Research Article

Properties of Functions with Symmetric Points Involving Subordination

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We study a new subclass of functions with symmetric points and derive an equivalent formulation of these functions in term of subordination. Moreover, we find coefficient estimates and discuss characterizations for functions belonging to this new class. We also obtain distortion and growth results. We relate our results with the existing literature of the subject.

1. Introduction and Definitions

Let $\mathcal{H}(\Delta)$ represent analytic functions f in the disc $\Delta := \{z : |z| < 1\}$ and $\mathcal{A} \subset \mathcal{H}(\Delta)$ be defined as:

$$\mathcal{A} := \{f \in \mathcal{H}(\Delta) : f(z) = z + a_2 z^2 + \dots (z \in \Delta)\}. \quad (1)$$

Let $\mathcal{Q}$ denote “Carathéodory functions” h such that $h(0) = 1$, $\operatorname{Re}(h(z)) > 0$ and $h(z) = 1 + h_1 z + h_2 z^2 + \dots, z \in \Delta$. The Möbius function $k_0(z) = (1+z)/(1-z) \in \mathcal{Q}$ or its rotation acts as an extremal function for the class $\mathcal{Q}$ and maps Δ onto $\operatorname{Re}(k_0(z)) > 0$. Recall that $\mathcal{Q}(\varepsilon) \subset \mathcal{Q}$, $0 \leq \varepsilon < 1$ consists of functions $h \in \mathcal{Q}$ such that $\operatorname{Re}(h(z)) > \varepsilon$ in Δ . For $f, g \in \mathcal{H}$, we say that the function f is subordinate to g and write $f \prec g$, if for

$$\Phi \in \mathcal{H}(\Delta), \text{ with } \Phi(0) = 0 \text{ and } |\Phi(z)| < 1, f(z) = g(\Phi(z)). \quad (2)$$

For a univalent function g , $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. For reference, see [1]. Applying subordination, Janowski [2] defined the class $\mathcal{Q}[A, A_1]$ for $-1 \leq A_1 < A \leq 1$. A function $h \in \mathcal{Q}[A, A_1]$, if

$$h(z) = \frac{1 + A\Phi(z)}{1 + A_1\Phi(z)} \prec \frac{1 + Az}{1 + A_1z} (z \in \Delta). \quad (3)$$

Geometrically, the image $h(\Delta)$ lies inside the disk centered on $\operatorname{Im}(z) = 0$, and diameter ends at $h(-1)$ and $h(1)$. Clearly, $\mathcal{Q}[A, A_1] \subset \mathcal{Q}((1-A)/(1-A_1))$. The class $\mathcal{Q}[A, A_1]$ is related with the class $\mathcal{Q}$ as: $h \in \mathcal{Q}$ iff, we write

$$\frac{(A+1)h(z) - (A-1)}{(A_1+1)h(z) - (A_1-1)} \in \mathcal{Q}[A, A_1]. \quad (4)$$

Also Λ_k , $k \geq 0$ is given by $\Lambda_k = \{\Phi = u + iv : u > k \sqrt{(u-1)^2 + v^2}\}$, represents various plane curves for the specific values of k . Let $\mathcal{S}$ be the class of complex-valued injective functions and $\mathcal{S}^*$ represents the class of starlike functions whereas $\mathcal{C}$ denotes the class of convex functions. A function $f \in \mathcal{S}$ is close-to-convex, if and only if there exists a function $g \in \mathcal{C}$ such that

$$\operatorname{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} \in \mathcal{Q}, z \in \Delta. \quad (5)$$

We denote the class of close-to-convex functions by $\mathcal{K}$. This class was introduced by Kaplan in [3]. Sakaguchi (see [4]) defined the class $\mathcal{S}_{\text{sp}}$ as:

Definition 1. Let $f \in \mathcal{S}$. Then, $f \in \mathcal{S}_{\text{SP}}$, if

$$\operatorname{Re} \left(\frac{2zf'(z)}{f(z) - f(-z)} \right) > 0, (z \in \Delta). \quad (6)$$

For $f \in \mathcal{A}$, $f \in \mathcal{C}_{\text{SP}}$ [5] iff $zf' \in \mathcal{S}_{\text{SP}}$, where $\mathcal{C}_{\text{SP}}$ is the class of convex functions with respect to symmetric points. Various authors studied the class $\mathcal{C}_{\text{SP}}$ and its subclasses, for detail, see [3, 6–8]. Obviously, it represents the univalent functions. Moreover, it includes the class of convex and odd starlike functions, see [4]. This and other classes are investigated in the literature of the subject; for example, see [9–14].

Definition 2. Let f be analytic in Δ defined by (1). We say that $f \in \mathcal{K}_{\text{SP}}(\varepsilon)$, $0 \leq \varepsilon < 1$, if for $g \in \mathcal{S}^*(1/2)$ we have

$$\operatorname{Re} \left(\frac{z^2 f'(z)}{g(z)g(-z)} \right) < -\varepsilon, (z \in \Delta). \quad (7)$$

For more details, see [15]. We see that $\mathcal{K}_{\text{SP}}(0) = \mathcal{K}_{\text{SP}}$, where $\mathcal{K}_{\text{SP}}$ is the class of functions defined in [16]. We study a new class $\mathcal{K}_{\text{SP}}(\varepsilon, \eta)$ involving $g \in \mathcal{S}^*(1/2)$.

Definition 3. Let $f \in \mathcal{A}$. Then, $f \in \mathcal{K}_{\text{SP}}(\varepsilon, \eta)$, $0 \leq \varepsilon < 1$, $0 \leq \eta \leq 1$ if for $g \in \mathcal{S}^*(1/2)$ we have

$$\operatorname{Re} \left(\frac{zf'(z) + \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} \right) < -\varepsilon (z \in \Delta), \quad (8)$$

where $G(z) = g(z)g(-z)/z$. By a simple calculations, we see that (8) is equivalent to

$$\left| \frac{zf'(z) + \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} + 1 \right| < \left| \frac{zf'(z) + \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} - 1 + 2\varepsilon \right|, (z \in \Delta). \quad (9)$$

From [16], we have the following lemma.

Lemma 4. For $g \in \mathcal{S}^*(1/2)$ such that

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, (z \in \Delta), \quad (10)$$

if we put

$$zG(z) = g(z)g(-z) = z^2 + \sum_{k=1}^{\infty} c_{k+1} z^{2k+1}, \quad (11)$$

where

$$c_k = 2b_{2k-1} - 2b_2 b_{2k-2} + \dots + (-1)^k 2b_{k-1} b_{k+1} + (-1)^{k+1} b_k^2, \quad (12)$$

then $G \in \mathcal{S}^*$.

Remark 5. Since $g \in \mathcal{S}^*(1/2)$, then Lemma 4 proves that $G \in \mathcal{S}^*$. Also from (8), we see that $\mathcal{K}_{\text{SP}}(\varepsilon, \eta)$ contains close-to-convex functions.

2. Main Results

In the following theorem, we have an equivalent formulation of condition (9) in terms of subordination.

Theorem 6. A function $f \in \mathcal{K}_{\text{SP}}(\varepsilon, \eta)$ ($0 \leq \varepsilon < 1$, $0 \leq \eta \leq 1$) iff for $g \in \mathcal{S}^*(1/2)$, we write

$$\frac{-zf'(z) - \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} < \frac{1 + (1-2\varepsilon)z}{1-z}, (z \in \Delta), \quad (13)$$

where $G(z) = g(z)g(-z)/z$.

Proof. Let $f \in \mathcal{K}_{\text{SP}}(\varepsilon, \eta)$. Then, for $g \in \mathcal{S}^*(1/2)$, we write

$$\left| \frac{zf'(z) + \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} + 1 \right| < \left| \frac{zf'(z) + \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} - 1 + 2\varepsilon \right|, (z \in \Delta), \quad (14)$$

or

$$\operatorname{Re} \left(\frac{-zf'(z) - \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} \right) > \varepsilon, (z \in \Delta), \quad (15)$$

where $G(z) = g(z)g(-z)/z$. Using subordination, we write

$$H(z) = \frac{-zf'(z) - \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} < \frac{1 + (1-2\varepsilon)z}{1-z}, z \in \Delta, \quad (16)$$

because $k_0(\varepsilon, z) = (1 + (1-2\varepsilon)z)/(1-z) \in \mathcal{S}$ and $H(0) = k_0(\varepsilon, 0) = 1$, where $G(z) = g(z)g(-z)/z$. Conversely, we assume that (13) holds. Then, there exists Φ with $\Phi(0) = 0$ and $|\Phi(z)| < 1$ such that

$$\frac{-zf'(z) - \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} = \frac{1 + (1-2\varepsilon)\Phi(z)}{1 - \Phi(z)}. \quad (17)$$

Hence, using $|\Phi(z)| < 1$, we obtain (9) equivalent to (8), so $f \in \mathcal{K}_{\text{SP}}(\varepsilon, \eta)$. $\square$

Now, we prove sufficient conditions for $f \in \mathcal{K}_{\text{SP}}(\varepsilon, \eta)$.

Theorem 7. Let $g \in \mathcal{S}^*(1/2)$ be a function given by (10) and $0 \leq \varepsilon < 1$, $0 \leq \eta \leq 1$. If f defined by (1) satisfies

$$2 \sum_{k=2}^{\infty} [k + \eta k(k-1)] |a_k| + (|1-2\varepsilon| + 1) \sum_{k=2}^{\infty} [2\eta(k-1) + 1] |c_k| \leq 2(1-\varepsilon), \quad (18)$$

where the coefficients c_k are given by (12), then $f \in \mathcal{K}_{\text{SP}}(\varepsilon, \eta)$.

In particular, if

$$\sum_{k=2}^{\infty} [k + \eta k(k-1)] |a_k| \leq 1 - \varepsilon, \quad (19)$$

then $f \in \mathcal{K}_{\text{SP}}(\varepsilon, \eta)$.

Proof. We set $F(\eta, z) = zf'(z) + \eta z^2 f''(z)$ for f given by (1), $zG(z) = g(z)g(-z)$ where g is defined by (10) and have

$$\begin{aligned} \Lambda &= |F(\eta, z) - (1 - \eta)G(z) + \eta zG'(z)| \\ &\quad - \left| F(\eta, z) - (1 - 2\eta) \left[(1 - \eta)G(z) + \eta zG'(z) \right] \right| \\ &= \left| \sum_{k=2}^{\infty} [k + \eta k(k-1)] a_k z^k - \sum_{k=2}^{\infty} [2\eta(k-1) + 1] c_k z^{2k-1} \right| \\ &\quad - \left| (2 - 2\varepsilon)z + \sum_{k=2}^{\infty} [k + \eta k(k-1)] a_k z^k \right. \\ &\quad \left. + (1 - 2\varepsilon) \sum_{k=2}^{\infty} [2\eta(k-1) + 1] c_k z^{2k-1} \right|. \end{aligned} \quad (20)$$

Hence, for $z \in \Delta$, we have the inequality

$$\begin{aligned} \Lambda &\leq \sum_{k=2}^{\infty} [k + \eta k(k-1)] |a_k| |z|^k + \sum_{k=2}^{\infty} [2\eta(k-1) + 1] |c_k| |z|^{2k-1} \\ &\quad - (2 - 2\varepsilon)|z| + \sum_{k=2}^{\infty} [k + \eta k(k-1)] |a_k| |z|^k \\ &\quad + |1 - 2\varepsilon| \sum_{k=2}^{\infty} [2\eta(k-1) + 1] |c_k| |z|^{2k-1} \\ &= -(2 - 2\varepsilon)|z| + \sum_{k=2}^{\infty} 2[k + \eta k(k-1)] |a_k| |z|^k \\ &\quad + (|1 - 2\varepsilon| + 1) \sum_{k=2}^{\infty} [2\eta(k-1) + 1] |c_k| |z|^{2k-1} \\ &\leq -(2 - 2\varepsilon) + \sum_{k=2}^{\infty} 2[k + \eta k(k-1)] |a_k| + (|1 - 2\varepsilon| + 1) \\ &\quad \cdot \sum_{k=2}^{\infty} [2\eta(k-1) + 1] |c_k| \leq 0. \end{aligned} \quad (21)$$

From these calculations, we see that $\Lambda < 0$. Also by (20), we can write

$$\begin{aligned} &\left| F(\eta, z) - \left[(1 - \eta)G(z) + \eta zG'(z) \right] \right| \\ &< \left| F(\eta, z) + (1 - 2\eta) \left[(1 - \eta)G(z) + \eta zG'(z) \right] \right|, \end{aligned} \quad (22)$$

which is equivalent to (9) and (8). Thus, $f \in \mathcal{K}_{\text{SP}}(\varepsilon, \eta)$, and it completes the proof. $\square$

The next theorem deals with the coefficient estimates $f \in \mathcal{K}_{\text{SP}}(\varepsilon, \eta)$.

Theorem 8. Let $0 \leq \varepsilon < 1$ and $0 \leq \eta \leq 1$. Suppose that f given by (1) and $g \in S^*(1/2)$ given by (10) are such that (8) holds. Then, for $k = 2, 3, \dots$ we have

$$\begin{aligned} &2k^2 \{1 + \eta(2k-1)\}^2 - 2(1 - \varepsilon)^2 \leq (1 - \varepsilon) \\ &\cdot \sum_{k=2}^n 2(2k-1) \eta_k^2 |a_{2k-1}| |c_{2k-1}| + (|2\varepsilon - 1| + 1) \eta_k^2 |c_{2k-1}|^2, \end{aligned} \quad (23)$$

where $\eta_k = 1 + 2\eta(k-1)$ and c_k is defined by (12). In particular, if $g(z) = z$, then

$$k(1 + \eta(2k-1)) |a_{2k}| \leq 1 - \varepsilon. \quad (24)$$

Proof. If $f \in \mathcal{K}_{\text{SP}}(\varepsilon, \eta)$ for some $g \in S^*(1/2)$, then (9) holds. Using Lemma 4, with $\alpha = \beta = 1$, we have

$$\frac{zf'(z) + \eta z^2 f''(z)}{(1 - \eta)G(z) + \eta zG'(z)} = \frac{1 + (2\varepsilon - 1)z\Phi(z)}{1 + z\Phi(z)}, \quad z \in \Delta, \quad (25)$$

where Φ is an analytic function in Δ , $|\Phi(z)| \leq 1$ for $z \in \Delta$, and G is given by (11). Then

$$\begin{aligned} &\left[\eta z^2 f''(z) + zf'(z) - (2\varepsilon - 1) \left\{ (1 - \eta)G(z) + \eta zG'(z) \right\} \right] \\ &\cdot z\Phi(z) = (1 - \eta)G(z) + \eta zG'(z) - zf'(z) - \eta z^2 f''(z). \end{aligned} \quad (26)$$

Now, $z\Phi(z) = \sum_{k=1}^{\infty} s^k z^k$. We see that $|\Phi(z)| \leq |z|$ for $z \in \Delta$. Thus

$$\begin{aligned} &\left[(2 - 2\varepsilon)z + \sum_{k=2}^{\infty} [k + \eta k(k-1)] a_k z^k - (2\varepsilon - 1) \sum_{k=2}^{\infty} [2\eta(k-1) + 1] c_k z^{2k-1} \right] \\ &\cdot \sum_{k=1}^{\infty} s^k z^k = \sum_{k=2}^{\infty} [2\eta(k-1) + 1] c_k z^{2k-1} - \sum_{k=2}^{\infty} [k + \eta k(k-1)] a_k z^k. \end{aligned} \quad (27)$$

Equating coefficients in (27), for $k \geq 2$, we can also write

$$\begin{aligned} &\left[(2 - 2\varepsilon)z + \sum_{k=1}^{n-1} 2k\eta_{2k} a_{2k} z^{2k} + \sum_{k=2}^n \eta_k [(2k-1)a_{2k-1} - (2\varepsilon - 1)c_{2k-1}] z^{2k-1} \right] \\ &\cdot z\Phi(z) = \sum_{k=2}^n \eta_k [c_{2k-1} - (2k-1)a_{2k-1}] z^{2k-1} - \sum_{k=1}^k 2k\eta_{2k} a_{2k} z^{2k} \\ &+ \sum_{k=2k+1}^{\infty} c_k z^k, \end{aligned} \quad (28)$$

where $\eta_k = 1 + 2\eta(k-1)$ and $\eta_{2k} = 1 + \eta(2k-1)$.

Then, we square and integrate along $|z| = r < 1$. After using the fact $|\Phi(z)| \leq |z| < 1$, we obtain

$$\begin{aligned} & \sum_{k=2}^n |\eta_k [c_{2k-1} - (2k-1)a_{2k-1}]|^2 r^{4k-2} \\ & + \sum_{k=1}^k |2k\eta_{2k}a_{2k}|^2 r^{4k} \\ & + \sum_{k=2k+1}^{\infty} |c_k|^2 r^{2k} < |2-2\varepsilon|^2 r^2 + \sum_{k=1}^{n-1} |2k\eta_{2k}a_{2k}|^2 r^{4k} \\ & + \sum_{k=2}^n |\eta_k [(2k-1)a_{2k-1} - (2\varepsilon-1)c_{2k-1}]|^2 r^{4k-2}. \end{aligned} \quad (29)$$

Letting $r \longrightarrow 1$, we have

$$\begin{aligned} & \sum_{k=2}^n |\eta_k [c_{2k-1} - (2k-1)a_{2k-1}]|^2 + \sum_{k=1}^k |2k\eta_{2k}a_{2k}|^2 < |2-2\varepsilon|^2 \\ & + \sum_{k=1}^{k-1} |2k\eta_{2k}a_{2k}|^2 + \sum_{k=2}^n |\eta_k [(2k-1)a_{2k-1} - (2\varepsilon-1)c_{2k-1}]|^2. \end{aligned} \quad (30)$$

Hence,

$$\begin{aligned} & 4k^2 \{1 + \eta(2k-1)\}^2 - 4(1-\varepsilon)^2 \\ & \leq \sum_{k=2}^n [|\eta_k [(2k-1)a_{2k-1} - (2\varepsilon-1)c_{2k-1}]|^2 - |\eta_k [c_{2k-1} - (2k-1)a_{2k-1}]|^2] \\ & \leq 2(1-\varepsilon) \sum_{k=2}^n 2\eta_k^2 |a_{2k-1}c_{2k-1}| + (|2\varepsilon-1|+1)\eta_k^2 |c_{2k-1}|^2. \end{aligned} \quad (31)$$

Thus, we have the inequality (23) which finishes the proof. $\square$

In the following theorem, we prove the growth and distortion theorems for f in the class $\mathcal{K}_{SP}(\varepsilon, \eta)$.

Theorem 9. *If $f \in \mathcal{K}_{SP}(\varepsilon, \eta)$, where $0 \leq \varepsilon < 1$ and $0 \leq \eta \leq 1$, then*

$$\chi_1(r, \varepsilon) \leq |f'(z)| \leq \chi_2(r, \varepsilon), \quad (32)$$

where

$$\begin{aligned} \chi_1(r, \varepsilon) &= \frac{1 - (1-2\varepsilon)r}{(1+r)(1+r^2)} - \frac{(-1)^{1/\eta}(1-\varepsilon)\{1 + (1+r) \operatorname{arctanh} r\}}{r^{1/\eta}(1+r)}, \\ \chi_2(r, \varepsilon) &= \frac{1 + (1-2\varepsilon)r}{(1-r)(1-r^2)} - \frac{(-1)^{1/\eta}(1-\varepsilon)}{4r^{1/\eta}} \\ & \quad \cdot \ln \left(\frac{1+r}{1-r} \right) - \frac{(1-\varepsilon)\{(-1)^{1/\eta}(1-\varepsilon) + 1\}}{2r^{1/\eta}(1-r)^2}. \end{aligned} \quad (33)$$

Also, we have

$$\chi_3(r, \varepsilon, \eta) \leq |f(z)| \leq \chi_4(r, \varepsilon, \eta) + \chi_5(r, \varepsilon, \eta), \quad (34)$$

where

$$\begin{aligned} \chi_3(r, \varepsilon, \eta) &= 1 - \varepsilon \ln \frac{1+r}{\sqrt{1+r^2}} \\ & \quad + \varepsilon \operatorname{arctan} r - \frac{\eta}{\eta-1} r^{-(1/\eta)+1} - (-1)^{1/\eta} \ln |1+r|, \\ \chi_4(r, \varepsilon, \eta) &= \frac{\varepsilon}{2} \ln \frac{1+r}{1-r} + \frac{(2-\varepsilon)r}{1-r} - \frac{\eta(-1)^{1/\eta}(1-\varepsilon)}{4(\eta-1)} \\ & \quad \cdot \left[\ln \left(\frac{1+r}{1-r} \right) r^{-(1/\eta)+1} - \ln \frac{|1+r|^{(-1)^{1/\eta}}}{|1-r|} \right], \\ \chi_5(r, \varepsilon, \eta) &= -\frac{1-\varepsilon}{2} \left[\frac{\eta}{\eta-1} [(-1)^{1/\eta} + 1] r^{-(1/\eta)+1} \right. \\ & \quad \left. - [(-1)^{1/\eta} + 1] \right] \ln |1-r|, \end{aligned} \quad (35)$$

and $|z| = r, 0 \leq r < 1$.

Proof. If $f \in \mathcal{K}_{SP}(\varepsilon, \eta)$, then for $g \in S^*(1/2)$, (8) holds. It follows from Lemma 4 that G in (11) is an odd starlike function. Then

$$\frac{r}{1+r^2} \leq |G(z)| \leq \frac{r}{1-r^2}, \quad |z| = r, 0 \leq r < 1, \quad (36)$$

$$\frac{1-r^2}{(1+r^2)^2} \leq |G'(z)| \leq \frac{1+r^2}{(1-r^2)^2}, \quad |z| = r, 0 \leq r < 1. \quad (37)$$

For detail, see [17]. From (8), we obtain a function h with real part greater than ε such that

$$\frac{zf'(z) + \eta z^2 f''(z)}{(1-\eta)G(z) + \eta z G'(z)} = h(z), \quad z \in \Delta. \quad (38)$$

It is known, see [18], that

$$\frac{1 - (1-2\varepsilon)r}{1+r} \leq |h(z)| \leq \frac{1 + (1-2\varepsilon)r}{1-r}, \quad |z| = r, 0 \leq r < 1. \quad (39)$$

Thus, from (36), (38), and (39), we obtain (32). From (32) for $z = te^{i\theta}$, we have

$$\begin{aligned}
|f(z)| &= \left| \int_0^z f'(s) ds \right| \leq \int_0^r |f'(te^{i\theta})| \\
&\cdot dt \leq \int_0^r \left[\frac{1 + (1-2\varepsilon)t}{(1-t)^2(1+t)} - \frac{(-1)^{1/\eta}(1-\varepsilon)}{4t^{1/\eta}} \right] \\
&\cdot \ln \left(\frac{1+r}{1-r} \right) - \frac{(1-\varepsilon)\{(-1)^{1/\eta}(1-\varepsilon)+1\}}{2t^{1/\eta}(1-t)^2} \Bigg] \\
&\cdot dt = \frac{\varepsilon}{2} \ln \frac{1+r}{1-r} + (2-\varepsilon) \frac{r}{1-r} - \frac{\eta(-1)^{1/\eta}(1-\varepsilon)}{4(\eta-1)} \\
&\cdot \left[\ln \left(\frac{1+r}{1-r} \right) r^{-(1/\eta)+1} - \ln \frac{|1+r|^{(-1)^{1/\eta}}}{|1-r|} \right] \\
&- \frac{1-\varepsilon}{2} \left[\frac{\eta}{\eta-1} [(-1)^{1/\eta}+1] r^{-(1/\eta)+1} - [(-1)^{1/\eta}+1] \right] \ln |1-r|.
\end{aligned} \tag{40}$$

This gives us the right-hand side of the inequality (34). To prove the left-hand side of the inequality (34), we must show that it holds for the nearest point $f(z_o)$ from zero, where $|z_o| = r$ and $0 < r < 1$. Moreover, we have $|f(z)| \geq |f(z_o)|$ for $|z| = r$. Since $f \in \mathcal{K}$, we know that the function f is univalent in the unit disc Δ . We conclude that the original image of the line segment

$$\begin{aligned}
|f(z)| &= \int_{f(C)} |d\Phi| = \int_C |f'(z)| |d(z)| \\
&\geq \int_0^r \left[\frac{1 - (1-2\varepsilon)t}{(1+t)(1+t^2)} - \frac{(-1)^{1/\eta}(1-\varepsilon)\{1 + (1+r) \operatorname{arctanh} t\}}{t^{1/\eta}(1+t)} \right] \\
&\cdot dt = (1-\varepsilon) \ln \frac{1+r}{\sqrt{1+r^2}} \\
&+ \varepsilon \operatorname{arctan} r - \frac{\eta}{\eta-1} r^{-(1/\eta)+1} - (-1)^{1/\eta} \ln |1+r|.
\end{aligned} \tag{41}$$

This finishes the proof of the inequality (34). $\square$

3. Conclusions

In this research, we studied a new subclass of functions with symmetric points and derived an equivalent formulation of these functions in terms of subordination. Moreover, we determined coefficient estimates and discussed characterizations for functions belonging to this new class. We also obtained distortion and growth results. We observed that our findings are related with the existing literature of the subject.

Data Availability

There is no data available.

Disclosure

The research is performed as part of the employment of the “Mirpur University of Science and Technology, Mirpur-10250 (AJK), Pakistan.”

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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