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Research Article

A Numerical Method for the Variable-Order Time-Fractional Wave Equations Based on the H2N2 Approximation

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Aiming at the initial boundary value problem of variable-order time-fractional wave equations in one-dimensional space, a numerical method using second-order central difference in space and H2N2 approximation in time is proposed. A finite difference scheme with second-order accuracy in space and $3-\gamma^*$ order accuracy in time is obtained. The stability and convergence of the scheme are further discussed by using the discrete energy analysis method. A numerical example shows the effectiveness of the results.

1. Introduction

In recent years, due to the non-locality of fractional calculus, more and more problems in physical science, electromagnetism, electrochemistry, diffusion and general transport theory can be described by the fractional calculus approach, among which the Riemann-Liouville fractional derivative and the Caputo fractional derivative are the most widely used [1–4]. At the same time, more and more researchers found that a variety of important dynamical problems exhibit fractional-order behavior that may vary with time, space, or other conditions. This phenomenon indicates that variable-order fractional calculus is a natural choice to provide an effective mathematical framework for the description of complex problems.

In 2020, Shen et al. proposed a new numerical approximation method—the H2N2 approximation [5] for the numerical differential formula of the Caputo fractional derivative of $\gamma \in (1,2)$ and applied it for the constant-order time-fractional wave equations in the following multidimensional space

$$\begin{cases} {}_0^C D_t^{\gamma} u(x,t) = \Delta u + q(x,t),\ x \in \Omega, t \in (0,T], \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega, t \in [0,T], \end{cases} \tag{1}$$

where $q(x,t), \varphi(x), \psi(x)$ are given sufficiently smooth functions, $\Omega = \prod_{j=1}^{d} (l^{(j)}, r^{(j)}) \subset R^d$, $\partial\Omega$ is the boundary of Ω, $x = (x^{(1)}, x^{(2)}, \cdots, x^{(d)}) \in \Omega$, $\Delta u = \sum_{j=1}^{d} \partial^2_{x^{(j)}} u$. When $x \in \partial\Omega$, $\varphi(x)$ and $\psi(x)$ satisfy consistency conditions $\varphi(x) = \psi(x) = 0$. It was proved that the proposed scheme has the accuracy of order of $(3-\gamma)$ in time and 2 in space, and it is clear that its theoretical analysis is similar to the L1 method applied in solving the constant-order time-fractional slow diffusion equations.

Motivated by the above literature [6–9], in this work, we consider the numerical solution of the following variable-order time-fractional wave equations in one-dimensional space

$${}_0^C D_t^{\gamma(t)} u(x,t) = u_{xx}(x,t) + f(x,t), x \in (0,L), t \in (0,T]. \tag{2}$$

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x), x \in (0,L). \tag{3}$$

$$u(0,t) = 0, u(L,t) = 0, t \in [0,T]. \tag{4}$$

where $1 < \gamma(t) < 2$, ${}_0^C D_t^{\gamma(t)} u(x,t)$ is the variable-order Caputo fractional derivative, $f(x,t), \varphi(x), \psi(x)$ are given sufficiently smooth functions and satisfy $\varphi(0) = \psi(0), \varphi(L) = \psi(L)$. Suppose its solution function $u \in C^{(4,3)}([0,L] \times [0,T])$.

The rest of this paper is organized as follows. In the next section, some necessary notations are introduced. In Section 3, the H2N2-based finite difference scheme for the variable-order time-fractional wave equations is derived. In Section 4, the stability and convergence of the difference scheme are studied. In Section 5, a numerical result is listed to verify the theoretical prediction and the effectiveness of the difference scheme. Finally, a brief conclusion is provided.

2. Preliminary Knowledge and Relevant Lemmas

Definition 1 (see [10]). Suppose the function $f(t)$ is defined on the interval $[0, T]$, $1 < \gamma(t) < 2$, then the variable-order Caputo fractional derivative is defined as

$$ {}_0^C D_t^{\gamma(t)} f(t) = \frac{1}{\Gamma(2-\gamma(t))} \int_0^t f''(s)(t-s)^{1-\gamma(t)} ds. \quad (5) $$

Next, mesh the solution intervals $[0, L]$ and $[0, T]$, take integers M and N, denote $h = L/M$, $\tau = T/N$, h and τ are called space step and time step, respectively. Denote $x_i = ih (0 \le i \le M)$, $t_k = k\tau (0 \le k \le N)$, $\Omega_h = \{x_i | 0 \le i \le M\}$, $\Omega_\tau = \{t_k | 0 \le k \le N\}$. Define the following grid function spaces

$$ \begin{aligned} U_h &= \{u \mid u = (u_0, u_1, \cdots, u_M)\}, \\ \widehat{U}_h &= \{u | u \in U_h, u_0 = u_M = 0\}. \end{aligned} \quad (6) $$

For grid function $u = \{u_i^k \mid 0 \le i \le M, 0 \le k \le N\}$ defined on $\Omega_h \times \Omega_\tau$, introduce the following notations

$$ \begin{aligned} \delta_x u_{i-1/2}^k &= \frac{1}{h}\left(u_i^k - u_{i-1}^k\right), \\ \delta_x^2 u_i^k &= \frac{1}{h^2}\left(u_{i+1}^k - 2u_i^k + u_{i-1}^k\right), \\ \delta_t u_i^{k+1/2} &= \frac{1}{\tau}\left(u_i^{k+1} - u_i^{k-1}\right), \\ \delta_t^2 u_i^k &= \frac{1}{\tau}\left(\delta_t u_i^{k+1/2} - \delta_t u_i^{k-1/2}\right). \end{aligned} \quad (7) $$

For any grid functions $u, v \in \widehat{U}_h$, denote the following notations

$$ \begin{aligned} (u, v) &= h \sum_{i=1}^{M-1} u_i v_i, \quad \|u\| = \sqrt{(u, u)}, \\ (\delta_x u, \delta_x v) &= h \sum_{i=0}^{M-1} (\delta_x u_{i+1/2})(\delta_x v_{i+1/2}), \\ \|u\|_\infty &= \max_{0 \le i \le M} |u_i|, \quad \|\delta_x u\| = \sqrt{(\delta_x u, \delta_x u)}. \end{aligned} \quad (8) $$

For any function $f(t)$ defined on the interval $[0, t_1]$, using the data $(t_0, f(t_0)), (t_1, f(t_1)), (t_0, f'(t_0))$ to make the quadratic Hermite interpolation polynomial of $f(t)$

$$ H_{2,0}(t) = f(t_0) + f'(t_0)(t-t_0) + \frac{1}{\tau}\left(\delta_t f^{1/2} - f'(t_0)\right)(t-t_0)^2. \quad (9) $$

Taking the twice derivative arrives at

$$ H_{2,0}''(t) = \frac{2}{\tau}\left(\delta_t f^{1/2} - f'(t_0)\right). \quad (10) $$

For any function $f(t)$ defined on the interval $[t_{k-1}, t_{k+1}]$ $(1 \le k \le N-1)$, using three points $(t_{k-1}, f(t_{k-1}))$, $(t_k, f(t_k))$, $(t_{k+1}, f(t_{k+1}))$ to make the quadratic Newton interpolation polynomial of $f(t)$

$$ \begin{aligned} N_{2,k}(t) = f(t_{k-1}) &+ \left(\delta_t f^{k-1/2}\right)(t - t_{k-1}) \\ &+ \frac{1}{2}\left(\delta_t^2 f^k\right)(t-t_{k-1})(t-t_k). \end{aligned} \quad (11) $$

Taking the second-order derivative yields

$$ N_{2,k}''(t) = \delta_t^2 f^k. \quad (12) $$

On the basis of the above interpolation polynomial, we next discuss the high-precision approximation formula of the variable-order Caputo fractional derivative.

Here, we denote $f^l = f(t_l)$, $\gamma_{n-1/2} = \gamma(t_{n-1/2})$, $t_{n-1/2} = t_n - \tau/2$. Suppose $f(t) \in C^3[t_0, t_n]$ and $1 < \gamma(t) < 2$, then at the half-grid point $t_{n-1/2}$, we have

$$ \begin{aligned} {}_0^C D_t^{\gamma_{n-1/2}} f(t_{n-1/2}) &= \frac{1}{\Gamma(2-\gamma_{n-1/2})}\left[\int_{t_0}^{t_{1/2}} f''(t)(t_{n-1/2}-t)^{1-\gamma_{n-1/2}} dt \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \int_{t_{k-1/2}}^{t_{k+1/2}} f''(t)(t_{n-1/2}-t)^{1-\gamma_{n-1/2}} dt \right] \\ &\approx \frac{1}{\Gamma(2-\gamma_{n-1/2})}\left[\int_{t_0}^{t_{1/2}} H_{2,0}''(t)(t_{n-1/2}-t)^{1-\gamma_{n-1/2}} dt \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \int_{t_{k-1/2}}^{t_{k+1/2}} N_{2,k}''(t)(t_{n-1/2}-t)^{1-\gamma_{n-1/2}} dt \right] = \frac{1}{\Gamma(2-\gamma_{n-1/2})} \\ &\quad \cdot \left[\int_{t_0}^{t_{1/2}} \frac{2}{\tau}\left(\delta_t f^{1/2} - f'(t_0)\right)(t_{n-1/2}-t)^{1-\gamma_{n-1/2}} dt \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \int_{t_{k-1/2}}^{t_{k+1/2}} \left(\delta_t^2 f^k\right)(t_{n-1/2}-t)^{1-\gamma_{n-1/2}} dt \right] = \frac{1}{\Gamma(2-\gamma_{n-1/2})} \\ &\quad \cdot \left[\frac{2}{\tau}\int_{t_0}^{t_{1/2}} (t_{n-1/2}-t)^{1-\gamma_{n-1/2}} dt \cdot \left(\delta_t f^{1/2} - f'(t_0)\right) \right. \\ &\quad \left. + \frac{1}{\tau}\sum_{k=1}^{n-1} \int_{t_{k-1/2}}^{t_{k+1/2}} (t_{n-1/2}-t)^{1-\gamma_{n-1/2}} dt \cdot \left(\delta_t f^{k+1/2} - \delta_t f^{k-1/2}\right)\right] \\ &= \frac{1}{\Gamma(2-\gamma_{n-1/2})}\left[b_{n-1}^{(n,\gamma_{n-1/2})}\left(\delta_t f^{1/2} - f'(t_0)\right) + \sum_{k=1}^{n-1} b_{n-k-1}^{(n,\gamma_{n-1/2})} \right. \\ &\quad \left. \cdot \left(\delta_t f^{k+1/2} - \delta_t f^{k-1/2}\right)\right] = \frac{1}{\Gamma(2-\gamma_{n-1/2})} \\ &\quad \cdot \left[b_0^{(n,\gamma_{n-1/2})}\delta_t f^{n-1/2} - \sum_{k=1}^{n-1}\left(b_{n-k-1}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n,\gamma_{n-1/2})}\right)\delta_t f^{k-1/2} \right. \\ &\quad \left. - b_{n-1}^{(n,\gamma_{n-1/2})} f'(t_0)\right] \equiv D_t^{\gamma_{n-1/2}} f(t_{n-1/2}). \end{aligned} \quad (13) $$

Here

$$b_{n-1}^{(n,\gamma_{n-1/2})} = \frac{2}{\tau}\int_{t_0}^{t_{1/2}} (t_{n-1/2}-t)^{1-\gamma_{n-1/2}} dt, \quad (14)$$

$$b_{n-k-1}^{(n,\gamma_{n-1/2})} = \frac{1}{\tau}\int_{t_{k-1/2}}^{t_{k+1/2}} (t_{n-1/2}-t)^{1-\gamma_{n-1/2}} dt, \quad (15)$$

where $1 \le k \le n-1$.

Then, it can be calculated that

$$b_k^{(n,\gamma_{n-1/2})} = \begin{cases} \dfrac{\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}}\left[(k+1)^{2-\gamma_{n-1/2}} - k^{2-\gamma_{n-1/2}}\right], & 0 \le k \le n-2, \\ 2\dfrac{\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}}\left[\left(n-\dfrac{1}{2}\right)^{2-\gamma_{n-1/2}} - (n-1)^{2-\gamma_{n-1/2}}\right], & k = n-1. \end{cases} \quad (16)$$

Denote

$$r_n = {}_0^C D_t^{\gamma_{n-1/2}} f(t_{n-1/2}) - D_t^{\gamma_{n-1/2}} f(t_{n-1/2}), \quad (17)$$

we have

$$|r_n| \le C_0 \max_{t_0 \le t \le t_n} |f'''(t)| \tau^{3-\gamma_{n-1/2}}. \quad (18)$$

Here,$C_0 = 1/8\Gamma(2-\gamma_{n-1/2}) + 1/12\Gamma(3-\gamma_{n-1/2}) + (\gamma_{n-1/2} - 1)/2\Gamma(4-\gamma_{n-1/2})$, the proof process is similar to Theorem 2.1 in Reference [5].

Lemma 2. *For any $n \ge 2$, according to $b_k^{(n,\gamma_{n-1/2})}$ defined by (14)–(15), we have*

$$\frac{\tau^{1-\gamma_{n-1/2}}}{(n-1/2)^{\gamma_{n-1/2}-1}} < b_{n-1}^{(n,\gamma_{n-1/2})} < b_{n-2}^{(n,\gamma_{n-1/2})} < \cdots < b_1^{(n,\gamma_{n-1/2})} < b_0^{(n,\gamma_{n-1/2})} = \frac{\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}}. \quad (19)$$

Proof. According to the formula (14)–(15), we have

$$b_{n-1}^{(n,\gamma_{n-1/2})} = \frac{2\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}}\left[\left(n-\frac{1}{2}\right)^{2-\gamma_{n-1/2}} - (n-1)^{2-\gamma_{n-1/2}}\right], \quad (20)$$

$$b_{n-k-1}^{(n,\gamma_{n-1/2})} = \frac{\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}}\left[(n-k)^{2-\gamma_{n-1/2}} - (n-k-1)^{2-\gamma_{n-1/2}}\right], \quad 1 \le k \le n-1. \quad (21)$$

When $k = n-1$, it can be obtained by calculation

$$b_0^{(n,\gamma_{n-1/2})} = \frac{\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}}. \quad (22)$$

From equations (20) and (21), we have

$$\begin{aligned} b_{n-1}^{(n,\gamma_{n-1/2})} &= 2\tau^{1-\gamma_{n-1/2}}\int_{n-1}^{n-1/2} \xi^{1-\gamma_{n-1/2}} d\xi, \\ b_k^{(n,\gamma_{n-1/2})} &= \tau^{1-\gamma_{n-1/2}}\int_k^{k+1} \xi^{1-\gamma_{n-1/2}} d\xi, \\ & 0 \le k \le n-2. \end{aligned} \quad (23)$$

Therefore, it can be obtained

$$b_{n-1}^{(n,\gamma_{n-1/2})} < b_{n-2}^{(n,\gamma_{n-1/2})} < \cdots < b_1^{(n,\gamma_{n-1/2})} < b_0^{(n,\gamma_{n-1/2})} = \frac{\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}}. \quad (24)$$

When $n \ge 2$, we have

$$\begin{aligned} \left(1-\frac{1}{2n-1}\right)^{2-\gamma_{n-1/2}} = 1 - \frac{2-\gamma_{n-1/2}}{2n-1} &+ \frac{(2-\gamma_{n-1/2})(1-\gamma_{n-1/2})}{2!}\left(-\frac{1}{2n-1}\right)^2 \\ &+ \frac{(2-\gamma_{n-1/2})(1-\gamma_{n-1/2})(-\gamma_{n-1/2})}{3!} \cdot \left(-\frac{1}{2n-1}\right)^3 + \cdots. \end{aligned} \quad (25)$$

From the above formula

$$\begin{aligned} &\left(n-\frac{1}{2}\right)^{2-\gamma_{n-1/2}} - (n-1)^{2-\gamma_{n-1/2}} - \frac{2-\gamma_{n-1/2}}{2(n-1/2)^{\gamma^{n-1/2}-1}} \\ &= \left(n-\frac{1}{2}\right)^{2-\gamma_{n-1/2}}\left[1 - \frac{2-\gamma_{n-1/2}}{2(n-1/2)} - \left(1-\frac{1}{2n-1}\right)^{2-\gamma_{n-1/2}}\right] \\ &= \left(n-\frac{1}{2}\right)^{2-\gamma_{n-1/2}}\left[-\frac{(2-\gamma_{n-1/2})(1-\gamma_{n-1/2})}{2!}\left(-\frac{1}{2n-1}\right)^2 \right. \\ &\quad \left. - \frac{(2-\gamma_{n-1/2})(1-\gamma_{n-1/2})(-\gamma_{n-1/2})}{3!}\left(-\frac{1}{2n-1}\right)^3 - \cdots\right] > 0. \end{aligned} \quad (26)$$

And when $n = 1$, we have

$$\left(\frac{1}{2}\right)^{2-\gamma_{n-1/2}} - \frac{2-\gamma_{n-1/2}}{2\cdot(1/2)^{\gamma_{n-1/2}-1}} = \frac{\gamma_{n-1/2}-1}{2^{2-\gamma_{n-1/2}}} > 0. \quad (27)$$

Therefore, it can be seen that

$$b_{n-1}^{(n,\gamma_{n-1/2})} > \frac{2\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}} \cdot \frac{2-\gamma_{n-1/2}}{2(n-1/2)^{\gamma_{n-1/2}-1}} = \frac{\tau^{1-\gamma_{n-1/2}}}{(n-1/2)^{\gamma_{n-1/2}-1}}. \quad (28)$$

To sum up, Lemma 2 is proved. □

Lemma 3 (see [11]). *If the function* $f \in C^4[x_{i-1}, x_{i+1}]$, $\lambda \in (x_{i-1}, x_{i+1})$, *there is*

$$f''(x_i) = \frac{f(x_{i-1}) - 2f(x_i) + f(x_{i+1})}{h^2} - \frac{h^2}{12} f^{(4)}(\lambda). \tag{29}$$

Lemma 4. *For any positive integer* m *and any* $\psi, V_1, V_2, \cdots, V_N \in \hat{U}_h$, *when*

$$(t_{n+1/2} - t)^{\gamma_{n+1/2}-1} \geq (t_{n-1/2} - t)^{\gamma_{n-1/2}-1}, t \in (0, t_{n-1/2}), t_{n+1/2} \leq T, \tag{30}$$

we have

$$\sum_{n=1}^{m}\left(b_0^{(n,\gamma_{n-1/2})} V^n - \sum_{k=1}^{n-1}\left(b_{n-k-1}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n,\gamma_{n-1/2})}\right) V^k - b_{n-1}^{(n,\gamma_{n-1/2})}\psi, V^n\right) \geq \frac{1}{2}\left(\sum_{k=1}^{m} b_{m-k}^{(m,\gamma_{m-1/2})}\left\|V^k\right\|^2 - \sum_{n=1}^{m} b_{n-1}^{(n,\gamma_{n-1/2})}\|\psi\|^2\right), \tag{31}$$

where $1 \leq m \leq N$.

Proof. On the basis of [12], it can be seen from the condition

$$\begin{aligned}
&\sum_{n=1}^{m}\left(b_0^{(n,\gamma_{n-1/2})} V^n - \sum_{k=1}^{n-1}\left(b_{n-k-1}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n,\gamma_{n-1/2})}\right) V^k - b_{n-1}^{(n,\gamma_{n-1/2})}\psi, V^n\right) \\
&= \sum_{n=1}^{m}\left(b_0^{(n,\gamma_{n-1/2})}\|V^n\|^2 - \sum_{k=1}^{n-1}\left(b_{n-k-1}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n,\gamma_{n-1/2})}\right)\left(V^k, V^n\right) - b_{n-1}^{(n,\gamma_{n-1/2})}(\psi, V^n)\right) \geq \sum_{n=1}^{m}\left[b_0^{(n,\gamma_{n-1/2})}\|V^n\|^2 - \frac{1}{2}\sum_{k=1}^{n-1}\right. \\
&\left.\cdot\left(b_{n-k-1}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n,\gamma_{n-1/2})}\right)\left(\left\|V^k\right\|^2 + \|V^n\|^2\right) - \frac{1}{2} b_{n-1}^{(n,\gamma_{n-1/2})}\left(\|\psi\|^2 + \|V^n\|^2\right)\right] \\
&= \frac{1}{2}\sum_{n=1}^{m}\left[\left(2b_0^{(n,\gamma_{n-1/2})} - \sum_{k=1}^{n-1}\left(b_{n-k-1}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n,\gamma_{n-1/2})}\right) - b_{n-1}^{(n,\gamma_{n-1/2})}\right)\|V^n\|^2\right. \\
&\left.- \sum_{k=1}^{n-1}\left(b_{n-k-1}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n,\gamma_{n-1/2})}\right)\left\|V^k\right\|^2 - b_{n-1}^{(n,\gamma_{n-1/2})}\|\psi\|^2\right] \\
&= \frac{1}{2}\sum_{n=1}^{m}\left[b_0^{(n,\gamma_{n-1/2})}\|V^n\|^2 - \sum_{k=1}^{n-1} b_{n-k-1}^{(n,\gamma_{n-1/2})}\left\|V^k\right\|^2 + \sum_{k=1}^{n-1} b_{n-k}^{(n,\gamma_{n-1/2})}\left\|V^k\right\|^2\right. \\
&\left.- b_{n-1}^{(n,\gamma_{n-1/2})}\|\psi\|^2\right] = \frac{1}{2}\sum_{n=1}^{m}\left[\sum_{k=1}^{n} b_{n-k}^{(n,\gamma_{n-1/2})}\left\|V^k\right\|^2 - \sum_{k=1}^{n-1} b_{n-k-1}^{(n,\gamma_{n-1/2})}\left\|V^k\right\|^2\right. \\
&\left.- b_{n-1}^{(n,\gamma_{n-1/2})}\|\psi\|^2\right] = \frac{1}{2}\left[\sum_{k=1}^{m}\sum_{n=k}^{m} b_{n-k}^{(n,\gamma_{n-1/2})}\left\|V^k\right\|^2 - \sum_{k=1}^{m-1}\sum_{n=k+1}^{m} b_{n-k-1}^{(n,\gamma_{n-1/2})}\left\|V^k\right\|^2\right. \\
&\left.- \sum_{n=1}^{m} b_{n-1}^{(n,\gamma_{n-1/2})}\|\psi\|^2\right] = \frac{1}{2}\left[b_0^{(m,\gamma_{m-1/2})}\|V^m\|^2\right. \\
&\left.+ \sum_{k=1}^{m-1}\left(\sum_{n=k}^{m} b_{n-k}^{(n,\gamma_{n-1/2})} - \sum_{n=k+1}^{m} b_{n-k-1}^{(n,\gamma_{n-1/2})}\right)\left\|V^k\right\|^2 - \sum_{n=1}^{m} b_{n-1}^{(n,\gamma_{n-1/2})}\|\psi\|^2\right] \\
&= \frac{1}{2}\left[b_0^{(m,\gamma_{m-1/2})}\|V^m\|^2 + \sum_{k=1}^{m-1} b_{m-k}^{(m,\gamma_{m-1/2})}\left\|V^k\right\|^2\right. \\
&\left.+ \sum_{k=1}^{m-1}\sum_{n=k}^{m-1}\left(b_{n-k}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n+1,\gamma_{n+1/2})}\right)\left\|V^k\right\|^2 - \sum_{n=1}^{m} b_{n-1}^{(n,\gamma_{n-1/2})}\|\psi\|^2\right].
\end{aligned} \tag{32}$$

When $\gamma(t)$ satisfies the following condition

$$b_{n-k}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n+1,\gamma_{n+1/2})} = \frac{1}{\tau}\int_{t_{k-3/2}}^{t_{k-1/2}}\left[(t_{n-1/2} - t)^{1-\gamma_{n-1/2}} - (t_{n+1/2} - t)^{1-\gamma_{n+1/2}}\right] dt, \tag{33}$$

namely

$$(t_{n+1/2} - t)^{\gamma_{n+1/2}-1} \geq (t_{n-1/2} - t)^{\gamma_{n-1/2}-1}. \tag{34}$$

Then, we have

$$\sum_{n=1}^{m}\left(b_0^{(n,\gamma_{n-1/2})} V^n - \sum_{k=1}^{n-1}\left(b_{n-k-1}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n,\gamma_{n-1/2})}\right)\cdot V^k - b_{n-1}^{(n,\gamma_{n-1/2})}\psi, V^n\right) \geq \frac{1}{2}\left(\sum_{k=1}^{m} b_{m-k}^{(m,\gamma_{m-1/2})}\left\|V^k\right\|^2 - \sum_{n=1}^{m} b_{n-1}^{(n,\gamma_{n-1/2})}\|\psi\|^2\right). \tag{35}$$

□

Remark 5. Consider the function

$$g(x, y) = x^y, \quad x > 0, \ y > 0. \tag{36}$$

We have

$$\begin{aligned}
\frac{\partial g(x, y)}{\partial x} &= yx^{y-1} > 0, \\
\frac{\partial g(x, y)}{\partial y} &= x^y Inx = \begin{cases} <0, & x \in (0, 1), \\ >0, & x > 1. \end{cases}
\end{aligned} \tag{37}$$

If $T \leq 1$ and $\gamma(t)$ is an non-increasing function on $[0, T]$, then $t_{n+1/2} - t \in (0, 1)$, $t_{n-1/2} - t \in (0, 1)$ and $\gamma_{n+1/2} \leq \gamma_{n-1/2}$, consequently

$$(t_{n+1/2} - t)^{\gamma_{n+1/2}-1} \geq (t_{n-1/2} - t)^{\gamma_{n-1/2}-1}, t \in (0, t_{n-1/2}), t_{n+1/2} \leq T. \tag{38}$$

(30) is valid.

If $T > 1$, $\gamma(t)$ is an non-increasing function on the interval $[0, 1]$ and $\gamma(t)$ is a constant on the interval $[1, T]$, (30) is also valid.

Lemma 6 (see [11]). *For any* $\varepsilon > 0$, $a, b \geq 0$, *there is*

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2. \tag{39}$$

Lemma 7 (see [11]). *For any grid function* $u \in \hat{U}_h$, *there is*

$$\|u\|_\infty \leq \frac{\sqrt{L}}{2}\|\delta_x u\|. \tag{40}$$

Lemma 8 (see [13]). *Suppose* $\{F^k \mid k \geq 0\}, \{G^k \mid k \geq 1\}$ *are two non-negative sequences,* $\{G^k\}$ *does not decrease with k, if*

$$F^k \leq C\tau \sum_{l=0}^{k} F^l + G^k, \quad k = 1, 2, \cdots, \tag{41}$$

where C is an non-negative constant, when $\tau \leq 2/3C$, *then*

$$F^k \leq \exp(3Ck\tau)\left(C\tau F^0 + 3G^k\right), \quad k = 1, 2, \cdots. \tag{42}$$

3. Establishment of the Difference Scheme

Denote $U_i^n = u(x_i, t_n), 0 \leq i \leq M, 0 \leq n \leq N; \varphi_i = \varphi(x_i), \psi_i = \psi(x_i)$, consider (2) at the point $(x_i, t_{n-1/2})$, we have

$$ {}_0^C D_t^{\gamma_{n-1/2}} u(x_i, t_{n-1/2}) = u_{xx}(x_i, t_{n-1/2}) + f(x_i, t_{n-1/2}), \quad 1 \leq i \leq M-1, \ 1 \leq n \leq N. \tag{43}$$

Applying (13) to approximate the temporal fractional derivative and central difference quotient (29) to approximate the spatial derivative, we can obtain

$$\begin{aligned} &\frac{1}{\Gamma(2-\gamma_{n-1/2})}\left[b_0^{(n,\gamma_{n-1/2})}\delta_t U_i^{n-1/2} - \sum_{k=1}^{n-1} \cdot \left(b_{n-k-1}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n,\gamma_{n-1/2})}\right)\delta_t U_i^{k-1/2} - b_{n-1}^{(n,\gamma_{n-1/2})}\psi_i\right] \\ &= \delta_x^2 U_i^{n-1/2} + f_i^{n-1/2} + R_i^{n-1/2}, \quad 1 \leq i \leq M-1, \ 1 \leq n \leq N. \end{aligned} \tag{44}$$

There exists a positive constant C_1 such that

$$\left|R_i^{n-1/2}\right| \leq C_1\left(\tau^{3-\gamma_{n-1/2}} + h^2\right), \quad 1 \leq i \leq M-1, \ 1 \leq n \leq N. \tag{45}$$

Noticing the initial and boundary value conditions (3) and (4), we have

$$\begin{cases} U_i^0 = \varphi_i, & 1 \leq i \leq M-1. \\ U_0^n = 0, \quad U_M^n = 0, & 0 \leq n \leq N. \end{cases} \tag{46}$$

Omitting the small term $R_i^{n-1/2}$ in the equation and replacing the grid function U_i^n by its numerical approximation u_i^n, we construct the difference scheme for solving the problems (2)–(4) as follows

$$\begin{aligned} &\frac{1}{\Gamma(2-\gamma_{n-1/2})}\left[b_0^{(n,\gamma_{n-1/2})}\delta_t u_i^{n-1/2} - \sum_{k=1}^{n-1} \cdot \left(b_{n-k-1}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n,\gamma_{n-1/2})}\right)\delta_t u_i^{k-1/2} - b_{n-1}^{(n,\gamma_{n-1/2})}\psi_i\right] \\ &= \delta_x^2 u_i^{n-1/2} + f_i^{n-1/2}, \quad 1 \leq i \leq M-1, \ 1 \leq n \leq N. \end{aligned} \tag{47}$$

$$u_i^0 = \varphi_i, \quad 1 \leq i \leq M-1. \tag{48}$$

$$u_0^n = 0, \quad u_M^n = 0, \ 0 \leq n \leq N. \tag{49}$$

4. Stability and Convergence of the Difference Scheme

Theorem 9. *Suppose* $\{u_i^n \mid 0 \leq i \leq M, 0 \leq n \leq N\}$ *is the solution of the following difference scheme*

$$\begin{aligned} &\frac{1}{\Gamma(2-\gamma_{n-1/2})}\left[b_0^{(n,\gamma_{n-1/2})}\delta_t u_i^{n-1/2} - \sum_{k=1}^{n-1} \cdot \left(b_{n-k-1}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n,\gamma_{n-1/2})}\right)\delta_t u_i^{k-1/2} - b_{n-1}^{(n,\gamma_{n-1/2})}\psi_i\right] \\ &= \delta_x^2 u_i^{n-1/2} + p_i^{n-1/2}, \quad 1 \leq i \leq M-1, \ 1 \leq n \leq N, \end{aligned} \tag{50}$$

$$u_i^0 = \varphi_i, \quad 1 \leq i \leq M-1, \tag{51}$$

$$u_0^n = 0, \quad u_M^n = 0, \ 0 \leq n \leq N, \tag{52}$$

where $p_i^{n-1/2}$ *is a given perturbation term, when* $\tau < 2/3c_0$, *it holds that*

$$\|\delta_x u^n\|^2 \leq \exp(3c_0 T)\left(c_0\tau\left\|\delta_x u^0\right\|^2 + 3Q^n\right), \quad 1 \leq n \leq N. \tag{53}$$

c_0 *and* Q^n *are given in (56) and (64), respectively.*

Proof. Taking an inner product (50) with $\Gamma(2-\gamma_{n-1/2})\delta_t u^{n-1/2}$ and summing n from 1 to m, we have

$$\begin{aligned} &\sum_{n=1}^{m}\left[b_0^{(n,\gamma_{n-1/2})}\left\|\delta_t u^{n-1/2}\right\|^2 - \sum_{k=1}^{n-1}\left(b_{n-k-1}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n,\gamma_{n-1/2})}\right)\left(\delta_t u^{k-1/2}, \delta_t u^{n-1/2}\right) - b_{n-1}^{(n,\gamma_{n-1/2})}\left(\psi, \delta_t u^{n-1/2}\right)\right] = \sum_{n=1}^{m}\Gamma(2-\gamma_{n-1/2})\left(\delta_x^2 u^{n-1/2}, \delta_t u^{n-1/2}\right) \\ &+ \sum_{n=1}^{m}\Gamma(2-\gamma_{n-1/2})\left(p^{n-1/2}, \delta_t u^{n-1/2}\right), \quad 1 \leq m \leq N. \end{aligned} \tag{54}$$

Noticing that

$$\begin{aligned}
&\sum_{n=1}^{m}\Gamma(2-\gamma_{n-1/2})\left(\delta_x^2u^{n-1/2},\delta_tu^{n-1/2}\right)\\
&=-\frac{1}{2\tau}\sum_{n=1}^{m}\Gamma(2-\gamma_{n-1/2})\left(\|\delta_xu^n\|^2-\left\|\delta_xu^{n-1}\right\|^2\right)\\
&=-\frac{1}{2\tau}\sum_{n=1}^{m}\Big\{\Gamma(2-\gamma_n)\|\delta_xu^n\|^2-\Gamma(2-\gamma_{n-1})\left\|\delta_xu^{n-1}\right\|^2\\
&\quad-[\Gamma(2-\gamma_n)-\Gamma(2-\gamma_{n-1/2})]\|\delta_xu^n\|^2-[\Gamma(2-\gamma_{n-1/2})\\
&\quad-\Gamma(2-\gamma_{n-1})]\left\|\delta_xu^{n-1}\right\|^2\Big\}\le-\frac{1}{2\tau}\sum_{n=1}^{m}\\
&\quad\cdot\left(\Gamma(2-\gamma_n)\|\delta_xu^n\|^2-\Gamma(2-\gamma_{n-1})\left\|\delta_xu^{n-1}\right\|^2\right)\\
&\quad+\frac{1}{2\tau}\sum_{n=1}^{m}[\Gamma(2-\gamma_n)-\Gamma(2-\gamma_{n-1/2})]\|\delta_xu^n\|^2\\
&\quad+\frac{1}{2\tau}\sum_{n=1}^{m}[\Gamma(2-\gamma_{n-1/2})-\Gamma(2-\gamma_{n-1})]\left\|\delta_xu^{n-1}\right\|^2\\
&\le-\frac{1}{2\tau}\sum_{n=1}^{m}\left(\Gamma(2-\gamma_n)\|\delta_xu^n\|^2-\Gamma(2-\gamma_{n-1})\left\|\delta_xu^{n-1}\right\|^2\right)\\
&\quad+\frac{1}{4}c_0\sum_{n=1}^{m}\left(\|\delta_xu^n\|^2+\left\|\delta_xu^{n-1}\right\|^2\right),
\end{aligned}\tag{55}$$

where

$$c_0=\max_{0\le t\le T}\left|\frac{d}{dt}\Gamma(2-\gamma(t))\right|.\tag{56}$$

Applying Lemma 4, we have

$$\begin{aligned}
&\frac{1}{2}\left(\sum_{k=1}^{m}b_{m-k}^{(m,\gamma_{m-1/2})}\left\|\delta_tu^{k-1/2}\right\|^2-\sum_{n=1}^{m}b_{n-1}^{(n,\gamma_{n-1/2})}\|\psi\|^2\right)\\
&\le-\frac{1}{2\tau}\sum_{n=1}^{m}\left(\Gamma(2-\gamma_n)\|\delta_xu^n\|^2-\Gamma(2-\gamma_{n-1})\left\|\delta_xu^{n-1}\right\|^2\right)\\
&\quad+\frac{1}{4}c_0\sum_{n=1}^{m}\left(\|\delta_xu^n\|^2+\left\|\delta_xu^{n-1}\right\|^2\right)+\sum_{n=1}^{m}\Gamma(2-\gamma_{n-1/2})\\
&\quad\times\left(p^{n-1/2},\delta_tu^{n-1/2}\right),\quad 1\le m\le N.
\end{aligned}\tag{57}$$

Then, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{m}b_{m-k}^{(m,\gamma_{m-1/2})}\left\|\delta_tu^{k-1/2}\right\|^2+\frac{1}{2\tau}\left(\Gamma(2-\gamma_m)\|\delta_xu^m\|^2-\left\|\delta_xu^0\right\|^2\right)\\
&\le\frac{1}{2}\sum_{n=1}^{m}b_{n-1}^{(n,\gamma_{n-1/2})}\|\psi\|^2+\sum_{n=1}^{m}\Gamma(2-\gamma_{n-1/2})\left(p^{n-1/2},\delta_tu^{n-1/2}\right)\\
&\quad+\frac{1}{2}c_0\sum_{n=0}^{m}\|\delta_xu^n\|^2,\quad 1\le m\le N.
\end{aligned}\tag{58}$$

By Lemma 2, noticing that

$$b_{m-k}^{(m,\gamma_{m-1/2})}>\frac{\tau^{1-\gamma_{m-1/2}}}{(m-1/2)^{\gamma_{m-1/2}-1}}=t_{m-1/2}^{1-\gamma_{m-1/2}},\ b_{n-1}^{(n,\gamma_{n-1/2})}<\frac{\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}}.\tag{59}$$

Then

$$\begin{aligned}
&\frac{1}{2}t_{m-1/2}^{1-\gamma_{m-1/2}}\sum_{k=1}^{m}\left\|\delta_tu^{k-1/2}\right\|^2+\frac{\Gamma(2-\gamma_m)}{2\tau}\|\delta_xu^m\|^2\\
&\le\frac{1}{2\tau}\left\|\delta_xu^0\right\|^2+\frac{1}{2}\sum_{n=1}^{m}\frac{\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}}\|\psi\|^2+\sum_{n=1}^{m}\Gamma(2-\gamma_{n-1/2})\\
&\quad\cdot\left(\frac{t_{m-1/2}^{1-\gamma_{m-1/2}}}{2\Gamma(2-\gamma_{n-1/2})}\left\|\delta_tu^{n-1/2}\right\|^2+\frac{\Gamma(2-\gamma_{n-1/2})}{2t_{m-1/2}^{1-\gamma_{m-1/2}}}\left\|p^{n-1/2}\right\|^2\right)\\
&\quad+\frac{1}{2}c_0\sum_{n=0}^{m}\|\delta_xu^n\|^2.
\end{aligned}\tag{60}$$

We use the Cauchy inequality for the inner product $(p^{n-1/2},\delta_tu^{n-1/2})$, the above equation can be simplified

$$\begin{aligned}
\frac{\Gamma(2-\gamma_m)}{2\tau}\|\delta_xu^m\|^2&\le\frac{1}{2\tau}\left\|\delta_xu^0\right\|^2+\frac{1}{2}\sum_{n=1}^{m}\frac{\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}}\|\psi\|^2\\
&\quad+\sum_{n=1}^{m}\frac{\Gamma^2(2-\gamma_{n-1/2})}{2t_{m-1/2}^{1-\gamma_{m-1/2}}}\left\|p^{n-1/2}\right\|^2\\
&\quad+\frac{1}{2}c_0\sum_{n=0}^{m}\|\delta_xu^n\|^2.
\end{aligned}\tag{61}$$

Multiplying by $2\tau/\Gamma(2-\gamma_m)$, then we have

$$\begin{aligned}
\|\delta_xu^m\|^2&\le\frac{1}{\Gamma(2-\gamma_m)}\left(\left\|\delta_xu^0\right\|^2+\tau\sum_{n=1}^{m}\frac{\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}}\|\psi\|^2\right.\\
&\quad\left.+\tau t_{m-1/2}^{\gamma_{m-1/2}-1}\sum_{n=1}^{m}\Gamma^2(2-\gamma_{n-1/2})\left\|p^{n-1/2}\right\|^2+\tau c_0\sum_{n=0}^{m}\|\delta_xu^n\|^2\right).
\end{aligned}\tag{62}$$

Note that Γ is decreasing on the interval $(0,1]$. Since $0<2-\gamma(t)\le 1$, $\Gamma(2-\gamma(t))^{-1}\le 1$. Then

$$\begin{aligned}
\|\delta_xu^m\|^2&\le\left\|\delta_xu^0\right\|^2+\tau\sum_{n=1}^{m}\frac{\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}}\|\psi\|^2\\
&\quad+\tau t_{m-1/2}^{\gamma_{m-1/2}-1}\sum_{n=1}^{m}\Gamma^2(2-\gamma_{n-1/2})\left\|p^{n-1/2}\right\|^2\\
&\quad+\tau c_0\sum_{n=0}^{m}\|\delta_xu^n\|^2.
\end{aligned}\tag{63}$$

Let

$$Q^m = \|\delta_x u^0\|^2 + \tau \sum_{n=1}^{m} \frac{\tau^{1-\gamma_{n-1/2}}}{2-\gamma_{n-1/2}} \|\psi\|^2 + \tau t_{m-1/2}^{\gamma_{m-1/2}-1} \sum_{n=1}^{m} \Gamma^2 \cdot (2-\gamma_{n-1/2}) \|p^{n-1/2}\|^2, \quad 1 \le m \le N. \tag{64}$$

Then

$$\|\delta_x u^m\|^2 \le \tau c_0 \sum_{n=0}^{m} \|\delta_x u^n\|^2 + Q^m. \tag{65}$$

It is easy to know Q^m does not decrease with m. According to Lemma 8, when $\tau < 2/3c_0$, we have

$$\|\delta_x u^m\|^2 \le \exp(3c_0 T)\left(c_0 \tau \|\delta_x u^0\|^2 + 3Q^m\right), \quad 1 \le m \le N. \tag{66}$$

Theorem 9 is proved. We can say that the difference scheme is stable. □

Theorem 10. *Assume $\{u(x_i, t_n)\}$ and $\{u_i^n\}$ are solutions of problems (2)–(4) and difference scheme (47)–(49), respectively. Denote*

$$e_i^n = u(x_i, t_n) - u_i^n, \quad 0 \le i \le M, \quad 0 \le n \le N. \tag{67}$$

Then, there exists a positive constant C_2, when $\tau < 2/3c_0$, $n\tau \le T$, such that

$$\|e^n\|_\infty \le \frac{\sqrt{L}}{2} C_2 \left(\tau^{3-\gamma^*} + h^2\right), \quad 0 \le n \le N, \tag{68}$$

where $C_2 = \sqrt{3TL}C_1 \exp(3c_0 T/2)$ and $\gamma^* = \max_{0\le t\le T} \gamma(t)$.

Proof. Subtracting (44) and (46) from (47)–(49), we obtain the system of error equations

$$\begin{cases} \dfrac{1}{\Gamma(2-\gamma_{n-1/2})}\left[b_0^{(n,\gamma_{n-1/2})} \delta_t e_i^{n-1/2} - \sum_{k=1}^{n-1}\left(b_{n-k-1}^{(n,\gamma_{n-1/2})} - b_{n-k}^{(n,\gamma_{n-1/2})}\right)\delta_t e_i^{k-1/2}\right] = \delta_x^2 e_i^{n-1/2} + R_i^{n-1/2}, \\ 1 \le i \le M-1, \quad 1 \le n \le N, \\ e_i^0 = 0, \quad 1 \le i \le M-1, \\ e_0^n = 0, \quad e_M^n = 0, \quad 0 \le n \le N. \end{cases} \tag{69}$$

Applying Theorem 9 and (45), it yields

$$\|\delta_x e^n\|^2 \le C_2^2 \left(\tau^{3-\gamma^*} + h^2\right)^2, \tag{70}$$

where $\gamma^* = \max_{0\le t\le T} \gamma(t)$.

Applying Lemma 7, it yields

$$\|e^n\|_\infty \le \frac{\sqrt{L}}{2} \|\delta_x e^n\| \le \frac{\sqrt{L}}{2} C_2 \left(\tau^{3-\gamma^*} + h^2\right). \tag{71}$$

The proof is ended. □

Table 1: Errors and temporal convergence orders, $M = 1000$.

$\gamma(t)$	τ	$E(h,\tau)$	Order_τ
$2 - t^2$	1/512	$8.7295e-4$	0
	1/1024	$4.2265e-4$	1.05
	1/2048	$2.0502e-4$	1.04
	1/4096	$9.9746e-5$	1.04
$1 + e^{-t}$	1/512	$4.0772e-4$	0
	1/1024	$1.8602e-4$	1.13
	1/2048	$8.5528e-5$	1.12
	1/4096	$3.9764e-5$	1.10
$\dfrac{6+\cos t}{4}$	1/512	$3.8173e-4$	0
	1/1024	$1.5945e-4$	1.26
	1/2048	$6.6746e-5$	1.26
	1/4096	$2.8186e-5$	1.24

5. Numerical Example

In order to verify the accuracy of the finite difference scheme, several different types of variable-order index $\gamma(t) \in (1,2)$ are used to solve the variable-order fractional wave equations (2)–(4) in 1D case. The scheme is implemented in MATLAB (R2019a).

Here, we take $L = \pi$, $T = 1$. The source term of equation (2)

$$f(x,t) = \left(\frac{6}{\Gamma(4-\gamma(t))} t^{3-\gamma(t)} + \frac{6}{\Gamma(3-\gamma(t))} t^{2-\gamma(t)} + t^3 + 3t^2 + 1\right) \sin x, \tag{72}$$

Table 2: Errors and spatial convergence orders, $N = 20000$.

$\gamma(t)$	h	$E(h, \tau)$	Order_h
$2 - t^2$	$\pi/5$	$3.2641e-2$	0
	$\pi/10$	$8.6512e-3$	1.92
	$\pi/20$	$2.1801e-3$	1.99
	$\pi/40$	$5.5869e-4$	1.96
$1 + e^{-t}$	$\pi/5$	$3.1901e-2$	0
	$\pi/10$	$8.4489e-3$	1.92
	$\pi/20$	$2.1209e-3$	1.99
	$\pi/40$	$5.3548e-4$	1.99
$\frac{6 + \cos t}{4}$	$\pi/5$	$3.0356e-2$	0
	$\pi/10$	$8.0401e-3$	1.92
	$\pi/20$	$2.0164e-3$	2.00
	$\pi/40$	$5.0707e-4$	1.99

the initial value

$$u(x, 0) = \sin x, \quad x \in [0, \pi], \tag{73}$$

the boundary value

$$u(0, t) = u(\pi, t) = 0, \quad t \in [0, 1], \tag{74}$$

the exact solution is given by

$$u(x, t) = \left(t^3 + 3t^2 + 1\right) \sin x. \tag{75}$$

Define the error of the numerical solution

$$E(h, \tau) = \max_{0 \le k \le N} \left\| U^k - u^k \right\|_\infty, \tag{76}$$

the temporal convergence order

$$\text{Order}_\tau = \log_2\left(\frac{E(h, 2\tau)}{E(h, \tau)}\right), \tag{77}$$

the spatial convergence order

$$\text{Order}_h = \log_2\left(\frac{E(2h, \tau)}{E(h, \tau)}\right). \tag{78}$$

Denote $M = 1000$, for different $\gamma(t) = 2 - t^2$, $1 + e^{-t}$, $(6 + \cos t)/4$. The time step τ is varied from $1/512$ to $1/4096$, where $N = 512, 1024, 2048, 4096$. Table 1 shows the errors and temporal convergence orders of the difference scheme (47)–(49). It can be seen from Table 1 that the difference scheme (47)–(49) has a precision of approximately $3 - \gamma^*$ order in time. The computational results are in good agreement with theoretical results.

Take a fixed and sufficiently small time step $\tau = 1/20000$, for different $\gamma(t) = 2 - t^2$, $1 + e^{-t}$, $(6 + \cos t)/4$, verify space step h from $\pi/5$ to $\pi/40$, where $N = 20000$, $M = 5, 10, 20, 40$. Table 2 shows the errors and spatial convergence orders of the difference scheme (47)–(49). It can be seen from Table 2 that the difference scheme (47)–(49) in the space has an accuracy of approximately 2 order, which is consistent with the theoretical results.

6. Conclusions

In this paper, we consider a numerical approximation method for the variable-order Caputo fractional derivative—H2N2 approximation, and give the corresponding calculation formula. Secondly, we use this formula to solve the one-dimensional variable-order time-fractional wave equations and discuss the stability and convergence of the equations by the discrete energy analysis method. Finally, a numerical example verifies the effectiveness of the scheme.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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