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Research Article

Relational Meir-Keeler Contractions and Common Fixed Point Theorems

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In this article, we prove some coincidence and common fixed point theorems under the relation-theoretic Meir-Keeler contractions in a metric space endowed with a locally finitely T-transitive binary relation. Our newly proved results generalize, extend, and sharpen some existing coincidence point as well as fixed point theorems existing in the literature. Moreover, we give some examples to affirm the efficacy of our results.

1. Introduction

Banach [1], a Polish mathematician, established the most successful result in fixed point theory, the Banach contraction principle (in short, BCP), in 1922, which says that a contraction mapping on a complete metric space has a unique fixed point. One of the noted generalizations of BCP comprising the concept of coincidence point (in short, CP) and common fixed point (in short, CFP) theorems was established by Jungck [2] in 1976. In succeeding years, many researchers introduced relatively weaker version of commuting mappings and developed exciting CFP results, see [3, 4].

On the other hand, generalizations of the underlying space have been trending since some decades. One of such important generalizations was initiated by Turinici [5, 6] in 1986, where he proved fixed point results in a partial ordered set. In this continuation, Alam and Imdad [7] generalized the BCP using a binary relation. Since then, many relation-theoretic fixed point theorems are being studied regularly, see [8, 9] and references therein.

Several researchers reported numerous fixed point results employing relatively more generalized contractions. One of such vital contractions was due to Meir and Keeler [10] in 1969, which was further extended by Rao and Rao [11]. In 2013, Patel et al. [12] established some CFP theorems for three and four self-mappings satisfying generalized Meir-Keeler α-contraction in metric spaces. Some generalizations of Meir-Keeler contraction in the framework of different types of spaces have also been reported, see [13–16]. Recently, Sk et al. [17] introduced the Meir-Keeler contraction in relation-theoretic sense and extended relation-theoretic contraction principle to relation-theoretic Meir-Keeler contraction principle.

In this paper, we prove some coincidence and common fixed point theorems using the relation-theoretic Meir-Keeler contraction in a metric space endowed with a locally finitely T-transitive binary relation. We also equip several examples to exhibit the significance of these new findings.

2. Preliminaries

We will go over some basic definitions in this section that will help us to prove our primary results. Throughout the paper, we pertain to $\mathbb{N} \cup \{0\}$ as $\mathscr{K}_0$, and empty set as $\varnothing$.

Definition 1 (see [18]). Let $\mathscr{X} \neq \varnothing$ be a set. A "binary relation" is a subset $\mathfrak{R}$ of $\mathscr{X}^2$. The subsets $\mathscr{X}^2$ and $\varnothing$ of $\mathscr{X}^2$ are called the "universal relation" and "empty relation," respectively.

Definition 2 (see [7]). Let $\mathscr{X} \neq \varnothing$ be a set with a binary relation $\mathfrak{R}$. If either $(\varrho, \sigma) \in \mathfrak{R}$ or $(\sigma, \varrho) \in \mathfrak{R}$ for $\varrho, \sigma \in \mathscr{X}$, then ϱ and σ are called as "$\mathfrak{R}$-comparative." $[\varrho, \sigma] \in \mathfrak{R}$ is the notion for it.

Definition 3 (see [18–23]). Let $\mathscr{X} \neq \varnothing$ be a set with a binary relation $\mathfrak{R}$. Then, the relation $\mathfrak{R}$ is called

(a) "amorphous" if $\mathfrak{R}$ has no precise attribute

(b) "reflexive" if $(\varrho, \varrho) \in \mathfrak{R} \forall \varrho \in \mathscr{X}$

(c) "symmetric" if $(\varrho, \sigma) \in \mathfrak{R} (\sigma, \varrho) \in \mathfrak{R}$

(d) "anti-symmetric" if $(\varrho, \sigma) \in \mathfrak{R}$ and $(\sigma, \varrho) \in \mathfrak{R} \varrho = \sigma$

(e) "transitive" if $(\varrho, \sigma) \in \mathfrak{R}$ and $(\sigma, \mathrm{w}) \in \mathfrak{R} (\varrho, \mathrm{w}) \in \mathfrak{R}$

(f) "complete", "connected" or "dichotomous" if $[\varrho, \sigma] \in \mathfrak{R} \forall \varrho, \sigma \in \mathscr{X}$

(g) "partial order" if $\mathfrak{R}$ is "reflexive", "anti-symmetric" and "transitive"

Definition 4 (see [18]). Let $\mathfrak{R}$ be a binary relation on a set $\mathscr{X} \neq \varnothing$. Then,

$$\mathfrak{R}^{-1} = \left\{ (\varrho, \sigma) \in \mathscr{X}^2 : (\sigma, \varrho) \in \mathfrak{R} \right\} \text{ and } \mathfrak{R}^s = \mathfrak{R} \cup \mathfrak{R}^{-1}, \quad (1)$$

are called inverse relation and symmetric closure of $\mathfrak{R}$, respectively.

Proposition 5 (see [7]). *Let $\mathscr{X} \neq \varnothing$ be a set with a binary relation $\mathfrak{R}$. Then, for $\varrho, \sigma \in \mathscr{X}$,*

$$(\varrho, \sigma) \in \mathfrak{R}^s \Longrightarrow [\varrho, \sigma] \in \mathfrak{R}. \quad (2)$$

Definition 6 (see [24]). Let $\mathscr{X} \neq \varnothing$ be a set with a binary relation $\mathfrak{R}$ and $\mathscr{S} \subseteq \mathscr{X}$. Then, the set $\mathfrak{R}|_{\mathscr{S}} = \mathfrak{R} \cap \mathscr{S}^2$ is defined as the restriction of $\mathfrak{R}$ to $\mathscr{S}$.

Definition 7 (see [7]). Let $\mathscr{X} \neq \varnothing$ be a set with a binary relation $\mathfrak{R}$. A sequence $\{\varrho_{\Bbbk}\} \subset \mathscr{X}$ is called $\mathfrak{R}$-preserving if

$$(\varrho_{\Bbbk}, \varrho_{\Bbbk+1}) \in \mathfrak{R} \quad \forall \Bbbk \in \mathscr{K}_0. \quad (3)$$

Definition 8 (see [7, 25]). Let T and H be two self-mappings on a set $\mathscr{X} \neq \varnothing$ and $\mathfrak{R}$ a binary relation on $\mathscr{X}$. Then,

(a) $\mathfrak{R}$ is said to be T-closed if

$\forall \varrho, \sigma \in \mathscr{X}, (\rho, \sigma) \in \mathfrak{R} \Longrightarrow (T(\varrho), T(\sigma)) \in \mathfrak{R}$

(b) $\mathfrak{R}$ is said to be (T, H)-closed if

$$\forall \varrho, \sigma \in \mathscr{X}, (H(\varrho), H(\sigma)) \in \mathfrak{R} \Longrightarrow (T(\varrho), T(\sigma)) \in \mathfrak{R} \quad (4)$$

Remark 9. Under $H = I$, the identity mapping on $\mathscr{X}$, the notion of (T, H)-closedness coincides with the notion of T-closedness of $\mathfrak{R}$.

Definition 10 (see [25]). Let $\mathscr{X} \neq \varnothing$ be a set with a metric d together with a binary relation $\mathfrak{R}$. If every $\mathfrak{R}$-preserving Cauchy sequence in $\mathscr{X}$ converges, we say $(\mathscr{X}, \mathrm{d})$ is $\mathfrak{R}$-complete.

Definition 11 (see [25]). Let $\mathscr{X} \neq \varnothing$ be a set with a metric d together with a binary relation $\mathfrak{R}$ and T a self-mapping on $\mathscr{X}$. If for any $\mathfrak{R}$-preserving sequence $\{\varrho_{\Bbbk}\} \subset \mathscr{X}$ converging to an element $\varrho \in \mathscr{X}$, we have $T(\varrho_{\Bbbk}) \xrightarrow{\mathrm{d}} T(\varrho)$, then the mapping T is said to be $\mathfrak{R}$-continuous.

Definition 12 (see [2]). Let $\mathscr{X} \neq \varnothing$ be a set with a metric d together with a binary relation $\mathfrak{R}$ and T, H two self-mappings on $\mathscr{X}$. Let $\{\varrho_{\Bbbk}\} \subset \mathscr{X}$ be a sequence satisfying $\lim_{\Bbbk \longrightarrow \infty} H(\varrho_{\Bbbk}) = \lim_{\Bbbk \longrightarrow \infty} T(\varrho_{\Bbbk})$. Then, the mappings T and H are compatible if $\lim_{\Bbbk \longrightarrow \infty} \mathrm{d}(HT(\varrho_{\Bbbk}), TH(\varrho_{\Bbbk})) = 0$.

Definition 13 (see [25]). Let $\mathscr{X} \neq \varnothing$ be a set with a metric d together with a binary relation $\mathfrak{R}$ and T, H two self-mappings on $\mathscr{X}$. Let $\{\varrho_{\Bbbk}\} \subset \mathscr{X}$ be a sequence such that $\{T(\varrho_{\Bbbk})\}$ and $\{H(\varrho_{\Bbbk})\}$ are $\mathfrak{R}$-preserving sequence satisfying $\lim_{\Bbbk \longrightarrow \infty} H(\varrho_{\Bbbk}) = \lim_{\Bbbk \longrightarrow \infty} T(\varrho_{\Bbbk})$. Then, the mappings T and H are "$\mathfrak{R}$-compatible" if $\lim_{\Bbbk \longrightarrow \infty} d(HT(\varrho_{\Bbbk}), TH(\varrho_{\Bbbk})) = 0$.

Remark 14 (see [25]). Let $\mathscr{X} \neq \varnothing$ be a set with a metric d together with a binary relation $\mathfrak{R}$. Then, the following relation holds:

$$\text{"commutativity} \Longrightarrow \text{compatibility} \Longrightarrow \mathfrak{R} - \text{compatibility} \Longrightarrow \text{weak compatibility"}. \quad (5)$$

Definition 15 (see [7, 25]). Let $\mathscr{X} \neq \varnothing$ be a set with a metric d together with a binary relation $\mathfrak{R}$ and T, H two self-mappings on $\mathscr{X}$. Consider the $\mathfrak{R}$-preserving sequence $\{\varrho_{\Bbbk}\} \subset \mathscr{X}$ such that $\varrho_{\Bbbk} \xrightarrow{d} \varrho$. Then,

(a) $\mathfrak{R}$ is called "d-self-closed" if there exists a subsequence $\{\varrho_{\Bbbk_p}\}$ of $\{\varrho_{\Bbbk}\}$ with $[\varrho_{\Bbbk_p}, \varrho] \in \mathfrak{R} \forall p \in \mathscr{K}_0$

(b) $\mathfrak{R}$ is called "$(H - d)$-self-closed" if there exists a subsequence $\{\varrho_{\Bbbk_p}\}$ of $\{\varrho_{\Bbbk}\}$ with $[H(\varrho_{\Bbbk_p}), H(\varrho)] \in \mathfrak{R}$ $\forall p \in \mathscr{K}_0$

Definition 16 (see [26–29]). Let $\mathscr{X} \neq \varnothing$ be set with a binary relation $\mathfrak{R}$ and T a self-mapping on $\mathscr{X}$

(a) If for any $\varrho, \sigma, \varsigma \in \mathcal{X}$, $(T(\varrho), T(\sigma)) \in \mathfrak{R}$ and $(T(\sigma), T(\varsigma)) \in \mathfrak{R} \Longrightarrow (T(\varrho), T(\varsigma)) \in \mathfrak{R}$, then $\mathfrak{R}$ is called "T-transitive"

(b) If for any $\varrho_0, \varrho_1, \cdots, \varrho_{\mathcal{K}} \in \mathcal{X}$ where $\mathcal{K}$ is a natural number ≥ 2, we have

$$(\varrho_{\ell-1}, \varrho_\ell) \in \mathfrak{R} \text{ for each } \ell (1 \leq \ell \leq \mathcal{K}) \Longrightarrow (\varrho_0, \varrho_{\mathcal{K}}) \in \mathfrak{R}, \quad (6)$$

then $\mathfrak{R}$ is called $\mathcal{K}$-transitive

(c) If for each denumerable subset $\mathcal{S}$ of $\mathcal{X}$, there exists $\mathcal{K} = \mathcal{K}(\mathcal{S}) \geq 2$, such that $\mathfrak{R}|_{\mathcal{S}}$ is $\mathcal{K}$-transitive, then $\mathfrak{R}$ is called "locally finitely transitive"

(d) If for each denumerable subset $\mathcal{S}$ of $T(\mathcal{X})$, there exists $\mathcal{K} = \mathcal{K}(\mathcal{S}) \geq 2$, such that $\mathfrak{R}|_{\mathcal{S}}$ is $\mathcal{K}$-transitive, then $\mathfrak{R}$ is called "locally finitely T-transitive"

Proposition 17 (see [29]). *Let $\mathcal{X}$ be a nonempty set, $\mathfrak{R}$ a binary relation on $\mathcal{X}$ and T a self-mapping on $\mathcal{X}$. Then,*

(a) $\mathfrak{R}$ is "T-transitive" $\Longleftrightarrow \mathfrak{R}|_{T\mathcal{X}}$ is "transitive"

(b) $\mathfrak{R}$ is "locally finitely T-transitive" $\Longleftrightarrow \mathfrak{R}|_{T\mathcal{X}}$ is "locally finitely transitive"

(c) $\mathfrak{R}$ is "transitive" $\Longrightarrow \mathfrak{R}$ is "finitely transitive" $\Longrightarrow \mathfrak{R}$ is "locally finitely transitive" $\Longrightarrow \mathfrak{R}$ is "locally finitely T-transitive"

(d) $\mathfrak{R}$ is "transitive" $\Longrightarrow \mathfrak{R}$ is "T-transitive" $\Longrightarrow \mathfrak{R}$ is "locally finitely T-transitive"

Definition 18 (see [23]). Let $\mathcal{X}$ be a nonempty set and $\mathfrak{R}$ a binary relation on $\mathcal{X}$. A subset $\mathcal{S}$ of $\mathcal{X}$ is called $\mathfrak{R}$-directed if for each $\varrho, \sigma \in \mathcal{S}$, there exists $\varsigma \in \mathcal{X}$ such that $(\varrho, \varsigma) \in \mathfrak{R}$ and $(\sigma, \varsigma) \in \mathfrak{R}$.

Definition 19 (see [24]). Let $\mathfrak{R}$ be a binary relation defined on a nonempty set $\mathcal{X}$. Then, for $\varrho, \sigma \in \mathcal{X}$, a finite sequence $\{\varrho_0, \varrho_1, \cdots, \varrho_p\} \subset \mathcal{X}$ satisfying the following conditions:

$$\begin{gathered} (\varrho_\ell, \varrho_{\ell+1}) \in \mathfrak{R} \text{ for each } \ell (0 \leq \ell \leq p-1), \\ \varrho_0 = \varrho \text{ and } \varrho_p = \sigma, \end{gathered} \quad (7)$$

is said to be a path of length p in $\mathfrak{R}$ from ϱ to σ.

Definition 20 (see [7]). Let $\mathfrak{R}$ be a binary relation on a nonempty set $\mathcal{X}$, and Y a subset of $\mathcal{X}$. If there exists a path in $\mathfrak{R}$ from ρ to σ for each $\varrho, \sigma \in Y$, then Y is called $\mathfrak{R}$-connected.

Lemma 21 (see [28]). *Let $\mathfrak{R}$ be a binary relation on a nonempty set $\mathcal{X}$, and $\{\varrho_{\Bbbk}\} \subset \mathcal{X}$ a sequence satisfying $(\varrho_{\Bbbk}, \varrho_{\Bbbk+1}) \in \mathfrak{R}$. Now, if for some natural number $\mathcal{K} \geq 2$, $\mathfrak{R}$ is $\mathcal{K}$-transitive on the set $L = \{\varrho_{\Bbbk} : \Bbbk \in \mathcal{K}_0\}$, then*

$$\left(\varrho_{\Bbbk}, \varrho_{\Bbbk+1+r(\mathcal{K}-1)}\right) \in \mathfrak{R} \text{ for all } \Bbbk, r \in \mathcal{K}_0. \quad (8)$$

3. Main Results

The first result in this section is on the existence of CP for two mapping T and H. For a nonempty set $\mathcal{X}$ and two self-mappings T and H on $\mathcal{X}$, the notations we use herein are as follows:

$$\begin{gathered} \Theta(T, H) = \{\rho \in \mathcal{X} : T(\varrho) = H(\varrho)\}, \\ \bar{\Theta}(T, H) = \{\bar{\varrho} \in \mathcal{X} : \bar{\varrho} = T(\varrho) = H(\varrho), \varrho \in \mathcal{X}\}. \end{gathered} \quad (9)$$

Theorem 22. *Let $\mathcal{X}$ be a nonempty set together with a metric d, $\mathfrak{R}$ a binary relation on $\mathcal{X}$ and T, H two self-mappings on $\mathcal{X}$. Suppose the following conditions hold:*

(a) $T(\mathcal{X}) \subset H(\mathcal{X})$

(b) $(\mathcal{X}, d)$ is $\mathfrak{R}$-complete

(c) there exists $\varrho_0 \in \mathcal{X}$ such that $(H(\varrho_0), T(\varrho_0)) \in \mathfrak{R}$

(d) $\mathfrak{R}$ is (T, H)-closed and locally finitely T-transitive

(e) T and H are $\mathfrak{R}$-compatible

(f) H is $\mathfrak{R}$-continuous

(g) T is $\mathfrak{R}$-continuous or $\mathfrak{R}$ is $(H-d)$-self-closed

(h) for every $\varepsilon > 0$ and $\varrho, \sigma \in \mathcal{X}$, there exists $\delta > 0$ such that

$$(H(\rho), H(\sigma)) \in \mathfrak{R} \text{ and } \varepsilon \leq d(H(\varrho), H(\sigma)) < \varepsilon + \delta \Longrightarrow d(T(\varrho), T(\sigma)) < \varepsilon \quad (10)$$

Then, T and H have a CP.

Proof. Assumption (c) confirms the existence of $\varrho_0 \in \mathcal{X}$ such that $(H(\varrho_0), T(\varrho_0)) \in \mathfrak{R}$. Now, if $H(\varrho_0) = T(\varrho_0)$ then nothing is left to be proved. Otherwise, by assumption (a), we can pick $\varrho_1 \in \mathcal{X}$ such that $T(\varrho_0) = H(\varrho_1)$. Again, there will be $\varrho_2 \in \mathcal{X}$ such that $H(\varrho_2) = T(\varrho_1)$. In this way, we construct a sequence $\{\varrho_{\Bbbk}\} \subset \mathcal{X}$ such that

$$H(\varrho_{\Bbbk+1}) = T(\varrho_{\Bbbk}) \quad \forall \Bbbk \in \mathcal{K}_0. \quad (11)$$

Now, we assert that $\{H(\varrho_{\Bbbk})\}$ is $\mathfrak{R}$-preserving, i.e.,

$$(H(\varrho_{\Bbbk}), H(\varrho_{\Bbbk+1})) \in \mathfrak{R} \quad \forall \Bbbk \in \mathcal{K}_0. \quad (12)$$

We will adopt the induction method to prove this fact. In view of assumption (c), equation (12) holds for $\Bbbk = 0$, i.e.,

$$(H(\varrho_0), H(\varrho_1)) \in \mathfrak{R}. \quad (13)$$

Now, suppose that equation (12) holds for $\Bbbk = p > 0$, i.e.,

$$\left(H(\varrho_p), H(\varrho_{p+1})\right) \in \mathfrak{R}. \quad (14)$$

Then, we have to show that

$$\left(H\left(\varrho_{p+1}\right), H\left(\varrho_{p+2}\right)\right) \in \mathfrak{R}. \quad (15)$$

In view of the fact that $\mathfrak{R}$ is (T, H)-closed, it is clear that

$$\left(H\left(\varrho_p\right), H\left(\varrho_{p+1}\right)\right) \in \mathfrak{R}\left(T\left(\varrho_p\right), T\left(\varrho_{p+1}\right)\right) \in \mathfrak{R}, \quad (16)$$

implying thereby

$$\left(H\left(\varrho_{p+1}\right), H\left(\varrho_{p+2}\right)\right) \in \mathfrak{R}, \quad (17)$$

which guarantees the fact that equation (2) holds for $\Bbbk = p + 1$. Therefore, $\{H(\varrho_\Bbbk)\}$ is $\mathfrak{R}$-preserving sequence. Notice that $\{T(\varrho_\Bbbk)\}$ is also a $\mathfrak{R}$-preserving sequence due to equation (1), i.e.,

$$(T(\varrho_\Bbbk), H(\varrho_{\Bbbk+1})) \in \mathfrak{R}. \quad (18)$$

Now, if there exists $n_0 \in \mathscr{K}$ such that $H(\varrho_{n_0}) = H(\varrho_{n_0+1})$, then, in view of equation (1), ϱ_{n_0} turns out to be a CP of T and H. As an alternative, consider that $H(\varrho_\Bbbk) \neq H(\varrho_{\Bbbk+1})$ for all $\Bbbk \in \mathscr{K}_0$, i.e., $d(H(\varrho_\Bbbk), H(\varrho_{\Bbbk+1})) \neq 0$.

Denote $\mu_\Bbbk := d(H(\varrho_\Bbbk), H(\varrho_{\Bbbk+1}))$. Now, in view of assumption (h), we get

$$\mu_{\Bbbk+1} = d(H(\varrho_{\Bbbk+1}), H\varrho_{\Bbbk+2}) = d(T(\varrho_\Bbbk), T(\varrho_{\Bbbk+1})) < d(H(\varrho_\Bbbk), H(\varrho_{\Bbbk+1})) = \mu_\Bbbk, \quad (19)$$

which gives

$$\mu_{\Bbbk+1} < \mu_\Bbbk. \quad (20)$$

Therefore, the sequence $\{\mu_\Bbbk\}$ is decreasing. As $\{\mu_\Bbbk\}$ is also bounded below by 0 (as a lower bound), we can find $r \geq 0$ satisfying

$$\lim_{\Bbbk \longrightarrow \infty} \mu_\Bbbk = r = \inf_{\Bbbk \in \mathscr{K}_0} \mu_\Bbbk. \quad (21)$$

Now, let us assume that $r > 0$. So, there will always be a $\delta(r) > 0$ such that

$$\begin{gathered}(H(\varrho), H(\sigma)) \in \mathfrak{R}, \\ r \leq d(H(\varrho), H(\sigma)) < r + \delta(r) \Longrightarrow d(T(\varrho), T(\sigma)) < r.\end{gathered} \quad (22)$$

Since $\{\mu_\Bbbk\}$ is decreasing sequence converging to r, there exists $p \in \mathscr{K}$ such that

$$r \leq d\left(H\left(\varrho_p\right), H\left(\varrho_{p+1}\right)\right) < r + \delta(r). \quad (23)$$

Thus, in view of assumption (h), we have

$$\mu_{p+1} = d\left(H\left(\varrho_{p+1}\right), H\varrho_{p+2}\right) < r, \quad (24)$$

which contradicts the fact that $r = \inf_{\Bbbk \longrightarrow \mathscr{K}_0} \mu_\Bbbk$. Hence, we conclude that

$$\lim_{\Bbbk \longrightarrow \infty} d(H(\varrho_\Bbbk), H(\varrho_{\Bbbk+1})) = 0. \quad (25)$$

Now, we establish that the sequence $\{H(\varrho_\Bbbk)\}$ is Cauchy. Utilizing equation (1), since $\{H(\varrho_\Bbbk)\} \subset T(\mathscr{X})$, we get that the range $\mathscr{S} = \{H(\varrho_\Bbbk): \Bbbk \in \mathscr{K}_0\}$ is a denumerable subset of $T(\mathscr{X})$. Hence, in view of assumption (d), there exist $\mathscr{K} = \mathscr{K}(\mathscr{S}) \geq 2$, such that $\mathfrak{R}|_\mathscr{S}$ is $\mathscr{K}$-transitive. Let $\varepsilon > 0$ be an arbitrary and fixed real number and let $\delta > 0$ corresponds to ε verifying the assumption (h). WLOG, we may consider that $\delta < \varepsilon$. In view of (2), there exists $n_0(\delta) \in \mathbb{N}$ satisfying

$$d(H(\varrho_\Bbbk), H(\varrho_{\Bbbk+1})) < \frac{\delta}{4\mathscr{K}} \quad \forall \Bbbk \geq n_0(\delta). \quad (26)$$

For all $\Bbbk \geq n_0(\delta)$ and for all $p (1 \leq p \leq \mathscr{K})$, using triangular inequality, we get

$$\begin{aligned}d\left(H(\varrho_\Bbbk), H\left(\varrho_{\Bbbk+\varrho}\right)\right) &\leq d(H(\varrho_\Bbbk), H(\varrho_{\Bbbk+1})) \\ &\quad + d(H(\varrho_{\Bbbk+1}), H\varrho_{\Bbbk+2}) \cdots + d\left(H\varrho_{\Bbbk+p-1}, H\varrho_{\Bbbk+p}\right) \\ &\leq \frac{\delta}{4\mathscr{K}} + \frac{\delta}{4\mathscr{K}} + \cdots + \frac{\delta}{4\mathscr{K}} = \frac{p\delta}{4\mathscr{K}}.\end{aligned} \quad (27)$$

Now, we claim that

$$d\left(H(\varrho_\Bbbk), H\varrho_{\Bbbk+p}\right) < \varepsilon + \frac{\delta}{2} \forall \Bbbk \geq n_0(\delta) \text{ and} \forall p \in \mathscr{K}. \quad (28)$$

This is demonstrated herein using the mathematical induction method. From (27), it is clear that (28) holds for all $p \in \{1, 2, 3, \cdots, \mathscr{K}\}$. Suppose that the conclusion holds for all $p \in \{1, 2, 3, \cdots, m\}$, where $m \geq \mathscr{K}$. We have to show that (28) holds for $\Bbbk = m + 1$ also. As $m \geq \mathscr{K}$, so $m - 1 \geq \mathscr{K} - 1 > 0$. By division algorithm, there exists unique integers μ and $\eta (0 \leq \eta \leq \mathscr{K} - 1)$ such that

$$\begin{gathered}m - 1 = (\mathscr{K} - 1)\mu + \eta \\ m = 1 + (\mathscr{K} - 1)\mu + \eta.\end{gathered} \quad (29)$$

Denoting $q =: 1 + (\mathscr{K} - 1)\mu$, the above equation reduces to

$$m = q + \eta, \quad (30)$$

so that

$$2 \leq \mathscr{K} \leq q \leq m = q + \eta. \quad (31)$$

Now, using (27), we get

$$d\left(H\left(\varrho_{\Bbbk+q+1}\right), H(\varrho_{\Bbbk+m+1})\right) = d\left(H\left(\varrho_{\Bbbk+q+1}\right), H\left(\varrho_{\Bbbk+q+\eta+1}\right)\right) \leq \frac{\eta\delta}{4\mathscr{K}}. \quad (32)$$

Now, using Lemma 21, we get

$$\left(H(\varrho_{\Bbbk}), H\left(\varrho_{\Bbbk+q}\right)\right) \in \mathfrak{R}. \tag{33}$$

As $q \in \{\mathscr{K}, \mathscr{K}+1, . \cdots, m\}$, using inductive hypothesis, we get

$$0 < d\left(H(\varrho_{\Bbbk}), H\left(\varrho_{\Bbbk+q}\right)\right) < \varepsilon + \frac{\delta}{2} < \varepsilon + \delta. \tag{34}$$

Using (33) and (34) and applying contractive condition (h), we get

$$d\left(H(\varrho_{\Bbbk+1}), H\left(\varrho_{\Bbbk+q+1}\right)\right) = d\left(T(\varrho_{\Bbbk}), T\left(\varrho_{\Bbbk+q}\right)\right) < \varepsilon. \tag{35}$$

Now, using triangular inequality, (25), (32), and (35), we get

$$\begin{aligned} d(H(\rho_{\Bbbk}), H\rho_{\Bbbk+m+1}) &\leq d(H(\varrho_{\Bbbk}), H(\varrho_{\Bbbk+1})) + d\left(H(\varrho_{\Bbbk+1}), H\left(\varrho_{\Bbbk+q+1}\right)\right) \\ &\quad + d\left(H\left(\varrho_{\Bbbk+q+1}\right), H(\varrho_{\Bbbk+m+1})\right) \\ &< \frac{\delta}{4\mathscr{K}} + \varepsilon + \frac{\eta\delta}{4\mathscr{K}} < \frac{\delta}{4\mathscr{K}} + \varepsilon + \frac{\delta}{4\mathscr{K}}(\mathscr{K}-1) \, as \mathscr{K} \\ &\geq 2 \text{ and } \eta < \mathscr{K} - 1 = \varepsilon + \frac{\delta}{4} < \varepsilon + \frac{\delta}{2}. \end{aligned} \tag{36}$$

Thus, by induction, (28) is verified. From (28), it embraces that the sequence $\{H(\varrho_{\Bbbk})\}$ is Cauchy. Now, the $\mathfrak{R}$-completeness property of $\mathscr{X}$ and $\mathfrak{R}$-preserving property of $\{H(\varrho_{\Bbbk})\}$ confirm the availability of an element $\varsigma \in \mathscr{X}$ such that

$$\lim_{\Bbbk \longrightarrow \infty} H(\varrho_{\Bbbk}) = \varsigma. \tag{37}$$

Also, from (11),

$$\lim_{\Bbbk \longrightarrow \infty} T(\varrho_{\Bbbk}) = \varsigma. \tag{38}$$

Now, by dint of the $\mathfrak{R}$-continuity of H, we acquire

$$\lim_{\Bbbk \longrightarrow \infty} H(H(\varrho_{\Bbbk})) = H\left(\lim_{\Bbbk \longrightarrow \infty} H(\varrho_{\Bbbk})\right) = H(\varsigma). \tag{39}$$

Utilizing (38) and $\mathfrak{R}$-continuity of H,

$$\lim_{\Bbbk \longrightarrow \infty} H(T(\varrho_{\Bbbk})) = H\left(\lim_{\Bbbk \longrightarrow \infty} T(\varrho_{\Bbbk})\right) = H(\varsigma). \tag{40}$$

Since $\{T(\varrho_{\Bbbk})\}$ and $\{H(\varrho_{\Bbbk})\}$ are $\mathfrak{R}$-preserving and

$$\lim_{\Bbbk \longrightarrow \infty} T(\varrho_{\Bbbk}) = \lim_{\Bbbk \longrightarrow \infty} H(\varrho_{\Bbbk}) = \varsigma, \tag{41}$$

by assumption (e),

$$\lim_{\Bbbk \longrightarrow \infty} d(HT(\varrho_{\Bbbk}), TH(\varrho_{\Bbbk})) = 0. \tag{42}$$

The next step is to establish that $\varsigma \in \Theta(T, H)$. From assumption (g), we first consider that T is "$\mathfrak{R}$-continuous." Using (12), (37), and $\mathfrak{R}$-continuity of T,

$$\lim_{\Bbbk \longrightarrow \infty} T(H(\varrho_{\Bbbk})) = T\left(\lim_{\Bbbk \longrightarrow \infty} H(\varrho_{\Bbbk})\right) = T(\varsigma). \tag{43}$$

Applying (40) and (42), we get

$$\begin{aligned} d(H(\varsigma), T(\varsigma)) &= d\left(\lim_{\Bbbk \longrightarrow \infty} HT(\varrho_{\Bbbk}), \lim_{\Bbbk \longrightarrow \infty} TH(\varrho_{\Bbbk})\right) \\ &= \lim_{\Bbbk \longrightarrow \infty} d(HT(\varrho_{\Bbbk}), TH(\varrho_{\Bbbk})) = 0, \end{aligned} \tag{44}$$

yielding thereby $H(\varsigma) = T(\varsigma)$, which establishes our claim.

Instead of $\mathfrak{R}$-continuity of T, we now suppose that $\mathfrak{R}$ is (H, d)-self-closed, based on assumption (g). Then, $\{H(\varrho_{\Bbbk})\}$ being $\mathfrak{R}$-preserving sequence guarantees the existence of a subsequence $\{H\varrho_{\Bbbk_p}\}$ such that $[H\varrho_{\Bbbk_p}, \varsigma] \in \mathfrak{R}$. If $H\varrho_{\Bbbk_{k_0}} = \varsigma$ for some $k_0 \in \mathscr{K}$, then using (11) and by the $\mathfrak{R}$-preserving property of $\{H(\varrho_{\Bbbk})\}$, we get $H(\varrho_{\Bbbk_{k_0}}) \in \Theta(T, H)$. Otherwise, suppose $H\varrho_{n_p} \neq \varsigma$, i.e., $d(H\varrho_{n_p}, \varsigma) \neq 0$ for all $p \in \mathscr{K}$. In this case, in view of assumption (h), assuming $\varepsilon = d(H\varrho_{\Bbbk_p}, \varsigma)$ and using assumption (h), we get

$$d\left(T\left(H\varrho_{n_p}\right), T(\varsigma)\right) < \varepsilon. \tag{45}$$

Using triangle inequality, we get

$$\begin{aligned} d(H(\varsigma), T(\varsigma)) \leq d&\left(H(\varsigma), HT\varrho_{\Bbbk_p}\right) \\ &+ d\left(HT\varrho_{\Bbbk_p}, TH\varrho_{\Bbbk_p}\right) \\ &+ d\left(TH\varrho_{\Bbbk_p}, T(\varsigma)\right). \end{aligned} \tag{46}$$

Now, using (40), (42), and (45) in the previous equation, we obtain

$$d(H(\varsigma), T(\varsigma)) = 0, \tag{47}$$

which establishes that $T(\varsigma) = H(\varsigma)$. □

It is clear that Theorem 22 solely considers the existence of a CP of T and H. As a result, we must add extra conditions to the hypothesis of Theorem 22 to obtain the uniqueness of point of coincidence, CP and CFPs. This is the purpose of our next theorems.

Theorem 23. *Assume that all of the criteria of Theorem 22 are met. Let the following condition holds additionally:*

(i) $T(\mathscr{X})$ *is* $\mathfrak{R}^s_{H(\mathscr{X})}$*-connected*

then T *and* H *have a unique point of coincidence.*

Proof. From Theorem 22, we get that $\Theta(T, H) \neq \emptyset$. Consider that $\varrho, \sigma \in \Theta(T, H)$. Then, there exist $\bar{\sigma}, \bar{\sigma} \in \mathcal{X}$ such that

$$T(\varrho) = H(\varrho) = \bar{\varrho} \text{ and } T(\sigma) = H(\sigma) = \bar{\sigma}. \tag{48}$$

It is now our goal to prove that $\bar{\varrho} = \bar{\sigma}$. Since $T(\varrho), T(\sigma) \in T(\mathcal{X}) \subseteq H(\mathcal{X})$, by assumption (i), there exists a path $\{H(\varsigma_0), H(\varsigma_1), H(\varsigma_2), \cdots, H(\varsigma_p)\}$ of some finite length p in $\mathfrak{R}^s_{H(\mathcal{X})}$ from $T(\rho)$ to $T(\sigma)$. Now, in view of (48), WLOG we can choose $\varsigma_0 = \varrho$ and $\varsigma_p = \sigma$. Thus, we have

$$[H(\varsigma_\ell), H(\varsigma_{\ell+1})] \in \mathfrak{R}_{H(\mathcal{X})} \text{ for each } \ell(0 \leq \ell \leq p-1). \tag{49}$$

Define the constant sequences $\varsigma^0_{\Bbbk} = \rho$ and $\varsigma^p_{\Bbbk}$, then in view of equation (48), we have $H(\varsigma^0_{\Bbbk+1}) = T(\varsigma^0_{\Bbbk}) = \bar{\varrho}$ and $H(\varsigma^p_{\Bbbk+1}) = T(\varsigma^p_{\Bbbk}) = \bar{\sigma}$ for all $\Bbbk \in \mathcal{K}_0$. Put $\varsigma^1_0 = \varsigma_1, \varsigma^2_0 = \varsigma_2, \varsigma^3_0 = \varsigma_3, \cdots, \varsigma^{p-1}_0 = \varsigma_{p-1}$. Now, since $T(\mathcal{X}) \subset H(\mathcal{X})$, we can define sequences $\{\varsigma^1_{\Bbbk}\}$, $\{\varsigma^2_{\Bbbk}\}$, ..., $\{\varsigma^{p-1}_{\Bbbk}\}$ such that $H(\varsigma^1_{\Bbbk+1}) = T(\varsigma^1_{\Bbbk})$, $H(\varsigma^2_{\Bbbk+1}) = T(\varsigma^2_{\Bbbk})$, ..., $H(\varsigma^{p-1}_{\Bbbk+1}) = T(\varsigma^{p-1}_{\Bbbk})$ for all $\Bbbk \in \mathcal{K}_0$. Hence, we have

$$H\left(\varsigma^\ell_{\Bbbk+1}\right) = T\left(\varsigma^\ell_{\Bbbk}\right) \forall \Bbbk \in \mathcal{K}_0 \text{ and for each } \ell(0 \leq \ell \leq p). \tag{50}$$

Now, we claim that

$$\left[H\left(\varsigma^\ell_{\Bbbk}\right), H\left(\varsigma^{\ell+1}_{\Bbbk}\right)\right] \in \mathfrak{R} \forall \Bbbk \in \mathcal{K}_0 \text{ and for each } \ell(0 \leq \ell \leq p-1). \tag{51}$$

This is demonstrated herein using the mathematical induction method. equation (51) holds for $\Bbbk = 0$ as a result of (49). Assume that equation (51) is true for $\Bbbk = r$, i.e.,

$$\left[H\left(\varsigma^\ell_r\right), H\left(\varsigma^{\ell+1}_r\right)\right] \in \mathfrak{R}. \tag{52}$$

As $\mathfrak{R}$ is (T, H)-closed, we obtain

$$\left[T\left(\varsigma^\ell_r\right), T\left(\varsigma^{\ell+1}_r\right)\right] \in \mathfrak{R}, \tag{53}$$

which on using (51) gives us that

$$\left[H\left(\varsigma^\ell_{r+1}\right), H\left(\varsigma^{\ell+1}_{r+2}\right)\right] \in \mathfrak{R}\ \Bbbk \in \mathcal{K}_0 \text{ and for each } \ell(0 \leq \ell \leq p-1). \tag{54}$$

Therefore, equation (51) holds. Now, for each $\Bbbk \in \mathcal{K}_0$ and for each $(0 \leq \ell \leq p-1)$, define

$$t^\ell_{\Bbbk} = d\left(H\left(\varsigma^\ell_{\Bbbk}\right), H\left(\varsigma^{\ell+1}_{\Bbbk}\right)\right). \tag{55}$$

We show that

$$\lim_{\Bbbk \longrightarrow \infty} t^\ell_{\Bbbk} = 0. \tag{56}$$

Now, we look at two scenarios in which ℓ is fixed. Firstly, suppose that

$$t^\ell_{n_0} = d\left(H\left(\varsigma^\ell_{n_0}\right), H\left(\varsigma^{\ell+1}_{n_0}\right)\right) = 0 \text{ for some } n_0 \in \mathcal{K}_0, \tag{57}$$

which gives rise to $H(\varsigma^\ell_{n_0}) = H(\varsigma^{\ell+1}_{n_0})$. Now applying (11), we have $t^\ell_{n_0+1} = 0$. Continuing this process, we get

$$\varsigma^\ell_{\Bbbk} = 0 \forall \Bbbk \geq n_0, \tag{58}$$

which establishes that $\lim_{\Bbbk \longrightarrow \infty} \varsigma^\ell_{\Bbbk} = 0$.

Alternatively, assume that $\varsigma^\ell_{\Bbbk} > 0 \forall \Bbbk \in \mathcal{K}_0$. For any $\varepsilon > 0$, assume $t^\ell_{\Bbbk} = d(H(\varsigma^\ell_{\Bbbk}), H(\varsigma^{\ell+1}_{\Bbbk})) = \varepsilon$. Then,

$$t^\ell_{\Bbbk+1} = d\left(H\left(\varsigma^\ell_{\Bbbk+1}\right), H\left(\varsigma^{\ell+1}_{\Bbbk+1}\right)\right) = d\left(T\left(\varsigma^\ell_{\Bbbk}\right), T\left(\varsigma^{\ell+1}_{\Bbbk}\right)\right) < \varepsilon = t^\ell_{\Bbbk}, \tag{59}$$

which gives

$$t^\ell_{\Bbbk+1} < t^\ell_{\Bbbk}. \tag{60}$$

As a result, the sequence $\{t^\ell_{\Bbbk}\}$ is decreasing. As $\{t^\ell_{\Bbbk}\}$ is also bounded below by 0 (as a lower bound), there exists $r \geq 0$ such that

$$\lim_{\Bbbk \longrightarrow \infty} t^\ell_{\Bbbk} = r = \inf_{\Bbbk \in \mathcal{K}_0} t^\ell_{\Bbbk}. \tag{61}$$

Now, we prove that $r = 0$. Assume, on the other hand that $r > 0$. So, there will always be a $\delta(r) > 0$ such that

$$(H(\varrho), H(\sigma)) \in \mathfrak{R} \text{ and } r \leq d(H(\varrho), H(\sigma)) < r + \delta(r) d(T(\varrho), T(\sigma)) < r. \tag{62}$$

Since $\{t^\ell_{\Bbbk}\}$ is decreasing sequence converging to r, there exists $p \in \mathcal{K}$ such that

$$r \leq d\left(H\left(\varsigma^\ell_p\right), H\left(\varsigma^{\ell+1}_p\right)\right) < r + \delta(r). \tag{63}$$

Thus, in view of assumption (h), we have

$$t^\ell_{p+1} = d\left(H\left(\varsigma^\ell_{p+1}\right), H\left(\varsigma^{\ell+1}_{p+1}\right)\right) < r, \tag{64}$$

which contradicts the fact that $r = \inf_{\Bbbk \longrightarrow \infty} t^\ell_{\Bbbk}$. Hence, we conclude that

$$\lim_{\Bbbk \longrightarrow \infty} t^\ell_{\Bbbk} = 0. \tag{65}$$

Thus, equation (56) holds $\forall \ell(0 \leq \ell \leq p-1)$. Now, in light of equation (56) and triangle inequality, we get

$$d(\bar{\varrho}, \bar{\sigma}) \leq t^0_{\Bbbk} + t^1_{\Bbbk} + \cdots + t^{p-1}_{\Bbbk} \longrightarrow 0 \text{ as } \Bbbk \longrightarrow \infty. \tag{66}$$

Therefore, $\bar{\varrho} = \bar{\sigma}$, which ends the proof. □

Theorem 24. *Assume that all of the criteria of Theorem 22 are met. Let the following condition holds additionally:*

(i) T and H are "weakly compatible"

then T and H have a unique CFP.

Proof. Assume $\rho \in \mathcal{X}$ such that $\varrho \in \Theta(T, H)$. Therefore, there exists $\bar{\rho} \in \mathcal{X}$ such that

$$H(\varrho) = T(\varrho) = \bar{\varrho}. \tag{67}$$

In light of the Remark 14, the concept $\mathfrak{R}$-compatibility coincides with the weak compatibility. Hence, $\bar{\varrho} \in \Theta(T, H)$. Utilizing $\varsigma = \bar{\varrho}$ in Theorem 23, we obtain $H(\varrho) = H(\bar{\varrho})$ yielding thereby

$$\bar{\varrho} = H(\bar{\varrho}) = T(\bar{\varrho}). \tag{68}$$

Hence, $\bar{\varrho}$ is a CFP of T and H.

Now, we assume that ϱ' is another CFP of T and H in order to assert the uniqueness. Applying Theorem 23, we get

$$\varrho' = H\left(\varrho'\right) = H(\bar{\varrho}) = \bar{\varrho}, \tag{69}$$

which finishes the proof. □

Theorem 25. *Assume that all of the criteria of Theorem 22 are met. Suppose either of the mappings T and H is one-to-one. Then, T and H have a unique CP.*

Proof. From Theorem 22, it is evident that $\Theta(T, H) \neq \varnothing$. Let, $\varrho, \sigma \in \Theta(T, H)$. Then, Theorem 23 permits us to write

$$T(\varrho) = H(\varrho) = T(\sigma) = H(\sigma). \tag{70}$$

Now, since T or H is one-to-one, we have, $\varrho = \sigma$ which finishes the proof. □

Theorem 22 has the following implication when we apply Proposition 17.

Corollary 26. *If either of the below conditions:*

(a) $\mathfrak{R}$ is "transitive"

(b) $\mathfrak{R}$ is "T-transitive"

(c) $\mathfrak{R}$ is "finitely transitive"

(d) $\mathfrak{R}$ is "locally finitely transitive"

is utilized in Theorem 22 instead of the locally finitely T-transitivity condition; then, the validity of Theorem 22 remains the same.

Corollary 27. *If either of the below conditions:*
(i'). $T(\mathcal{X})$ is $\mathfrak{R}^s$-directed
(i''). $\mathfrak{R}|_{T(\mathcal{X})}$ is complete

holds in place of condition (i) of Theorem 23, then the validity of Theorem 23 remains the same.

Proof. If condition (i') is satisfied, then, for each $\varrho, \sigma \in T(\mathcal{X})$, we get $\varsigma \in \mathcal{X}$ satisfying $[\rho, \varsigma] \in \mathfrak{R}$ and $[\sigma, \varsigma] \in \mathfrak{R}$. Notice that the sequence $\{\varrho, \varsigma, \sigma\}$ works as a path of length 2 in $\mathfrak{R}^s$ from ρ to σ, which establishes the fact that $T(\mathcal{X})$ is $\mathfrak{R}^s$-connected. Now, applying Theorem 23, we obtain the uniqueness of point of coincidence.

Alternately, from assumption (i''), we get $[\varrho, \sigma] \in \mathfrak{R} \forall \varrho, \sigma \in T(\mathcal{X})$, which assents that $\{\rho, \sigma\}$ constitutes a path of length 1 in $\mathfrak{R}^s$. As a result, $T(\mathcal{X})$ is $\mathfrak{R}^s$-connected, which wrap up the proof when Theorem 23 is applied. □

Under $H = I$, the identity map, we obtain the following result which is proved by Sk et al. [17].

Corollary 28 (see [17]). *Let $(\mathcal{X}, d)$ be a $\mathfrak{R}$-complete metric space endowed with a binary relation $\mathfrak{R}$ on $\mathcal{X}$ and T a self-mapping on $\mathcal{X}$. Suppose that the following conditions hold:*

(a) there exists $\varrho_0 \in \mathcal{X}$ such that $(\varrho_0, T\varrho_0) \in \mathfrak{R}$,

(b) $\mathfrak{R}$ is T-closed and locally finitely T-transitive

(c) either T is $\mathfrak{R}$-continuous or $\mathfrak{R}$ is d-self-closed

(d) for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(\varrho, \sigma) \in \mathfrak{R} \text{ and } \varepsilon \leq d(\varrho, \sigma) < \varepsilon + \delta \Longrightarrow d(T(\varrho), T(\sigma)) < \varepsilon \tag{71}$$

Then, T has a fixed point. Further, if we impose an additional hypothesis:

(e) $T(\mathcal{X})$ is $\mathfrak{R}^s$-connected

then T has a unique fixed point.

Remark 29. Under the universal relation $\mathfrak{R}$ and $H = I$, the identity map, Theorem 22, and Theorem 23 reduce to the classical fixed point theorem of Meir and Keeler [10].

Remark 30. Under partial order the relation $\mathbb{R} = {}^\circ$, and $H = I$, the identity map, Theorem 22, and Theorem 23 reduces to fixed point theorem of Harjani et al. [30].

4. Examples

Now, we equip two examples to show how important our results are in comparison to other results in the literature.

Example 1. Let $\mathcal{X} = \{(0, 1), (1, 0), (1, 1), (0, 0)\} \subset \mathbb{R}^2$ together with the usual Euclidean metric d. Consider the following relation endowed with $\mathcal{X}$:

$$\mathfrak{R} = \{((1, 1), (0, 0))\}. \tag{72}$$

Then, $(\mathcal{X}, d)$ is a $\mathfrak{R}$-complete metric space. Now consider that $T, H : \mathcal{X} \longrightarrow \mathcal{X}$ are two mappings defined by

$$T(1,0) = (0,1)\,;\, T(0,1) = (1,0)\,;\, T(1,1) = (1,1)\,;\, T(0,0) = (0,0),$$
$$H(0,1) = (1,0)\,;\, H(0,0) = (0,1)\,;\, H(1,1) = (1,1)\,;\, H(1,0) = (0,0). \tag{73}$$

Notice that for $\varepsilon = d((0,1),(1,0)) = \sqrt{2}$, we have

$$d(T(0,1), T(1,0)) = d((1,0),(0,1)) = \sqrt{2} < \varepsilon, \tag{74}$$

which is absurd. Further, $((1,1),(0,0)) \in \mathfrak{R}$ and $d((1,1),(0,0)) = \sqrt{2}$ but the inequality

$$d(T(1,1), T(0,0)) = d((1,1,),(0,0)) = \sqrt{2} < \varepsilon, \tag{75}$$

does not hold. Hence, the existing theorems cannot be applied for this example. Now, assume that $\varepsilon = d(H(1,1), H(1,0)) = d((1,1),(0,0)) = \sqrt{2}$. Then, the inequality

$$d(T(1,1), T(1,0)) = d((1,1),(0,1)) = 1 < \varepsilon, \tag{76}$$

holds. As a result, assumption (h) of Theorem 22 holds. It can also be seen that all of the conditions of Theorem 22 are met using regular calculation. Therefore, T and H have a CP, namely, $(1,1)$.

Although it does not satisfy Theorem 23, the CP of T and H in Example 1 is unique, proving that condition (i) of Theorem 23 is not a necessary condition for the uniqueness of CPs.

Example 2. Let $\mathcal{X} = \{(0,1),(1,0),(1,1),(0,0)\} \subset \mathbb{R}^2$ together with the usual Euclidean metric d. Consider the following relation endowed with $\mathcal{X}$,

$$\mathfrak{R} = \{(\varrho, \sigma) : \varrho, \sigma \in \{(0,1),(1,1)\}\}. \tag{77}$$

Then, $(\mathcal{X}, d)$ is a $\mathfrak{R}$-complete metric space. Now consider that $T, H : \mathcal{X} \longrightarrow \mathcal{X}$ are two mappings defined by

$$T(1,0) = (1,0)\,;\, T(0,1) = (0,1)\,;\, T(1,1) = (1,0)\,;\, T(0,0) = (0,1),$$
$$H(1,0) = (1,0)\,;\, H(0,1) = (0,1)\,;\, H(1,1) = (0,1), H(0,0) = (1,1). \tag{78}$$

Now, for $\varepsilon = d(H(0,1), H(0,0)) = 1$, we have

$$d(T(0,1), T(0,0)) = d((0,1),(0,1)) = 0 < \varepsilon, \tag{79}$$

holds. As a result, assumption (h) of Theorem 22 holds. It can also be seen that all of the conditions of Theorem 22 are met using regular calculation. Therefore, T and H have CPs, namely, $(0,1),(1,0)$. The availability of more than one fixed point certifies the eminence of Theorem 23.

Notice that for $\varepsilon = d((0,1),(1,0)) = \sqrt{2}$, we have

$$d(T(0,1), T(1,0)) = d((1,0),(0,1)) = \sqrt{2} < \varepsilon, \tag{80}$$

which is absurd. Further, $((0,1),(1,1)) \in \mathfrak{R}$ and $d((0,1),(1,1)) = 1$ but the inequality

$$d(T(0,1), T(1,1)) = d((0,1,),(1,0)) = \sqrt{2} < \varepsilon, \tag{81}$$

does not hold. Hence, the existing theorems cannot be applied for this example.

5. Conclusion

In this paper, we have established some coincidence point theorems for two mappings employing the relation-theoretic Meir-Keeler contractions in a metric space endowed with a class of transitive binary relation. Our findings have also led to the deduction of certain related fixed point results. Furthermore, some examples are given to demonstrate the significant progress made in this area.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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