [29]). *Denote by V_L^- the negative eigenspace of the self-adjoint operator L. Let $\mathscr{L}(E^-)$ be the set of invertible self-adjoint operators whose negative eigenspace is commensurable with E^-. Then, $\mathscr{L}(E^-)$ is relatively closed in the set of invertible operators. The function*

$$\mathscr{L}(E^-) \ni L \mapsto E^+ - \dim\ V_L^- \in Z \tag{34}$$

is continuous.

Lemma 13 (see [32]). *Assume that L is a self-adjoint Fredholm operator, and that K is a self-adjoint compact operator. Then, the negative eigenspaces of $L + K$ and of L are commensurable.*

3.2. The E^+-Cohomology. Let τ_{E^+} be the weakest topology on E such that all the bounded linear functions and the quotient projection $\pi : E \longrightarrow E/E^+$ are continuous. Equivalently, τ_{E^+} is the product topology between the weak topology of E^+ and the strong topology of E^-. If V^- is commensurable with E^-, then the topologies τ_{E^+} and τ_{V^+} coincide. Thus, the topology τ_{E^+} depends only on the commensurability class of E^+.

Definition 14. A subset X of E is E^+-locally compact if $X \cap \pi^{-1}(\alpha)$ is τ_{E^+}-locally compact, for every finite dimensional subspace α of E/E^+.

Obviously, every bounded τ_{E^+}-closed subset X of E is E^+-locally compact.

Definition 15. An E^+-pair (X, A) is a topological pair of subset of E such that X and A are τ_{E^+}-closed and E^+-locally compact.

Definition 16. An τ_{E^+}-continuous map $\Phi : (X, A) \mapsto (Y, B)$ is an E^+-morphism if

(1) It has the form

$$\Phi(x) = Tx + K(x), \tag{35}$$

where K maps bounded sets into τ_{E^+}-precompact sets and T is a linear automorphism of E such that $TE^+ = E^+$.

(2) $\Phi^{-1}(U)$ is bounded for every bounded U

Definition 17. An τ_{E^+}-continuous map $\Psi : ([0, 1] \times X, [0, 1] \times A) \mapsto (Y, B)$ is an E^+-homotopy if

(1) It has the form

$$\Psi(t, x) = T_t x + K(t, x), \tag{36}$$

where K maps bounded sets into τ_{E^+}-precompact sets and T_t is a linear automorphism of E such that $T_t E^+ = E^+$.

(2) $\Psi^{-1}(U)$ is bounded for every bounded U

Theorem 18 (see [29]). *There exists a generalised cohomology $H^*_{E^+}$, with coefficients in Z_2, which acts on the category of E^+-pairs and E^+-morphisms. More precisely,*

(1) *Controvariant Functoriality. If $I : (X, A) \mapsto (X, A)$ is the identity map, $H^*_{E^+}(I)$ is the identity homomorphism on $H^*_{E^+}(X, A)$. If $\Phi : (X, A) \mapsto (Y, B)$ and Φ' : $(Y, B) \mapsto (Z, C)$ are E^+-morphisms, then $H^*_{E^+}(\Phi' \circ \Phi) = H^*_{E^+}(\Phi) \circ H^*_{E^+}(\Phi')$.*

(2) *Homotopy Invariance. If two E^+-morphisms Φ, Φ' are E^+-homotopic, then $H^*_{E^+}(\Phi) = H^*_{E^+}(\Phi')$.*

(3) *Strong Excision. If X and Y are τ_{E^+}-closed E^+-locally compact subsets of E and $i : (X, X \cap Y) \rightarrowtail (X \cup Y, Y)$ is the inclusion map, then $H^*_{E^+}(i)$ is an isomorphism.*

(4) *Naturality of the Coboundary. Given three τ_{E^+}-closed and E^+-locally compact sets X, Y, Z such that $X \subset Y \subset Z$, there exists a family of coboundary homomorphisms*

$$\delta^q_{E^+}(Z, Y, Z) \colon H^q_{E^+}(Y, X) \longrightarrow H^{q+1}_{E^+}(Z, Y). \tag{37}$$

If $\Phi : (Z, Y) \mapsto (Z', Y')$ is a E^+-morphism such that $\Phi(X) \subset X' \subset Y'$, then

$$H^{q+1}_{E^+} \circ \delta^q_{E^+}\left(Z', Y', X'\right)() = \delta^q_{E^+}(Z, Y, X) \circ H^q_{E^+}\left(\Phi|_{(Y,X)}\right). \tag{38}$$

(5) *Long Exact Sequence. Given three τ_{E^+}-closed and E^+-locally compact sets X, Y, Z such that $X \subset Y \subset Z$, denote by $i(Y, X) \rightarrowtail (Z, X)$ and $j : (Z, X) \rightarrowtail (Z, Y)$ the inclusion maps. Then, the following sequence of homomorphisms is exact*

$$\cdots \longrightarrow H^q_{E^+}(Z, X) \overset{H^q_{E^+}(i)}{\longrightarrow} H^q_{E^+}(Y, X) \overset{\delta^q_{E^+}}{\longrightarrow} H^{q+1}_{E^+}(Z, Y) \overset{H^{q+1}_{E^+}(j)}{\longrightarrow} H^{q+1}_{E^+}(Z, X) \longrightarrow \cdots. \tag{39}$$

(6) *Dimension Property. Let V be a closed subspace of E, commensurable with E^-. Let B be a closed ball in V and ∂B be its relative boundary in V. Then,*

$$H^q_{E^+}(B, \partial B) = \begin{cases} Z_2, \text{ if } q = E^+ - \dim\ V, \\ 0, \text{ otherwise}. \end{cases} \tag{40}$$

Proposition 19 (see [29]). *Assume that $A \subset Y \subset X$ are τ_{E^+}-closed E^+-locally compact subsets of E.*

(1) *If there exists a E^+-homotopy*

$$\Psi : ([0, 1] \times X, [0, 1] \times A) \mapsto (X, A), \tag{41}$$

such that $\Phi_0 = id$, $\Phi_1(X) \subset Y$, and $\Psi_t(Y) \subset Y$ for every t, then

$$H^*_{E^+}(X, A) \cong H^*_{E^+}(Y, A). \tag{42}$$

(2) *If there exists a E^+-homotopy*

$$\Phi : ([0, 1] \times X, [0, 1] \times Y) \mapsto (X, Y), \tag{43}$$

such that $\Phi_0 = id$, $\Phi_1(Y) \subset A$, and $\Phi_t(A) \subset A$ for every t, then

$$H^*_{E^+}(X, Y) \cong H^*_{E^+}(X, A). \tag{44}$$

3.3. The E^+-Conley Theory. If J is a twice continuously differentiable real valued function on E. Let K be the critical set of J, i.e., the set $K = \{x \in E \mid dJ(x) = 0\}$. Denote by J^a the set $\{x \in E \mid J(x) \le a\}$.

Recall that the Palais-Smale sequence (denoted by (PS) sequence) is a sequence $\{x_n\}$ of elements of E such that $J(x_n)$ is bounded and $\|\nabla J(x_n)\|$ converges to zero.

We assume that

(A1) Each sublevel J^a is τ_{E^+}-closed and τ_{E^+}-locally compact

(A2) Every bounded (PS) sequence has a converging subsequence

(A3) ∇J is globally Lipschitz

(A4) The flow η defined by

$$\begin{cases} \dfrac{\partial}{\partial t}\eta(t, x) = -J'(\eta(t, x)), \\ \eta(0, x) = x, \end{cases} \tag{45}$$

is an E^+-homotopy.

Since ∇J is globally Lipschitz, the flow η exists globally. If $t \in R$ and $x \in E$, we will use both notations $\eta(t, x) = \eta_t(x)$. Besides, (A3) implies that J is bounded on bounded sets. Recall that an isolated critical set is an isolated subset of K.

Definition 20. Let K_0 be a compact isolated critical set. An E^+-index pair for K_0 is a topological pair (X, A) in E such that

(1) (X, A) is a bounded E^+-pair

(2) $(X, A) \cap K = K_0 \subset \mathrm{Int}(X)$ and $(K \setminus K_0) \cap X \subset \mathrm{Int}_X(A)$. Here $\mathrm{Int}(X)$ is the interior part of X with respect

to the topology of E, and $\mathrm{Int}_X(A)$ is the interior part of A with respect to the topology induced on X by the strong topology of E

(3) A is strongly positively invariant with respect to X; if $x \in A$ and $\eta(t, x) \in X$ for $t > 0$, then $\eta(t, x) \in A$

(4) A is an exit set for X: if $x \in X$ and $\eta([0, t] \times \{x\})$ is not contained in X, then there exists $t^* \in [0, t]$ such that $\eta(t^*, x) \in A$

Moreover, an E^+-index pair (X, A) for K_0 is called strict if $K \cap X = K_0$.

Definition 21. An elementary critical set is a compact isolated critical set K_0 such that J is constant on K_0.

Lemma 22 (see [29]). *Let K_0 be an elementary critical set and let U be a neighborhood of K_0 which is τ_{E^+}-closed. Then, there exists a strict E^+-index pair (X, A) for K_0 such that $X \subset U$.*

Lemma 23 (see [29]). *Let K_0 be a compact isolated critical set. Let (X, A) be an E^+-index pair for K_0 and (X_0, A_0) a strict E^+-index pair for K_0 such that $X_0 \subset X$. Then,*

$$H^*_{E^+}(X, A) \cong H^*_{E^+}(X_0, A_0). \tag{46}$$

Lemma 24 (see [29]). *Let K_0 be an elementary critical set. If (X, A) and (Y, B) are E^+-index pairs for K_0, then*

$$H^*_{E^+}(X, A) \cong H^*_{E^+}(Y, B). \tag{47}$$

We recall that the E^+-Poincaré polynomial of an E^+-pair (X, A) is defined as

$$P_{E^+}(X, A) = \sum_{q \in Z} \dim H^q_{E^+}(X, A)\lambda^q. \tag{48}$$

In general, $P_{E^+}(X, A)$ is a formal Laurent series, whose coefficients are nonnegative integers or $+\infty$.

Definition 25. Let K_0 be an elementary critical set of J. The E^+-critical group of K_0 is the vector space

$$c^q_{E^+}(K_0) = H^q_{E^+}(X, A), \tag{49}$$

where (X, A) is an E^+-index pair for K_0. The E^+-Morse polynomial of K_0 is the E^+-Poincaré polynomial of (X, A):

$$M_{E^+}(K_0) = \sum_{q \in Z} \dim c^q_{E^+}(K_0)\lambda^q. \tag{50}$$

Definition 26. Assume that $f \in C^2(E)$. An elementary critical set K_0 is called E^+-nondegenerate critical manifold if it is a finite dimensional compact C^1-manifold embedded in E and for every $x \in K_0$:

(1) x is a weak nondegenerate critical point, and the kernel of $D^2 f(x)$ coincides with the tangent space of K_0 in x

(2) The negative eigenspace of $D^2 f(x)$ is commensurable with E^-

Proposition 11 easily implies that if the E^+-nondegenerate critical manifold K_0 is connected, the E^+-Morse index $E^+ - m(x)$ is constant on K_0. In this case, we set

$$E^+ - m(K_0) = E^+ - m(K_0 ; J) = E^+ - m(x ; J), \text{ for } x \in K_0. \tag{51}$$

Proposition 27 (see [29]). *If K_0 is a connected E^+-nondegenerate critical manifold of a functional of class C^2, then*

$$M_{E^+}(K_0) = P(K_0)\lambda^{m(K_0)}, \tag{52}$$

where $P(K_0)$ is the standard Poincaré polynomial of K_0.

Now we state a result which follows immediately from Morse-Conley relations.

Theorem 28 (see [29]). *Let K be an isolated critical set of J, and let (X, A) be an E^+-index pair for K. Assume that $K = K_1 \cup \cdots \cup K_m$, where $K_1, \cdots, K_m$ are pairwise disjoint elementary critical sets. Then, there exists a Laurent series Q with positive or infinite coefficients such that*

$$\sum_{i=1}^{m} M_{E^+}(K_i) = P_{E^+}(X, A) + (1 + \lambda)Q(\lambda). \tag{53}$$

If K consists of E^+-nondegenerate critical manifolds, the above relation is an equality between true Laurent polynomials with finite coefficients.

In the case of a Morse function, we have the following immediate corollary.

Corollary 29 (see [32]). *Assume that $x_1, \cdots, x_k$ are nondegenerate critical points of J with finite E^+-Morse index. If (X, A) is an E^+-index pair for the critical set $\{x_1, \cdots, x_k\}$, there exists a Laurent polynomial Q with positive coefficients such that*

$$\sum_{i=1}^{k} \lambda^{E^+ - m(x_i, J)} = P_{E^+}(X, A) + (1 + \lambda)Q(\lambda). \tag{54}$$

Now we state a useful lemma.

Lemma 30 (see [29]). *Let $E = V^+ \oplus V^-$ be an orthogonal splitting such that V^+ is commensurable with E^+ (and therefore, V^- is commensurable with E^-). Let P^+ and P^- be the orthogonal projections onto V^+ and V^-, respectively. Assume that there exists $\bar{R}$ such that for every $R > \bar{R}$, the following* *inequalities hold:*

$$\frac{d}{dt}\|P^{-}\eta(t,x)\|^{2}\Big|_{t=0} > 0\forall x \in B_{V^{+}}(R)\times \partial B_{V^{-}}(R), \quad (55)$$

$$\frac{d}{dt}\|P^{+}\eta(t,x)\|^{2}\Big|_{t=0} < 0\forall x \in \partial B_{V^{+}}(R)\times B_{V^{-}}(R). \quad (56)$$

Then, the critical set K of J is compact, and

$$(X,A)=\left(B_{V^{+}}(\bar{R})\times B_{V^{-}}(\bar{R}), B_{V^{+}}(\bar{R})\times \partial B_{V^{-}}(\bar{R})\right) \quad (57)$$

is an E^{+}-index pair for K.

4. Systems with Nonresonant at Infinity

In this section, we consider system (8) satisfying $(F1)$-$(F4)$. If $z(t)$ is a 2π-periodic solutions of (8), we can linearise system (8) near z obtaining the 2π-periodic system

$$\dot{x}(t) = -D^{2}F(t,z(t))x\left(t-\frac{\pi}{2}\right), \quad (58)$$

where D^2F is the Hessian of F with respect to variable x.

Definition 31. A 2π-periodic solution z of system (8) is called 2π-resonant if the linearised system (58) has nontrivial 2π-periodic solutions. Otherwise, it is called 2π-nonresonant.

The corresponding functional is quadratic

$$J(x)=\frac{1}{2}\langle Ax,x\rangle+\frac{1}{2}\langle B_{z}x,x\rangle, \quad (59)$$

where B_z is the self-adjoint operator such that

$$\langle B_{z}x,y\rangle=\int_{0}^{2\pi}\left(D^{2}F(t,z(t))x(t),y(t)\right)dt. \quad (60)$$

Lemma 5 implies that B_z is a compact operator. Let W_z^{+}, W_z^{0}, and W_z^{-} be the positive, kernel, and negative eigenspaces of $A+B_z$.

The linearisation of (8) at infinity is

$$\dot{x}(t)=-B_{\infty}(t)x\left(t-\frac{\pi}{2}\right). \quad (61)$$

Definition 32. The linear system (61) is said 2π-resonant if it has nontrivial 2π-periodic solutions. Otherwise, it is said 2π-nonresonant. The asymptotically linear system (8) is said 2π-resonant at infinity if system (61) is 2π-resonant, 2π-nonresonant at infinity in the opposite case.

Let B_∞ be the self-adjoint operator on E defined by extending bilinear form

$$\langle B_{\infty}x,y\rangle=\int_{0}^{2\pi}(B_{\infty}(t)x(t),y(t))dt, \quad (62)$$

and then, B_∞ is a compact operator. Define

$$\phi(x)=\int_{0}^{2\pi}G(t,x(t))dt. \quad (63)$$

Therefore,

$$J(x)=\frac{1}{2}\langle (A+B_{\infty})x,x\rangle+\phi(x). \quad (64)$$

Let V_∞^{+}, V_∞^{0}, V_∞^{-} be the positive, kernel, and negative eigenspaces of $A+B_\infty$, respectively. Let P^{+}, P^{0}, P^{-} be the orthogonal projections onto V_∞^{+}, V_∞^{0}, V_∞^{-}, respectively.

Now, we want to check that (A1)-(A4) hold under assumptions (F1)-(F4), so that it is possible to apply the E^{+}-Conley theory to J.

Lemma 33 (see [32]). *Let V be a finite codimensional subspace of E^{+}. If $\sigma,\theta,\lambda^{+},\lambda^{-}$ are positive constants, the functional*

$$h(u)=\lambda^{+}\|P_{V}u\|^{\sigma}-\lambda^{-}\|P_{V^{\perp}}u\|^{\theta} \quad (65)$$

has E^{+}-locally compact sublevels.

Lemma 34. *Assume that F satisfies (F2) and (F3). Then, the sublevels of J are $\tau_{E^{+}}$-closed and also E^{+}-locally compact. So (A1) holds.*

Proof. Notice that the variational functional is

$$J(x)=\frac{1}{2}\langle Ax,x\rangle+\varphi(x). \quad (66)$$

Its quadratic part is lower semicontinuous and convex on E^{+} and thus weakly lower semicontinuous on E^{+}. Since it is strongly continuous on E^{-}, it is $\tau_{E^{+}}$-lower semicontinuous. By $(F3)$, F has quadratic growth. It follows that φ is continuous on L^{p}, if p is large enough. By Proposition 3, φ is weakly continuous on E, and therefore, it is also $\tau_{E^{+}}$-continuous. We conclude that J is $\tau_{E^{+}}$-lower semicontinuous.

By $(F3)$, $\varphi(x)\geq -C_{1}\|x\|_{L^{2}}^{2}-C_{2}$.Since E embeds compactly into L^{2}, for every $\varepsilon>0$, we can find a finite codimensional subspace V of E^{+} such that

$$\|x\|_{L^{2}}^{2}\leq \varepsilon\|x\|^{2},\forall x\in V. \quad (67)$$

Then,

$$J(x) = \frac{1}{2}\|P_{V^+}x\|^2 - \frac{1}{2}\|P_{V^-}x\|^2 + \varphi(x) \geq \frac{1}{2}\|P_{V^+}x\|^2 - \frac{1}{2}\|P_{V^-}x\|^2 - C_1\|x\|_{L^2}^2 - C_2 \geq \frac{1}{2}\|P_V x\|^2 + \frac{1}{2}\left\|P_{V^\perp}|_{V^+}x\right\|^2 - \frac{1}{2}\|P_{V^-}x\|^2 - C_1\left(\varepsilon\|P_V x\|^2 + \|P_{V^\perp}x\|^2\right) - C_2 \geq \left(\frac{1}{2} - C_1\varepsilon\right)\|P_V x\|^2 - \left(\frac{1}{2} + C_1\right)\|P_{V^\perp}x\|^2 - C_2. \tag{68}$$

Since V is finite codimensional in V^+, Lemma 33 implies that the right side of (68) has E^+-locally compact sublevels. Then, J also has E^+-locally compact sublevels. □

Lemma 35. *Assume that F satisfies (F2) and (F3). Then, all the bounded (PS) sequences are precompact. So (A2) holds.*

Proof. Assume that $\{x_n\} \subset E$ is a bounded (PS) sequence. The boundedness of $\{x_n\}$ yields that there exists a subsequence $\{x_{n_k}\}$ which converges weakly to some x_0.

According to Lemma 4, φ' is completely continuous. Then, there exists a subsequence $\{x_{n_k}\}$ such that the sequence $\varphi'(x_{n_k})$ converges strongly. Since $J'(x_{n_k}) = Ax_{n_k} + \varphi'(x_{n_k}) \longrightarrow 0$, then Ax_{n_k} converges strongly. Together with the fact that A is invertible implies that x_{n_k} must converge. Thus, (A2) is satisfied. □

Lemma 36. *Assume that F satisfies* $(F2)$ *and* $(F3)$*. Then,* ∇J *is globally Lipschitz, and (A3) holds.*

Proof. Since $\|J'(x) - J'(y)\| \leq \|Ax - Ay\| + \|\varphi'(x) - \varphi'(y)\|$, we need to only check that the second part in the right side of inequality is globally Lipschitz. By a directly computation, we have

$$\left\|\varphi'(x) - \varphi'(y)\right\| \leq \sup_{\|z\|=1}\int_0^{2\pi} |f(t, x(t)) - f(t, y(t)), z(t))| dt \leq \|f(t, x(t)) - f(t, y(t))\|_{L^2} \sup_{\|z\|=1}\|z(t)\|_{L^2} \leq M_1\|x - y\|, \tag{69}$$

where M_1 is a positive constant. The last inequality is guaranteed by $(F3)$ and Proposition 3. This finishes the proof of this lemma. □

Lemma 37. *Assume that F satisfies (F2) and (F3); then, the flow defined by (48) is an* E^+*-homotopy, and (A4) holds.*

Proof. By induction, we can define a sequence of flows

$$\begin{cases} \eta(0, x) = x, \\ \eta_n(t, x) = x - \int_0^t J'(\eta_{n-1}(s, x))ds, n \geq 1. \end{cases} \tag{70}$$

It is a standard fact in the theory of ordinary differential equations in Banach spaces that J' is locally Lipschitz, and η_n converges uniformly on the bounded subsets of $R \times E$ to the solution $\eta : R \times E \longrightarrow E$ of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}\eta(t, x) = -J'(\eta(t, x)), \\ \eta(0, x) = x. \end{cases} \tag{71}$$

Since J' maps bounded sets into bounded sets, so does η_n. Therefore, also η maps bounded sets into bounded sets.

Since J' is τ_{E^+}-continuous, also η_n is τ_{E^+}-continuous. Thus, both $P_{E^-} \circ \eta_n$ and $g_y^n(t, x) = \langle \eta_n(t, x), y \rangle, y \in E$ are τ_{E^+}-continuous. Moreover, $P_{E^-} \circ \eta_n$ and $\{g_y^n\}$ converge uniformly on bounded sets to $P_{E^-} \circ \eta$ and g_y, respectively. Therefore, $P_{E^-} \circ \eta$ and g_y are τ_{E^+}-continuous, for every $y \in E$. Then, η is τ_{E^+}-continuous.

Since η solves the following nonhomogeneous equation

$$\frac{\partial}{\partial t}\eta(t, x) + L\eta(t, x) = -\varphi'(\eta(t, x)), \tag{72}$$

it can be represented as

$$\eta(t, x) = e^{-tA}x - \int_0^t e^{(s-t)A}\varphi'(\eta(s, x))ds. \tag{73}$$

e^{-tA} is a continuous path of invertible operators which preserve the splitting $E = E^+ \oplus E^-$. Set

$$K(t, x) = -\int_0^t e^{(s-t)A}\varphi'(\eta(s, x))ds. \tag{74}$$

If $X \subset E$ is bounded and $T > 0$, $\eta(X \times [-T, T])$ is bounded, as we have shown before. Therefore, $\varphi'(\eta([-T, T] \times X))$ is τ_{E^+}-continuous, and we conclude that $K([-T, T] \times X)$ is τ_{E^+}-continuous.

Finally, since $\eta(t, \cdot)$ is a diffeomorphism whose inverse is $\eta(t, \cdot)$,

$$\left(\eta|_{[-T,T]\times E}\right)^{-1}(X) = \eta([-T, T] \times X) \tag{75}$$

must be bounded for every bounded X. □

Proposition 38. *Assume that F satisfies (F1)-(F3). If the system (8) is* 2π*-nonresonant at infinity, the critical set K is* *compact and*

$$\left(\overline{B_{V_\infty^+}}(R) \times \overline{B_{V_\infty^-}}(R), \overline{B_{V_\infty^+}}(R) \times \partial B_{V_\infty^-}(R)\right) \tag{76}$$

is an E^+-index pair for K, provided R is large enough.

Proof. Since the system is 2π-nonresonant at infinity, the operator $A + B_\infty$ is invertible and $V_\infty^0 = \{0\}$. Let $\gamma > 0$ be such that $A + B_\infty \geq \gamma I$ on V_∞^+ and $A + B_\infty \leq -\gamma I$ on V_∞^-.

Lemma 35 implies that all the bounded (PS) sequences are precompact: in order to prove that K is compact, it is enough to show that it is bounded.

Now we check conditions (55) and (56) of Lemma 30.

Let $x \in \overline{B_{V_\infty^+}}(R) \times \partial B_{V_\infty^-}(R))$. Then,

$$\begin{aligned} \frac{d}{dt}\|P_\infty^- \eta(t,x)\|^2\Big|_{t=0} &= -2 < J'(x), P_\infty^- x > \\ &= -2\langle (A + B_\infty)P_\infty^- x, P_\infty^- x\rangle \\ &\quad - 2\left\langle \phi'(x), P_\infty^- x\right\rangle \geq 2\gamma\|P_\infty^- x\|^2 \\ &\quad - 2\left\langle \phi'(x), P_\infty^- x\right\rangle. \end{aligned} \tag{77}$$

By (F3), there exists r such that

$$|g(t,x)| \leq \frac{\gamma}{4}|x| \text{ if } |x| \geq r. \tag{78}$$

Let $M_2 > 0$ be such that

$$|g(t,x)| \leq M_2 \text{ if } |x| \leq r. \tag{79}$$

Then,

$$\begin{aligned} \left|\left\langle \phi'(x), y\right\rangle\right| &\leq \int_{\{t \mid |x(t)| \geq r\}} |g(t,x)||y(t)|dt \\ &\quad + \int_{\{t \mid |x(t)| \leq r\}} |g(t,x)||y(t)|dt \leq \frac{\gamma}{4}\|x\|_{L^2}\|y\|_{L^2} \\ &\quad + M_2\|y\|_{L^1} \leq \frac{\gamma}{4}\|x\|\|y\| + M_2\|y\|. \end{aligned} \tag{80}$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\|P_\infty^- \eta(t,x)\|^2\Big|_{t=0} &\geq 2\gamma\|P_\infty^- x\|^2 - 2\left\langle \phi'(x), P_\infty^- x\right\rangle \\ &\geq 2\gamma\|P_\infty^- x\|^2 - \frac{\gamma}{2}\|x\|\|P_\infty^- x\| \\ &\quad - 2M_2\|P_\infty^- x\| \geq \frac{3\gamma}{2}\|P_\infty^- x\|^2 \\ &\quad - 2M_2\|P_\infty^- x\| - \frac{\gamma}{2}\|P_\infty^- x\|\|P_\infty^+ x\| \\ &\geq \gamma R^2 - 2M_2 R, \end{aligned} \tag{81}$$

which is positive when R is large enough. A similar discussion can prove (56). Then, we can use Lemma 30. □

Proof of Theorem 39. Notice that $(F3)$ and $(F4)$ imply that F has quadratic growth. Therefore, Lemmas 34–37 show that we can apply the E^+-Conley theory to the functional J.

By Proposition 38, if R is large enough, the pair

$$\left(\overline{B_{V_\infty^+}}(\bar{R}) \times \overline{B_{V_\infty^-}}(\bar{R}), \overline{B_{V_\infty^+}}(\bar{R}) \times \partial B_{V_\infty^-}(\bar{R})\right) \tag{82}$$

is an E^+-index pair for the critical set of J, which is compact. Then, Theorem 18 is

$$\sum_{i=1}^{k} \lambda^{E^+ - m(x_i;J)} = P_{E^+}(X, A) + (1+\lambda)Q(\lambda). \tag{83}$$

Then, both (48) and Definition 10 imply

$$\begin{aligned} \lambda^{E^+ - m(x_i;J)} &= \lambda^{E^+ - \dim\ W^-}, \\ P_{E^+}(X, A) &= \lambda^{E^+ - \dim\ V_\infty^-}. \end{aligned} \tag{84}$$

Theorem 1 follows from the above two equalities. □

5. Systems with Resonant at Infinity

In this section, we want to study the existence of periodic solutions of asymptotically linear systems which is resonant at infinity.

Since we consider resonant system in this section, the variational functional does not satisfy (PS) condition. So we first make a perturbation. We prove that the perturbable functional satisfies (PS) condition. Since critical points of first perturbable functional may not to be nondegenerate, we make a second perturbation and critical points of the secondly perturbable functional are nondegenerate. Finally, we prove the critical points of the secondly perturbable functional are the critical points of the original functional.

Let $\theta \in C^\infty(R)$ be a nondecreasing function such that $\theta(s) = 0$ for $s \leq 0$ and $\theta(s) = s$ for $s \geq 1$. For $R > \|x_0\|$, we define two new functionals on E, J_{R+} and J_{R-}, of class of C^2 as

$$J_{R\pm}(x) = J(x) \pm \theta\left(\left\|P_\infty^0 x\right\|^2 - R^2\right). \tag{85}$$

The gradients of these functionals are

$$J'_{R\pm}(x) = (A + B_\infty)x + \phi'(x) \pm 2\theta'\left(\left\|P_\infty^0 x\right\|^2 - R^2\right)P_\infty^0 x. \tag{86}$$

Critical points x of $J_{R\pm}$ such that $\|P_\infty^0 x\| \leq R$ are also critical points of J, and thus 2π-periodic solutions of system (8). Let us check that the perturbed functionals $J_{R\pm}$ satisfy the (PS) condition.

Lemma 40. *For every $R > 0$, every sequence $\{x_n\} \subset E$ such that $J'_{R\pm}(x_n) \longrightarrow 0$ contains a convergent subsequence.*

Proof. Let $x_n = x_n^+ + x_n^0 + +x_n^- \in V_\infty^+ \oplus V_\infty^0 \oplus V_\infty^-$. Since the proofs of two cases are the same, we only prove this lemma for J_{R+}. Multiplying both sides of (86) by x^+ and integrating over $[0, 2\pi]$, we have

$$\left\langle J_{R+}'(x_n), x_n^+ \right\rangle = \langle (A + B_\infty)x_n, x_n^+ \rangle + \left\langle \phi'(x_n), x_n^+ \right\rangle + \left\langle 2\theta'\left(\left\|P_\infty^0 x_n\right\|^2 - R^2\right)P_\infty^0 x_n, x_n^+ \right\rangle. \quad (87)$$

By $(F6)$ and Proposition 3, there exist two positive constants M and a_1 such that

$$|\phi'(x_n)| \le M_3, \|x\|_{L^1} \le a_1\|x\|, \forall x \in E. \quad (88)$$

According to the assumption, there are small enough $\varepsilon > 0$ such that

$$\varepsilon\|x_n^+\| \ge \gamma\|x_n^+\|^2 - M_3 a_1\|x_n^+\|, \quad (89)$$

which implies that the sequence $\{x_n^+\}$ is bounded. Arguing similarly, we can prove that the sequence $\{x_n^-\}$ is bounded.

Next, we prove the sequence $\{x_n^0\}$ is bounded. Since

$$\begin{aligned} o(1)\|x_n^0\| &= \left\langle J_{R+}'(x_n), x_n^0 \right\rangle = \left\langle (A + B_\infty)x_n, x_n^0 \right\rangle \\ &+ \left\langle \phi'(x_n), x_n^0 \right\rangle + \left\langle 2\theta'\left(\|P_\infty^0 x_n\|^2 - R^2\right)P_\infty^0 x_n, x_n^0 \right\rangle \\ &= \left\langle \phi'(x_n), x_n^0 \right\rangle + 2\theta'\left(\left\|P_\infty^0 x_n\right\|^2 - R^2\right)\left\|x_n^0\right\|^2. \end{aligned} \quad (90)$$

Suppose that $\|\theta\| \longrightarrow +\infty$ as $n \longrightarrow \infty$. For large n, we have $\theta'(\|P_\infty^0 x_n\|^2 - R^2) = 1$. It follows that

$$o(1)\left\|x_n^0\right\| = <\phi'(x_n), x_n^0> +2\left\|x_n^0\right\|^2. \quad (91)$$

Subsequently, we have

$$\begin{aligned} 2\left\|x_n^0\right\|^2 &= o(1)\left\|x_n^0\right\| - \left\langle \phi'(x_n), x_n^0 \right\rangle \le o(1)\left\|x_n^0\right\| \\ &+ \left|\left\langle \phi'(x_n), x_n^0 \right\rangle\right| \le o(1)\left\|x_n^0\right\| + M_3 a_1\left\|x_n^0\right\|, \end{aligned} \quad (92)$$

which contradicts with the fact that $\{x_n^0\}$ is unbounded. Thus, the sequence $\{x_n^0\}$ is bounded and so is $\{x_n\}$. Up to subsequence, it converges weakly to x_0. Standard arguments show that this convergence is strong. □

Now we can check that critical points of $J_{R+}'(x_n)$ have priori boundedness in V_∞^+ and V_∞^-.

Lemma 41. *There exists $N > 0$, independent of R, such that, for every critical point x of $J_{R\pm}'$, we have $\|P_\infty^\pm x\| \le N$.*

Proof. Let x be a critical point of J_{R+}'. Set $x = x^+ + x^0 + x^- \in V_\infty^+ \oplus V_\infty^0 \oplus V_\infty^-$. Multiplying both sides of (86) by x^+ and integrating over $[0, 2\pi]$, we have

$$\langle (A + B_\infty)x, x^+ \rangle + \left\langle \phi'(x), x^+ \right\rangle = 0. \quad (93)$$

So

$$\gamma\|x^+\|^2 \le |\langle (A + B_\infty)x, x^+ \rangle| = \left|\left\langle \phi'(x), x^+ \right\rangle\right| \le M_3 a_1\|x^+\|, \quad (94)$$

and thus, $\|x^+\|$ must be bounded. Similarly, we can prove the boundedness of $\|x^-\|$. □

Lemma 42. *Assume that (F1)-(F3), (F5), and (F6) hold. For every $N > 0$, there exists $Q > 0$, independent of R, such that the following property holds: for every critical point x of $J_{R\pm}'$, if $\|P_\infty^+ x\| \le N$, $\|P_\infty^- x\| \le N$, and $\|P_\infty^0 x\| \ge Q$, we have*

$$\begin{aligned} E^+ - \dim\ V_x^- &\ge E^+ - \dim\ V_\infty^-, \\ E^+ - \dim\ V_x^- \oplus V_x^0 &\le E^+ - \dim\ V_\infty^- \oplus V_\infty^0, \end{aligned} \quad (95)$$

where V_x^0 and V_x^- are the kernel and negative eigenspace of $D^2 J_{R\pm}(x)$.

Proof. We firstly prove the second inequality of this lemma. If $y \in V_\infty^+$, we have

$$\begin{aligned} J_{R+}''(x)[y, y] &= J''(x)[y, y] + 2\theta'\left(\left\|P_\infty^0 x\right\|^2 - R^2\right)\langle P_\infty^0 y, y \rangle \\ &+ 4\theta''\left(\left\|P_\infty^0 x\right\|^2 - R^2\right)\langle P_\infty^0 x, y \rangle^2 \\ &= J''(x)[y, y] = \langle (A - B_\infty)y, y \rangle \\ &- \int_0^{2\pi}\left(G''(t, x)y, y\right)dt \ge \gamma\|y\|^2 \\ &- \int_0^{2\pi}\left(G''(t, x)y, y\right)dt. \end{aligned} \quad (96)$$

We claim that $\int_0^{2\pi}(G''(t, x)y, y)dt < \gamma\|y\|^2$. By $(F5)$ and $(F6)$, there exist $M_4 > 0$, large enough $r > 0$ and small enough $\varepsilon > 0$ such that

$$|G''(t, x)| \le M_4, \forall (t, x) \in [0, 2\pi] \times \mathbb{R}^n, |G''(t, x)| \le \varepsilon, \forall |x| > r. \quad (97)$$

Set $\Gamma_1 = \{t \in [0, 2\pi] \mid |x(t)| \le r\}$. By Proposition 3, there exists $a_i > 0$ such that $\|y\|_{L^i}^2 \le a_i\|y\|^2$, where $\|\cdot\|_{L^i}$ denotes the L^i-norm and $i = 2, 4$. Computing directly, we obtain

$$\int_0^{2\pi}\left(G''(t, x)y, y\right)dt \le \varepsilon a_2\|y\|^2 - M_4 a_4\|y\|^2 \text{meas}(\Gamma_1)^{1/2}. \quad (98)$$

By assumptions $\|P_\infty^+ x + P_\infty^- x\| \le \sqrt{2}N$. Let $s > 0$ be such that for every $x \in V_\infty^+ \oplus V_\infty^-$ with ,$\|x\| \le \sqrt{2}N$, there holds

$$\text{meas}(\{t \in [0, 2\pi] \mid \mid x(t) \mid \ge s\}) \le \varepsilon^2. \quad (99)$$

Then,

$$\begin{aligned} \text{meas}(\Gamma_1) \le & \text{meas}(\{t \in [0, 2\pi] \mid \mid P_\infty^+ x(t) + P_\infty^- x(t) \mid \ge s\}) \\ & + \text{meas}\left(\left\{t \in [0, 2\pi] \mid \mid P_\infty^0 x(t) \mid \le r + s\right\}\right). \end{aligned} \quad (100)$$

The subspace V_∞^0 is finite dimensional. Subsequently, there exists $q > 0$ such that

$$|y(t)| > r + s, \forall y \in V_\infty^0 \text{ such that } \|y\|_\infty \ge q. \quad (101)$$

Since $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent on V_∞^0, there exists $l > 1$ such that

$$\frac{1}{l}\|y\|_\infty \le \|y\| \le l\|y\|_\infty, \forall y \in V_\infty^0. \quad (102)$$

Denote by $Q = lq$. If $\|P_\infty^0 x\| \ge Q$, then $\|P_\infty^0 x\|_\infty \ge q$. It follows that

$$\text{meas}(\Gamma_1)^{1/2} \le \varepsilon. \quad (103)$$

Substituting (103) into (98), we get

$$\int_0^{2\pi} \left(G''(t, x)y, y\right) dt \le \varepsilon(a_2 + M_4 a_4)\|y\|^2. \quad (104)$$

The arbitrary of ε induces that $\int_0^{2\pi}(G''(t, x)y, y)dt \le \gamma\|y\|^2$. Therefore, if x is a critical point of $J_{R+}'(x)$, then for all $y \in V_\infty^+ \setminus \{0\}$, we have

$$J_{R+}'{}'(x)[y, y] > 0. \quad (105)$$

Subsequently, $J_{R+}'{}'(x)$ is strictly positive on V_∞^+. Therefore, $E^+ - \dim\ V_x^- \oplus V_x^0 \le E^+ - \dim\ V_\infty^- \oplus V_\infty^0$.

Repeating the above argument, we conclude that $J_{R-}'{}'(x)$ is strictly negative on V_∞^-. Thus, $E^+ - \dim\ V_x^- \ge E^+ - \dim\ V_\infty^-$. □

Lemma 43. *The maps J_{R+}' and J_{R-}' are proper Fredholm maps of index 0.*

Proof. Notice that $J_{R\pm}'$ can be written as

$$J_{R\pm}'(x) = \left(A + B_\infty \pm 2P_\infty^0\right)x + \phi'(x) \pm 2\left[\theta'\left(\left\|P_\infty^0 x\right\|^2 - R^2\right) - 1\right]P_\infty^0 x. \quad (106)$$

The first term is an invertible linear operator. The last two terms are compact operators. Assuming that $\{J_{R\pm}'(x_n)\}$ converges, it is easy to prove that $\{x_n\}$ has a converging subsequence. Thus, $J_{R\pm}'$ are proper operator. Moreover, the differentials of the last two terms are compact self-adjoint operators. Therefore, $D^2 J_{R\pm}$ is a Fredholm operator of index 0, for every $u \in E$. □

Consider the constant Q, fixed in Lemma 42. We assume, by contradiction, that the functionals $J_{Q\pm}'$ have no critical points x such that $\|P_\infty^0 x\| \le Q$, apart from x_0.

According to Lemma 43, the critical set of $J_{Q\pm}$ is compact. Since $J_{Q\pm}$ has no critical points x such that $\|P_\infty^0 x\| = Q$, there exists $\varepsilon > 0$ such that there are no critical points x such that

$$Q \le \left\|P_\infty^0 x\right\| \le Q + \varepsilon. \quad (107)$$

As we can see, the perturbable functionals satisfy (PS) condition. However, the critical points of $J_{R\pm}(x)$ maybe not to be a nondegenerate set. Thus, we need a second perturbation to grantee the critical points of perturbable functional are nondegenerate.

Let $\tilde{\omega} \in C^\infty(R)$ be a nondecreasing function such that $\tilde{\omega}(s) = 0$ for $s \le Q^2$ and $\tilde{\omega}(s) = 1$ for $s \ge (Q + \varepsilon)^2$. Set $\omega(x) = \tilde{\omega}(\|P_\infty^0 x\|^2)$ for all $x \in E$. ω is a smooth functional on E. $\omega'(x)$ always belongs to V_∞^0, and there exists a constant D such that $\|\omega'(x)\| \le D$ for all $x \in E$. For $z \in E$, we define two new functional $h_{z\pm}$ on E as

$$h_{z\pm}(x) = J_{Q\pm}(x) - \omega(x) < x, z > . \quad (108)$$

The gradients of these functional are

$$h_{z\pm}'(x) = J_{Q\pm}'(x) - \omega'(x) < x, z > -\omega(x)z. \quad (109)$$

Lemma 44. *If $\|z\|$ is small enough, for every critical point x of $h_{z\pm}$, there holds*

$$\|P_\infty^\pm x\| \le N = \frac{2M_3}{\gamma}. \quad (110)$$

Proof. From (86) and (109), we get

$$\begin{aligned} 0 = dh_{z\pm}(x)[P_\infty^+ x] = & \langle (A + B_\infty)P_\infty^+ x, P_\infty^+ x\rangle + d\phi(x)[P_\infty^+ x] \\ & - \omega(x)\langle z, P_\infty^+ x\rangle \ge \gamma\|P_\infty^+ x\|^2 - M_3\|P_\infty^+ x\| - \|z\|\|P_\infty^+ x\|. \end{aligned} \quad (111)$$

Therefore,

$$\|P_\infty^+ x\| \le \frac{M_3 + \|z\|}{\gamma}, \quad (112)$$

and it is enough to assume $\|z\| \le M_3$. By a similar argument, we estimate the bound for $\|P_\infty^- x\|$. □

Lemma 45. *If $\|z\|$ is small enough, there are no critical points x of $h_{z\pm}$ such that*

$$Q \le \|P_\infty^0 x\| \le Q + \varepsilon. \quad (113)$$

Proof. By Lemma 44, we can assume that $\|x\|$ is small enough that

$$\|P_{\infty}^{\pm}x\| \le N, \tag{114}$$

for every critical point x of h_{z+}.

Since J_{Q+} has no critical point x with $Q \le \|P_{\infty}^{0}x\| \le Q+\varepsilon$, by Lemma 43, there exists $m>0$ such that

$$\|J_{Q+}'(x)\| \ge m \text{ if } Q \le \|P_{\infty}^{0}x\| \le Q+\varepsilon. \tag{115}$$

Therefore, if x is a critical point of h_{z+} with $Q \le \|P_{\infty}^{0}x\| \le Q+\varepsilon$, we have

$$\begin{aligned} 0 = \left\|h_{z\pm}'(x)\right\| &= \left\|J_{Q+}'(x) - \omega'(x)\langle x, z\rangle - \omega(x)z\right\| \\ &\ge \left\|J_{Q+}'(x)\right\| - \|z\|\left(\left\|\omega'(x)\right\|\|x\| + \|\omega(x)\|\right) \\ &\ge m - \|z\|[D(2N+Q+\varepsilon)+1]. \end{aligned} \tag{116}$$

This implies that

$$\|z\| \ge \frac{m}{D(2N+Q+\varepsilon)+1}, \tag{117}$$

which is contradiction, provided $\|z\|$ is small enough. □

Now we can work with second perturbation functionals. Firstly, we check some proposition of those functionals.

Lemma 46. *Every bounded (PS) sequence for $h_{z\pm}$ admits a converging subsequence; thus, (A2) holds.*

Proof. Let $\{x_n\}$ be a bounded (PS) sequence for $H_{z\pm}$. Up to its subsequence, it converges weakly. Since ω is weakly continuous and ω' is completely continuous, by (109), $\{J_{Q+}'\}$ converges strongly. By Lemma 43, $\{x_n\}$ converges strongly. □

Lemma 47. *The functionals $h_{z\pm}$ satisfy the assumption (A1), (A3), and (A4).*

Proof. Since

$$h_{z\pm}(x) = J(x) \pm \theta\left(\|P_{\infty}^{0}x\|^2 - R^2\right) \pm \omega(x)\langle x, y\rangle, \tag{118}$$

Lemma 34 implies that J is τ_{E^+}-lower semicontinuous. The last two terms are weakly continuous and thus also τ_{E^+}-continuous. So the sublevels of $h_{z\pm}$ are τ_{E^+}-closed. Arguing similarly as in the proof of Lemma 34, the sublevels of $h_{z\pm}$ are E^+-locally compact and (A1) holds.

Computing directly, we obtain

$$h_{z\pm}'(x) = J'(x) \pm 2\theta'\left(\|P_{\infty}^{0}x\|^2 - Q^2\right)P_{\infty}^{0}x \pm \omega'(x)\langle x, z\rangle \pm. \tag{119}$$

According to Lemma 36, J' is globally Lipschitz. Since θ and ω are both C^{∞} functions, the second and third items are globally Lipschitz. We need only the last item is globally Lipschitz. But,

$$\|\omega(x)z - \omega(\bar{x})z\| \le D\|z\|\|x-\bar{x}\|, \tag{120}$$

which yields that ω is globally Lipschitz when z is fixed. Thus, (A3) holds.

Since $h_{z\pm}$ can be seen as the sum of the quadratic form $1/2\langle (A-B_{\infty})x, x\rangle$ and of a function which is continuous from the weak to the strong topology of E, then (A4) follows from Proposition 15.1 in [28]. □

Now we are in a position to use E^+-Conley index theory. We claim that the critical points of $h_{z\pm}$ are compact and prove neighborhoods of zero are their E^+-Conley index pairs.

Lemma 48. *The critical set K^+ of h_{z+} is compact, and*

$$\left(B_{V_{\infty}^{+}\oplus V_{\infty}^{0}}(R) \times B_{V_{\infty}^{-}}(R), B_{V_{\infty}^{+}\oplus V_{\infty}^{0}}(R) \times \partial B_{V_{\infty}^{-}}(R)\right) \tag{121}$$

is an E^+-index pair for K^+, with respect to the functional h_{z+}, provided R is large enough and $\|y\|$ is small enough. The same conclusion also works where the E^+-index pair is replaced by

$$\left(B_{V_{\infty}^{+}}(R) \times B_{V_{\infty}^{-}\oplus V_{\infty}^{0}}(R), B_{V_{\infty}^{+}}(R) \times \partial B_{V_{\infty}^{-}\oplus V_{\infty}^{0}}(R)\right). \tag{122}$$

Proof. We just prove the first claim, since the second one can be proved with analogous arguments.

Let η^+ be the flow of the vector field $-h_{z+}'$. We use Lemma 30 to prove this lemma. In order to use Lemma 30, we should check conditions (55) and (56).

Let $x \in B_{V_{\infty}^{+}\oplus V_{\infty}^{0}}(R) \times \partial B_{V_{\infty}^{-}}(R)$. Then,

$$\begin{aligned} \left.\frac{d}{dt}\|P_{\infty}^{-}\eta^{+}(t,x)\|^2\right|_{t=0} &= -2\left\langle h_{z+}'(x), P_{\infty}^{-}x\right\rangle \\ &= -2\langle (A+B_{\infty})P_{\infty}^{-}x, P_{\infty}^{-}x\rangle \\ &\quad - 2\left\langle \phi'(x), P_{\infty}^{-}x\right\rangle - \omega(x)\langle y, P_{\infty}^{-}x\rangle \\ &\ge 2\gamma\|P_{\infty}^{-}x\|^2 - (2C+\|y\|)\|P_{\infty}^{-}x\| \\ &= R(2\gamma R - 2C - \|y\|). \end{aligned} \tag{123}$$

Then, (55) holds when $R > (2C - \|y\|)/2\gamma$.

Now let $x \in \partial B_{V_\infty^+ \oplus V_\infty^0}(R) \times B_{V_\infty^-}(R)$. Then,

$$\begin{aligned}\frac{d}{dt}\|(P_\infty^+ + P_\infty^0)\eta^+(t,x)\|^2\Big|_{t=0} &= -2\left\langle h_{z+}'(x), (P_\infty^+ + P_\infty^0)x\right\rangle \\ &= -2\langle (A+B_\infty)P_\infty^+ x, P_\infty^+ x\rangle - 2\left\langle \phi'(x), (P_\infty^+ + P_\infty^0)x\right\rangle \\ &\quad - 4\theta'\left(\|P_\infty^0 x\|^2 - Q^2\right)\|P_\infty^0 x\|^2 - \omega(x)\left\langle z, (P_\infty^+ + P_\infty^0)x\right\rangle \\ &\quad - \left\langle \omega'(x), (P_\infty^+ + P_\infty^0)x\right\rangle\langle x, z\rangle \le -2\gamma\|P_\infty^+ x\|^2 \\ &\quad - 4\theta'\left(\|P_\infty^0 x\|^2 - Q^2\right)\|P_\infty^0 x\|^2 + (2C + \|z\|)\|(P_\infty^+ + P_\infty^0)x\| \\ &\quad + D\|z\|\|x\|\|P_\infty^0 x\|.\end{aligned} \tag{124}$$

Since $\|x\| \le 2R$ and $\|(P_\infty^+ + P_\infty^0)x\| = R$, we have that

$$\|P_\infty^+ x\|^2 \ge \frac{1}{2}R^2 \text{ or} \|P_\infty^0 x\|^2 \ge \frac{1}{2}R^2. \tag{125}$$

In the first case, noticing that $\|y\|$ is small enough, we have

$$\frac{d}{dt}\left\|\left(P_\infty^+ + P_\infty^0\right)\eta^+(t,x)\right\|^2\Big|_{t=0} \le -\gamma R^2 + 2CR = -R(\gamma R - 2C), \tag{126}$$

which is negative for $R > 2C/\gamma$. In the second case, we can assume that $\|P_\infty^0 x\|^2 \ge 1/2R^2 > Q^2 + 1 > Q^2$. So $\theta'(\|P_\infty^0 x\|^2 - Q^2) = 1$. We have

$$\begin{aligned}\frac{d}{dt}\left\|\left(P_\infty^+ + P_\infty^0\right)\eta^+(t,x)\right\|^2\Big|_{t=0} &\le -2R^2 + (2C + \|z\|)R \\ &\quad + \sqrt{2}D\|z\|R^2 = -R\left[\left(2 - \sqrt{2}D\|z\|\right)R - 2C - \|z\|\right],\end{aligned} \tag{127}$$

which is negative if

$$\begin{aligned}\|z\| &< \frac{2}{\sqrt{2}D}, \\ R &> \frac{2C + \|z\|}{2 - \sqrt{2}D\|z\|}.\end{aligned} \tag{128}$$

Then, (56) holds. Applying Lemma 30, we prove this lemma. □

Proof of Theorem 49. We claim that we can choose z_0 and z_1, such that $\|z_0\|$ and $\|z_1\|$ are so small that these of Lemmas 44, 45, and 48 hold and such that all critical points of $h_{z\pm}$ are nondegenerate.

Since we assume the system is 2π-nonresonant periodic solution, the critical point x_0 is nondegenerate. Lemma 45 implies all the other critical points x such that $\|P_\infty^0 x\| > Q + \varepsilon$. If y is in a neighborhood of such a critical point x, then

$$h_{z\pm}'(y) = J_{Q\pm}'(y) - z. \tag{129}$$

Therefore,

$$D^2 h_{z\pm}(y) = D^2 J_{Q\pm}(y). \tag{130}$$

$J_{Q\pm}'$ is a continuously differentiable Fredholm map of index 0. By an infinite dimensional Sard-Smale theorem, the set of its critical values has first category. Therefore, we can choose z_0 and z_1, such that $\|z_0\|$ and $\|z_1\|$ are so small that the theses of Lemmas 44, 45, and 48 hold and such that y^+ and y^- are regular values for J_{Q+}' and J_{Q-}', respectively.

If x is a critical point of h_{z_0+} different from x_0, then

$$J_{Q\pm}'(y) = z_0, \tag{131}$$

and the linear map $D^2 h_{z_0+}(x) = D^2 J_{Q+}(x)$ is invertible. The same result holds for h_{z_1-}.

Lemmas 46 and 47 imply that we can use E^+-Conley index theory and get the following Morse relations:

$$\lambda^{E^+ - m\left(x_0, h_{z_0+}\right)} + \sum_\mu \lambda^{E^+ - m\left(x, h_{z_0+}\right)} = \lambda^{E^+ - \dim\ V_\infty^-} + (1+\lambda)Q(\lambda), \tag{132}$$

where the sum is taken over the finite set of all critical points x of h_{z_0+} such that $\|P_\infty^0 x\| > Q$. If $E^+ - \dim\ W_{x_0} < E^+ - \dim\ V_\infty^- - 1$, there is existence of a critical point x' such that

$$|E^+ - m\left(x', h_{z_0+}\right) - E^+ - m\left(x_0, h_{z_0+}\right)| = 1. \tag{133}$$

Therefore,

$$E^+ - m\left(x', h_{z_0+}\right) \le E^+ - m\left(x_0, h_{z_0+}\right) + 1 < E^+ - \dim\ V_\infty^-. \tag{134}$$

But by Lemmas 42 and 44, every critical point x has E^+-Morse index $E^+ - m(x, h_{z_0+}) \ge E^+ - \dim\ V_\infty^-$ which is a contradiction.

If $E^+ - m(x, h_{z_1-}) > E^+ - \dim\ V_\infty^- + E^+ - \dim\ V_\infty^0 + 1$, arguing similarly as the previews case can result a contradiction which finishes our proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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