Research Article

The Research of Sequence Shadowing Property and Regularly Recurrent Point on the Double Inverse Limit Space

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According to the definition of sequence shadowing property and regularly recurrent point in the inverse limit space, we introduce the concept of sequence shadowing property and regularly recurrent point in the double inverse limit space and study their dynamical properties. The following results are obtained: (1) Regularly recurrent point sets of the double shift map $\sigma_f \circ \sigma_g$ are equal to the double inverse limit space of the double self-map $f \circ g$ in the regularly recurrent point sets. (2) The double self-map $f \circ g$ has sequence shadowing property if and only if the double shift map $\sigma_f \circ \sigma_g$ has sequence shadowing property. Thus, the conclusions of sequence shadowing property and regularly recurrent point are generalized to the double inverse limit space.

1. Introduction

Inverse limit space plays an important role in topological dynamical systems, and a series of research results have been obtained (see [1–4]). With the gradual deepening of problem research, it is found that the finite symbols in the inverse limit space have been difficult to solve the practical problems in life. Therefore, scholars began to try to extend the inverse limit space to the double inverse limit space and study the dynamical properties of Lipschitz tracking property, topological transitivity, and nonwandering points in the double inverse limit space. The relevant results are shown in [5–8]. Up to now, there are few new results in the double inverse limit space and it needs to be further studied. In addition, it is known that shadowing property and regularly recurrent point have always been the focus of topological dynamical systems. At present, many scholars have studied them in the inverse limit space and obtained valuable research results (see [9–15]). But there are few research results about them in the double inverse limit space. In this paper, firstly, the new concepts of sequence shadowing property and regularly recurrent point are given in double inverse

limit space. Secondly, by using the method of mutual disclosure of the double self-map $f \circ g$ and the double shift map $\sigma_f \circ \sigma_g$, we can obtain the following theorem.

Theorem 1. *Let (X, d) be a compact metric space, $f : X \rightarrow X$ be a continuous map, $g : X \rightarrow X$ be a continuous map, and $f \circ g = g \circ f$. Then, we have that*

$$RR(\sigma_f \circ \sigma_g) = \limleftarrow \langle RR(f \circ g), f \circ g \rangle. \quad (1)$$

Theorem 2. *Let (X, d) be a compact metric space, $f : X \rightarrow X$ be a homeomorphic map, $g : X \rightarrow X$ be a homeomorphic map, and $f \circ g = g \circ f$. Then, the double self-map $f \circ g$ has sequence shadowing property if and only if the double shift map $\sigma_f \circ \sigma_g$ has sequence shadowing property.*

The results enrich the conclusions of sequence shadowing property and regularly recurrent point in the double inverse limit space.

2. Regularly Recurrent Point in the Double Inverse Limit Space

Definition 3. Let (X, d) be a metric space and f be a continuous map from X to X . f is said to be a homeomorphic map if f is one-to-one and the map f and f^{-1} are continuous.

Definition 4 (see [7]). Let (X, d) be a compact metric space, $f : X \rightarrow X$ be a continuous map, $g : X \rightarrow X$ be a continuous map and $f \circ g = g \circ f$. $X_{f \circ g}$ is said to be the double inverse limit spaces of X if we write

$$X_{f \circ g} = \left\{ \bar{x} = \{x_{i,j}\}_{i,j=-\infty}^{+\infty} : f^m g^n(x_{i,j}) = x_{i-m,j-n}, \forall m, n \geq 0, \forall i, j \in \mathbb{Z} \right\}, \quad (2)$$

where we write $X_{f \circ g} = \lim_{\leftarrow} \langle X, f \circ g \rangle$.

The metric $\bar{d}$ in $X_{f \circ g}$ is defined by

$$\bar{d}(\bar{x}, \bar{y}) = \max_{i,j \in \mathbb{Z}} \left\{ \frac{d(x_{i,j}, y_{i,j})}{2^{|i|+|j|}} \right\}, \quad (3)$$

where $\bar{x} = \{x_{i,j}\}_{i,j=-\infty}^{+\infty} \in X_{f \circ g}$ and $\bar{y} = \{y_{i,j}\}_{i,j=-\infty}^{+\infty} \in X_{f \circ g}$.

The double shift map $\sigma_{f \circ g} : X_{f \circ g} \rightarrow X_{f \circ g}$ is defined by

$$\sigma_f^m \sigma_g^n \left(\{x_{i,j}\}_{i,j=-\infty}^{+\infty} \right) = \{x_{i-m,j-n}\}_{i,j=-\infty}^{+\infty}. \quad (4)$$

Thus, $(X_{f \circ g}, \bar{d})$ is compact metric space, and the double shift map $\sigma_{f \circ g}$ is homeomorphic. In addition, we have

$$\sigma_f \circ \sigma_g = \sigma_g \circ \sigma_f. \quad (5)$$

For any $i, j \in \mathbb{Z}$, the projection map $\pi_{i,j} : X_{f \circ g} \rightarrow X$ is defined by

$$\pi_{i,j} \left(\{x_{i,j}\}_{i,j=-\infty}^{+\infty} \right) = x_{i,j}. \quad (6)$$

Then, the map $\pi_{i,j}$ is a continuous, surjective, and open map.

Definition 5. Let (X, d) be a metric space and f be a continuous map from X to X . A point $x \in X$ is called to be a regularly recurrent point if each open set $x \in U$, there exists positive integer $m > 0$ such that for any positive integer $k > 0$, we have $f^{km}(x) \in U$, denoted by $RR(f)$ the regularly recurrent point set of the map f .

Remark 6. According to the concept of regularly recurrent point of the map f , we will give the concept of regularly recurrent point of the double map $f \circ g$.

Definition 7. Let (X, d) be a compact metric space, $f : X \rightarrow X$ be a continuous map, $g : X \rightarrow X$ be a continuous

map, and $f \circ g = g \circ f$. A point $x \in X$ is called to be a regularly recurrent point if each open set $x \in U$, there exists positive integer $n, m > 0$ such that for any positive integer $k, l > 0$, we have $f^{kn} \circ g^{lm}(x) \in U$, denoted by $RR(f \circ g)$ the regularly recurrent points set of the double map $f \circ g$.

In order to prove the main theorem in this section, we will give two lemmas below.

Lemma 8 (see [8]). Let (X, d) be a compact metric space, $f : X \rightarrow X$ be a continuous map, $g : X \rightarrow X$ be a continuous map, and $f \circ g = g \circ f$. Then, for any positive integer $i, j \in \mathbb{Z}$ and nonnegative integer $m, n \geq 0$, we have

$$\pi_{i,j} \circ \sigma_f^n \circ \sigma_g^m = f^n \circ g^m \circ \pi_{i,j}. \quad (7)$$

Lemma 9. Let (X, d) be a compact metric space, $f : X \rightarrow X$ be a continuous map, $g : X \rightarrow X$ be a continuous map, and $f \circ g = g \circ f$. Then, we have that

$$f \circ g(RR(f \circ g)) \subset RR(f \circ g). \quad (8)$$

Proof. Suppose $x \in RR(f \circ g)$. Let U be an any open set containing the point $f \circ g(x)$. Then, $(f \circ g)^{-1}(U)$ is an open set containing the point x . According to $x \in RR(f \circ g)$, there exists positive integer $n, m > 0$ such that for any positive integer $k, l > 0$, we have

$$f^{kn} \circ g^{lm}(x) \in (f \circ g)^{-1}(U). \quad (9)$$

Thus, $f^{kn} \circ g^{lm}(f \circ g(x)) \in U$. So $f \circ g(x) \in RR(f \circ g)$. Hence, $f \circ g(RR(f \circ g)) \subset RR(f \circ g)$. $\square$

Remark 10. The regularly recurrent point set $RR(f \circ g)$ is invariant to the double map $f \circ g$. So we can study the double inverse limit space of the double self-map $f \circ g$ in the regularly recurrent point set $RR(f \circ g)$. Now let us give the proof process of Theorem 11.

Theorem 11. Let (X, d) be a compact metric space, $f : X \rightarrow X$ be a continuous map, $g : X \rightarrow X$ be a continuous map, and $f \circ g = g \circ f$. Then, we have

$$RR(\sigma_f \circ \sigma_g) = \lim_{\leftarrow} \langle RR(f \circ g), f \circ g \rangle. \quad (10)$$

Proof. Suppose $\bar{x} = \{x_{i,j}\}_{i,j=-\infty}^{+\infty} \in RR(\sigma_f \circ \sigma_g)$. For any integer $i, j \in \mathbb{Z}$, let U be an any open set containing the point $x_{i,j}$. Thus, $\pi_{i,j}^{-1}(U)$ is an open set containing the point $\bar{x}$. According to $\bar{x} \in RR(\sigma_f \circ \sigma_g)$, there exists positive integer $n, m > 0$ such that for any positive integer $k, l > 0$, we have that

$$\sigma_f^{kn} \circ \sigma_g^{lm}(\bar{x}) \in \pi_{i,j}^{-1}(U). \quad (11)$$

Thus, we have that

$$\pi_{i,j} \circ \sigma_f^{kn} \circ \sigma_g^{lm}(\bar{x}) \in U. \quad (12)$$

By Lemma 8, we can get that

$$f^{kn} \circ g^{lm} \circ \pi_{i,j}(\bar{x}) \in U. \quad (13)$$

That is,

$$f^{kn} \circ g^{lm}(x_{i,j}) \in U. \quad (14)$$

So $x_{i,j} \in \text{RR}(f \circ g)$. Hence, $\text{RR}(\sigma_f \circ \sigma_g) \subset \limleftarrow < \text{RR}(f \circ g), f \circ g >$.

Suppose $\bar{y} = \{y_{i,j}\}_{i,j \in -\infty}^{+\infty} \in \limleftarrow < \text{RR}(f \circ g), f \circ g >$. Then, for any integer $i, j \in \mathbb{Z}$, we have

$$y_{i,j} \in \text{RR}(f \circ g). \quad (15)$$

Let V be an any open set containing the point $\bar{y}$. Then, $\pi_{i,j}(V)$ is an open set containing the point $y_{i,j}$. Thus, there exists positive integer $t_1, t_2 > 0$ such that for any positive integer $p, s > 0$, we have that

$$f^{pt_1} \circ g^{st_2}(y_{i,j}) \in \pi_{i,j}(V). \quad (16)$$

Thus, we have that

$$\pi_{i,j}^{-1} \circ f^{pt_1} \circ g^{st_2}(y_{i,j}) \in V. \quad (17)$$

By Lemma 8, we can get that

$$\sigma_f^{pt_1} \circ \sigma_g^{st_2} \circ \pi_{i,j}^{-1}(y_{i,j}) \in V. \quad (18)$$

Thus, we have that

$$\sigma_f^{pt_1} \circ \sigma_g^{st_2}(\bar{y}) \in V. \quad (19)$$

So $\bar{y} \in \text{RR}(\sigma_f \circ \sigma_g)$. Hence, $\limleftarrow < \text{RR}(f \circ g), f \circ g > \subset \text{RR}(\sigma_f \circ \sigma_g)$. This completes the proof. $\square$

3. Sequence Shadowing Property in the Double Inverse Limit Space

Definition 12 (see [4]). Let (X, d) be a metric space and f be a continuous map from X to X . The sequence $\{x_i\}_{i=0}^{\infty}$ is called δ -pseudo orbit of f if for any $i \geq 0$, we have $d(f(x_i), x_{i+1}) < \delta$.

Definition 13 (see [4]). Let (X, d) be a metric space and f be a continuous map from X to X . The sequence $\{x_i\}_{i=0}^{\infty}$ is said to be ε -shadowed by the point x in X if for any $i \geq 0$, we have $d(f^i(x), x_i) < \varepsilon$.

Definition 14 (see [4]). Let (X, d) be a metric space and f be a continuous map from X to X . The map f has a sequence shadowing property if each $\varepsilon > 0$, there exists $\delta > 0$ such that any δ -pseudo orbit $\{x_i\}_{i=0}^{\infty}$ of f , there exists a point x in X

and nonnegative integer sequence $\{n_i\}_{i=0}^{\infty}$ such that the sequence $\{x_{n_i}\}_{i=0}^{\infty}$ is ε -shadowed by the point x .

Remark 15. By Definitions 12–14, we will give the concept of sequence shadowing property of the double map $f \circ g$.

Definition 16. Let (X, d) be a compact metric space, $f : X \rightarrow X$ be a continuous map, $g : X \rightarrow X$ be a continuous map, and $f \circ g = g \circ f$. The sequence $\{x_{i,j}\}_{i,j=-\infty}^{+\infty}$ is called δ -pseudo orbit of the double map $f \circ g$ if for any integer $i, j \in \mathbb{Z}$, we have $d(f \circ g(x_{i,j}), x_{i+1,j+1}) < \delta$.

Definition 17. Let (X, d) be a compact metric space, $f : X \rightarrow X$ be a continuous map, $g : X \rightarrow X$ be a continuous map, and $f \circ g = g \circ f$. The sequence $\{x_{i,j}\}_{i,j=-\infty}^{+\infty}$ is said to be ε -shadowed by the point x in X if for any positive integer $i, j \in \mathbb{Z}$, we have

$$d(f^i \circ g^j(x), x_{i,j}) < \varepsilon. \quad (20)$$

Definition 18. Let (X, d) be a compact metric space, $f : X \rightarrow X$ be a continuous map, $g : X \rightarrow X$ be a continuous map, and $f \circ g = g \circ f$. The double map $f \circ g$ has sequence shadowing property if each $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ -pseudo orbit $\{x_{i,j}\}_{i,j=-\infty}^{+\infty}$ of $f \circ g$, there exists a point x in X and increasing positive integer sequence $\{n_i\}_{i=-\infty}^{+\infty}$ and $\{m_j\}_{j=-\infty}^{+\infty}$ such that the sequence $\{x_{n_i, m_j}\}_{i,j=-\infty}^{+\infty}$ is ε -shadowed by the point $x \in X$.

Now we begin to prove Theorem 19.

Theorem 19. Let (X, d) be a compact metric space, $f : X \rightarrow X$ be a homeomorphic map, $g : X \rightarrow X$ be a homeomorphic map, and $f \circ g = g \circ f$. Then, the double map $f \circ g$ has sequence shadowing property if and only if the double shift map $\sigma_f \circ \sigma_g$ has sequence shadowing property.

Proof. Suppose the double map $f \circ g$ has the sequence shadowing property. Since X is compact metric space, it is bounded. Write $M = \text{diam}(X)$. Then, for any $\varepsilon > 0$, there exists positive integer $m_1 > 0$ and $n_1 > 0$ such that

$$\begin{aligned} \frac{M}{2^{m_1}} &< \varepsilon, \\ \frac{M}{2^{n_1}} &< \varepsilon. \end{aligned} \quad (21)$$

Since the map f and g is uniformly continuous, it follows that for any $0 \leq i \leq 2m_1$ and $0 \leq j \leq 2n_1$, there exists $0 < \delta_1 < \varepsilon$ such that $d(u, v) < \delta_1$ implies

$$d(f^i \circ g^j(u), f^i \circ g^j(v)) < \varepsilon. \quad (22)$$

Note that the double map $f \circ g$ has the sequence shadowing property, and it follows that there exists $0 < \delta_2 < \delta_1$

such that for any δ_2 -pseudo orbit $\{x_{ij}\}_{i,j=-\infty}^{+\infty}$ of $f \circ g$, there exists a point $x \in X$ and increasing positive integer sequence $\{n_i\}_{i=-\infty}^{+\infty}$ and $\{m_i\}_{i=-\infty}^{+\infty}$ such that the sequence $\{x_{n_i, m_i}\}_{i,j=-\infty}^{+\infty}$ is δ_1 -shadowed by the point x . Let $\{\bar{x}^{s,t}\}_{s,t=-\infty}^{+\infty}$ be $\delta_2/2^{m_1+n_1}$ -pseudo orbit of the double map $\sigma_f \circ \sigma_g$ where $\bar{x}^{s,t} = \{x_{i,j}^{s,t}\}_{i,j=-\infty}^{+\infty} \in X_{f \circ g}$. Then, for any integer $s, t \in \mathbb{Z}$, we have that

$$\bar{d}(\sigma_f \circ \sigma_g(\bar{x}^{s,t}), \bar{x}^{s+1, t+1}) < \frac{\delta_2}{2^{m_1+n_1}}. \quad (23)$$

According to the definition of the metric $\bar{d}$, for any integer $s, t \in \mathbb{Z}$, we can get that

$$\frac{d(f \circ g(x_{m_1, n_1}^{s,t}), x_{m_1, n_1}^{s+1, t+1})}{2^{m_1+n_1}} < \frac{\delta_2}{2^{m_1+n_1}}. \quad (24)$$

So we have that

$$d(f \circ g(x_{m_1, n_1}^{s,t}), x_{m_1, n_1}^{s+1, t+1}) < \delta_2. \quad (25)$$

Thus, $\{x_{m_1, n_1}^{s,t}\}_{s,t=-\infty}^{+\infty}$ is δ_2 -pseudo orbit of the double map $f \circ g$ in X . Hence, there exists $x \in X$ and increasing positive integer sequence $\{p_s\}_{s=-\infty}^{+\infty}$ and $\{l_t\}_{t=-\infty}^{+\infty}$ such that for any integer $s, t \in \mathbb{Z}$, we have that

$$d(f^{p_s} \circ g^{l_t}(x), x_{m_1, n_1}^{p_s, l_t}) < \delta_1. \quad (26)$$

By (22), for any $0 \leq i \leq 2m_1$ and $0 \leq j \leq 2n_1$, we can get

$$d(f^i \circ g^j(f^{p_s} \circ g^{l_t}(x)), f^i \circ g^j(x_{m_1, n_1}^{p_s, l_t})) < \varepsilon. \quad (27)$$

Let $\bar{y} = \{f^{m_1-h} g^{n_1-k}(x)\}_{h,k=-\infty}^{+\infty}$ and $y_{h,k} = f^{m_1-h} g^{n_1-k}(x)$. For any $i \geq 0$ and $j \geq 0$, we have that

$$\begin{aligned} f^i \circ g^j(y_{h,k}) &= f^i \circ g^j(f^{m_1-h} g^{n_1-k}(x)) \\ &= f^{m_1-(h-i)} \circ g^{n_1-(k-j)}(x) = y_{h-i, k-j}. \end{aligned} \quad (28)$$

So the point $\bar{y}$ is in $X_{f \circ g}$. It is easy to know that

$$\bar{d}(\sigma_f^{p_s} \circ \sigma_g^{l_t}(\bar{y}), \bar{x}^{p_s, l_t}) = \max_{h,k \in \mathbb{Z}} \left\{ \frac{d(f^{p_s} \circ g^{l_t}(f^{m_1-h} g^{n_1-k}(x)), x_{h,k}^{p_s, l_t})}{2^{|h|+|k|}} \right\}. \quad (29)$$

When $|h| > m_1$ or $|k| > n_1$, we can, respectively, get that

$$\begin{aligned} \frac{d(f^{p_s} \circ g^{l_t}(f^{m_1-h} g^{n_1-k}(x)), x_{h,k}^{p_s, l_t})}{2^{|h|+|k|}} &< \frac{M}{2^{m_1}} < \varepsilon, \\ \frac{d(f^{p_s} \circ g^{l_t}(f^{m_1-h} g^{n_1-k}(x)), x_{h,k}^{p_s, l_t})}{2^{|h|+|k|}} &< \frac{M}{2^{n_1}} < \varepsilon. \end{aligned} \quad (30)$$

When $|h| > m_1$ and $|k| > n_1$, for any $0 \leq i \leq 2m_1$ and $0 \leq j \leq 2n_1$, we have that

$$\begin{aligned} &\frac{d(f^{p_s} \circ g^{l_t}(f^{m_1-h} g^{n_1-k}(x)), x_{h,k}^{p_s, l_t})}{2^{|h|+|k|}} \\ &= \frac{d(f^i \circ g^j(f^{p_s} \circ g^{l_t}(x)), f^i \circ g^j(x_{m_1, n_1}^{p_s, l_t}))}{2^{|m_1-i|+|n_1-j|}}, \end{aligned} \quad (31)$$

where $m_1 - h = i$ and $n_1 - k = j$.

By (27), we can obtain that

$$\frac{d(f^i \circ g^j(f^{p_s} \circ g^{l_t}(x)), f^i \circ g^j(x_{m_1, n_1}^{p_s, l_t}))}{2^{|m_1-i|+|n_1-j|}} < \varepsilon. \quad (32)$$

Thus, for any $h, k \in \mathbb{Z}$ we can get that

$$\frac{d(f^{p_s} \circ g^{l_t}(f^{m_1-h} g^{n_1-k}(x)), x_{h,k}^{p_s, l_t})}{2^{|h|+|k|}} < \varepsilon. \quad (33)$$

So for any integer $s, t \in \mathbb{Z}$, we have that

$$\bar{d}(\sigma_f^{p_s} \circ \sigma_g^{l_t}(\bar{y}), \bar{x}^{p_s, l_t}) < \varepsilon. \quad (34)$$

Hence, the double shift map $\sigma_f \circ \sigma_g$ has the sequence shadowing property.

Suppose the double shift map has the sequence shadowing property. For each $\eta > 0$, there exists $\delta_3 > 0$ such that for any δ_3 -pseudo orbit $\{y_{i,j}\}_{i,j=-\infty}^{+\infty}$ of $\sigma_f \circ \sigma_g$, there exists a point $y \in X$ and increasing positive integer sequence $\{n_i\}_{i=-\infty}^{+\infty}$ and $\{m_i\}_{i=-\infty}^{+\infty}$ such that the sequence $\{x_{n_i, m_i}\}_{i,j=-\infty}^{+\infty}$ is η -shadowed by the point y . Let $m_2 > 0$ and $n_2 > 0$ such that

$$\begin{aligned} \frac{M}{2^{m_2}} &< \delta_3, \\ \frac{M}{2^{n_2}} &< \delta_3. \end{aligned} \quad (35)$$

Since the map f and g is uniformly continuous, it follows that for any $-m_2 \leq s \leq m_2$ and $-n_2 \leq t \leq n_2$, there exists $0 < \delta_4 < \delta_3$ such that $d(u, v) < \delta_4$ implies

$$d(f^s \circ g^t(u), f^s \circ g^t(v)) < \delta_3. \quad (36)$$

Let $\{y_{i,j}\}_{i,j=-\infty}^{+\infty}$ be δ_4 -pseudo orbit of the double map $f \circ g$. Then, for any $i, j \in \mathbb{Z}$, we have that

$$d(f \circ g(y_{i,j}), y_{i+1, j+1}) < \delta_4. \quad (37)$$

By (36), when $-m_2 \leq s \leq m_2$ and $-n_2 \leq t \leq n_2$, we have that

$$d(f^{s+1} \circ g^{t+1}(y_{i,j}), f^s \circ g^t(y_{i+1, j+1})) < \delta_3. \quad (38)$$

For every $y_{i,j}$, let $\bar{y}^{i,j} = \{f^{-h}g^{-k}(y_{i,j})\}_{h,k=-\infty}^{\infty}$. It is easy to know that $\bar{y}^{i,j} \in X_{f \circ g}$

$$\begin{aligned} & \bar{d}(\sigma_f \circ \sigma_g(\bar{y}^{i,j}), (\bar{y}^{i+1,j+1})) \\ &= \max_{h,k \in \mathbb{Z}} \left\{ \frac{d(f \circ g(f^{-h}g^{-k}(y_{i,j})), f^{-h}g^{-k}(y_{i+1,j+1}))}{2^{|h|+|k|}} \right\}. \end{aligned} \quad (39)$$

When $|h| > m_2$ or $|k| > n_2$, we can, respectively, get that

$$\begin{aligned} & \frac{d(f \circ g(f^{-h}g^{-k}(y_{i,j})), f^{-h}g^{-k}(y_{i+1,j+1}))}{2^{|h|+|k|}} < \frac{M}{2^{m_2}} < \delta_3, \\ & \frac{d(f \circ g(f^{-h}g^{-k}(y_{i,j})), f^{-h}g^{-k}(y_{i+1,j+1}))}{2^{|h|+|k|}} < \frac{M}{2^{n_2}} < \delta_3. \end{aligned} \quad (40)$$

When $|h| > m_2$ and $|k| > n_2$, for any $-m_2 \leq s \leq m_2$ and $-n_2 \leq t \leq n_2$, we have that

$$\begin{aligned} & \frac{d(f \circ g(f^{-h}g^{-k}(y_{i,j})), f^{-h}g^{-k}(y_{i+1,j+1}))}{2^{|h|+|k|}} \\ &= \frac{d(f^{s+1} \circ g^{t+1}(y_{i,j}), f^s \circ g^t(y_{i+1,j+1}))}{2^{|-s|+|-t|}}, \end{aligned} \quad (41)$$

where $-h = s$ and $-k = t$.

By (38), we can get that

$$\frac{d(f^{s+1} \circ g^{t+1}(y_{i,j}), f^s \circ g^t(y_{i+1,j+1}))}{2^{|-s|+|-t|}} < \delta_3. \quad (42)$$

So for any integer $h, k \in \mathbb{Z}$, we have that

$$\frac{d(f \circ g(f^{-h}g^{-k}(y_{i,j})), f^{-h}g^{-k}(y_{i+1,j+1}))}{2^{|h|+|k|}} < \delta_3. \quad (43)$$

Thus for any integer $i, j \in \mathbb{Z}$, we have that

$$\bar{d}(\sigma_f \circ \sigma_g(\bar{y}^{i,j}), (\bar{y}^{i+1,j+1})) < \delta_3. \quad (44)$$

Hence, $\{\bar{y}^{i,j}\}_{i,j=-\infty}^{+\infty}$ is δ_3 -pseudo orbit of the double map $\sigma_f \circ \sigma_g$ in $X_{f \circ g}$. So there exists a point $\bar{z} = \{z_{h,k}\}_{h,k=-\infty}^{+\infty} \in X_{f \circ g}$ and increasing positive integer sequence $\{p_i\}_{i=-\infty}^{+\infty}$ and $\{l_j\}_{j=-\infty}^{+\infty}$ such that for any integer $i, j \in \mathbb{Z}$, we have that

$$\bar{d}(\sigma_f^{p_i} \circ \sigma_g^{l_j}(\bar{z}), \bar{y}^{p_i, l_j}) < \eta. \quad (45)$$

According to the definition of the metric $\bar{d}$, we can get that

$$d(f^{p_i} \circ g^{l_j}(z_{00}), y_{p_i, l_j}) < \eta. \quad (46)$$

Hence, the map $f \circ g$ has the sequence shadowing property. Thus, we end the proof. $\square$

4. Conclusion

Firstly, we give the new concepts of sequence shadowing property and regularly recurrent point of double inverse limit space in this paper. Secondly, let (X, d) be a compact metric space, $f : X \rightarrow X$ be a homeomorphic map, $g : X \rightarrow X$ be a homeomorphic map, and $f \circ g = g \circ f$. By using the method of mutual disclosure of the double self-map $f \circ g$ and the double shift map $\sigma_f \circ \sigma_g$, we can obtain the following results: (1) $\text{RR}(\sigma_f \circ \sigma_g) = \limleftarrow < \text{RR}(f \circ g), f \circ g >$; (2) the double self-map $f \circ g$ has sequence shadowing property if and only if the double shift map $\sigma_f \circ \sigma_g$ has sequence shadowing property. These results enrich the conclusions of sequence has shadowing property and regularly recurrent point in double inverse limit space. Most importantly, it provided the theoretical basis and scientific foundation for the application of sequence shadowing property in computational mathematics and biological mathematics.

Data Availability

The data used to support the findings of this study are included within references [1–15] in the article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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