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Research Article

Yosida Approximation Iterative Methods for Split Monotone Variational Inclusion Problems

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In this paper, we present two iterative algorithms involving Yosida approximation operators for split monotone variational inclusion problems (S_pMVIP). We prove the weak and strong convergence of the proposed iterative algorithms to the solution of S_pMVIP in real Hilbert spaces. Our algorithms are based on Yosida approximation operators of monotone mappings such that the step size does not require the precalculation of the operator norm. To show the reliability and accuracy of the proposed algorithms, a numerical example is also constructed.

1. Introduction

Variational inequality which was brought into existence by Hartman and Stampacchia [1] plays an important role as mathematical model in physics, economics, optimization, networking structural analysis, and medical images. In 1994, Censor and Elfving [2] first presented the split feasibility problems (in short, SFP) for modeling in medical image reconstruction. From the last two decades, SFP has been implemented widely in intensity-modulation therapy treatment planning and other branches of applied sciences (see, e.g., [3–5]). Censor et al. [6] combined the VIP and SFP and presented a new type of variational inequality problem called split variational inequality problem (in short, SVIP) as follows:

$$\text{Find } x^* \in C \text{ such that } x^* \in \mathrm{VIP}(V_1; C) \text{ and } Ax^* \in \mathrm{VIP}(V_2; Q), \tag{1}$$

where C and Q are closed, convex subsets of Hilbert spaces H_1 and H_2, respectively, $A : H_1 \longrightarrow H_2$ is a bounded linear operator, $V_1 : H_1 \longrightarrow H_1$ and $V_2 : H_2 \longrightarrow H_2$ are two operators, $\mathrm{VIP}(V_1; C) = \{y \in C : \langle V_1(y), x - y\rangle \geq 0, \forall x \in C\}$ and $\mathrm{VIP}(V_2; Q) = \{z \in Q : \langle g(z), x - z\rangle \geq 0, \forall x \in Q\}$.

Moudafi [7] generalized SVIP into split monotone variational inclusion problem (in short, S_pMVIP) as follows:

$$\text{Find } x^* \in H_1 \text{ such that } x^* \in \mathrm{VI}(V_1, G_1; H_1) \text{ and } Ax^* \in \mathrm{VI}(V_2, G_2; H_2), \tag{2}$$

where $G_1 : H_1 \longrightarrow 2^{H_1}$ and $G_2 : H_2 \longrightarrow 2^{H_2}$ are set-valued mappings on Hilbert spaces H_1 and H_2, respectively, $\mathrm{VI}(V_1, G_1; H_1) = \{y \in H_1 : 0 \in V_1(y) + G_1(y)\}$ and $\mathrm{VI}(V_2, G_2; H_2) = \{z \in H_2 : 0 \in V_2(z) + G_2(z)\}$.

Moudafi [7] formulated the following iterative algorithm to find the solution of S_pMVIP. Let $\lambda > 0$, select an arbitrary starting point $x_0 \in H_1$, and compute

$$x_{n+1} = U[x_n + \gamma A^*(W - I)Ax_n], \tag{3}$$

where A^* is an adjoint operator of A, $\gamma \in (0, 1/L)$ with L being a spectral radius of operator A^*A, $U = R_\lambda^{G_1}(I - \lambda V_1) = (I + \lambda G_1)^{-1}(I - \lambda V_1)$ and $W = R_\lambda^{G_2}(I - \lambda V_2) = (I + \lambda G_2)^{-1}(I - \lambda V_2)$.

Let $N_C(x) = \{z \in H_1 : \langle z, y - x\rangle \le 0, \forall y \in C\}$ and $N_Q(x) = \{w \in H_2 : \langle w, y - x\rangle \le 0, \ \forall y \in Q\}$ be normal cones to the closed and convex sets C and Q, respectively. If $G_1 = N_C$ and $G_2 = N_Q$, then S_pMVIP reduces to S_pVIP. If $V_1 = V_2 = 0$, then S_pMVIP reduces to the split variational inclusion problem (in short, S_pVIP) for set-valued maximal monotone mappings, introduced and studied by Byrne et al. [8]:

$$\text{Find } x^* \in H_1 \text{ such that } x^* \in \mathrm{VI}(G_1 ; H_1) \text{ and } Ax^* \in \mathrm{VI}(G_2 ; H_2), \tag{4}$$

where $\mathrm{VI}(G_1 ; H_1) = \{y \in H_1 : 0 \in G_1(y)\}$ and $\mathrm{VI}(G_2 ; H_2) = \{z \in H_2 : 0 \in G_2(z)\}$, G_1, G_2 are the same as in (2). We denote the solution set of S_pVIP by Δ. Moreover, Byrne et al. [8] presented the following iterative algorithm to find the solution of S_pVIP. Let $\lambda > 0$, and select a starting point $x_0 \in H_1$. Then, compute

$$x_{n+1} = R_\lambda^{G_1}\left[x_n + \gamma A^*\left(R_\lambda^{G_2} - I\right)Ax_n\right], \tag{5}$$

where A^* is the adjoint operator of A, $L = \|A^*A\|$, $\gamma \in (0, 2/L)$ and $R_\lambda^{G_1}$, $R_\lambda^{G_2}$ are the resolvents of monotone mappings G_1, G_2, respectively. It can be easily seen that x^* solves S_pVIP if and only if $x^* = R_\lambda^{G_1}[x^* + \gamma A^*(I - R_\lambda^{G_2})Ax^*]$. Kazmi and Rizwi [9] proposed the following iterative method for approximating the common solutions of S_pVIP and fixed point problem of a nonexpansive mapping:

$$\begin{aligned} y_n &= R_\lambda^{G_1}\left[x_n + \gamma A^*\left(R_\lambda^{G_2} - I\right)Ax_n\right], \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)Sy_n, \end{aligned} \tag{6}$$

where f is a contraction and S is nonexpansive mapping. Later, Sitthithakerngkiet et al. [10] studied the common solutions of S_pVIP and a fixed point of an infinite family of nonexpansive mappings and introduced the following iterative method:

$$\begin{aligned} y_n &= R_\lambda^{G_1}\left[x_n + \gamma A^*\left(R_\lambda^{G_2} - I\right)Ax_n\right], \\ x_{n+1} &= \alpha_n \xi u + \beta_n x_n + [(1 - \beta_n)I - \alpha_n D]W_n y_n, \quad \forall n \ge 1, \end{aligned} \tag{7}$$

where $u \in H_1$ is a given point and W_n is W-mapping which is generated by an infinite family of nonexpansive mappings. Similar results related to S_pVIP can be found in [11–17].

The common figure among the above-explained iterative methods is that they used the resolvent of associated monotone mappings; secondly, the step size depends on the operator norm $\|A^*A\|$. To avoid this obstacle, self-adaptive step size iterative algorithms have been introduced (see, for example, [18–24]). Lopez et al. [20] introduced a relaxed method for solving split feasibility problem with self-adaptive step size. Recently, Dilshad et al. [25] proposed two iterative algorithms to solve S_pVIP in which the precalculation of the operator norm $\|A^*A\|$ is not required. They studied the weak and strong convergence of the proposed methods to approximate the solution of S_pVIP with the step size $\gamma_n = (\|x_n - R_\lambda^{G_1}x_n\|^2 + \|A^*(I - R_\lambda^{G_2})Ax_n\|^2)/(\|x_n - R_\lambda^{G_1}x_n + A^*(I - R_\lambda^{G_2})Ax_n\|^2)$, which do not depend upon the precalculated operator norm.

The resolvent of a maximal monotone operator G is defined as $J_\lambda^G = (I + \lambda G)^{-1}$, where λ is a positive real number. A resolvent operator of maximal monotone operator is single valued and firmly nonexpansive. Due to the fact that the zeros of maximal monotone operator are the fixed point sets of resolvent operator, the resolvent associated with a set-valued maximal monotone operator plays an important role to find the zeros of monotone operators. Following Byrne's iterative method (5), which is mainly based on the resolvents of monotone mappings, many researchers introduced and studied various iterative methods for S_pVIP (see, for example, [7–9, 18, 25, 26] and references therein).

Yosida approximation operator for a monotone mapping G and parameter $\lambda > 0$ is defined as $J_\lambda^G = (1/\lambda)(I - R_\lambda^G)$. It is well known that set-valued monotone operator can be regularized into a single-valued monotone operator by the process known as the Yosida approximation. Yosida approximation is a tool to solve a variational inclusion problem using nonexpansive resolvent operator and has been used to solve various variational inclusions and system of variational inclusions in linear and nonlinear spaces (see, for example, [18, 25–30]).

Due to the fact that the zero of Yosida approximation operator associated with monotone operator G is the zero of inclusion problem $0 \in G(x)$ and inspired by the work of Moudafi, Byrne, Kazmi, and Dilshad et al., our motive is to propose two iterative methods to solve S_pMVIP. The rest of the paper is organized as follows.

The next section contains some fundamental results and preliminaries. In Section 3, we describe two iterative algorithms using Yosida approximation of monotone mappings G_1 and G_2. Section 4 is devoted to the study of weak and strong convergence of the proposed iterative methods to the solution of S_pMVIP. In the last section, we give a numerical example in support of our main results and show the convergence of sequence obtained from the proposed algorithm to the solution of S_pMVIP.

2. Preliminaries

Let H be a real Hilbert space endowed with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. The strong and weak convergence of a sequence $\{x_n\}$ to x is denoted by $x_n \longrightarrow x$ and $x_n \rightharpoonup x$, respectively. The operator $T : H \longrightarrow H$ is said to be a contraction if $\forall x, y \in H, \|T(x) - T(y)\| \leq \kappa\|x - y\|, \kappa \in (0, 1)$; if $\kappa = 1$, then T is called nonexpansive and firmly nonexpansive if $\forall x, y \in H, \|T(x) - T(y)\|^2 \leq \langle x - y, Tx - Ty\rangle$; T is called τ-inverse strongly monotone if there exists $\tau > 0$ such that $\langle T(x - T(y), x - y\rangle \geq \tau\|T(x) - T(y)\|^2$.

For some $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{8}$$

$P_C x$ is called the projection of x onto $C \subset H$, which satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{9}$$

Moreover, $P_C x$ is also characterized by the fact that

$$P_C x = z \Leftrightarrow \langle x - z, y - z\rangle \geq 0, \quad y \in C. \tag{10}$$

In Hilbert spaces, the following equality and inequality hold for all $x, y, z \in H, \alpha, \beta, \gamma \in [0, 1]$ such that $\alpha + \beta + \gamma = 1$

$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 = {} & \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 \\ & - \beta\gamma\|y - z\|^2 - \gamma\alpha\|x - z\|^2,\end{aligned} \tag{11}$$

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{12}$$

Let $G : H \longrightarrow 2^H$ be a set-valued operator. The graph of G is defined by $\{(x, y): y \in G(x)\}$, and inverse of G is denoted by $G^{-1} = \{(y, x): y \in G(x)\}$. A set-valued mapping G is said to be monotone if $\langle u - v, x - y\rangle \geq 0$, for all $u \in G(x), v \in G(y)$. A monotone operator G is called a maximal monotone if there exists no other monotone operator such that its graph properly contains the graph of G.

Lemma 1 (see [31]). *If $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \beta_n)a_n + \delta_n, \quad n \geq 0, \tag{13}$$

where $\{\beta_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty}\beta_n = \infty$

(ii) $\limsup_{n\longrightarrow\infty} \delta_n/\beta_n \leq 0$ *or* $\limsup_{n\longrightarrow\infty} |\delta_n| < \infty$

then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2 (see [32]). *Let H be a Hilbert space. A mapping $F : H \longrightarrow H$ is τ -inverse strongly monotone if and only if $I - \tau F$ is firmly nonexpansive, for $\tau > 0$.*

Lemma 3 (see [33]). *Let H be a Hilbert space and $\{x_n\}$ be a bounded sequence in H. Assume there exists a nonempty subset $C \subset H$ satisfying the properties*

(i) $\lim_{n\longrightarrow\infty}\|x_n - z\|$ *exists for every* $z \in C$

(ii) $\omega_w(x_n) \subset C$

Then, there exists $x^* \in C$ such that $\{x_n\}$ converges weakly to x^*.

Lemma 4 (see [34]). *Let Γ_n be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence Γ_{n_k} of Γ_n such that $\Gamma_{n_k} < \Gamma_{n_k+1}$ for all $k \geq 0$. Also, consider the sequence of integers $\{\sigma(n)\}_{n\geq n_0}$ defined by*

$$\sigma(n) = \max\{k \leq n : \Gamma_k \leq \Gamma_{k+1}\}. \tag{14}$$

Then, $\{\sigma(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\longrightarrow\infty}\sigma(n) = \infty$ and for all $n \geq n_0$,

$$\max\left\{\Gamma_{\sigma(n)}, \Gamma_{(n)}\right\} \leq \Gamma_{\sigma(n)+1}. \tag{15}$$

3. Yosida Approximation Iterative Methods

Let $V_1 : H_1 \longrightarrow H_1, V_2 : H_2 \longrightarrow H_2$ be single-valued monotone mappings and $G_1 : H_1 \longrightarrow 2^{H_1}, G_2 : H_2 \longrightarrow 2^{H_2}$ be set-valued mappings such that $V_1 + G_1 : H_1 \longrightarrow 2^{H_1}$ and $V_2 + G_2 : H_2 \longrightarrow 2^{H_2}$ are set-valued maximal monotone mappings; $R_{\lambda_1}^{V_1+G_1}$, $R_{\lambda_2}^{V_2+G_2}$ and $J_{\lambda_1}^{V_1+G_1}$, $J_{\lambda_2}^{V_2+G_2}$ are the resolvents and Yosida approximation operators of $V_1 + G_1$ and $V_2 + G_2$, respectively. We propose the following iterative methods to approximate the solution of $\mathrm{S_pMVIP}$.

Algorithm 1. For an arbitrary x_0, compute the $n + 1^{\text{th}}$ iteration as follows:

$$\begin{aligned} u_n &= x_n - \gamma_n J_{\lambda_1}^{V_1+G_1}(x_n), \\ x_{n+1} &= u_n - \mu_n A^* J_{\lambda_2}^{V_2+G_2}(Au_n), \end{aligned} \tag{16}$$

where γ_n and μ_n are defined as

$$\gamma_n = \begin{cases} \dfrac{\tau_n\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|}{\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\| + \left\|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|}, & \text{if } \left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\| + \left\|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\| \neq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

$$\mu_n = \begin{cases} \dfrac{\tau_n \left\| J_{\lambda_2}^{V_2+G_2}(Au_n) \right\|}{\left\| J_{\lambda_1}^{V_1+G_1}(u_n) \right\| + \left\| A^* J_{\lambda_2}^{V_2+G_2}(Au_n) \right\|}, & \text{if} \left\| J_{\lambda_1}^{V_1+G_1}(u_n) \right\| + \left\| A^* J_{\lambda_2}^{V_2+G_2}(Au_n) \right\| \neq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{18}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$ and $\theta = \min \{2\lambda_1, 2\lambda_2\}$ such that $\tau_n \in (0, \theta)$.

Algorithm 2. For an arbitrary x_0, compute the $n + 1^{\text{th}}$ iteration as follows:

$$\begin{aligned} u_n &= x_n - \gamma_n J_{\lambda_1}^{V_1+G_1}(x_n), \\ v_n &= u_n - \mu_n A^* J_{\lambda_2}^{V_2+G_2}(Au_n), \\ x_{n+1} &= (1 - \beta_n)u_n + \alpha_n(v_n - u_n). \end{aligned} \tag{19}$$

where γ_n and μ_n are defined as

$$\begin{aligned} \gamma_n &= \begin{cases} \dfrac{\tau_n \left\| J_{\lambda_1}^{V_1+G_1}(x_n) \right\|}{\left\| J_{\lambda_1}^{V_1+G_1}(x_n) \right\| + \left\| A^* J_{\lambda_2}^{V_2+G_2}(Ax_n) \right\|}, & \text{if} \left\| J_{\lambda_1}^{V_1+G_1}(x_n) \right\| + \left\| A^* J_{\lambda_2}^{V_2+G_2}(Ax_n) \right\| \neq 0, \\ 0, & \text{otherwise}, \end{cases} \\ \mu_n &= \begin{cases} \dfrac{\tau_n \left\| J_{\lambda_2}^{V_2+G_2}(Au_n) \right\|}{\left\| J_{\lambda_1}^{V_1+G_1}(u_n) \right\| + \left\| A^* J_{\lambda_2}^{V_2+G_2}(Au_n) \right\|}, & \text{if} \left\| J_{\lambda_1}^{V_1+G_1}(u_n) \right\| + \left\| A^* J_{\lambda_2}^{V_2+G_2}(Au_n) \right\| \neq 0, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{20}$$

where $\alpha_n, \beta_n \in (0, 1)$, $\lambda_1 > 0$, $\lambda_2 > 0$, and $\theta = \min \{2\lambda_1, 2\lambda_2\}$ such that $\tau_n \in (0, \theta)$.

4. Main Results

We assume that the problem S_pMVIP is consistent and the solution set is denoted by Δ.

First, we prove following lemmas, which are used in the proof of our main results.

Lemma 5. *Let $V_1 : H_1 \longrightarrow H_1$ be single-valued monotone mappings and $G_1 : H_1 \longrightarrow 2^{H_1}$ be set-valued mappings such that $V_1 + G_1 : H_1 \longrightarrow 2^{H_1}$ be set-valued maximal monotone mapping. If $R_{\lambda_1}^{V_1+G_1}$ and $J_{\lambda_1}^{V_1+G_1}$ are the resolvent and Yosida approximation operators of $V_1 + G_1$, respectively, then for $\lambda_1 > 0$, following are equivalent:*

(i) $x^* \in H_1$ is the solution of $(V_1 + G_1)^{-1}(0)$

(ii) $R_{\lambda_1}^{V_1+G_1}(x^*) = x^*$

(iii) $J_{\lambda_1}^{V_1+G_1}(x^*) = 0$

Proof. The proof is trivial which is an immediate consequence of definitions of resolvent and Yosida approximation operator of maximal monotone mapping $V_1 + G_1$.□ □

Theorem 6. *Let H_1, H_2 be real Hilbert spaces; $V_1 : H_1 \longrightarrow H_1$, $V_2 : H_2 \longrightarrow H_2$ be single-valued monotone mappings, $G_1 : H_1 \longrightarrow 2^{H_1}$, $G_2 : H_2 \longrightarrow 2^{H_2}$ be set-valued maximal monotone mappings such that $V_1 + G_1$ and $V_2 + G_2$ are maximal monotone, and $A : H_1 \longrightarrow H_2$ be a bounded linear operator. Assume that $\theta = \min \{2\lambda_1, 2\lambda_2\}$ such that $\inf \tau_n(\theta - \tau_n) > 0$. Then, the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a point $z \in \Delta$.*

Proof. Let $z \in \Delta$. Since the Yosida approximation operator $J_{\lambda_1}^{V_1+G_1}$ is λ_1-inverse strongly monotone, for $\lambda_1 > 0$, then by Algorithm 1 and (12), we have

$$\begin{aligned} \|u_n - z\|^2 &= \left\| x_n - \gamma_n J_{\lambda_1}^{V_1+G_1}(x_n) - z \right\|^2 \\ &= \|x_n - z\|^2 + \gamma_n^2 \left\| J_{\lambda_1}^{V_1+G_1}(x_n) \right\|^2 \\ &\quad - 2\gamma_n \left\langle J_{\lambda_1}^{V_1+G_1}(x_n), x_n - z \right\rangle \\ &\leq \|x_n - z\|^2 + \gamma_n^2 \left\| J_{\lambda_1}^{V_1+G_1}(x_n) \right\|^2 \\ &\quad - 2\gamma_n\lambda_1 \left\| J_{\lambda_1}^{V_1+G_1}(x_n) \right\|^2 = \|x_n - z\|^2 \\ &\quad + \left(\gamma_n^2 - 2\gamma_n\lambda_1\right) \left\| J_{\lambda_1}^{V_1+G_1}(x_n) \right\|^2. \end{aligned} \tag{21}$$

Now, using (17), we estimate that

From (21) and (22), we get

$$\begin{aligned}\left(\gamma_n^2-2\gamma_n\lambda_1\right)\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|^2 &= \left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|^2\left[\frac{\tau_n{}^2\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|^2}{\left(\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|\right)^2}-\frac{2\tau_n\lambda_1\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|}{\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|}\right]\\ &= \left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|^3\left[\frac{\tau_n{}^2\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|-2\lambda_1\tau_n\left(\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|\right)}{\left(\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|\right)^2}\right]\\ &\le \left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|^3\left[\frac{\left(\tau_n{}^2-2\lambda_1\tau_n\right)\left(\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|\right)}{\left(\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|\right)^2}\right]\\ &= \frac{\left(\tau_n{}^2-2\lambda_1\tau_n\right)\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|}.\end{aligned} \tag{22}$$

$$\|u_n-z\|^2\le\|x_n-z\|^2+\frac{\left(\tau_n{}^2-2\lambda_1\tau_n\right)\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|}. \tag{23}$$

Since $J_{\lambda_2}^{V_2+G_2}$ is λ_2-inverse strongly monotone and using (12), we estimate

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \left\|u_n-\mu_nJ_{\lambda_2}^{V_2+G_2}(Au_n)-z\right\|^2\\ &\le \|u_n-z\|^2+\mu_n^2\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^2\\ &\quad -2\mu_n\left\langle J_{\lambda_2}^{V_2+G_2}(Au_n),u_n-z\right\rangle\\ &= \|u_n-z\|^2+\mu_n^2\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^2\\ &\quad -2\mu_n\left\|J_{\lambda_1}^{V_1+G_1}(Au_n)\right\|^2=\|u_n-z\|^2\\ &\quad +\left(\mu_n^2-2\mu_n\lambda_2\right)\left\|J_{\lambda_1}^{V_2+G_2}(Au_n)\right\|^2.\end{aligned} \tag{24}$$

By (18), it turns out that

$$\begin{aligned}\left(\mu_n^2-2\mu_n\lambda_2\right)\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^2 &= \left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^2\left[\frac{\tau_n{}^2\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^2}{\left(\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|\right)^2}-\frac{2\tau_n\lambda_2\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|}{\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|}\right]\\ &= \left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^3\left[\frac{\tau_n{}^2\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|-2\lambda_2\tau_n\left(\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|\right)}{\left(\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|\right)^2}\right]\\ &= \left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^3\left[\frac{\left(\tau_n{}^2-2\lambda_2\tau_n\right)\left(\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|\right)}{\left(\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\|+\|\left\|A^*J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|\right)^2}\right]\\ &= \frac{\left(\tau_n{}^2-2\lambda_2\tau_n\right)\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|}.\end{aligned} \tag{25}$$

It follows from (24) and (25) that

$$\|x_{n+1}-z\|^2 \le \|u_n-z\|^2 + \frac{\left(\tau_n{}^2 - 2\lambda_2\tau_n\right)\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\| + \left\|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|}. \tag{26}$$

Combining (23) and (26), we get

$$\begin{aligned}\|x_{n+1}-z\|^2 \le \|x_n-z\|^2 &- \frac{\tau_n(2\lambda_1-\tau_n)\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\| + \left\|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|} \\ &- \frac{\tau_n(2\lambda_2-\tau_n)\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\| + \left\|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|},\end{aligned} \tag{27}$$

$$\le \|x_n - z\|, \tag{28}$$

which implies that $\{x_n\}$ is Fejér monotone with respect to Δ and hence bounded, which assures that $\lim_{n\longrightarrow\infty}\|x_n-z\|$ exists for all $z\in\Delta$. Keeping in mind that $\theta=\min\{2\lambda_1,2\lambda_2\}$, from (27), we have

$$\sum_{n=1}^{\infty}\tau_n(\theta-\tau_n)\left[\frac{\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\| + \left\|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|} + \frac{\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\| + \left\|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|}\right] < \infty. \tag{29}$$

Due to the assumption that $\inf \tau_n(\theta-\tau_n)>0$ and the properties of convergent series, we conclude that

$$\lim_{n\longrightarrow\infty}\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\| = \lim_{n\longrightarrow\infty}\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\| = 0. \tag{30}$$

Hence, there exist constants K_1 and K_2 such that

$$\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\| \le K_1, \left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\| \le K_2. \tag{31}$$

By Algorithm 1 and (30), we get

$$\|x_{n+1}-x_n\| \le \|x_{n+1}-u_n\| + \|u_n-x_n\| \le K_1\gamma_n + K_2\mu_n \longrightarrow 0, \quad \text{as } n\longrightarrow\infty. \tag{32}$$

Let$\{x^*\}\in\omega_w(x_n)$and$\{x_{n_k}\}$be a subsequence of$\{x_n\}$that converges weakly to$\{x^*\}$, which implies that$\{x_{n_k}\}$and$\{u_{n_k}\}$ also converge to$\{x^*\}$. Recall that $J_{\lambda_1}^{V_1+G_1}$ is λ_1-inverse strongly monotone and $\{x_{n_k}\}$ converges to x^*, and using (30), we get

$$\left\langle J_{\lambda_1}^{V_1+G_1}(x_{n_k}) - J_{\lambda_1}^{V_1+G_1}(x^*), x_{n_k}-x^*\right\rangle \ge \lambda_1\left\|J_{\lambda_1}^{V_1+G_1}(x_{n_k}) - J_{\lambda_1}^{V_1+G_1}(x^*)\right\|^2. \tag{33}$$

Taking limit $k\longrightarrow\infty$, we obtain $J_{\lambda_1}^{V_1+G_1}(x^*)=0$.

Replacing $J_{\lambda_1}^{V_1+G_1}$ by $J_{\lambda_2}^{V_2+G_2}A$, x_{n_k} by Au_{n_k} with the same arguments, we get $J_{\lambda_2}^{V_2+G_2}A(x^*)=0$. This completes the proof.□ □

Theorem 7. *Let H_1, H_2 be real Hilbert spaces; $V_1: H_1 \longrightarrow H_1$, $V_2: H_2\longrightarrow H_2$ be single-valued monotone mappings, $G_1: H_1\longrightarrow 2^{H_1}$, $G_2: H_2\longrightarrow 2^{H_2}$ be set-valued maximal monotone mappings such that V_1+G_1 and V_2+G_2 are maximal monotone, and $A: H_1\longrightarrow H_2$ be a bounded linear operator. If $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0,1)$ and $\theta=\min\{2\lambda_1,2\lambda_2\}$ such that $\tau_n\in(0,\theta)$ and*

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\beta_n = 0,\\ &\sum_{n=0}^{\infty}\beta_n=\infty,\\ &\lim_{n\longrightarrow\infty}(1-\alpha_n)\alpha_n>0,\\ &\inf_n \tau_n(\theta-\tau_n)>0,\end{aligned} \tag{34}$$

then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $z=P_\Delta(0)$.

Proof. Let $z=P_\Delta(0)$; then, from (23) and (26) of the proof of Theorem 6, we have

$$\|u_n-z\|^2 \le \|x_n-z\|^2 + \frac{\left(\tau_n{}^2 - 2\lambda_1\tau_n\right)\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\| + \left\|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|}, \tag{35}$$

$$\|v_n-z\|^2 \le \|u_n-z\|^2 + \frac{\left(\tau_n{}^2 - 2\lambda_2\tau_n\right)\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\| + \left\|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|}. \tag{36}$$

Since $\tau_n\le\min\{2\lambda_1,2\lambda_2\}$, we get $\|v_n-z\|\le\|u_n-z\|\le\|x_n-z\|$. From Algorithm 2, we have

$$\begin{aligned}\|x_{n+1}-z\| &= \|(1-\beta_n)u_n + \alpha_n(v_n-u_n) - z\|\\ &\le (1-\alpha_n-\beta_n)\|u_n-z\| + \|+\alpha_n\|v_n-z\| + \beta_n\|-z\|\\ &\le (1-\beta_n)\|x_n-z\| + \beta_n\|z\| \le \max\{\|x_n-z\|,\|z\|\}\\ &\le\cdots\le\max\{\|x_0-z\|,\|z\|\},\end{aligned} \tag{37}$$

which implies that the sequence $\{x_n\}$ is bounded and hence, the sequences $\{u_n\}$,$\{v_n\}$,$\{J_{\lambda_1}^{V_1+G_1}(u_n)\}$ and $\{A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\}$

are also bounded. Now,

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|(1-\beta_n)u_n+\alpha_n(v_n-u_n)-z\|^2\\ &\le (1-\alpha_n-\beta_n)\|u_n-z\|^2+\alpha_n\|v_n-z\|^2\\ &\quad+\beta_n\|z\|^2-\alpha_n(1-\alpha_n-\beta_n)\|v_n-u_n\|^2.\end{aligned} \tag{38}$$

Combining (35), (36), and (38), we obtain

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le (1-\alpha_n-\beta_n)\left[\|x_n-z\|^2+\frac{(\tau_n{}^2-2\lambda_1\tau_n)\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|}\right]\\ &\quad+\alpha_n\left[\|u_n-z\|^2+\frac{(\tau_n{}^2-2\lambda_2\tau_n)\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|}\right]\\ &\quad+\beta_n\|z\|^2-\alpha_n(1-\alpha_n-\beta_n)\|v_n-u_n\|^2\le\|x_n-z\|^2\\ &\quad+\beta_n(-\|x_n-z\|^2+\|z\|^2)-\alpha_n(1-\alpha_n-\beta_n)\|v_n-u_n\|^2\\ &\quad-\frac{(1-\alpha_n-\beta_n)(\tau_n{}^2-2\lambda_1\tau_n)\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|}\\ &\quad-\frac{\alpha_n(\tau_n{}^2-2\lambda_2\tau_n)\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|}.\end{aligned} \tag{39}$$

We discuss the two possible cases.

Case 1. If the sequence$\{\|x_n-z\|\}$ is nonincreasing, then there exists a number $k\ge0$ such that $\|x_{n+1}-z\|\le\|x_n-z\|$, for each $n\ge k$. Then, $\lim_{n\longrightarrow\infty}\|x_n-z\|$ exists and hence, $\lim_{n\longrightarrow\infty}(\|x_{n+1}-z\|-\|x_n-z\|)=0$. Thus, it follows from (39) that

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\|v_n-u_n\|=0,\ \lim_{n\longrightarrow\infty}\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|=0,\\ &\lim_{n\longrightarrow\infty}\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|=0.\end{aligned} \tag{40}$$

From (40), we conclude that $\lim_{n\longrightarrow\infty}\gamma_n=\lim_{n\longrightarrow\infty}\mu_n=0$ and $\lim_{n\longrightarrow\infty}\|u_n-x_n\|=0$. We observe from Algorithm 2 that $x_{n+1}-u_n=\alpha_n(v_n-u_n)+\gamma_nu_n\longrightarrow0$; thus,

$$\begin{aligned}\|x_{n+1}-x_n\|&\le\|x_{n+1}-u_n\|+\|u_n-x_n\|\le\|v_n-u_n\|\\ &\quad+\gamma_n\|u_n\|+\|u_n-x_n\|\longrightarrow0\quad\text{as n}\longrightarrow\infty.\end{aligned} \tag{41}$$

This shows that the sequence $\{x_n\}$ is asymptotically regular. By Theorem 6, we have that $\omega_w(x_n)\subset\Delta$. Setting$z_n=(1-\alpha_n)u_n+\alpha_nv_n$and rewriting$x_{n+1}=(1-\beta_n)z_n+\alpha_n\beta_n(v_n-u_n)$, we have

$$\begin{aligned}\|z_n-z\|&=\|(1-\alpha_n)u_n+\alpha_nv_n-z\|\\ &\le(1-\alpha_n)\|u_n-z\|+\alpha_n\|v_n-z\|\le\|x_n-z\|.\end{aligned} \tag{42}$$

From (42) and Algorithm 2, we get

$$\begin{aligned}\|x_{n+1}-z\|^2&=\|(1-\beta_n)(z_n-z)+\beta_n(\alpha_n(v_n-u_n)-z)\|^2\\ &\le(1-\beta_n)^2\|z_n-z\|^2+2\beta_n\langle\alpha_n(v_n-u_n)-z,x_{n+1}-z\rangle\\ &\le(1-\beta_n)\|x_n-z\|^2+2\beta_n\{\alpha_n\langle v_n-u_n,x_{n+1}-z\rangle\\ &\quad+\langle -z,x_{n+1}-z\rangle\},\end{aligned} \tag{43}$$

or

$$a_{n+1}=(1-\beta_n)a_n+b_n, \tag{44}$$

where $a_n=\|x_n-z\|, b_n=2\beta_n\{\alpha_n\langle v_n-u_n,x_{n+1}-z\rangle+\langle -z, x_{n+1}-z\rangle\}$.

Since $\omega_w(x_n)\subset\Delta$ and $z=P_\Delta(0)$, then using (40), we get

$$\begin{aligned}\lim\sup_{n\longrightarrow\infty}\frac{b_n}{\beta_n}&=\lim\sup_{n\longrightarrow\infty}\{2\alpha_n\langle v_n-u_n,x_{n+1}-z\rangle\\ &\quad+\langle -z,x_{n+1}-z\rangle\}=\lim\sup_{n\longrightarrow\infty}\langle -z,x_{n+1}-z\rangle\le0.\end{aligned} \tag{45}$$

Thus, by Lemma 1, we obtain $x_n\longrightarrow z$.

Case 2. If the sequence $\{\|x_n-z\|\}$ is not nonincreasing, we can select a subsequence $\{\|x_{n_k}-z\|\}$ of $\{\|x_n-z\|\}$ such that $\|x_{n_k}-z\|\le\|x_n-z\|$ for all $k\in\mathbb{N}$. In this case, we define a subsequence of positive integers $\sigma(n)\longrightarrow\infty$ with the properties

$$\begin{aligned}&\left\|x_{\sigma(n)}-z\right\|<\left\|x_{\sigma(n)+1}-z\right\|,\\ &\max\left\{\left\|x_{\sigma(n)}-z\right\|,\|x_n-z\|\right\}\le\left\|x_{\sigma(n)+1}-z\right\|.\end{aligned} \tag{46}$$

If $\|x_{n+1}-z\|>\|x_n-z\|$ for some $n\ge0$, then it follows from (39) that

$$\begin{aligned}&\alpha_n(1-\alpha_n-\beta_n)\|v_n-u_n\|^2+\frac{(1-\alpha_n-\beta_n)(\tau_n{}^2-2\lambda_1\tau_n)\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|}\\ &\quad+\frac{\alpha_n(\tau_n{}^2-2\lambda_2\tau_n)\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|^3}{\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\|+\left\|A^*J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|}\le\beta_n(\|z\|^2-\|x_n-z\|^2).\end{aligned} \tag{47}$$

Replacing n by $\sigma(n)$ and taking limit $n\longrightarrow\infty$, we get the following relation for the subsequences $\{x_{\sigma(n)}\}$, $\{u_{\sigma(n)}\}$, and $\{v_{\sigma(n)}\}$:

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\left\|v_{\sigma(n)}-u_{\sigma(n)}\right\|=0,\\ &\lim_{n\longrightarrow\infty}\left\|J_{\lambda_1}^{V_1+G_1}\left(x_{\sigma(n)}\right)\right\|=0,\\ &\lim_{n\longrightarrow\infty}\left\|J_{\lambda_2}^{V_2+G_2}\left(Au_{\sigma(n)}\right)\right\|=0\end{aligned} \tag{48}$$

Thus, we have $\|x_{\sigma(n+1)} - x_{\sigma(n)}\| \longrightarrow 0$, as $n \longrightarrow \infty$ and $\omega_w(x_{\sigma(n)}) \subset \Delta$. It is remaining to show that $x_n \longrightarrow z$.

Replacing n by $\sigma(n)$ in (47), using $\|x_{\sigma(n)} - z\| < \|x_{\sigma(n)+1} - z\|$ and boundedness of $\|x_n - z\|$, we have

$$\left\|x_{\sigma(n)} - z\right\|^2 \leq M\left\|v_{\sigma(n)} - u_{\sigma(n)}\right\| + 2\left\langle -z, x_{\sigma(n)+1} - z\right\rangle. \quad (49)$$

Since $z = P_\Delta(0)$, $\omega(x_{\sigma(n)}) \subset \Delta$ with using $\|v_{\sigma(n)} - u_{\sigma(n)}\| \longrightarrow 0$ and $\|x_{\sigma(n)+1} - x_{\sigma(n)}\| \longrightarrow 0$, we have

$$\limsup_{n\longrightarrow\infty}\left\langle -z, x_{\sigma(n)+1} - z\right\rangle = \limsup_{n\longrightarrow\infty}\left\langle -z, x_{\sigma(n)} - z\right\rangle = \max_{r\in\omega_w(x_\sigma(n))}\langle -z, r - z\rangle\Big\} \leq 0. \quad (50)$$

From (49) and (52), we conclude that $x_{\sigma(n)} \longrightarrow z$ and

$$\|x_n - z\| \leq \left\|x_{\sigma(n)+1} - z\right\| \leq \left\|x_{\sigma(n)+1} - x_{\sigma(n)}\right\| + \left\|x_{\sigma(n)} - z\right\| \longrightarrow 0, \quad (51)$$

that is, $x_n \longrightarrow z$. This complete the proof.□ □

For $\tau_n = 1$, we have the following result for the convergence of Algorithm 2.

Corollary 8. *Let H_1, H_2, V_1, V_2, G_1, G_2, and A, A^* be the same as defined in Theorem 7. If $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0, 1)$ and assuming that $\lambda_1 > 1/2$ and $\lambda_2 > 1/2$ satisfying*

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\beta_n = 0, \sum_{n=0}^{\infty}\beta_n = \infty,\\&\lim_{n\longrightarrow\infty}(1 - \alpha_n)\alpha_n > 0,\end{aligned} \quad (52)$$

then the sequence $\{x_n\}$ generated by Algorithm 2 (with $\tau_n = 1$) converges strongly to $z = P_\Delta(0)$.

For $\beta_n = 0$, we have the following corollary for the convergence of Algorithm 2.

Corollary 9. *Let H_1, H_2, V_1, V_2, G_1, G_2, and A, A^* be the same as defined in Theorem 7. If $\{\alpha_n\}$ is a sequence in $(0, 1$) and assuming that $\theta = \min\{2\lambda_1, 2\lambda_2\}$ such that $\tau_n \in (0, \theta$) and*

$$\begin{aligned}&\lim_{n\longrightarrow\infty}(1 - \alpha_n)\alpha_n > 0,\\&\inf_n \tau_n(\theta - \tau_n) > 0,\end{aligned} \quad (53)$$

then the sequence $\{x_n\}$ generated by the iterative method

$$\begin{aligned}u_n &= x_n - \gamma_n J_{\lambda_1}^{V_1+G_1}(x_n),\\v_n &= u_n - \mu_n A^* J_{\lambda_2}^{V_2+G_2}(Au_n),\\x_{n+1} &= (1 - \alpha_n)u_n + \alpha_n v_n,\end{aligned} \quad (54)$$

where γ_n and μ_n are defined as in Algorithm 2 (with $\tau_n = 1$), converges strongly to $z \in \Delta$.

For $\tau_n = 1$ and $\beta_n = 0$, we have the following corollary for the convergence of Algorithm 2.

Corollary 10. *Let H_1, H_2, V_1, V_2, G_1, G_2, and A, A^* be the same as defined in Theorem 7. If $\{\alpha_n\}$ be a sequence in $(0, 1)$ and assuming that*

$$\begin{aligned}&\lim_{n\longrightarrow\infty}(1 - \alpha_n)\alpha_n > 0,\\&\lambda_1 > \frac{1}{2},\\&\lambda_2 > \frac{1}{2},\end{aligned} \quad (55)$$

then the sequence $\{x_n\}$ generated by the iterative method

$$\begin{aligned}u_n &= x_n - \gamma_n J_{\lambda_1}^{V_1+G_1}(x_n),\\v_n &= u_n - \mu_n A^* J_{\lambda_2}^{V_2+G_2}(Au_n),\\x_{n+1} &= (1 - \alpha_n)u_n + \alpha_n v_n,\end{aligned} \quad (56)$$

where γ_n and μ_n are defined in Algorithm 2 (with $\tau_n = 1$), converges strongly to $z \in \Delta$.

5. Numerical Example

Let $H_1 = H_2 = \mathbb{R}$ and $V_1 = V_2 = 0$; $G_1 : \mathbb{R} \longrightarrow \mathbb{R}$, $G_2 : \mathbb{R} \longrightarrow \mathbb{R}$ are defined as $G_1(x) = 2x + 3$ and $G_2(x) = 2(x + 1)$, respectively. One can easily check that G_1 and G_2 are monotone and the Yosida approximation operator of G_1 and G_2 for $\lambda_1 = \lambda_2 = 1$ is computed as

$$\begin{aligned}J_{\lambda_1}^{V_1+G_1}(x) &= \frac{2x + 3}{3},\\J_{\lambda_2}^{V_2+G_2}(x) &= \frac{2x + 2}{3}.\end{aligned} \quad (57)$$

Let $A : H_1 \longrightarrow H_2$ be defined as $A(x) = 2x/3$, then, for $\tau_n = (2 - (e^{1/n}/2)) \in (0, 2)$, we compute the step size as

$$\begin{aligned}\gamma_n &= \frac{\tau_n\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\|}{\left\|J_{\lambda_1}^{V_1+G_1}(x_n)\right\| + \left\|A^* J_{\lambda_2}^{V_2+G_2}(Ax_n)\right\|} = \left(2 - \frac{e^{1/n}}{2}\right)\frac{9}{13},\\\mu_n &= \frac{\tau_n\left\|J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|}{\left\|J_{\lambda_1}^{V_1+G_1}(u_n)\right\| + \left\|A^* J_{\lambda_2}^{V_2+G_2}(Au_n)\right\|} = \left(2 - \frac{e^{1/n}}{2}\right)\frac{6}{13}.\end{aligned} \quad (58)$$

Then, for $\alpha_n = (2 - (e^{1/n}/3))$ and two different values of (β_n) (for example, $\beta_n = 1/(n + 5)$ and $\beta_n = 1/(n + 10)$) and for arbitrary x_0 (for example, $x_0 = -2$ and $x_0 = 0$), we compute the iterative sequences from Algorithm 2 as follows:

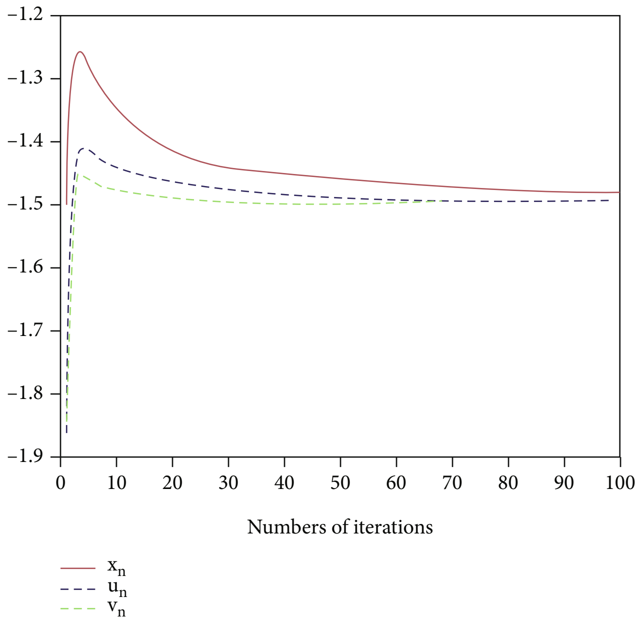

FIGURE 1: Convergence of iterative sequences $\{u_n\}$, $\{v_n\}$, and $\{x_n\}$ to $z = -1.5$ for $\beta_n = 1/n + 5$ and $x_0 = -2$.

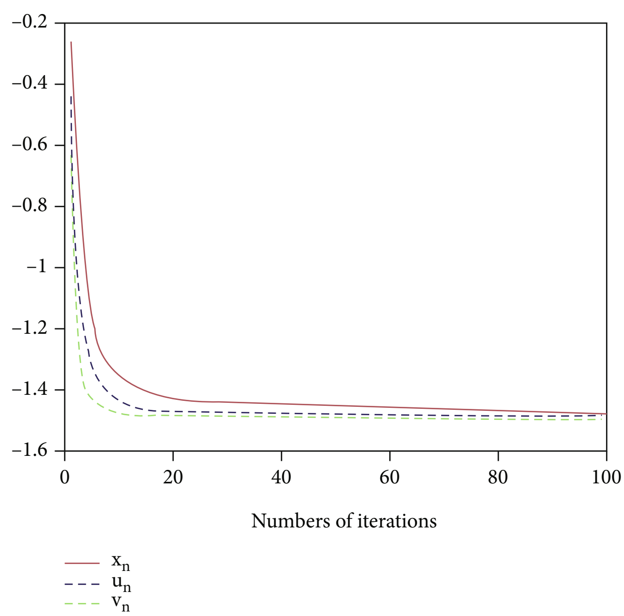

FIGURE 2: Convergence of iterative sequences $\{u_n\}$, $\{v_n\}$, and $\{x_n\}$ to $z = -1.5$ for $\beta_n = 1/n + 10$ and $x_0 = 0$.

$$u_n = x_n - \left(2 - \frac{e^{1/n}}{2}\right)\frac{3}{13}(2x_n + 3),$$
$$v_n = u_n - \left(2 - \frac{e^{1/n}}{2}\right)\frac{8}{117}(2x_n + 3), \qquad (59)$$
$$x_{n+1} = \left(1 - \frac{1}{n+5}\right)u_n + \left(2 - \frac{e^{1/n}}{3}\right)(v_n - u_n).$$

In Figures 1 and 2, we show that the obtained sequences $\{u_n\}$, $\{v_n\}$, and $\{x_n\}$ converge to $z = -(3/2)$ for randomly selected arbitrary values of $x_0 = -2$ and 0.

6. Conclusions

We have proposed two iterative algorithms for S_pMVIP which are mainly based on the Yosida approximation operators. Since the zero of Yosida approximation of monotone mapping $V_1 + G_1$ is the solution of $(V_1 + G_1)^{-1}(0)$, we used the Yosida approximations of monotone mappings $V_1 + G_1$ and $V_2 + G_2$ to solve S_pMVIP. We proved the weak and strong convergence of the composed iterative algorithms to investigate the solution of S_pMVIP under some suitable assumptions such that the estimation of step size does not require any prior calculation of the operator norm $\|A^*A\|$. To show the accuracy and efficiency of our algorithms, we have present a numerical example and showed the convergence using different parameters.

Data Availability

We claim that this work is a theoretical result, and there is no available data source.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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