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Retraction

Retracted: Certain Structure of Lagrange's Theorem with the Application of Interval-Valued Intuitionistic Fuzzy Subgroups

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] D. A. Kattan, M. Amin, and A. Bariq, "Certain Structure of Lagrange's Theorem with the Application of Interval-Valued Intuitionistic Fuzzy Subgroups," *Journal of Function Spaces*, vol. 2022, Article ID 3580711, 9 pages, 2022.

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Research Article

Certain Structure of Lagrange's Theorem with the Application of Interval-Valued Intuitionistic Fuzzy Subgroups

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This paper presents the concept of an interval-valued intuitionistic fuzzy subgroup defined on interval-valued intuitionistic fuzzy sets. We study some of the fundamental algebraic properties of interval-valued intuitionistic fuzzy cosets and interval-valued intuitionistic fuzzy normal subgroup of a given group. This idea is used to describe the interval-valued intuitionistic fuzzy order and index of interval-valued intuitionistic fuzzy subgroup. We have created numerous algebraic properties of interval-valued intuitionistic fuzzy order of an element. We also prove the interval-valued intuitionistic fuzzification of Lagrange's theorem.

1. Introduction

The introduction of interval-valued intuitionistic fuzzy sets is based on the ideas of intuitionistic fuzzy sets and interval-valued fuzzy sets (IVIFSs). Zadeh [1] was the first to propose the concept of a fuzzy set in 1965. Rosenfeld [2] utilized this concept in 1971 to establish the concept of fuzzy groups. In the year 2000, Lee [3] described bipolar-valued fuzzy sets and their fundamental operations. In 2004, Lee [4] conducted a comparison of interval-valued fuzzy sets, IFSs, and bipolar fuzzy sets.

In 2009, Park et al. [5] investigated the IVIFS correlation coefficient and its application to multi-attribute group decision-making situations. In 2013, Chen and Li [6] used IVIFSs to evaluate students' answer scripts. In 2013, Meng et al. [7] used an interval-valued intuitionistic fuzzy Choquet integral with respect to a generalized Shapley index to address the multi-criteria group decision-making problem. In 2013, Ye [8] used intuitionistic fuzzy setting and interval-valued intuitionistic fuzzy setting to construct multi-attribute group decision-making procedures with unknown weights. In 2013, Zhang et al. [9] proposed an interval-valued intuitionistic fuzzy multi-attribute group decision-making method based on correlation coefficients. In 2014, Chen [10] presented using IVIFSs a prioritized aggregation operator-based approach to multi-criteria decision making. In 2014, Jin et al. [11] developed an interval-valued intuitionistic fuzzy continuous weighted entropy and used it to multi-criteria fuzzy group decision making. In 2014, Li [12] used interval-valued intuitionistic fuzzy information to solve decision-making difficulties in company financial performance assessment.

In 2014, Liu et al. [13] published a multi-attribute large-group decision-making method based on an interval-valued intuitionistic fuzzy principal component analysis model. In 2015, Chen and Chiou [14] published a multi-attribute decision-making method using IVIFSs. In 2015, Gupta et al. [15] developed a mixed solution technique for multi-criteria group decision making in an interval-valued intuitionistic fuzzy environment employing entropy/cross entropy. In 2015, Liu et al. [16] extended the Einstein aggregation procedures based on interval-valued intuitionistic fuzzy numbers and proved their use in group decision making.

In 2017, Chen and Huang [17] used interval-valued intuitionistic fuzzy values and linear programming to examine the multi-attribute decision-making problem. In 2017, Xian et al. [18] used IVIFSs and a weighted averaging operator to make group decisions. Shuaib et al. [19] characterized on r-interval-valued intuitionistic fuzzification of Lagrange's theorem of r-intuitionistic fuzzy subgroups in 2017. Mu et al. [20] developed the concept of interval-valued intuitionistic fuzzy Zhengyuan aggregation operators and its application to multi-attribute decision-making problems in 2018. In 2018, Zhang [21] proposed the geometric Bonferroni means of interval-valued intuitionistic fuzzy numbers and their use in multi-attribute group decision making. In 2018, Khan and Abdullah [22] defined an interval-valued Pythagorean fuzzy grey relational analysis approach for multi-attribute decision making with incomplete weight information for multi-attribute decision making with missing weight information. In 2018, Xu [23] proposed a consensus model for interval-valued intuitionistic multi-attribute group decision making with few changes. In 2018, Gupta et al. [24] introduced the notion of multi-attribute group decision making in an interval-valued intuitionistic fuzzy environment using an extended TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) technique. In 2018, Qin et al. [25] proposed a novel technique based on ordered weighted averaging distance operators for interval-value intuitionistic fuzzy multi-criteria decision making with immediate probability. The VIKOR technique for industrial robot selection was presented by Narayanamoorthy et al. [26]. It is based on an interval-valued intuitionistic hesitant fuzzy entropy. Alolaiyan et al. [27] proposed the concept of t-intuitionistic fuzzification of Lagrange's theorem of t-intuitionistic fuzzy subgroups in 2019. Hosinzadeh et al. [28] proposed an artificial intelligence-based prediction way to describe the flow of a Newtonian liquid/gas on a permeable flat surface in 2021. Ghasemi et al. [29] proposed a dual-phase-lag (DPL) transient non-Fourier heat transfer analysis of functional graded cylindrical material under axial heat flux.

This paper is organized as follows. Section 2 contains basic definitions of interval-valued intuitionistic fuzzy order of an element of interval-valued intuitionistic fuzzy subgroup and the related result which are very useful to build up the consequent investigation of this paper. We construct the algebraic properties of interval-valued intuitionistic fuzzy order of an element of interval-valued intuitionistic fuzzy subgroup of a finite cyclic group in Section 3. In Section 4, we extend the study of this notion to introduce interval-valued intuitionistic left cosets and index of interval-valued intuitionistic fuzzy subgroups. Moreover, we develop Lagrange's theorem by using the notion of interval-valued fuzzy information and establish some key fundamental algebraic aspects.

2. Interval-Valued Intuitionistic Fuzzy Order of an Element of Interval-Valued Intuitionistic Fuzzy Subgroup

This section reviews some fundamental concept of IVIFSs and interval-valued intuitionistic fuzzy subgroup along with the relevant results.

Definition 1. Let X be non-empty set. An interval-valued fuzzy set M defined on X is given by $= \{(m, [M^L, M^U]), \forall m \in X\}$, where M^L and M^U are two fuzzy sets of X such that $M^L \leq M^U$, for all $m \in X$.

On the other hand, an interval-valued fuzzy set (IVFS) of X is specified as $M_X\colon X \longrightarrow \text{Int}([0,1])$, where $\text{Int}([0,1])$ is the set of all intervals within $[0,1]$, and is expressed as $M = [M^L, M^U]$ such that $M^L \leq M^U$.

Definition 2. Let K be an ordinary set. Then, $P\colon K \longrightarrow [0,1] \times [0,1]$ is designed by $P = \{(m, M_P(m), N_P(m)) | m \in K\}$, where $M_P\colon K \longrightarrow \text{Int}[0,1]$ is designed by $M_P = [M_P^L, M_P^U]$, where $M_P^L \leq M_P^L$ and $N_P\colon K \longrightarrow \text{Int}[0,1]$ is designed by $N_P = [N_P^L, N_P^L]$, where $N_P^L \leq N_P^L$ and $\text{Sup}M_P(m) + \text{Sup}N_P(m) \leq 1$.

Definition 3. An IVIFS P of group $\mathbb{G}$ is known as an IVIFSG of group $\mathbb{G}$ if it satisfies the following axioms: $\mu_{P^U}(mn^{-1}) \geq \min\{\mu_{P^U}(m), \mu_{P^U}(n)\}$, $\mu_{P^L}(mn^{-1}) \geq \min\{\mu_{P^L}(m), \mu_{P^L}(n)\}$ and $\nu_{P^U}(mn^{-1}) \leq \max\{\nu_{P^U}(m), \nu_{P^U}(n)\}$, $\nu_{P^L}(mn^{-1}) \leq \max\{\nu_{P^L}(m), \nu_{P^L}(n)\}, \forall m, n \in \mathbb{G}$.

Theorem 1. *Let P an IVIFS$\mathbb{G}$ of a group $\mathbb{G}$ and $m \in \mathbb{G}$; then, $\mu_{P^U}(mn) = \mu_{P^U}(m), \mu_{P^L}(mn) = \mu_{P^L}(m)$ and $\nu_{P^U}(mn) = \nu_{P^U}(m)$, $\nu_{P^L}(mn) = \nu_{P^L}(m)$ for all $n \in \mathbb{G}$ if and only if $\mu_{P^U}(n) = \mu_{P^U}(e), \mu_{P^L}(n) = \mu_{P^L}(e)$ and $\nu_{P^U}(n) = \nu_{P^U}(e)$, $\nu_{P^L}(n) = \nu_{P^L}(e)$.*

Proof. Assume that $\mu_{P^U}(mn) = \mu_{P^U}(m), \mu_{P^L}(mn) = \mu_{P^L}(m)$ and $\nu_{P^U}(mn) = \nu_{P^U}(m), \nu_{P^L}(mn) = \nu_{P^L}(m)$ for all $n \in \mathbb{G}$. By replacing n by e, we have the required result.

Conversely, if $\mu_{P^U}(n) = \mu_{P^U}(e), \mu_{P^L}(n) = \mu_{P^L}(e)$. Since P is IVIF$\mathbb{G}$, $\mu_{P^U}(n) \leq \mu_{P^U}(e), \mu_{P^L}(n) \leq \mu_{P^L}(e)$ and $\nu_{P^U}(n) \geq \nu_{P^U}(e), \nu_{P^L}(n) \geq \nu_{P^L}(e)$ for all $n \in \mathbb{G}$. Now $\mu_{P^U}(mn^{-1}) \geq \min\{\mu_{P^U}(m), \mu_{P^U}(n)\}$, $\mu_{P^L}(mn^{-1}) \geq \min\{\mu_{P^L}(m), \mu_{P^L}(n)\}$.

We have

$$\mu_{P^U}(mn) \geq \mu_{P^U}(n), \mu_{P^L}(mn) \geq \mu_{P^L}(n). \quad (1)$$

But $\mu_{P^U}(n) = \mu_{P^U}(nmm^{-1}) \geq \min\{\mu_{P^U}(m), \mu_{P^U}(nm)\}$, $\mu_{P^L}(n) = \mu_{P^L}(nmm^{-1}) \geq \min\{\mu_{P^L}(m), \mu_{P^L}(nm)\}$, and this shows that

$$\mu_{P^U}(n) \geq \mu_{P^U}(nm), \mu_{P^L}(n) \geq \mu_{P^L}(nm), \text{for all } m \in \mathbb{G}. \quad (2)$$

From (1) and (2), we have $\mu_{P^U}(mn) = \mu_{P^U}(n)$, $\mu_{P^L}(mn) = \mu_{P^L}(n)$.

Similarly, we can show that $\nu_{P^U}(nm) = \nu_{P^U}(n), \nu_{P^L}(nm) = \nu_{P^L}(n)$.

Definition 4. Let P be an IVIFS$\mathbb{G}$ of a group $\mathbb{G}$ and m_1 be an element of the group. The interval-valued intuitionistic fuzzy right coset of IVIFS$\mathbb{G}P$ of $\mathbb{G}$ is defined as

$$\mu_{m_1 P}(g) = \left[\mu_P^U\left(gm_1^{-1}\right), \mu_P^L\left(gm_1^{-1}\right)\right] = \mu_P\left(gm_1^{-1}\right) \quad \text{for all } g \in \mathbb{G},$$
$$\nu_{m_1 P}(g) = \left[\nu_P^U\left(gm_1^{-1}\right), \nu_P^L\left(gm_1^{-1}\right)\right] = \nu_P\left(gm_1^{-1}\right). \tag{3}$$

Definition 5. An IVIFS$\mathbb{G}$ P is known as an IVIFNS$\mathbb{G}$ of group $\mathbb{G}$, if $\mu_{P^U}(mm^{-1}n) = \mu_{P^U}(n)$, $\mu_{P^L}(mm^{-1}n) = \mu_{P^L}(n)$ and $\nu_{P^U}(n) = \nu_{P^U}(nmm^{-1})$, $\nu_{P^L}(n) = \nu_{P^L}(nmm^{-1})$, for all $m, n \in \mathbb{G}$.

Definition 6. Consider an IVIFS$\mathbb{G}$ P of a group $\mathbb{G}$, which is finite and $m \in \mathbb{G}$. Then, the interval-valued intuitionistic fuzzy order (IVIFO) of m is named as $\text{IVIFO}_P(m)$ and is defined as

$\text{IVIFO}_P(m) = |Q(m)|$, where

$$Q(m) = \{r \in \mathbb{G}: \mu_{P^U}(r) \geq \mu_{P^U}(m), \mu_{P^L}(r) \geq \mu_{P^L}(m) \text{ and } \nu_{P^U}(r) \leq \nu_{P^U}(m), \nu_{P^L}(r) \leq \nu_{P^L}(m)\}. \tag{4}$$

The algebraic information can be observed in the following example.

Example 1. Let $\mathbb{G} = \{e, a, a^2, b, ab, a^2b\}$ be a symmetric group of order 6. Then, an IVIFS$\mathbb{G}P$ of $\mathbb{G}$ is defined as

$$P = \left\{\langle e, [0.6, 1], [0, 0.4] >, < a, [0.6, 1], [0, 0.4] >, < a^2, [0.6, 1], [0, 0.4] >, < b, [0.3, 0.35], [0.5, 0.55] >, < ab, [0.3, 0.35], [0.5, 0.55] >, < a^2b, [0.3, 0.35], [0.5, 0.55]\rangle\right\}. \tag{5}$$

Clearly, $\text{IVIFO}_P(e) = \text{IVIFO}_P(a) = 3, \text{IVIFO}_P(a^2) = 3, \text{IVIFO}_P(b) = \text{IVIFO}_P(ab) = \text{IVIFO}_P(a^2b) = 6$.

Theorem 2. *$Q(m)$ forms a subgroup of $\mathbb{G}$.*

Proof. As $m \in Q(m)$, $Q(m)$ is a non-empty set. By Definition 6, for arbitrary two elements $r, q \in Q(m)$, we have $\mu_{P^U}(r) \geq \mu_{P^U}(m), \mu_{P^L}(r) \geq \mu_{P^L}(m)$ and $\nu_{P^U}(r) \leq \nu_{P^U}(m), \nu_{P^L}(r) \leq \nu_{P^L}(m)$. Also, $\mu_{P^U}(q) \geq \mu_{P^U}(m), \mu_{P^L}(q) \geq \mu_{P^L}(m)$ and $\nu_{P^U}(q) \leq \nu_{P^U}(m), \nu_{P^L}(q) \leq \nu_{P^L}(m)$

Since P is an IVIFS$\mathbb{G}$, $\mu_{P^U}(rq^{-1}) \geq \min\{\mu_{P^U}(r), \mu_{P^U}(q)\}$, $\mu_{P^L}(rq^{-1}) \geq \min\{\mu_{P^L}(r), \mu_{P^L}(q)\}$ and $\nu_{P^U}(rq^{-1}) \leq \max\{\nu_{P^U}(r), \nu_{P^U}(q)\}, \nu_{P^L}(rq^{-1}) \leq \max\{\nu_{P^L}(r), \nu_{P^L}(q)\}$ which implies that $\mu_{P^U}(rq^{-1}) \geq \mu_{P^U}(m), \mu_{P^L}(rq^{-1}) \geq \mu_{P^L}(m)$ and $\nu_{P^U}(rq^{-1}) \leq \nu_{P^U}(m), \nu_{P^L}(rq^{-1}) \leq \nu_{P^L}(m)$. Thus, $rq^{-1} \in Q(m)$. Consequently, $Q(m)$ is a subgroup of $\mathbb{G}$.

Corollary 1. *Assume that there exists an IVIFS$\mathbb{G}P$ of a group $\mathbb{G}$; then, the IVIFO of any element of P divides $\mathbb{G}$'s order.*

Proof. By Theorem 2 and Lagrange's theorem, anyone can show that the IVIFO of any element of IVIFS$\mathbb{G}$ always divides group $\mathbb{G}$'s order.

Theorem 3. *Let P be an IVIFS$\mathbb{G}$ of a group $\mathbb{G}$ and $e \neq m \in \mathbb{G}$. Then, $\text{IVIFO}_P(e) \leq \text{IVIFO}_P(m)$.*

Proof. Let $r \in Q(m)$; then, $\mu_{P^U}(r) = \mu_{P^U}(m), \mu_{P^L}(r) = \mu_{P^L}(m)$ and $\nu_{P^U}(r) = \nu_{P^U}(m), \nu_{P^L}(r) = \nu_{P^L}(m)$. This means that $\mu_{P^U}(r) \geq \min\{\mu_{P^U}(m)\}$, $\mu_{P^L}(r) \geq \min\{\mu_{P^L}(m)\}$ and $\nu_{P^U}(r) \leq \max\{\nu_{P^U}(m)\}, \nu_{P^L}(r) \leq \max\{\nu_{P^L}(m)\}$, for all $m \in \mathbb{G}$. Thus, $r \in Q(m)$. Consequently, $Q(e) \subseteq Q(m)$ and $\text{IVIFO}_P(e) \leq \text{IVIFO}_P(m)$.

The next result produces a relation between the IVIFO of any element of P and the order of that element in $\mathbb{G}$.

Theorem 4. *Let P be an IVIFS$\mathbb{G}$ of a group $\mathbb{G}$ and $m \in \mathbb{G}$; then, $O(m)$ divides $\text{IVIFO}_P(m)$.*

Proof. Assume that $O(m) = l$ and consider a subgroup $K = < m: m^l = e >$ of $\mathbb{G}$. In view of Definition 6, we have $m^2 \in Q(m)$, and similarly, we can have $m^3, m^4, m^5, \ldots, m^l \in Q(m)$. This indicates that $K \subseteq Q(m)$. Consequently, K forms subgroup of $Q(m)$ and $|K|$ divides $|Q(m)|$. This means that $|K|$ divides $\text{IVIFO}_P(m)$, andt therefore $O(m)$ divides $\text{IVIFO}_P(m)$.

Definition 7. The IVIFO of IVIFS$\mathbb{G}P$ of $\mathbb{G}$ is denoted by $\text{IVIFO}(P)$ and can be obtained by computing the greatest common divisor of the IVIFO of all elements of P.

Example 2. Let $\mathbb{G} = \{e, a, a^2, b, ab, a^2b\}$ be a symmetric group of order 6. An IVIFS$\mathbb{G}P$ of $\mathbb{G}$ is defined as

$$P = \left\{\langle e, [0.6, 1], [0, 0.4] >, < a, [0.4, 0.5], [0.3, 0.4] >, < a^2, [0.4, 0.5], [0.3, 0.4] >, < b, [0.3, 0.35], [0.5, 0.55] >, < ab, [0.3, 0.35], [0.5, 0.55] >, < a^2b, [0.3, 0.35], [0.5, 0.55]\rangle\right\}. \tag{6}$$

Clearly, $\mathrm{IVIFO}_P(e) = 1$, $\mathrm{IVIFO}_P(a) = 3$, $\mathrm{IVIFO}_P(a^2) = 3$, $\mathrm{IVIFO}_P(b) = \mathrm{IVIFO}_P(ab) = \mathrm{IVIFO}_P(a^2b) = 6$.

The IVIFO of P in $\mathbb{G}$ is 1.

In the following result, we prove the condition that $\mu_{P^U}(m^l) \geq \mu_{P^U}(m)$, $\mu_{P^L}(m^l) \geq \mu_{P^L}(m)$ and $\nu_{P^U}(m^l) \leq \nu_{P^U}(m)$, $\nu_{P^L}(m^l) \leq \nu_{P^L}(m)$.

Theorem 5. *Let P an IVIFS$\mathbb{G}$ of a group $\mathbb{G}$ and $m \in \mathbb{G}$; then, $\mu_{P^U}(m^l) \geq \mu_{P^U}(m)$, $\mu_{P^L}(m^l) \geq \mu_{P^L}(m)$ and $\nu_{P^U}(m^l) \leq \nu_{P^U}(m)$, $\nu_{P^L}(m^l) \leq \nu_{P^L}(m)$ where l is an integer.*

Proof. This result is clear for $l = 0$ and 1. For $l = 2$,

$$\begin{aligned}\mu_{P^U}\left(m^2\right) &\geq \mu_{P^U}(m \cdot m)\\ &\geq \min\{\mu_{P^U}(m), \mu_{P^U}(m)\}\\ &= \mu_{P^U}(m).\end{aligned} \tag{7}$$

Assume the statement is true for $n < l$.

Now,

$$\begin{aligned}\mu_{P^U}\left(m^{n+1}\right) &= \mu_{P^U}(m^n \cdot m)\\ &\geq \min\{\mu_{P^U}(m^n), \mu_{P^U}(m)\}\\ &= \mu_{P^U}(m),\end{aligned} \tag{8}$$

which completes the induction.

If $l < 0$, then

$$\begin{aligned}\mu_{P^U}\left(m^l\right) &= \mu_{P^U}\left(m^l\right)^{-1}\\ &= \mu_{P^U}\left(m^{-l}\right) \geq \mu_{P^U}(m).\end{aligned} \tag{9}$$

Similarly, $\mu_{P^L}(m^l) \geq \mu_{P^L}(m)$.

Therefore, we can easily prove $\nu_{P^U}(m^l) \leq \nu_{P^U}(m)$, $\nu_{P^L}(m^l) \leq \nu_{P^L}(m)$, for any integer l.

Remark 1. If $(O(m), l) = 1$, then $\mu_{P^U}(m^l) = \mu_{P^U}(m)$, $\mu_{P^L}(m^l) = \mu_{P^L}(m)$ and $\nu_{P^U}(m^l) = \nu_{P^U}(m)$, $\nu_{P^L}(m^l) = \nu_{P^L}(m)$, for any integer l.

Theorem 6. *Let $\mathrm{IVIFO}_P(m) = r$ and $(r, s) = 1$, $r, s \in Z$ and $m \in \mathbb{G}$. Then, $\mu_{P^U}(m^s) = \mu_{P^U}(m)$, $\mu_{P^L}(m^s) = \mu_{P^L}(m)$ and $\nu_{P^U}(m^s) = \nu_{P^U}(m)$, $\nu_{P^L}(m^s) = \nu_{P^L}(m)$.*

Proof. We are aware that if $(r, s) = 1$, then $ar + bs = 1$, for $a, b \in Z$. So,

$$\begin{aligned}\mu_{P^U}(m) &= \mu_{P^U}\left(m^{ar+bs}\right)\\ &= \mu_{P^U}\left(\left(m^r\right)^a, \left(m^s\right)^b\right)\\ &\geq \min\left\{\mu_{P^U}\left(\left(m^r\right)^a\right), \mu_{P^U}\left(\left(m^s\right)^b\right)\right\}\\ &= \min\left\{\mu_{P^U}(e), \mu_{P^U}\left(m^s\right)\right\} \geq \mu_{P^U}\left(m^s\right).\end{aligned} \tag{10}$$

But $\mu_{P^U}(m^s) \geq \mu_{P^U}(m)$.

Consequently, $\mu_{P^U}(m^s) = \mu_{P^U}(m)$.

Similarly, we can easily prove for the lower case.

Therefore, we can prove $\nu_{P^U}(m^s) = \nu_{P^U}(m)$, $\nu_{P^L}(m^s) = \nu_{P^L}(m)$.

Theorem 7. *Let $r, s \in Z$ such that $\mu_{P^U}(m^s) = \mu_{P^U}(e)$, $\mu_{P^L}(m^s) = \mu_{P^L}(e)$ and $\nu_{P^U}(m^r) = \nu_{P^U}(e)$, $\nu_{P^L}(m^r) = \nu_{P^L}(e)$, for all $m \in \mathbb{G}$. Then, both r and s divide $\mathrm{IVIFO}_P(m)$.*

Proof. Let m be a non-identity element and $\mathrm{IVIFO}_P(m) = x$. Suppose s does not divide x; then, $(s, x) = 1$.

By Theorem 6, we have $\mu_{P^U}(m^s) = \mu_{P^U}(m)$, $\mu_{P^L}(m^s) = \mu_{P^L}(m)$. But $\mu_{P^U}(m^s) = \mu_{P^U}(e)$, $\mu_{P^L}(m^s) = \mu_{P^L}(e)$, so $m = e$.

As such, it is a contradiction, and thus s divides $\mathrm{IVIFO}_P(m)$.

Similarly, we can easily prove r divides $\mathrm{IVIFO}_P(m)$.

Theorem 8. *If $\mathrm{IVIFO}_P(m) = r$, then $\mathrm{IVIFO}_P(m^s) = \mathrm{IVIFO}_P(m)/(r, s)$ for some integer s.*

Proof. Suppose that $\mathrm{IVIFO}_P(m^s) = y$.

Consider

$$\begin{aligned}\mu_{P^U}\left(\left(m^s\right)^{r/d}\right) &= \mu_{P^U}\left(\left(m^r\right)^{s/d}\right)\\ &\geq \mu_{P^U}\left((e)^{s/d}\right)\\ &= \mu_{P^U}(e).\end{aligned} \tag{11}$$

Similarly, we can easily prove for the lower case.

Therefore, $\nu_{P^U}((m^s)^{r/d}) = \nu_{P^U}(e)$. We can also prove for the lower limit.

By Theorem 7, we have that r/d divides y.

Moreover, since $(r, s) = d$, $ar + bs = d$, for some $a, b \in Z$. Now,

$$\begin{aligned}\mu_{P^U}\left(m^{y\,d}\right) &= \mu_{P^U}\left(m^{y(ar+bs)}\right)\\ &= \mu_{P^U}\left(m^{yar} m^{ybs}\right)\\ &\geq \min\left\{\mu_{P^U}\left(\left(m^r\right)^{ay}\right), \mu_{P^U}\left(\left(m^{ys}\right)^b\right)\right\}\\ &\geq \min\left\{\mu_{P^U}\left(m^r\right), \mu_{P^U}\left(m^{ys}\right)\right\}\\ &\geq \min\left\{\mu_{P^U}\left(m^r\right), \mu_{P^U}\left(\left(m^s\right)^y\right)\right\}\\ &= \min\left\{\mu_{P^U}(e), \mu_{P^U}(e)\right\}\\ &= \mu_{P^U}(e).\end{aligned} \tag{12}$$

We know that $\mu_{P^U}((mn)^{xy}) \leq \mu_{P^U}(e)$, and hence $\mu_{P^U}((mn)^{xy}) = \mu_{P^U}(e)$.

Similarly, we can easily prove for the lower case.

Therefore, $\nu_{P^U}(m^{y\,d}) = \nu_{P^U}(e)$, $\nu_{P^L}(m^{y\,d}) = \nu_{P^L}(e)$. By applying Theorem 7, we get $y\ d/r$.

Consequently, $y = r/d$.

Theorem 9. *Let P be an IVIFS$\mathbb{G}$ of a group $\mathbb{G}$ and $m \in \mathbb{G}$; then, $\mathrm{IVIFO}_P(m^{-1}) = \mathrm{IVIFO}_P(m)$.*

Proof. Since P is IVIFS$\mathbb{G}$ of $\mathbb{G}$, $\mu_{P^U}(m) = \mu_{P^U}(m^{-1})$, $\mu_{P^L}(m) = \mu_{P^L}(m^{-1})$ and $\nu_{P^U}(m) = \nu_{P^U}(m^{-1})$, $\nu_{P^L}(m) = \nu_{P^L}(m^{-1})$, for all $m \in \mathbb{G}$. This means that $Q(m) = Q(m^{-1})$; as such, $|Q(m)| = |Q(m^{-1})|$. In addition, we know that $\mathrm{IVIFO}_P(a) = |Q(a)|$, for all a. Therefore, $\mathrm{IVIFO}_P(m^{-1}) = \mathrm{IVIFO}_P(m)$.

In the following theorem, we illustrate another form of IVIFO of elements of IVIFNS$\mathbb{G}$.

Theorem 10. *Let P be an IVIFS$\mathbb{G}$P of a group $\mathbb{G}$ and $n \in \mathbb{G}$ be any fixed element; then, $\mathrm{IVIFO}_P(mnm^{-1}) = \mathrm{IVIFO}_P(n)$ for all $m \in \mathbb{G}$.*

Proof. By Definition 5, we have $\mu_{P^U}(mnm^{-1}) = \mu_{P^U}(n)$, $\mu_{P^L}(mnm^{-1}) = \mu_{P^L}(n)$ and $\nu_{P^U}(n) = \nu_{P^U}(mnm^{-1})$, $\nu_{P^L}(n) = \nu_{P^L}(mnm^{-1})$. So, $Q(n) = Q(mnm^{-1})$.

Consequently, $\text{IVIFO}_P(mnm^{-1}) = \text{IVIFO}_P(n)$.

Theorem 11. *Let P be an IVIFS𝔾 of a group 𝔾; then, $IVIFO_P(mn) = IVIFO_P(nm)$, for all $m, n \in \mathbb{G}$.*

Proof. Since $\text{IVIFO}_P(mn) = \text{IVIFO}_P(n^{-1}n)(mn) = \text{IVIFO}_P(n^{-1}(nm)n)$, by Theorem 10, $\text{IVIFO}_P(n^{-1}(nm)n) = \text{IVIFO}_P(nm)$.

So, we have $\text{IVIFO}_P(mn) = \text{IVIFO}_P(nm)$.

Theorem 12. *Let $IVIFO_P(m) = x$, for all $m \in \mathbb{G}$. If $i \equiv j$ (mod$x$), where $i, j \in Z$, then $IVIFO_P(m^i) = IVIFO_P(m^j)$.*

Proof. Assume that $\text{IVIFO}_P(m^i) = s$ and $\text{IVIFO}_P(m^j) = r$. Since $i = j + qx$ for some $q \in Z$,

$$\begin{aligned}\mu_{P^U}\left(\left(m^i\right)^r\right) &= \mu_{P^U}\left(\left(m^{j+qx}\right)^r\right)\\ &= \mu_{P^U}\left(\left(m^j\right)^r\left(m^x\right)^{qr}\right)\\ &\geq \min\left\{\mu_{P^U}\left(m^j\right)^r, \mu_{P^U}\left(\left(m^x\right)^{qr}\right)\right\}\\ &\geq \min\left\{\mu_{P^U}(e), \mu_{P^U}(e)\right\}\\ &= \mu_{P^U}(e).\end{aligned} \tag{13}$$

As such, s/r. Similarly, we can prove $\mu_{P^L}((m^i)^r) = \mu_{P^L}(e)$ and r/s. Hence, $\text{IVIFO}_P(m^i) = \text{IVIFO}_P(m^j)$.

Theorem 13. *Assume that for all $m, n \in \mathbb{G}$ $(IVIFO_P(m), IVIFO_P(n)) = 1$, $mn = nm$ and $P(mn) = P(e)$. Then, $P(m) = P(n) = P(e)$.*

Proof. Proof. Suppose that $\text{IVIFO}_P(m) = x$ and $\text{IVIFO}_P(n) = y$. By Theorem 5, we have $\mu_{P^U}(m^y n^y) = \mu_{P^U}(e)$, $\mu_{P^L}(m^y n^y) = \mu_{P^L}(e)$. By Theorem 7, we have $\mu_{P^U}(m^y) = \mu_{P^U}(n^y) = \mu_{P^U}(e)$, $\mu_{P^L}(m^y) = \mu_{P^L}(n^y) = \mu_{P^L}(e)$.

Similarly, we can easily prove for the non-membership function.

Theorem 14. *If $(IVIFO_P(m), IVIFO_P(n)) = 1$ and $mn = nm$ for all $m, n \in \mathbb{G}$, then $IVIFO_P(mn) = [IVIFO_P(m)] \times [IVIFO_P(n)]$.*

Proof. Proof. Suppose $\text{IVIFO}_P(mn) = z$, $\text{IVIFO}_P(m) = x$ and $\text{IVIFO}_P(n) = y$. Now consider

$$\begin{aligned}\mu_{P^U}\left((mn)^{xy}\right) &= \mu_{P^U}\left(m^{xy}n^{xy}\right)\\ &\geq \min\left\{\mu_{P^U}\left((m^x)^y, \mu_{P^U}\left((n^y)^x,\right)\right)\right\}\\ &= \min\left\{\mu_{P^U}(e), \mu_{P^U}(e)\right\}\\ &\geq \mu_{P^U}(e).\end{aligned} \tag{14}$$

We know that

$$\mu_{P^U}\left((mn)^{xy}\right) \leq \mu_{P^U}(e). \tag{15}$$

From (14) and (15), we have $\mu_{P^U}((mn)^{xy}) = \mu_{P^U}(e)$.
Similarly, $\mu_{P^L}((mn)^{xy}) = \mu_{P^L}(e)$.
Likewise, $\nu_{P^U}((mn)^{xy}) = \nu_{P^U}(e)$, $\nu_{P^L}((mn)^{xy}) = \nu_{P^L}(e)$.
By Theorem 7, we have the relation

$$\frac{xy}{z}. \tag{16}$$

Since $(x, y) = 1$, x/z or y/z.
Assume that x/z; then,

$$\text{IVIFO}_P\left(m^z\right) = \frac{x}{(x, z)}. \tag{17}$$

By using Theorem 7,

$$\text{IVIFO}_P\left(n^z\right) = \frac{y}{(y, z)}. \tag{18}$$

By equations (17) and (18), we have $(\text{IVIFO}_P(m^z), \text{IVIFO}_P(n^z)) = 1$.

From Theorem 13 and equations (17) and (18), we have $P(m) = P(n) = P(e)$. This means that

$$\frac{z}{xy}. \tag{19}$$

From (16) and (19), we get the result.

Remark 2. Let P and Q be any two IVIFS𝔾 of group 𝔾. If $P \subseteq Q$ and $P(e) = Q(e)$, then $\text{IVIFO}_P(m)/\text{IVIFO}_Q(m)$ for all $m \in \mathbb{G}$.

Theorem 15. *If P and Q are any two IVIFS𝔾 of a group 𝔾 such that $P \subseteq Q$ and $P(e) = Q(e)$, then $IVIFO_P(m)/IVIFO_Q(m)$.*

Proof. As IVIFO(P) and IVIFO(Q) are finite, the IVIFO of every element of P and Q is finite. Let M and N be the sets consisting of IVIFO s of elements in P and Q, respectively. Remark 2 gives that $\text{IVIFO}_P(m)$ divides $\text{IVIFO}_Q(m)$ for all $m \in \mathbb{G}$. Then, gcd of every elements of M divides the gcd of every elements of N. As a result, $\text{IVIFO}_P(m)/\text{IVIFO}_Q(m)$.

3. Properties of IVIFO of Elements in IVIFSG in a Finite Cyclic Group

This section examines the IVIFO of elements of IVIFS𝔾 in cyclic groups and their elementary properties.

Lemma 1. *Assume that there exists an IVIFS𝔾P of a cyclic group 𝔾 and m, n are any two generators of 𝔾; then, $IVIFO_P(m) = IVIFO_P(n)$.*

Proof. Assume that $O(\mathbb{G}) = r$. Since m and n are two generators of 𝔾, $m^r = n^r = e$.

Since for some $s \in Z$ we have $n = m^s$, $(r, s) = 1$. Next, by Theorem 6, $\text{IVIFO}_P(m) = \text{IVIFO}_P(n) = \text{IVIFO}_P(m^s)$.

Theorem 16. *Let P be an IVIFS$\mathbb{G}$ on a finite cyclic group $\mathbb{G}$. The following results hold for all $m, n \in \mathbb{G}$.*

(1) If $O(m) = O(n)$, then $IVIFO_P(m) = IVIFO_P(n)$.

(2) If $O(m)$ divides $O(n)$, then $IVIFO_P(m)$ divides $IVIFO_P(n)$.

Proof. Let x be a generator of $\mathbb{G}$; then, $m = x^r, n = x^s$ and $\text{IVIFO}_P(x) = u$ where $r, s, u \in Z$. We have $O(m) = n/(n, r)$ and $O(n) = n/(n, s)$. In view of Theorem 8, we have $\text{IVIFO}_P(m) = u/(u, r)$ and $\text{IVIFO}_P(n) = u/(u, s)$. From Theorem 3, we have n/u.

(1) Since $O(m) = O(n)$, then $O(x^r) = O(x^s)$. This shows that $(r, n) = (s, n)$. From the above relation, we have $(r, u) = (s, u)$. Consequently, $\text{IVIFO}_P(m) = \text{IVIFO}_P(n)$.

(2) Since $O(m)$ divides $O(n)$, $(s, n)/(r, n)$. This implies that $(s, u)/(r, u)$ In addition, as n/u, $\text{IVIFO}_P(m)$ divides $\text{IVIFO}_P(n)$.

Corollary 2. *Let P be an IVIFS$\mathbb{G}$ of a finite cyclic group $\mathbb{G}$ of order q. If $IVIFO_P(n) = IVIFO_P(m)$, then $P(m) = P(n)$ for all $m, n \in \mathbb{G}$.*

Corollary 3. *Let P an IVIFS$\mathbb{G}$ of a group $\mathbb{G}$ of order q. If $IVIFO_P(n)$ divides $IVIFO_P(m)$, then $\mu_{P^U}(n) \geq \mu_{P^U}(m)$, $\mu_{P^L}(n) \geq \mu_{P^L}(m)$ and $\nu_{P^U}(n) \leq \nu_{P^U}(m)$, $\nu_{P^U}(n) \leq \nu_{P^U}(m)$.*

Theorem 17. *Let P be an IVIFS$\mathbb{G}$ of a group $\mathbb{G}$ and $K = < x >$ be a cyclic subgroup of $\mathbb{G}$. For all $m, n \in K$, if $O(m)$ divides $O(n)$, then $\mu_{P^U}(n) \geq \mu_{P^U}(m)$, $\mu_{P^L}(n) \geq \mu_{P^L}(m)$ and $\nu_{P^U}(n) \leq \nu_{P^U}(m)$, $\nu_{P^L}(n) \leq \nu_{P^L}(m)$.*

Proof. Suppose $O(m) = r$ and $O(n) = qr$ for some $q \in N$. Let $m = x^u$ and $n = x^v$ for some $u, v \in N$. It follows that $x^{ur} = e = x^{vqr}$. Thus, $m = n^q$. As such, $\mu_{P^U}(m) = \mu_{P^U}(n^q) \geq \mu_{P^U}(n)$. Similarly, we can prove for the lower limit of a non-membership function.

Likewise, $\nu_{P^U}(n) \leq \nu_{P^U}(m)$, $\nu_{P^U}(n) \leq \nu_{P^U}(m)$.

In the following example, we show that Theorem 17 is not valid for all $m, n \in \mathbb{G}$.

Example 3. Let $\mathbb{G} = \{e, 1, 2, 3, 4, 5\}$ be a group of order 6. Then, and IVIFS$\mathbb{G}P$ of $\mathbb{G}$ is defined as

$$P = \{\langle e, [0.6, 1], [0, 0.4] >, < 1, [0.4, 0.5], [0.3, 0.4] >, < 2, [0.4, 0.5], [0.3, 0.4] >, < 3, [0.3, 0.35], [0.5, 0.55] >,\\ < 4, [0.3, 0.35], [0.5, 0.55] >, < 5, [0.3, 0.35], [0.5, 0.55]\rangle\}. \tag{20}$$

We know that $O(4) = 3$ and $O(1) = 6$ in $\mathbb{G}$.

Clearly, $O(4)$ divides $O(1)$, but $\mu_{P^U}(1) \geq \mu_{P^U}(4)$, $\mu_{P^L}(1) \geq \mu_{P^L}(4)$ and $\nu_{P^U}(1) \leq \nu_{P^U}(4)$, $\nu_{P^L}(1) \leq \nu_{P^L}(4)$.

4. Interval-Valued Intuitionistic Fuzzification of Lagrange's Theorem

This part recapitulates the concept pertaining to the index of IVIFS$\mathbb{G}$. In addition, interval-valued intuitionistic fuzzification of Lagrange's theorem of IVIFS$\mathbb{G}$ is studied.

Theorem 18. *Assume that there exists an IVIFS$\mathbb{G}P$ of a group $\mathbb{G}$ and ϖ is the set of all interval-valued intuitionistic fuzzy left cosets (IVIFLC) of $\mathbb{G}$ by P. Then, ϖ forms a group with*

$$(mP)o(nP) = (mn)P, \quad \text{for all } m, n \in \mathbb{G}. \tag{21}$$

Define a mapping P: $\varpi \longrightarrow [0, 1]$ by

$$P(mP) = P(m), \quad \text{for all } m \in \mathbb{G}. \tag{22}$$

Then, $P^\wedge$ is an IVIFS$\mathbb{G}$ of ϖ.

Proof. Let $m, n, m_o, n_o \in \mathbb{G}$ such that

$$mP = m_o P \text{ and } nP = n_o P. \tag{23}$$

Then, we must show that

$$(mP)o(nP) = (m_o P)o(n_o P),\\ (mn)P = (m_o n_o)P. \tag{24}$$

By Definition 4,

$$\mu_{(mn)P^U}(u) = \mu_{P^U}\left(un^{-1}m^{-1}\right), \quad \text{for all } u \in \mathbb{G},\\ \mu_{(m_o n_o)P^U}(u) = \mu_{P^U}\left(un_o^{-1}m_o^{-1}\right), \quad \text{for all } u \in \mathbb{G}. \tag{25}$$

Now,

$$\begin{aligned}\mu_{P^U}\left(un^{-1}m^{-1}\right) &= \mu_{P^U}\left(un_o^{-1}n_o n^{-1}m^{-1}\right)\\ &= \mu_{P^U}\left(un_o^{-1}m_o^{-1}m_o n_o n^{-1}m^{-1}\right)\\ &\geq \min\left[\mu_{P^U}\left(un_o^{-1}m_o^{-1}\right), \mu_{P^U}\left(m_o n_o n^{-1}m^{-1}\right)\right].\end{aligned} \tag{26}$$

Using Definition 4 in (23) gives

$$\mu_{P^U}\left(um^{-1}\right) = \mu_{P^U}\left(um_o^{-1}\right), \quad \text{for all } u \in \mathbb{G}, \tag{27}$$

and

$$\mu_{P^U}\left(un^{-1}\right) = \mu_{P^U}\left(un_0^{-1}\right), \quad \text{for all } u \in \mathbb{G}. \tag{28}$$

Now, replace u by $m_o n_o n^{-1}$ in (27), and we have

$$\mu_{P^U}\left(m_o n_o n^{-1}m^{-1}\right) = \mu_{P^U}\left(m_o n_o n^{-1}m_o^{-1}\right). \tag{29}$$

Substitute u with n_o in (28), and we have

$$= \mu_{P^U}\left(n_o n^{-1}\right) = \mu_{P^U}(e). \tag{30}$$

But $\mu_{P^U}(e) \geq \mu_{P^U}(u n_o^{-1} m_o^{-1})$. Since P is IVIFS$\mathbb{G}$, $\mu_{P^U}(u) \geq \mu_{P^U}(e)$ and $\nu_{P^U}(e) \geq \nu_{P^U}(u)$ for all $u \in \mathbb{G}$. Thus, (26) now yields

$$\mu_{P^U}\left(u n^{-1} m^{-1}\right) \geq \mu_{P^U}\left(u n_o^{-1} m_o^{-1}\right). \tag{31}$$

Similarly, $\mu_{P^U}(u n^{-1} m^{-1}) \leq \mu_{P^U}(u n_o^{-1} m_o^{-1})$.

This shows that $\mu_{P^U}(u n^{-1} m^{-1}) = \mu_{P^U}(u n_o^{-1} m_o^{-1})$.

Consequently, $\mu_{(mn)P^U}(u) = \mu_{(m_o n_o)P^U}(u)$ for all $u \in \mathbb{G}$.

The lower case can be proved in the same way.

Similarly, we can show that

$$\begin{aligned} \nu_{(mn)P^U}(u) &= \nu_{(m_o n_o)P^U}(u), \nu_{(mn)P^L}(u) \\ &= \nu_{(m_o n_o)P^L}(u), \quad \text{for all } u \in \mathbb{G}. \end{aligned} \tag{32}$$

This shows that this is a well-defined composition.

We can view that the inverse of mP is $m^{-1}P$, for $m \in \mathbb{G}$.

Hence, ϖ is a group.

Now, let $P^\wedge(mP), P^\wedge(nP) \in P^\wedge$ where $mP, nP \in \varpi$.

Consider

$$\begin{aligned} \mu_{P^U}^{\wedge}\left(\mu_{mP^U} o \mu_{nP^U}\right) &= \mu_{P^U}^{\wedge}\left(\mu_{mnP^U}\right) = \mu_{P^U}(mn) \\ &\geq \min\{\mu_{P^U}(m), \mu_{P^U}(n)\} \\ &= \min\{\mu_{P^U}^{\wedge}(\mu_{mP^U}), \mu_{P^U}^{\wedge}(\mu_{nP^U})\}. \end{aligned} \tag{33}$$

Similarly, the lower case can be established.

As such,

$$\begin{aligned} \nu_{P^U}^{\wedge}\left(\nu_{mP^U} o \nu_{nP^U}\right) &\leq \max\{\nu_{P^U}^{\wedge}(\nu_{mP^U}), \nu_{P^U}^{\wedge}(\nu_{nP^U})\}, \\ \nu_{P^L}^{\wedge}\left(\nu_{mP^L} o \nu_{nP^L}\right) &\leq \max\{\nu_{P^L}^{\wedge}(\nu_{mP^L}), \nu_{P^L}^{\wedge}(\nu_{nP^L})\}. \end{aligned} \tag{34}$$

Moreover,

$$\begin{aligned} \mu_{P^U}^{\wedge}\left(\mu_{m^{-1}P^U}\right) &= \mu_{P^U}\left(m^{-1}\right) \\ &= \mu_{P^U}(m) \\ &= \mu_{P^U}^{\wedge}\left(\mu_{mP^U}\right). \end{aligned} \tag{35}$$

The lower case can be proved in the same way.

Similarly,

$\nu_{P^U}^{\wedge}(\nu_{m^{-1}P^U}) = \nu_{P^U}^{\wedge}(\nu_{mP^U})$ and $\nu_{P^L}^{\wedge}(\nu_{m^{-1}P^L}) = \nu_{P^L}^{\wedge}(\nu_{mP^L})$.

This shows that $P^\wedge$ is a IVIFS$\mathbb{G}$ of ϖ.

Definition 8. Assume that there exists an IVIFS$\mathbb{G}P$ of a finite group $\mathbb{G}$; define a mapping $P^\wedge: \varpi \longrightarrow [0,1]$ by

$P^\wedge(mP) = P(m)$, for all $m \in \mathbb{G}$, which is called an interval-valued intuitionistic fuzzy quotient group.

Theorem 19. *Let P be an IVIFNS$\mathbb{G}$. Then, establish a homomorphism t from $\mathbb{G}$ to ϖ defined by $t(m) = mP$ for all $m \in \mathbb{G}$ with kernel $\mathbb{G} = \{x \in \mathbb{G}: \mu_{P^U}(x) = \mu_{P^U}(e), \mu_{P^L}(x) = \mu_{P^L}(e)$ and $\nu_{P^U}(x) = \nu_{P^U}(e), \nu_{P^L}(x) = \nu_{P^L}(e)\}$.*

Proof. Let $m, n \in \mathbb{G}$. Then,

$$t(mn) = (mn)P = (mP)o(nP) = t(m)t(n), \tag{36}$$

which indicates that t is a natural homomorphism.

Moreover,

$$\begin{aligned} \text{Ker.}\, t &= \{x \in \mathbb{G}: t(x) = P\} \\ &= \{x \in \mathbb{G}: xP = P\}. \\ &= \{x \in \mathbb{G}: (xP)(y) = P(y), \quad \text{for all } y \in \mathbb{G}\} \\ &\{x \in \mathbb{G}: \mu_{xP^U}(y) = \mu_{P^U}(y), \mu_{xP^L}(y) = \mu_{P^L}(y) \text{ and } \nu_{xP^U}(y) = \nu_{P^U}(y), \nu_{xP^L}(y) = \nu_{P^L}(y), \quad \text{for all } y \in \mathbb{G}\}. \end{aligned} \tag{37}$$

In view of Definition 4, we have

$$\text{Ker.t} = \left\{x \in \mathbb{G}: \mu_{P^U}\left(yx^{-1}\right) = \mu_{P^U}(y), \mu_{P^L}\left(yx^{-1}\right) = \mu_{P^L}(y) \text{ and } \nu_{P^U}\left(yx^{-1}\right) = \nu_{P^U}(y), \nu_{P^L}\left(yx^{-1}\right) = \nu_{P^L}(y), \quad \text{for all } y \in \mathbb{G}\right\}. \tag{38}$$

Using Theorem 2 in the above relation yields $\mu_{P^U}(x) = \mu_{P^U}(e), \mu_{P^L}(x) = \mu_{P^L}(e)$ and $\nu_{P^U}(x) = \nu_{P^U}(e)$, $\nu_{P^L}(x) = \nu_{P^L}(e)$.

Consequently,

$$\text{Ker.t} = \{x \in \mathbb{G}: \mu_{P^U}(x) = \mu_{P^U}(e), \mu_{P^L}(x) = \mu_{P^L}(e) \text{ and } \nu_{P^U}(x) = \nu_{P^U}(e), \nu_{P^L}(x) = \nu_{P^L}(e)\}. \tag{39}$$

Remark 3. Note that $|\mathrm{Ker.t}| = \mathrm{IVIFO}(P)$.

Definition 9. Let P be an IVIFS$\mathbb{G}$ of finite group $\mathbb{G}$. Then, $|\varpi|$ is called the index of IVIFS$\mathbb{G}P$ and is denoted by $[\mathbb{G}\colon P]$.

Theorem 20 (interval-valued intuitionistic fuzzification of Lagrange's theorem). *If P is an IVIFS$\mathbb{G}$ of a finite group $\mathbb{G}$, then the index of IVIFS$\mathbb{G}$ of $\mathbb{G}$ divides the order of $\mathbb{G}$.*

Proof. By Theorem 19, we have a homomorphism t from $\mathbb{G}$ to ϖ, where $\varpi = \{mP, \text{ for all } m \in \mathbb{G}\}$.

As $\mathbb{G}$ is finite, it is trivial that ϖ is also finite.

Define

$$\hat{H} = \{x \in \mathbb{G}\colon xP = eP\}. \tag{40}$$

By Theorem 19, we have $\hat{H} = \{x \in \mathbb{G}\colon P(x) = P(e)\}$.

Now we partition group $\mathbb{G}$ into disjoint union of cosets.

Consider

$$\mathbb{G} = m_1\hat{H} \cup m_2\hat{H} \cup \ldots \cup m_k\hat{H}, \tag{41}$$

where $m_1\hat{H} = \hat{H}$. Now we prove that for each coset $m_j\hat{H}$ in relation (41), there exists an IVIF coset in ϖ; also, its corresponding counterpart is injective. Take a coset $m_j\hat{H}$. Let $\hbar \in \hat{H}$; then,

$$\begin{aligned} t\left(m_j\hbar\right) &= m_j\hbar P \\ &= m_jP\hbar P \\ &= m_jPeP = m_jP. \end{aligned} \tag{42}$$

Thus, ϖ maps every element of $m_j\hat{H}$ into the interval-valued intuitionistic fuzzy coset m_jP.

Now, we give a relation $\mathfrak{t}$ between set $\left\{m_j\hat{H}\colon 1 \le j \le k\right\}$ and set ϖ by

$$\mathcal{T}\left(m_j\mathbb{G}\right) = m_jP, \quad 1 \le j \le k. \tag{43}$$

The correspondence $\mathfrak{t}$ is injective.

For this, let $m_iP = m_lP$; then, $m_l^{-1}m_iP = eP$.

By using (40), we have $m_l^{-1}m_i \in \hat{H}$; this means that $m_i\hat{H} = m_l\hat{H}$, and hence $\mathfrak{t}$ is injective.

It is clear from the above discussion that $[\mathbb{G}\colon \hat{H}]$ and $[\mathbb{G}\colon P]$ are equal, since $[\mathbb{G}\colon \hat{H}]$ divides $O(\mathbb{G})$.

Corollary 4. *Assume that there exists an IVIFS$\mathbb{G}P$ of a finite group $\mathbb{G}$; then, $\mathrm{IVIFO}(P)$ divides the order of $\mathbb{G}$.*

The index of IVIFS$\mathbb{G}P$ of a finite group $\mathbb{G}$ can be obtained from the following relation.

Remark 4. $[\mathbb{G}\colon P] = |\mathbb{G}|/\mathrm{IVIFO}(P)$.

The algebraic information can be observed in the following examples.

Example 4. Let $\mathbb{G} = \{e, 1, 2, 3, 4, 5\}$ be a group of order 6. The IVIFS$\mathbb{G}P$ of $\mathbb{G}$ is defined as

$$\begin{aligned} P = \{&\langle e, [0.6, 1], [0, 0.4] >, < 1, [0.4, 0.5], [0.3, 0.4] >, < 2, [0.4, 0.5], [0.3, 0.4] >, < 3, [0.3, 0.35], [0.5, 0.55] >, \\ &< 4, [0.3, 0.35], [0.5, 0.55] >, < 5, [0.3, 0.35], [0.5, 0.55]\rangle\}. \end{aligned} \tag{44}$$

The set of all interval-valued intuitionistic fuzzy left cosets of $\mathbb{G}$ by P is given by

$$\varpi = \{P_e(g), P_1(g), P_2(g), P_3(g), P_4(g), P_5(g)\}. \tag{45}$$

This means that $[\mathbb{G}\colon P] = \mathrm{Card}(\varpi) = 6$.

Example 5. Let $\mathbb{G} = \{1, i, -i, -1\}$ be a cyclic group of order 4. The IVIFS$\mathbb{G}P$ of $\mathbb{G}$ is defined as

$$\begin{aligned} P = \{&\langle 1, [0.8, 0.9], [0, 0.4] >, \\ &< -1, [0.7, 0.8], [0.3, 0.4] >, \\ &< i, [0.5, 0.6], [0.5, 0.55] >, \\ &< -i, [0.5, 0.6], [0.5, 0.55]\rangle\}. \end{aligned} \tag{46}$$

The set of all interval-valued intuitionistic fuzzy left cosets of $\mathbb{G}$ by P is given by

$$\varpi = \{P_{-1}(g), P_i(g)\}. \tag{47}$$

This means that $[\mathbb{G}\colon P] = \mathrm{Card}(\varpi) = 2$.

5. Conclusion

In this article, we have fostered the idea of IVIFO of an element and have demonstrated the basic algebraic characteristic of these phenomena. Besides, we have created numerous algebraic properties of interval-valued intuitionistic fuzzy order of an element and have presented the interval-valued intuitionistic fuzzification of Lagrange theorems.

Data Availability

The data used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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