

Research Article

Dynamics of Hidden Attractors in Four-Dimensional Dynamical Systems with Power Law

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Four-dimensional continuous chaotic models with Caputo fractional derivative are presented. Fixed point theory is used to investigate the existence and uniqueness of complex systems. The dynamical properties are studied, including the Lyapunov exponent, phase portrait, and time series analysis. The hyperchaotic nature of each system is revealed by the positive exponents. The numerical method is introduced to describe the influence of the order of the Caputo fractional derivative. The phase portraits are presented to investigate the behavior and effect of some key parameters and fractional orders on model dynamics. The systems approach fixed point attractors for fractional-order and increase the visibility of the attractor by decreasing fractional order. This means that a change in fractional order has a significant impact on the dynamics of the models. When the order of the derivative is equal to one, both systems oscillate frequently. However, as the fractional order is reduced, the system oscillations decrease as compared to the integer-order, and the system moves towards its fixed point, reveals the hidden attractors inherent in the system, and enabling it to develop to a stable state more efficiently.

1. Introduction

Fractional calculus (FC) is widely used to model different natural phenomena in several fields of science and engineering. The nonlocal nature of fractional order derivatives and integrals is shown to be a significant component in fractional calculus for a wide range of applications [1]. Contiguous data is used to approximate the integer-order derivative of a function at a particular point, but the fractional derivative requires a completely unique situation from the beginning. The nonlocality of the fractional derivative is important in describing the storage and hereditary aspects in the system [2–5]. As a result, models that use fractional derivatives are more realistic than integer-order. The fractional derivatives' second advantage is that it can model transitional processes. The integer order derivative cannot model the genuine occurrences in several physical problems, such as fluid diffusion in porous media, as the processes are intermediate. Recently, several novel fractional operators have been sug-

gested with different types of kernels. The Caputo fractional operator is one of the most important definition in fractional calculus [6, 7].

Since the analytical techniques are unable to solve many fractional-order dynamic systems, therefore, a wide range of numerical algorithms have been developed to find an approximate solution to fractional-order systems [8–11]. Differential equations are particularly used to describe a variety of natural phenomena. Dynamical systems are one of the most important and rapidly increasing subjects that make extensive use of differential equations. The dynamical system can describe each point in phase space across time. If the mathematical model is a dynamical system that represents a real-world situation, its state may be predicted at any moment t . Fractional-order calculus is one of the useful tools for understanding complex dynamical systems with nonlinear properties [12, 13].

A chaotic system is the combination of differential equations in which any two solutions for two relatively similar

conditions are drastically different at any given time. For the reasons stated above, many chaotic systems have been introduced to produce chaos. These systems produce attractors, which are a set of invariant points in phase space. In 1963, Lorenz was the first to introduce chaos [14]. Since then, chaos has attracted the curiosity of many academics and research groups throughout the world. Over the last few decades, chaos has been successfully applied in communications, control applications such as managing irregular behavior in devices and systems [15], communication privacy, and synchronisation of same or different systems that lead to data encryption, chaotic band radio, and secure communications [16].

Although it has been shown that fractional-order chaotic systems are better to realize physical objects due to their dynamic nature, the majority of chaotic systems are of integer order. Fractional-order chaotic systems have been shown to be more appropriate for modelling nonlinear systems than integer-order because their future state is determined not only by their current state but also by the conditions of their previous states. Because fractional calculus can improve the complexity and precision of chaos, researchers are interested in fractional order multiwing chaotic systems. Many scholars have examined a wide range of chaotic systems with fractional-order operators and have found this area to be a useful tool for analysing the system's complexities [17–20].

According to Liouville's theorem, for the conservative dynamical systems, the volume of phase space is conservative and the flow is incompressible. Lichtenberg and Lieberman constructed a system with coexisting quasiperiodic and coexisting chaotic flows [21]. 3D conservative chaotic systems were constructed by Cang et al. [22]; meanwhile, conservative flows and the dynamics were also studied. Furthermore, conservative chaotic mapping has been revealed in [23, 24]. Hidden chaotic attractors and hidden periodic oscillations have been investigated in drilling systems [25], phase-locked loops, and in aviation control system [26, 27]. The attraction basin for hidden attractor's is not connected to any nearby regions of equilibrium, which shows that these attractors are different from self-excited attractors. Most of the work done in understanding of hidden attractors is with integer-order derivatives, though the investigations of hidden attractors in a chaotic system can be done very efficiently using noninteger derivatives. The investigation of hidden chaotic attractors in fractional-order systems is the best way to go deeper into a new intriguing and underexplored subject with great importance. Therefore, it has a great importance to realize the hidden attractors with fractional derivatives [28]. The investigation fractional-order dynamical systems with hidden attractors having one stable equilibrium [29], no-equilibria [30], a line or surfaces of equilibria [31], and even fractional-order hyper-chaotic systems [32] have been reported in the literature. Also, different families of hidden attractors have been studied extensively [33].

Motivated from the above works, we consider two different conservative four-dimensional chaotic systems with fractional order Caputo derivative. We study the hidden

attractors present in these two systems for particular values of the control parameters. The numerical algorithm used for the solution has many advantages. This method is consistent, stable, and convergent [34]. Here, we consider two different four-dimensional dynamical systems [35] with Caputo fractional derivative as

$$\begin{aligned} {}^{\mathcal{C}}\mathbf{D}_t^\alpha x(t) &= dxy, {}^{\mathcal{C}}\mathbf{D}_t^\alpha y(t) = xz, {}^{\mathcal{C}}\mathbf{D}_t^\alpha z(t) \\ &= -xy + cw, {}^{\mathcal{C}}\mathbf{D}_t^\alpha w(t) = -dxy - cz, \end{aligned} \quad (1)$$

$$\begin{aligned} {}^{\mathcal{C}}\mathbf{D}_t^\alpha x(t) &= myw, {}^{\mathcal{C}}\mathbf{D}_t^\alpha y(t) = yz, {}^{\mathcal{C}}\mathbf{D}_t^\alpha z(t) \\ &= -y^2 + nw, {}^{\mathcal{C}}\mathbf{D}_t^\alpha w(t) = -mxy - nz. \end{aligned} \quad (2)$$

2. Preliminaries

Some important and helpful definitions and lemmas [36, 37] are stated below.

Definition 1. The integral of function defined by Riemann-Liouville, $h \in L([0, 1], R)$ of order $0 < n < 1$ as

$$\int_1^n h(t) = \frac{1}{\Gamma(n)} \int_1^1 (t - \alpha)^{n-1} h(\alpha) d(\alpha), \quad (3)$$

in such a way that the integral in the right side exists.

Definition 2. Fractional derivative in Caputo sense for $\bar{h} \in C[0, b]$ for order $0 < \alpha < 1$ as

$${}^{\mathcal{C}}\mathbf{D}_t^\alpha \bar{h}(t) = \left(\frac{1}{\Gamma(1 - \omega)} \int_0^t \frac{\widehat{h}(\alpha)}{(t - \alpha)^\omega}, 0 < \omega < 1, \right. \\ \left. \frac{d\widehat{h}}{dt}, \omega \right) = 1. \quad (4)$$

Lemma 3. The given results satisfied the problems related to noninteger order

$$\begin{cases} {}^{\mathcal{C}}\mathbf{D}_t^\alpha \bar{h}(t) = x(t), 0 < \omega \leq 1, \\ \bar{h}(0) = \bar{h}_0, \end{cases} \quad (5)$$

$$\bar{h}(t) = \bar{h}_0 + \frac{1}{\Gamma(\omega)} \int_0^t (t - \alpha)^{\omega-1} x(\alpha) d\alpha. \quad (6)$$

3. Existence Theory

Using fixed point theory, we investigate the existence and uniqueness of the solutions of the considered fractional-order systems. For the sake of convenience, consider the system (1) in the form

$$\begin{aligned} {}^{\mathcal{C}}\mathbf{D}_t^\alpha x(t) &= \Pi_1(t, x, y, z, w), {}^{\mathcal{C}}\mathbf{D}_t^\alpha y(t) = \Pi_2(t, x, y, z, w), {}^{\mathcal{C}}\mathbf{D}_t^\alpha z(t) \\ &= \Pi_3(t, x, y, z, w), {}^{\mathcal{C}}\mathbf{D}_t^\alpha w(t) = \Pi_4(t, x, y, z, w). \end{aligned} \quad (7)$$

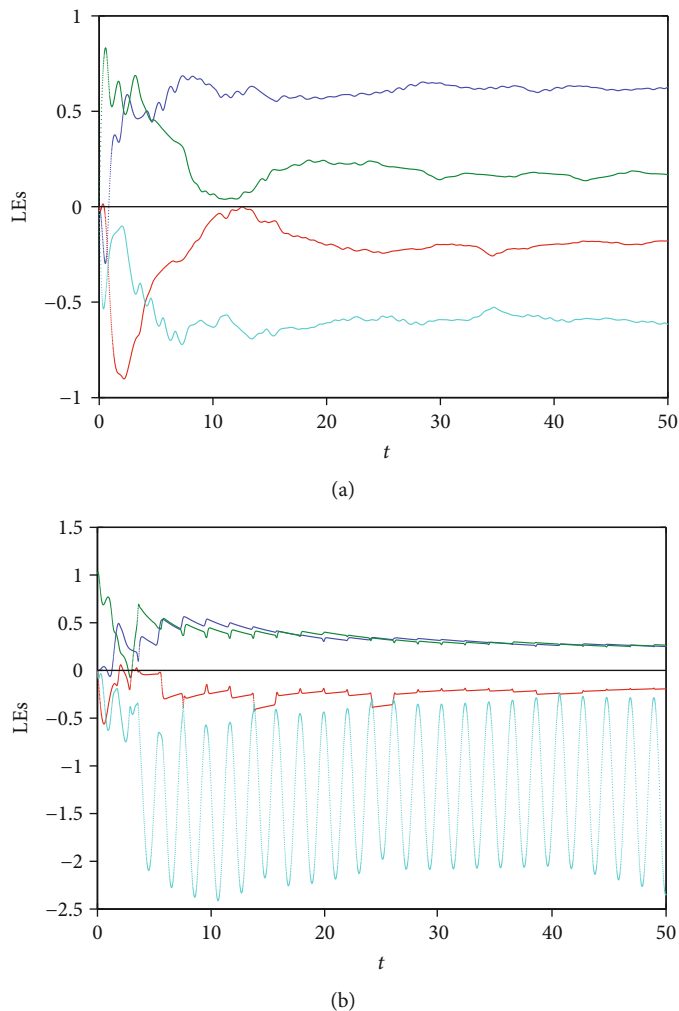


FIGURE 1: The Lyapunov spectra versus time (t) with fractional order $\varpi = 0.98$: (a) Lyapunov spectra of system 1 with $c = 2, d = 5$, (b) Lyapunov spectra of system 2 with $m = 2, n = 5$.

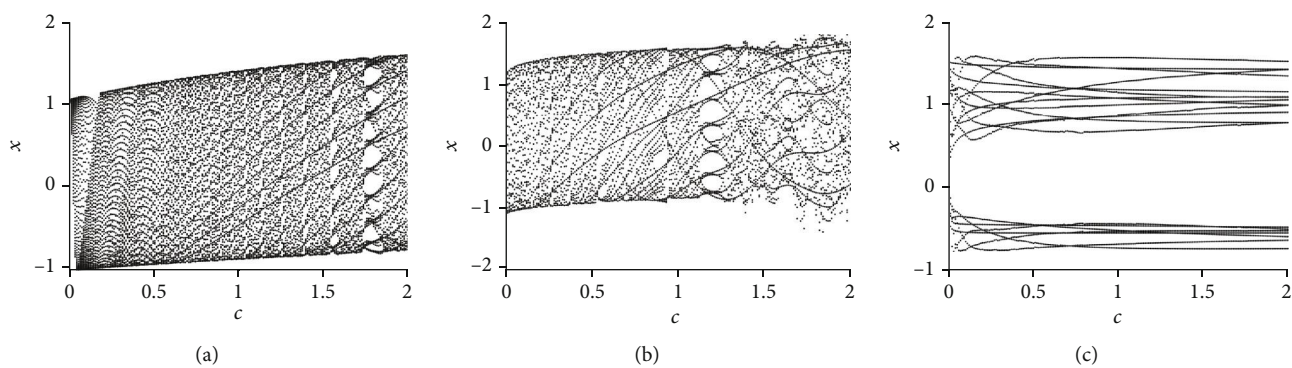


FIGURE 2: The bifurcations in system (1) versus parameter c with fractional orders (a) $\varpi = 1$, (b) $\varpi = 0.99$, and (c) $\varpi = 0.98$.

For

Consider system 1 in the form

$$\begin{aligned}
 \Pi_1(t, x, y, z, w) &= myw, \Pi_2(t, x, y, z, w) = yz, \Pi_3(t, x, y, z, w) \\
 &= -y^2 + nw, \Pi_4(t, x, y, z, w) = -mxy - nz.
 \end{aligned}
 \tag{8}$$

$$\begin{cases}
 \mathcal{D}_t^\varpi(\alpha(t)) = \psi(t, \alpha(t)), \\
 \alpha(0) = \alpha_0 \geq 0,
 \end{cases}
 \tag{9}$$

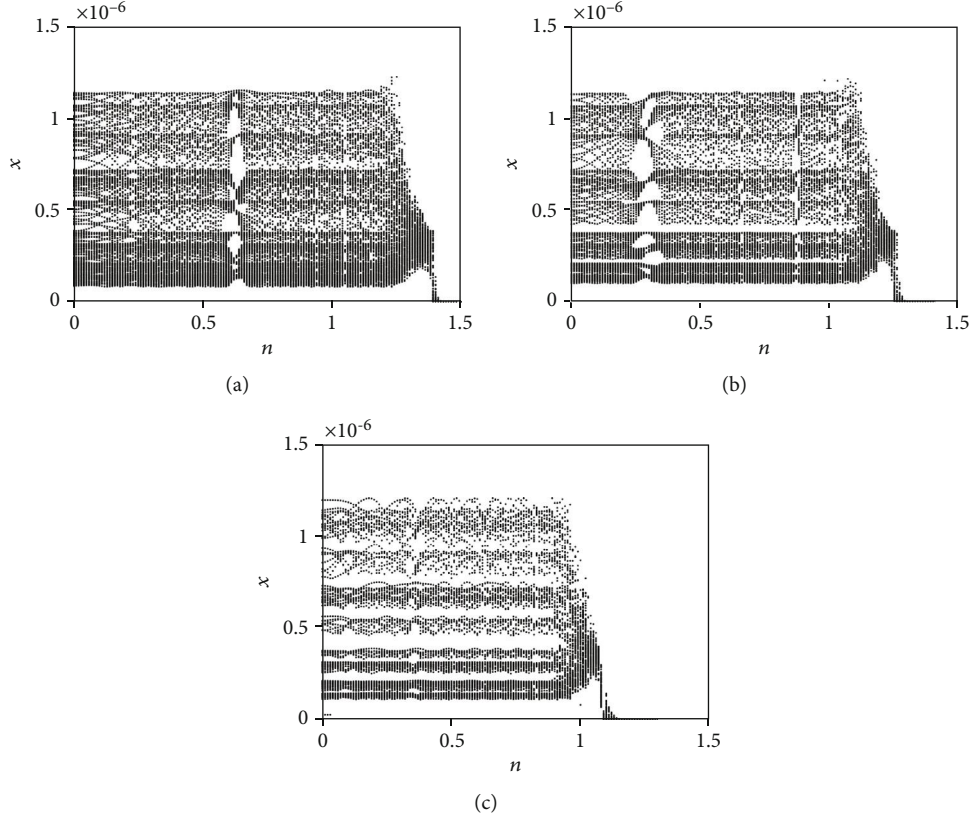


FIGURE 3: The bifurcations in system (2) versus parameter n with fractional orders (a) $\omega = 1$, (b) $\omega = 0.99$, and (c) $\omega = 0.98$.

if

$$\begin{cases} \alpha(t) = (x, y, z, w)^T, \\ \alpha_0 = (x(0), y(0), z(0), w(0))^T, \\ \psi(t, \alpha(t)) = (\Pi_i(t, x, y, z, w))^T, i = 1, 2, 3, 4, \end{cases} \quad (10)$$

where $(\star)^T$ represents the transpose. Using Lemma 3, the solution of Eq. (9) can be written as

$$\alpha(t) = \alpha_0 + \frac{1}{\Gamma(\omega)} \int_0^t (t - \phi)^{\omega-1} \psi(\phi, \alpha(\phi)) d\phi. \quad (11)$$

Let us define a Banach space $B = L^4$ with the norm $\|\alpha\| = \sup_{t \in [0, b]} |\alpha(t)|$, and define the operator $\Theta : B \rightarrow B$ as

$$\Theta[\alpha(t)] = \alpha_0 + \frac{1}{\Gamma(\omega)} \int_0^t (t - \phi)^{\omega-1} \psi(\phi, \alpha(\phi)) d\phi. \quad (12)$$

Consider that $\alpha(t, \phi(t))$ satisfies the growth and Lipschitz conditions.

There exist constants $M_\psi > 0$ and $C_\psi > 0$, the given growth condition satisfies

$$|\psi(t, \alpha(t))| \leq M_\psi + C_\psi |\alpha|, t \in [0, b]. \quad (13)$$

TABLE 2: Linear stability analysis of system (2).

E_i	Eigen values (λ_{pq} , $p = 1, 2, 3, 4, 5$ and $q = 1, 2, 3$).	Stability
[0,0,0,0]	$\lambda_{11,12} = \mp 5i, \lambda_{13,14} = 0$	Unstable
[1,0,0,0]	$\lambda_{21,22} = \pm 5i, \lambda_{22} = -5i, \lambda_{23,24} = 0$	Unstable, undamped

TABLE 1: Linear stability analysis of system (1).

E_i	Eigen values (λ_{pq} , $p = 1, 2, 3, 4, 5$ and $q = 1, 2, 3$).	Stability
[0,0,0,0]	$\lambda_{11,12} = \pm 2i, \lambda_{13,14} = 0$	Unstable
[0,1,0,0]	$\lambda_{21} = 2, \lambda_{22,23} = \mp 2i, \lambda_{24} = 0$	Unstable spiral
[0,0,1,0]	$\lambda_{31,32} = \mp 2i, \lambda_{33,34} = 0$	Unstable

There exist a constant $L_\psi > 0$ such that for each $\alpha_1, \alpha_2 \in C([J, R])$,

$$|\psi(t, \alpha_1(t)) - \psi(t, \alpha_2(t))| \leq L_\psi |\alpha_1(t) - \alpha_2(t)|. \quad (14)$$

For the uniqueness of the solution of system (1), we use of the following lemma and theorems.

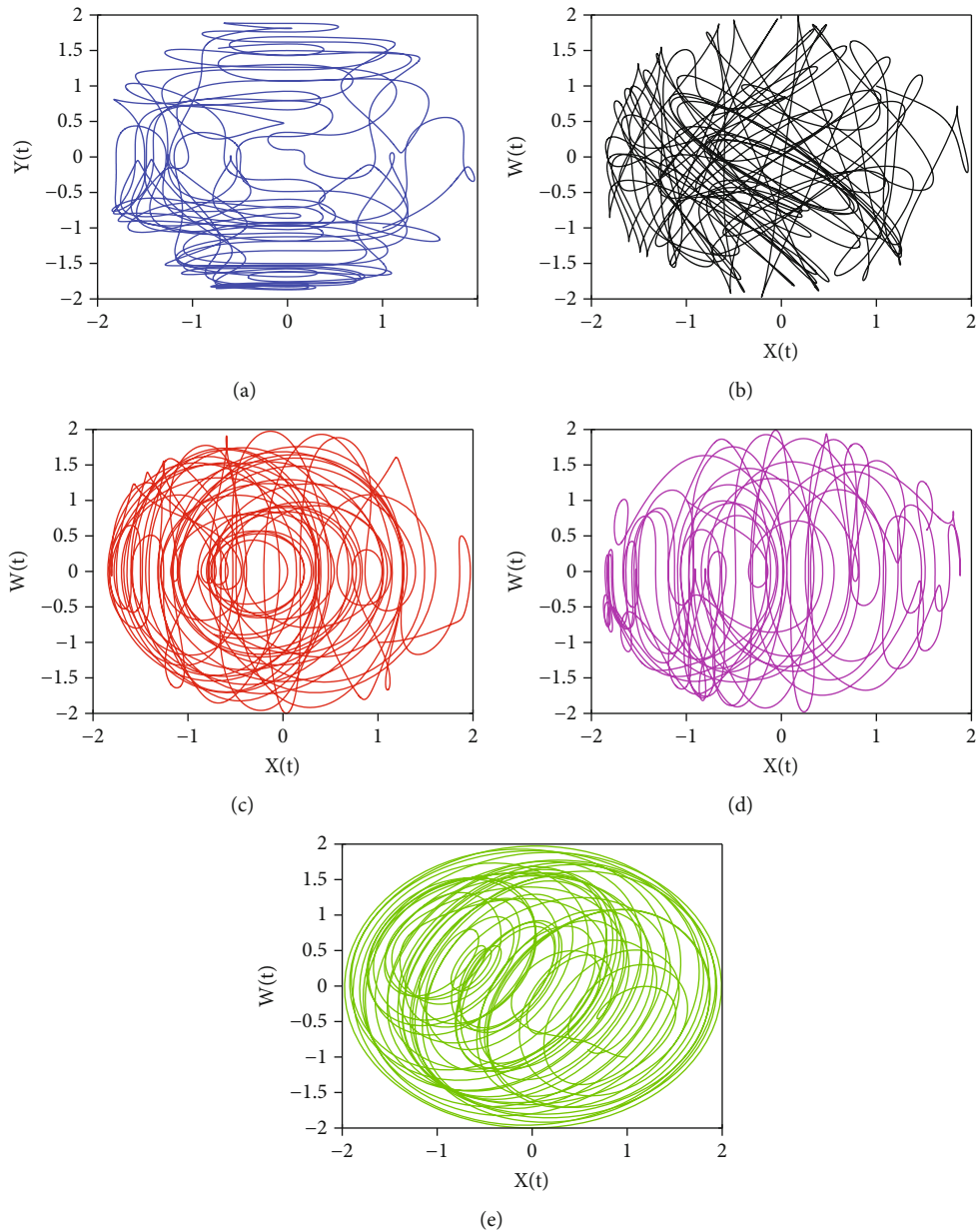


FIGURE 4: The behavior of different state variables of the system 1 with fractional order $\omega = 1$ and the parameters are considered as $c = d = 2$.

Lemma 4 (see [38]). *If B represents a Banach space and $\Theta : B \rightarrow B$ be completely-continuous such that $\mathbf{X}_\rho = \{\alpha \in B : \alpha = \eta\Theta\alpha, \eta \in [0, 1]\}$ is bounded, then, Θ has at least one fixed point.*

Theorem 5. *Suppose C_1 holds, also consider that $\psi : [0, b] \times B \rightarrow R$ be a continuous function, then, system (1) has at least one solution.*

Proof. Suppose that $\mathbf{X}_\rho = \{\alpha \in B : \|\alpha\| \leq \rho\}$. Where $\mathbf{X}_\rho \neq \emptyset$ closed and convex subset of B . For the continuity of Θ , consider $\{\alpha_n\}$ is a sequence in \mathbf{X}_ρ , such that $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$. Suppose for $t \in [0, b]$, we have

$$\begin{aligned} \|\Theta[\alpha_n] - \Theta[\alpha]\| &= \sup_{t \in [0, b]} |\Theta[\alpha_n(t)] - \Theta[\alpha(t)]| \\ &= \sup_{t \in [0, b]} \left| \int_0^t \frac{(t-\phi)^{\omega-1}}{\Gamma(\omega)} [\psi(\phi, \alpha_n(\phi)) - \psi(\phi, \alpha(\phi))] d\phi \right| \\ &\leq \frac{L_\psi}{\Gamma(\omega)} \sup_{t \in [0, b]} \int_0^t (t-\phi)^{\omega-1} |\alpha_n(\phi) - \alpha(\phi)| d\phi. \end{aligned} \tag{15}$$

As ψ continuous, so by Lebesgue dominant convergence theorem, we have

$$\|\Theta[\alpha_n] - \Theta[\alpha]\| \rightarrow 0, \text{ as } n \rightarrow \infty. \tag{16}$$

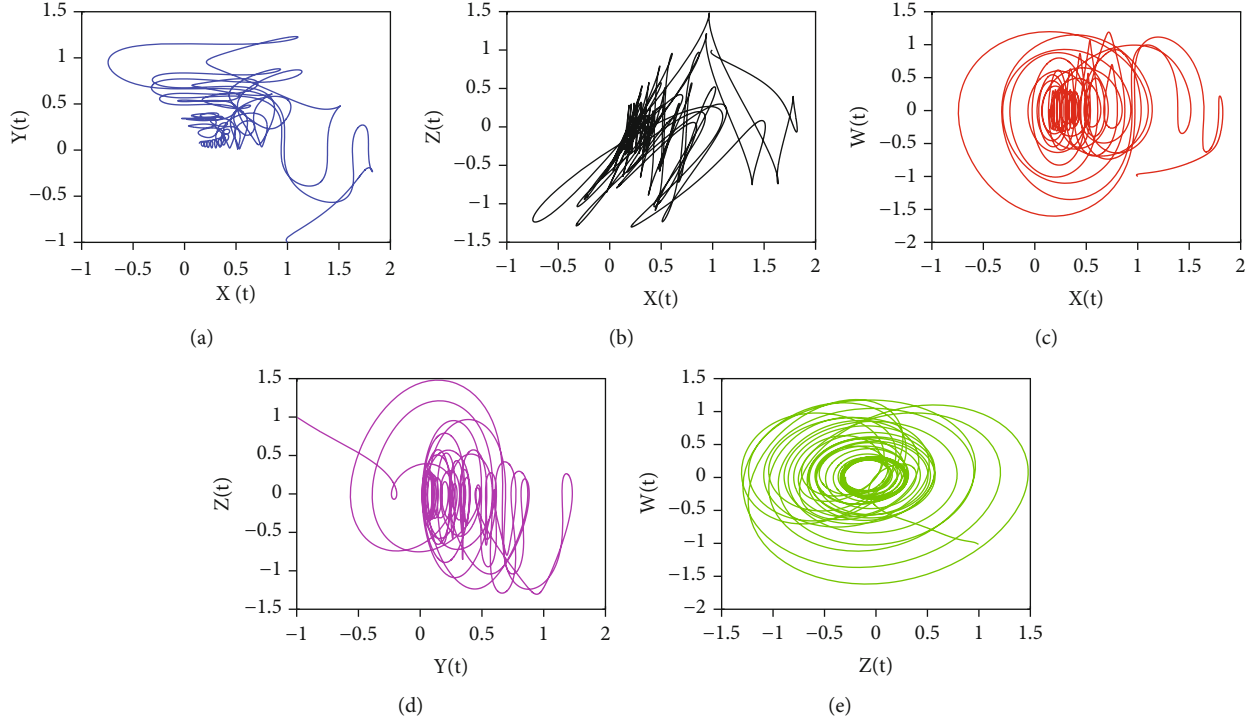


FIGURE 5: The behavior of different state variables of system 1 with fractional order $\omega = 0.99$ and the parameters are considered as $c = d = 2$.

Hence, Θ is continuous. Moreover, to show that Θ is bounded or $\Theta(X_\rho) \subset X_\rho$, taking

$$\begin{aligned} \|\Theta[\alpha]\| &= \sup_{t \in [0, b]} |\Theta[\alpha(t)]| = \sup_{t \in [0, b]} \left| \int_0^t \frac{(t-\phi)^{\omega-1}}{\Gamma(\omega)} [\psi(\phi, \alpha(\phi))] d\phi \right| \\ &\leq \frac{b^\omega}{\Gamma(\omega+1)} [M_\psi + C_\psi \|\alpha\|] \leq \rho. \end{aligned} \quad (17)$$

Since $\|\Theta[\alpha]\| \leq \rho$, it shows that Θ is bounded. Now to show that Θ is relatively compact, we need to prove that Θ is equi-continuous operator. For this, let us take $t_1, t_2 \in [0, b]$, so we have

$$\begin{aligned} |\Theta[\alpha(t_2)] - \Theta[\alpha(t_1)]| &\leq \int_0^{t_1} \frac{1}{\Gamma(\omega)} [(t_1 - \phi)^{\omega-1} - (t_2 - \phi)^{\omega-1}] |\psi(\phi, \alpha(\phi))| d\phi \\ &\quad + \int_{t_1}^{t_2} (t_2 - \phi)^{\omega-1} |\psi(\phi, \alpha(\phi))| d\phi \\ &\leq \frac{(M_\psi + C_\psi \rho)}{\Gamma(\omega)} \left[\int_0^{t_1} [(t_1 - \phi)^{\omega-1} - (t_2 - \phi)^{\omega-1}] d\phi \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2 - \phi)^{\omega-1} d\phi \right] \\ &\leq \frac{(M_\psi + C_\psi \rho)}{\Gamma(\omega+1)} (t_1^\omega - t_2^\omega + (t_2 - t_1)^\omega). \end{aligned} \quad (18)$$

Hence, we see that $\|\Theta[\alpha(t_2)] - \Theta[\alpha(t_1)]\| \rightarrow 0$ as $t_2 \rightarrow t_1$, which show Θ is equi-continuous. Using Arzelà theorem, the operator Θ has at least one fixed point. Hence, the considered system has at least one solution. \square

Next, we need to show the uniqueness of the solution to system (1). For this, we use of the following theorem.

Theorem 6. Let C_2 holds, then the solution of the considered system (1) is unique if $L_\psi \sigma < 1$, where

$$\sigma = \frac{a^\omega}{\Gamma(\omega+1)}. \quad (19)$$

Proof. For the proof of the above theorem let $\alpha_1, \alpha_2 \in B$, then from the definition of Θ , we have

$$\begin{aligned} |\Theta[\alpha_1(t)] - \Theta[\alpha_2(t)]| &\leq \frac{1}{\Gamma(\omega)} \int_0^t (t-\phi)^{\omega-1} |\psi(\phi, \alpha_1(\phi)) - \psi(\phi, \alpha_2(\phi))| d\phi \\ &\leq \frac{L_\psi}{\Gamma(\omega)} \int_0^t (t-\phi)^{\omega-1} |\alpha_1(\phi) - \alpha_2(\phi)| d\phi \\ &\leq \frac{L_\psi b^\omega}{\Gamma(\omega+1)} \|\alpha_1 - \alpha_2\|. \end{aligned} \quad (20)$$

On the other hand

$$|\Theta[\alpha_1(t)] - \Theta[\alpha_2(t)]| \leq L_\psi \sigma \|\alpha_1 - \alpha_2\|. \quad (21)$$

Hence, by the Banach contraction principle, the considered system (1) has a unique solution in B . \square

4. Lyapunov Spectra

This section presents the Lyapunov spectra of the models under consideration. It is observed that there exist two

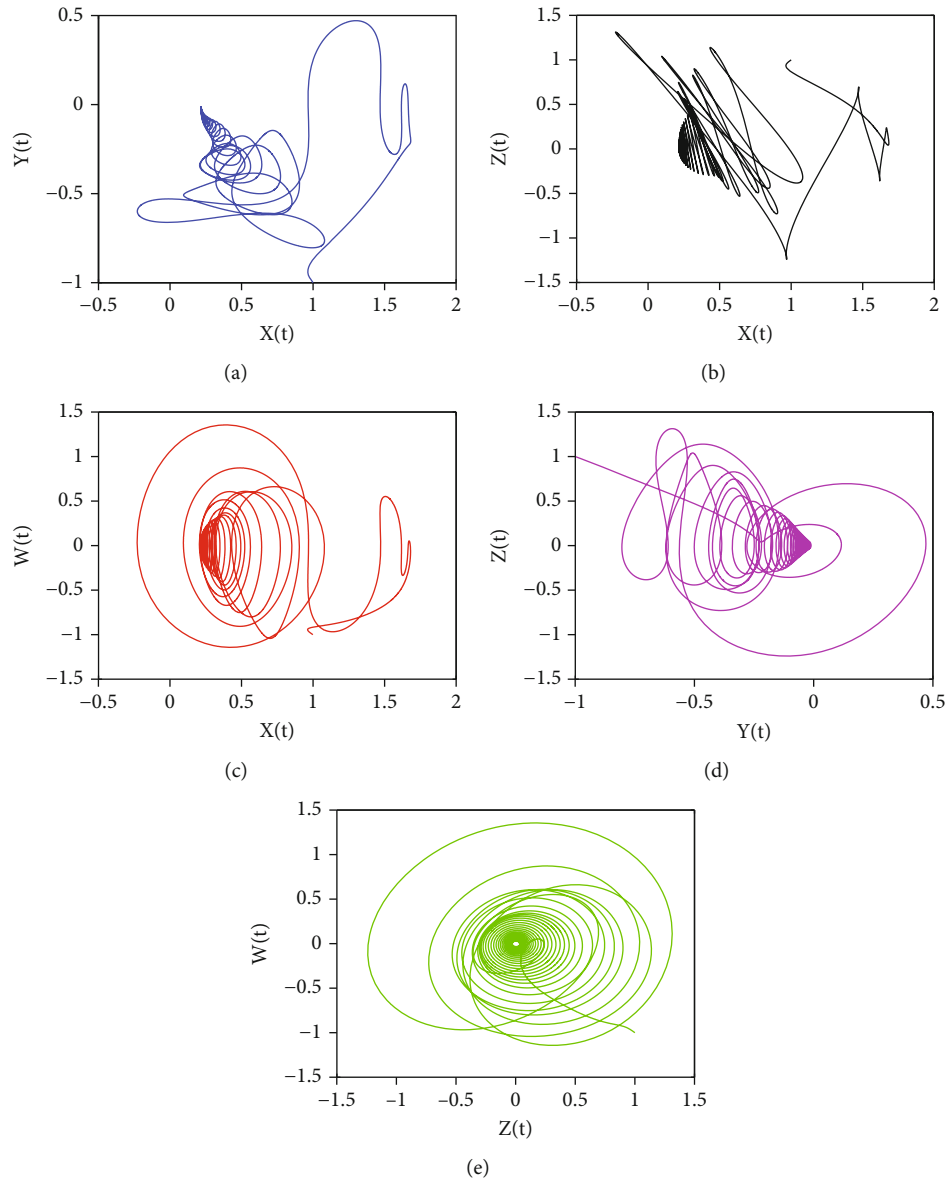


FIGURE 6: The behavior of different state variables of system 1 with fractional order $\omega = 0.98$ and the parameters are considered as $c = d = 2$.

positive Lyapunov exponents for models 1 and 2 with fractional order $\omega = 0.98$ as shown in Figure 1. The existence of two positive different exponents gives the hyperchaotic nature of the systems. For more details on the technique applied here for the analysis of Lyapunov exponents, we refer to [39]. The Lyapunov exponents obtained for model (1) are $\mathcal{L}_1 = 0.6257, \mathcal{L}_2 = 0.1684, \mathcal{L}_3 = -0.1790, \mathcal{L}_4 = -0.6152$, while the exponents obtained for model (2) are $\mathcal{L}_1 = 0.2503, \mathcal{L}_2 = 0.2649, \mathcal{L}_3 = -0.1926, \mathcal{L}_4 = -2.3488$.

5. Bifurcation

The bifurcation demonstrates a topological or qualitative change during the evolution of the dynamical system by

fluctuating the bifurcation parameter gradually. A bifurcation diagram can illustrate the presence of a limit cycle, periodic orbit, or chaotic orbit. It delivers a graphical intimation of the system's solutions through a parametric range. In order to study the bifurcations in system (1) and system (2), we consider the time $t = 100$, step-size $h = 0.01$, and different fractional orders. The bifurcations in the system (1) are projected in Figure 2, while bifurcations in the system (2) are projected in Figure 3.

6. Equilibria and Linear Stability Analysis

Consider that $\dot{q} = g(q)$ $q \in R^n$ possesses an equilibrium point r , i.e., $g(r) = 0$. Then, $q = r$ is a solution for all t . Sometimes,

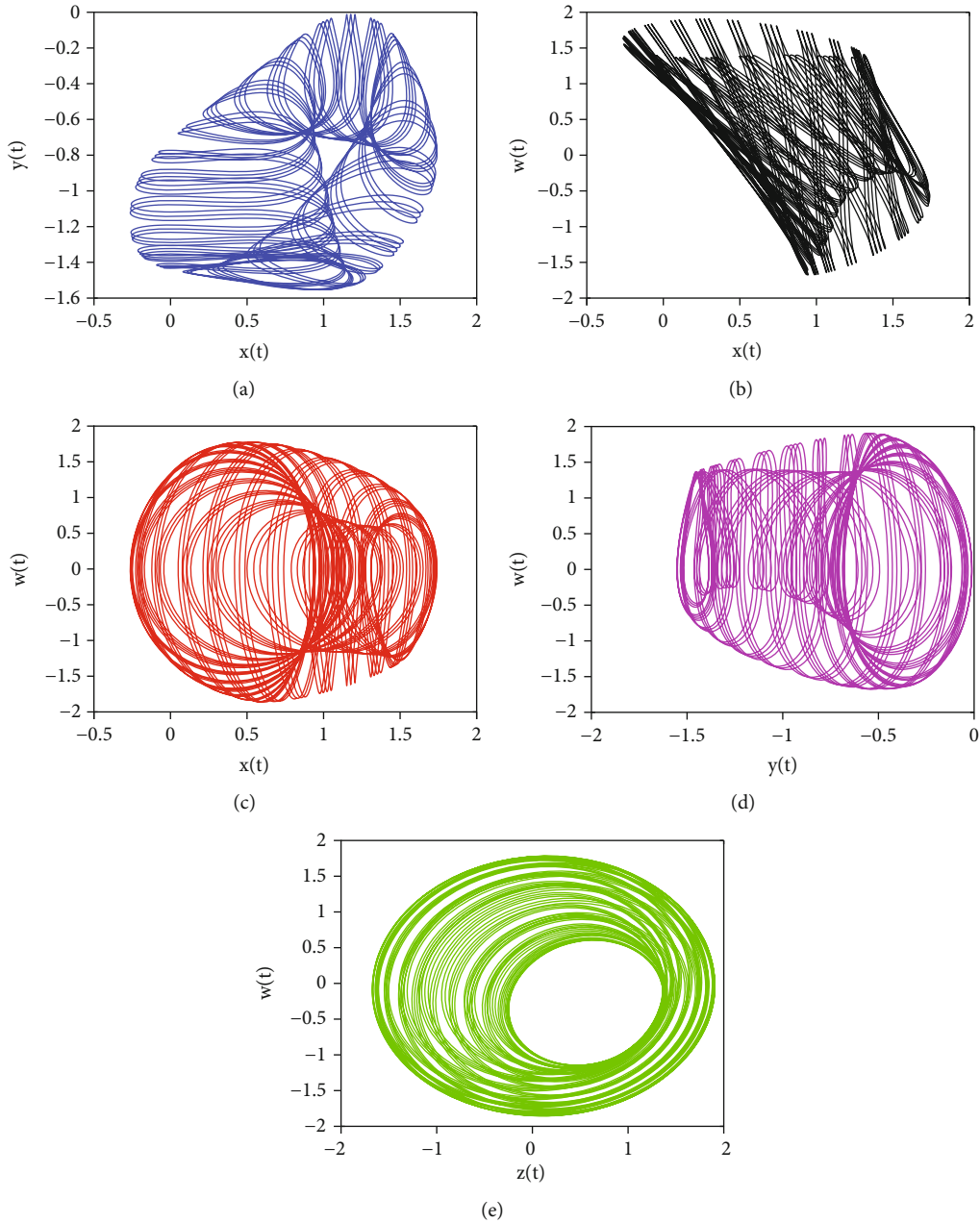


FIGURE 7: The behavior of different state variables of system 1 with fractional order $\omega = 1$ and the parameters are considered as $c = 2, d = 5$.

it is important to have the information whether the solution of a system is stable, i.e., whether it persists essentially unchanged on the infinite interval $[0, \infty)$ under small changes in the initial data. Therefore, one uses the stability based on the equilibrium points to understand the evolution of the system with time, whether the system is stable or unstable. To study the stability of the considered system, we first find the equilibrium points, then we calculate the eigenvalues from the Jacobian matrices of both systems under study on the basis of which we study the stability of the systems. Since we know that if a hidden oscillation attracts all nearby oscillations, then, it is called a hidden attractor. For our considered systems, we see that the equi-

librium point E_0 attracts the trajectories towards it, which shows that this is a hidden fixed point attractor.

6.1. Equilibria and Stability Analysis of Model (1). The equilibrium points for system (1) can be obtained by taking L.H.S of system (1) as 0.

$$\begin{cases} 0 = dxy, \\ 0 = xz, \\ 0 = -xy + cw, \\ 0 = -dxy - cz. \end{cases} \quad (22)$$

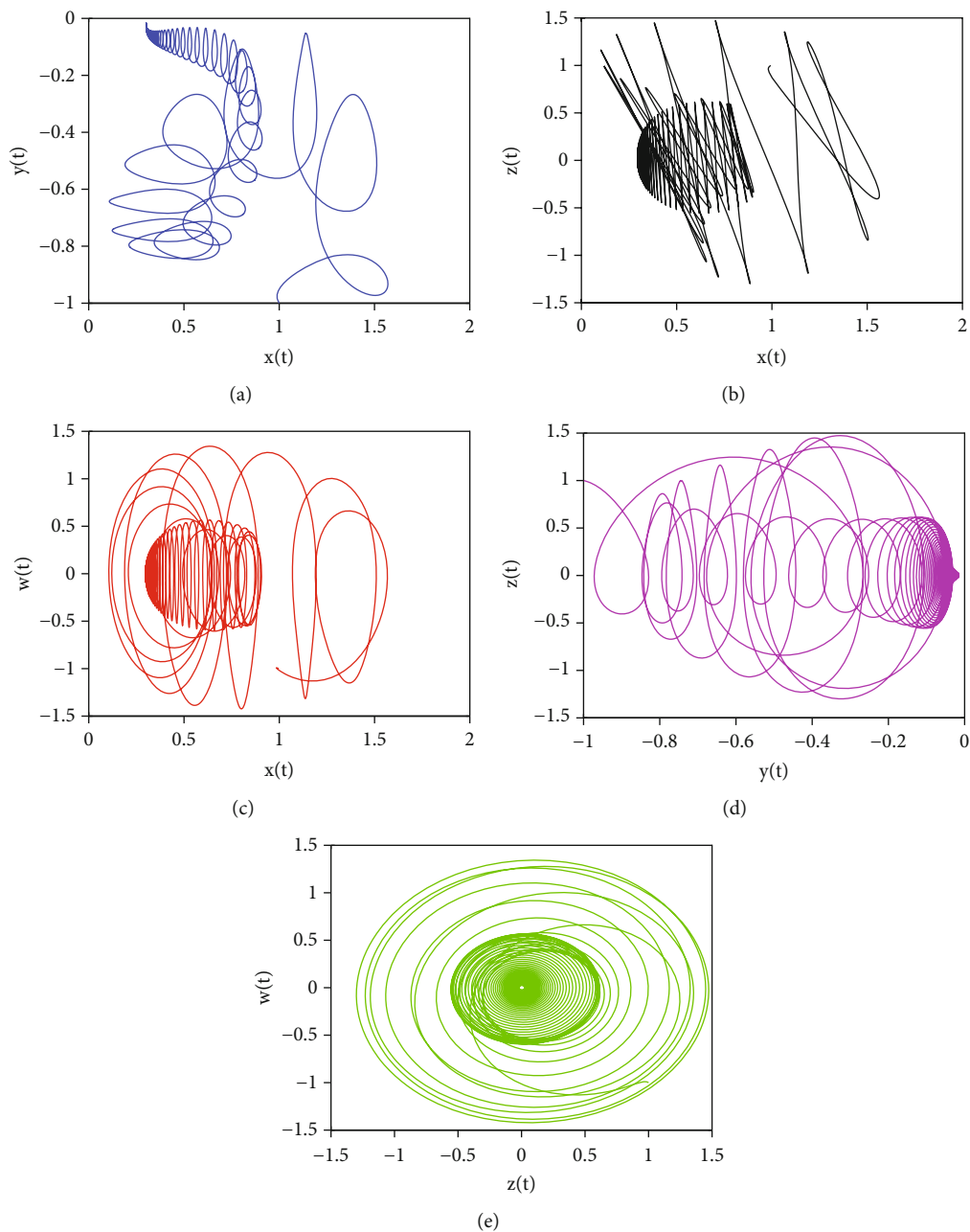


FIGURE 8: The behavior of different state variables of system 1 with fractional order $\varpi = 0.99$ and the parameters are considered as $c = 2, d = 5$

So, the equilibrium points obtained are

$$\begin{pmatrix} E_0 = (0,0,0,0), \\ E_1 = (0, z,0,0), \\ E_2 = (0, 0, z, 0). \end{pmatrix} \quad (23)$$

$$\begin{bmatrix} -yd & xd & 0 & 0 \\ z & 0 & x & 0 \\ -y & -x & 0 & c \\ -yd & -xd & -c & 0 \end{bmatrix}. \quad (24)$$

The Jacobian of system (1) can be written as

The linear stability analysis of system (1) based on the equilibrium points is presented in Table 1.

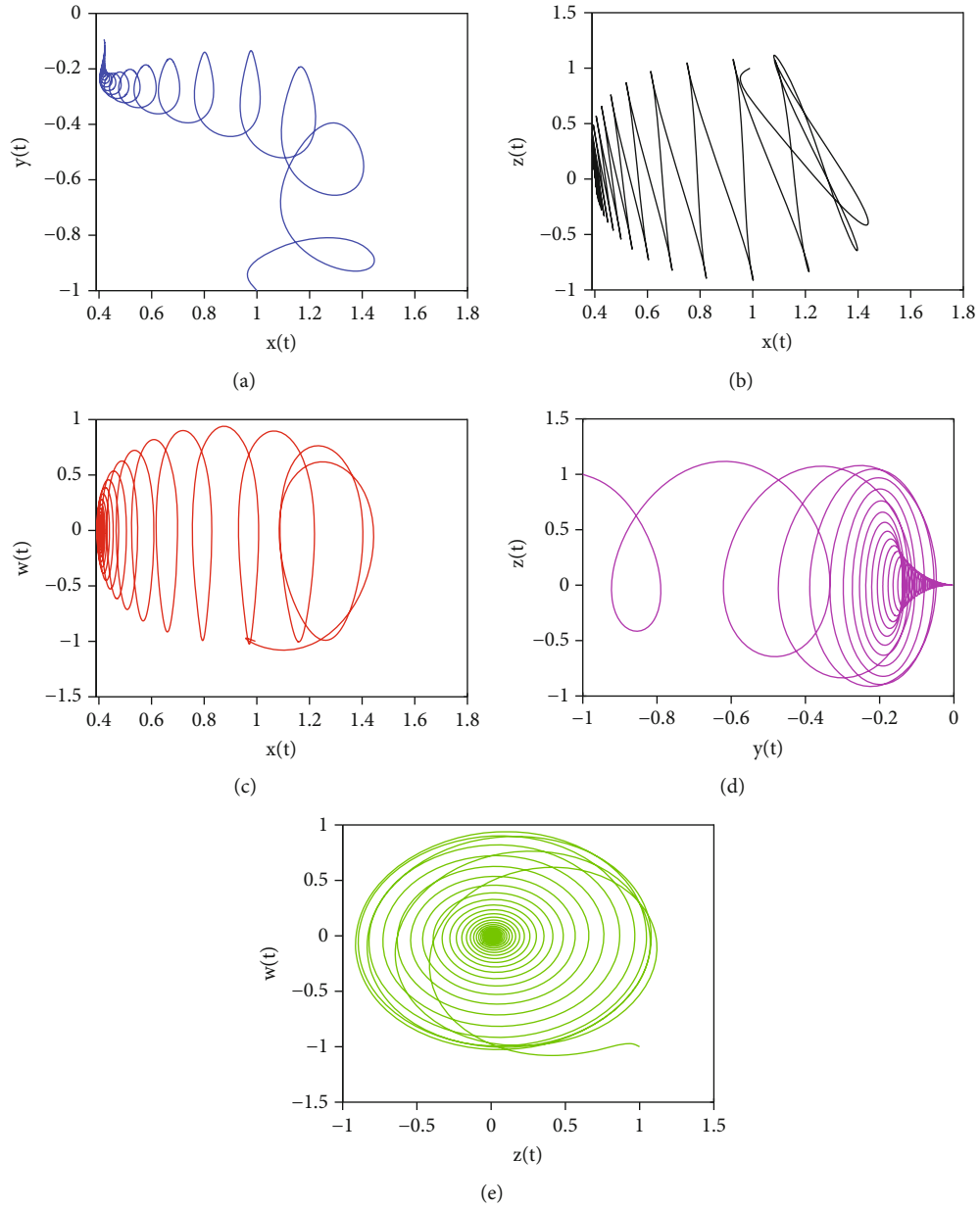


FIGURE 9: The behavior of different state variables of system 1 with fractional order $\bar{\omega} = 0.98$ and the parameters are considered as $c = 2, d = 5$.

6.2. *Equilibria and Stability Analysis of Model (2).* The equilibrium points for system (2) can be obtained by taking L.H.S of system (2) as 0.

$$\begin{pmatrix} E_0 = (0,0,0,0), \\ E_1 = (z,0,0,0). \end{pmatrix} \quad (26)$$

$$\begin{cases} 0 = myw, \\ 0 = yz, \\ 0 = -y^2 + nw, \\ 0 = -mxy - nz. \end{cases} \quad (25)$$

So, the equilibrium points obtained are

The Jacobian of system (2) can be written as

$$\begin{bmatrix} 0 & wm & 0 & ym \\ 0 & z & y & 0 \\ 0 & -2y & 0 & n \\ -ym & -xm & -n & 0 \end{bmatrix}. \quad (27)$$

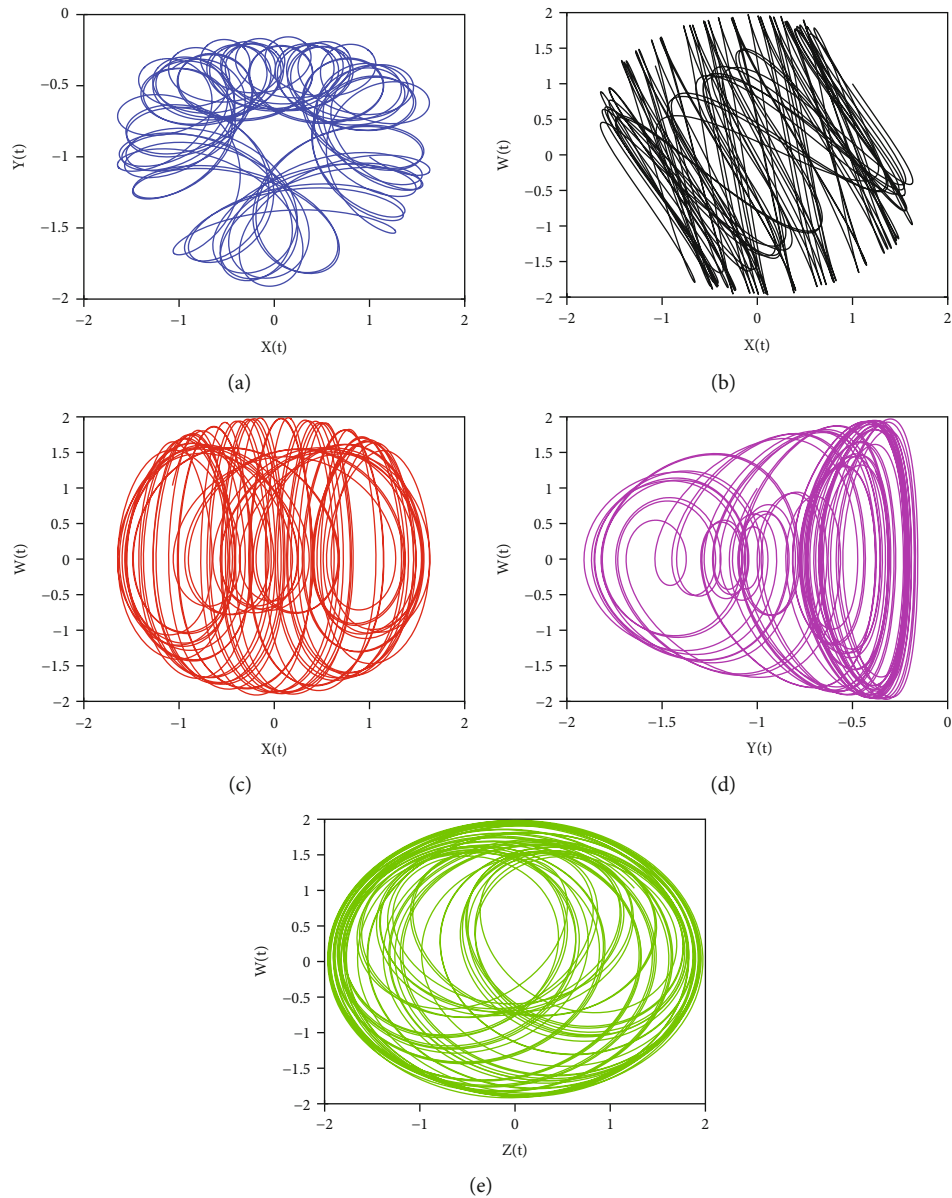


FIGURE 10: The behavior of different state variables of system 2 with fractional order $\omega = 1$ and the parameters are considered as $m = n = 5$.

The linear stability analysis of system (2) based on the equilibrium points is presented in Table 2. where

7. Numerical Scheme

In this section, we present the numerical solutions to the considered models (1) and (2). To solve model the considered systems numerically, we first apply fractional integral to model (1), gives

$$\begin{aligned}
 x(t) &= x(0) + I^\alpha \mathcal{F}_1(t, x), y(t) = y(0) + I^\alpha \mathcal{F}_2(t, y), z(t) \\
 &= z(0) + I^\alpha \mathcal{F}_3(t, z), w(t) = w(0) + I^\alpha \mathcal{F}_4(t, w),
 \end{aligned}
 \tag{28}$$

$$\begin{aligned}
 \mathcal{F}_1(t, x) &= myw, \mathcal{F}_2(t, y) = yz, \mathcal{F}_3(t, z) \\
 &= -y^2 + nw, \mathcal{F}_4(t, w) = -mxy - nz.
 \end{aligned}
 \tag{29}$$

Replacing t with t_n in the above equations, we obtain

$$\begin{aligned}
 x(t_n) &= x(0) + I^\alpha \mathcal{F}_1(t_n, x), y(t_n) \\
 &= y(0) + I^\alpha \mathcal{F}_2(t_n, y), z(t_n) \\
 &= z(0) + I^\alpha \mathcal{F}_3(t_n, z), w(t_n) \\
 &= w(0) + I^\alpha \mathcal{F}_4(t_n, w).
 \end{aligned}
 \tag{30}$$

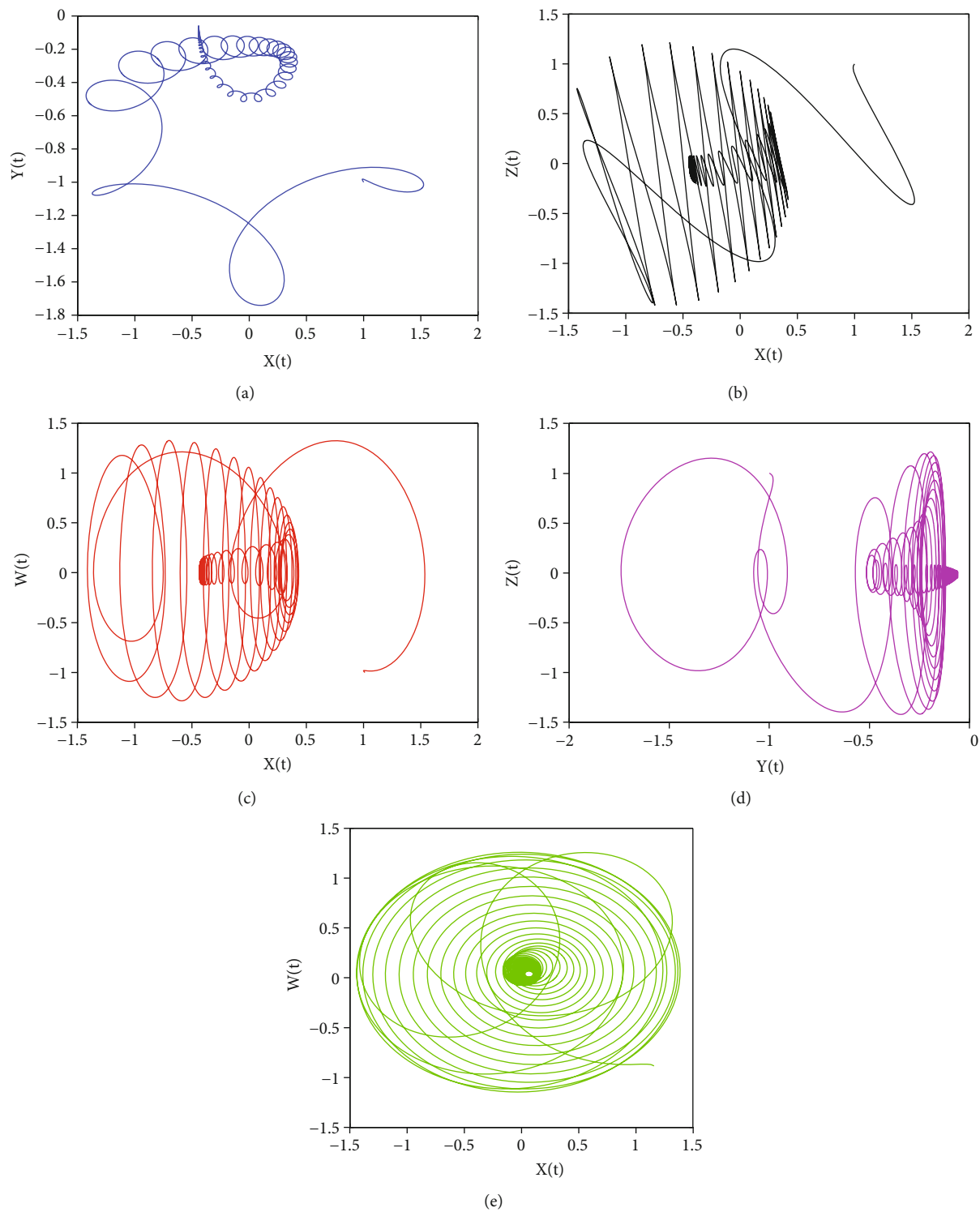


FIGURE 11: The behavior of different state variables of system 2 with fractional order $\omega = 0.99$ and the parameters are considered as $m = n = 5$

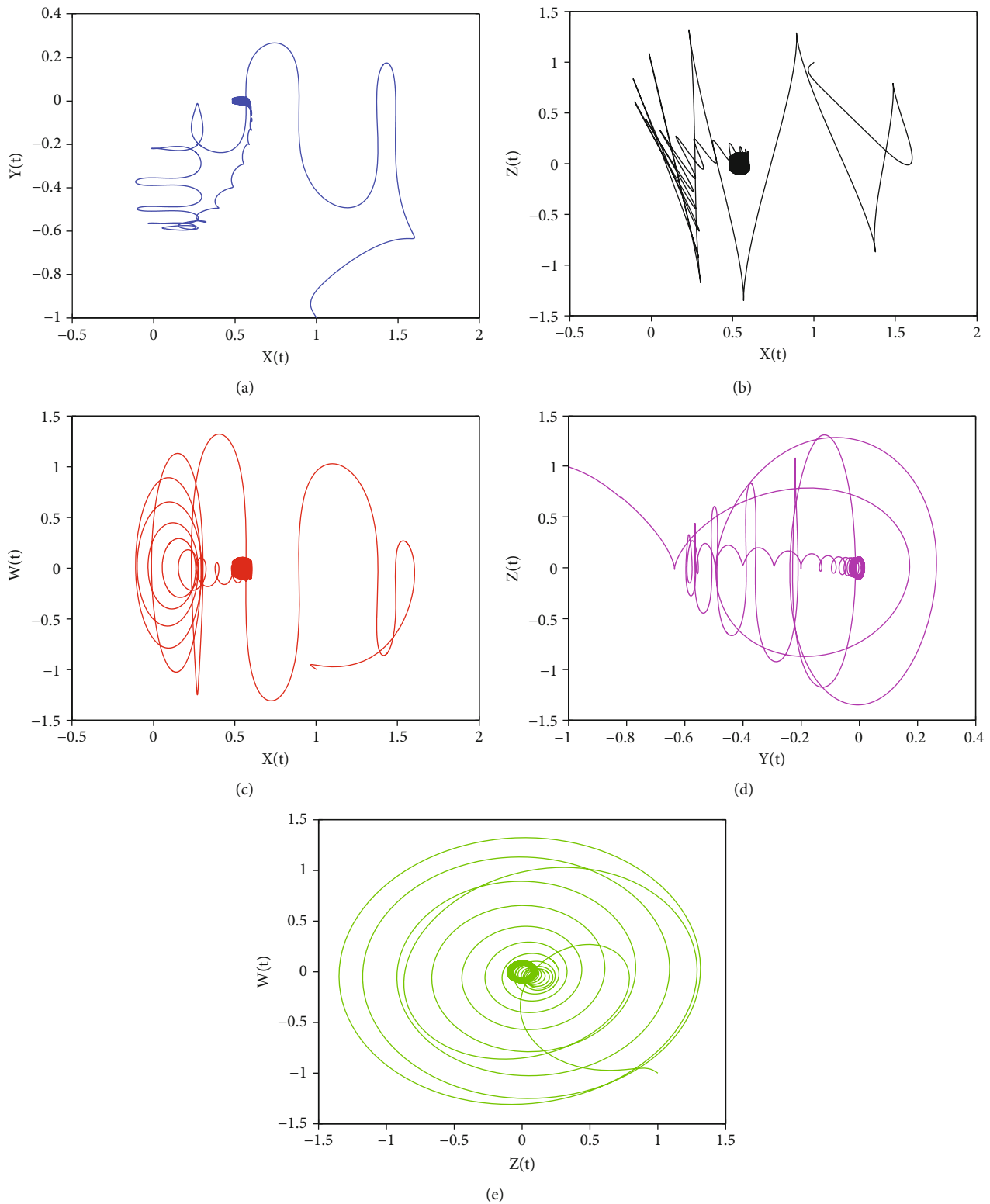


FIGURE 12: The behavior of different state variables of system 2 with fractional order $\omega = 0.98$ and the parameters are considered as $m = n = 5$

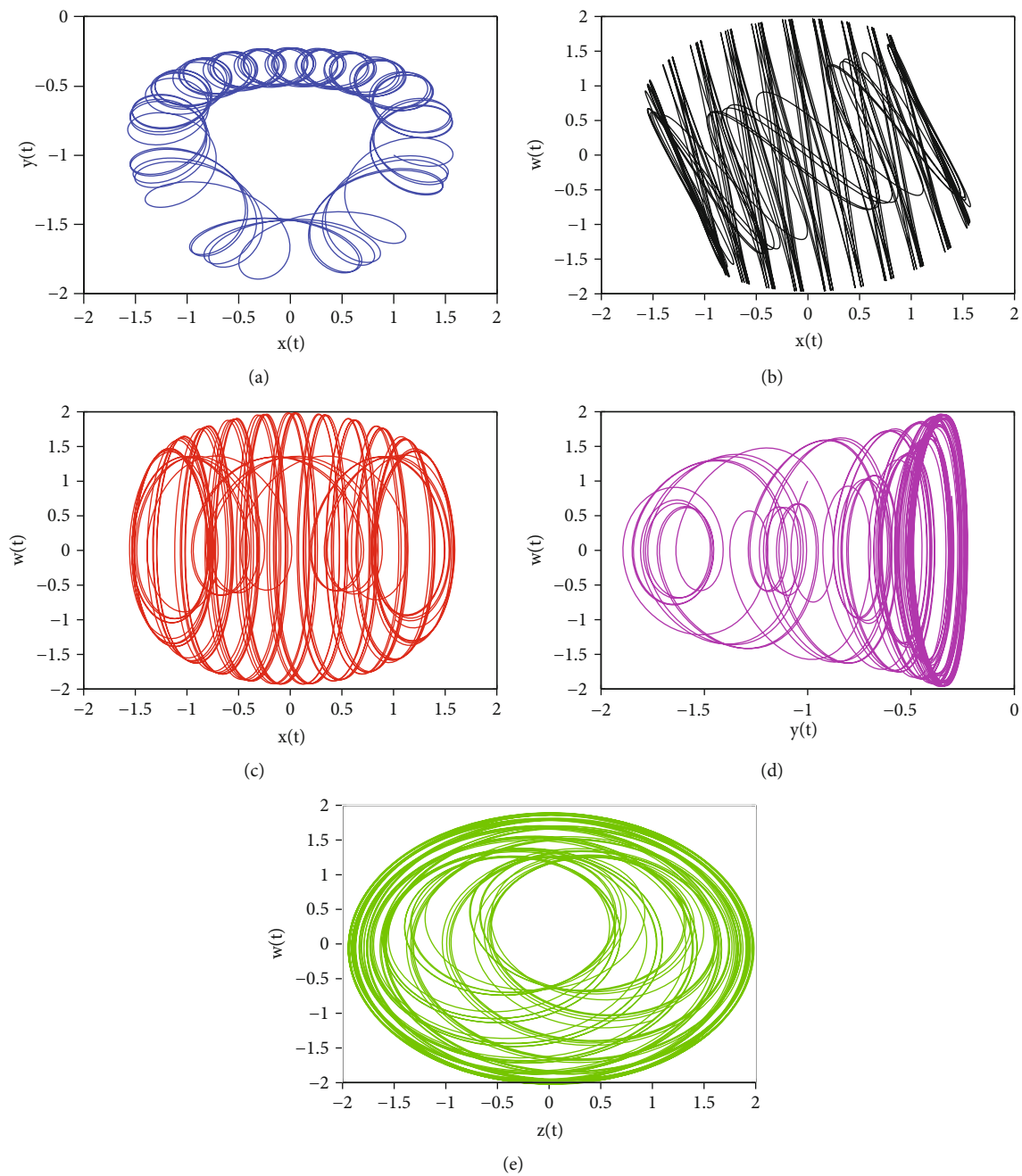


FIGURE 13: The behavior of different state variables of system 2 with fractional order $\omega = 1$.

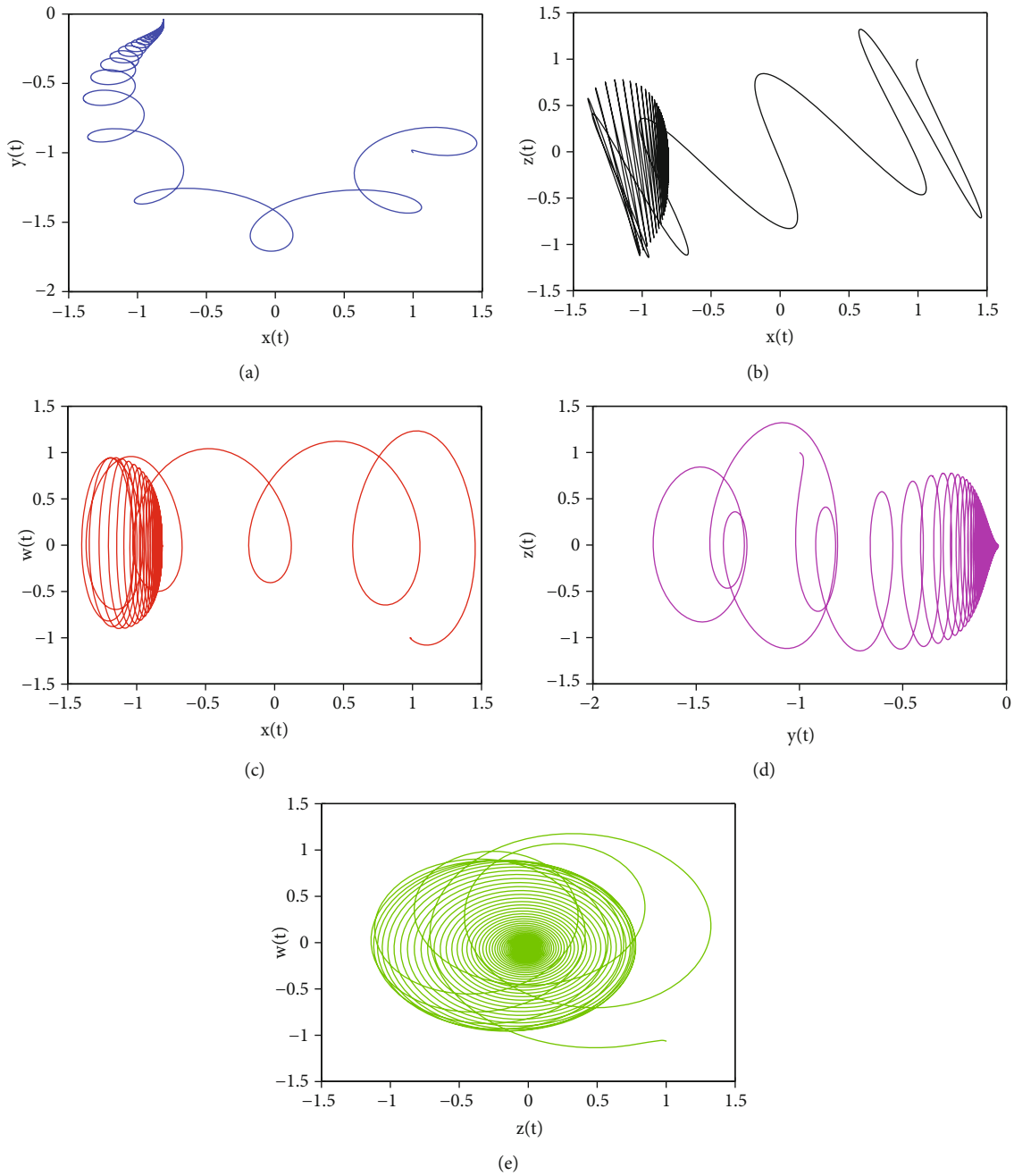


FIGURE 14: The behavior of different state variables of system 2 with fractional order $\varpi = 0.99$ and the parameters are considered as $m = 2, n = 5$.

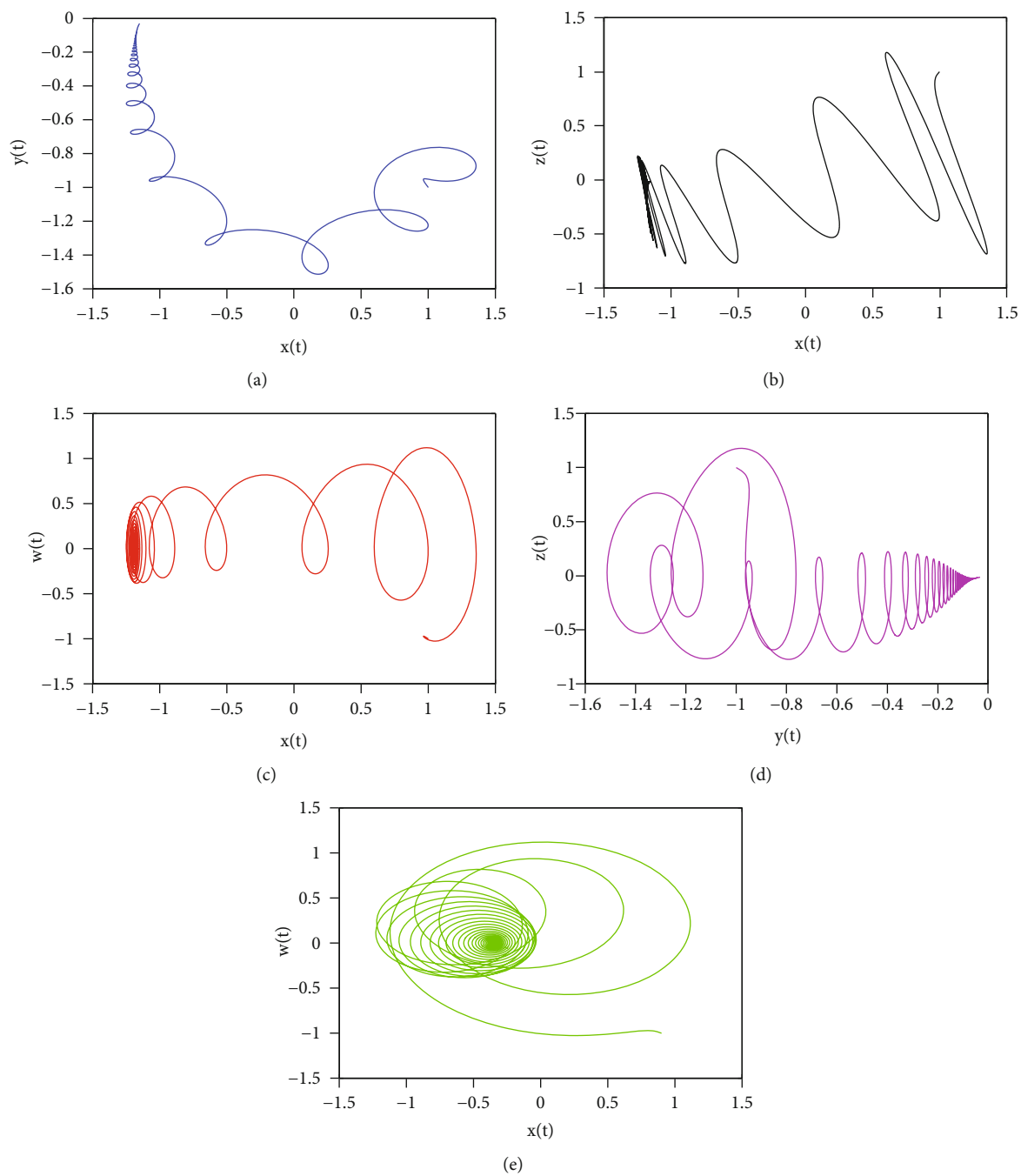


FIGURE 15: The behavior of different state variables of system 2 with fractional order $\varpi = 0.98$ and the parameters are considered as $m = 2$, $n = 5$.

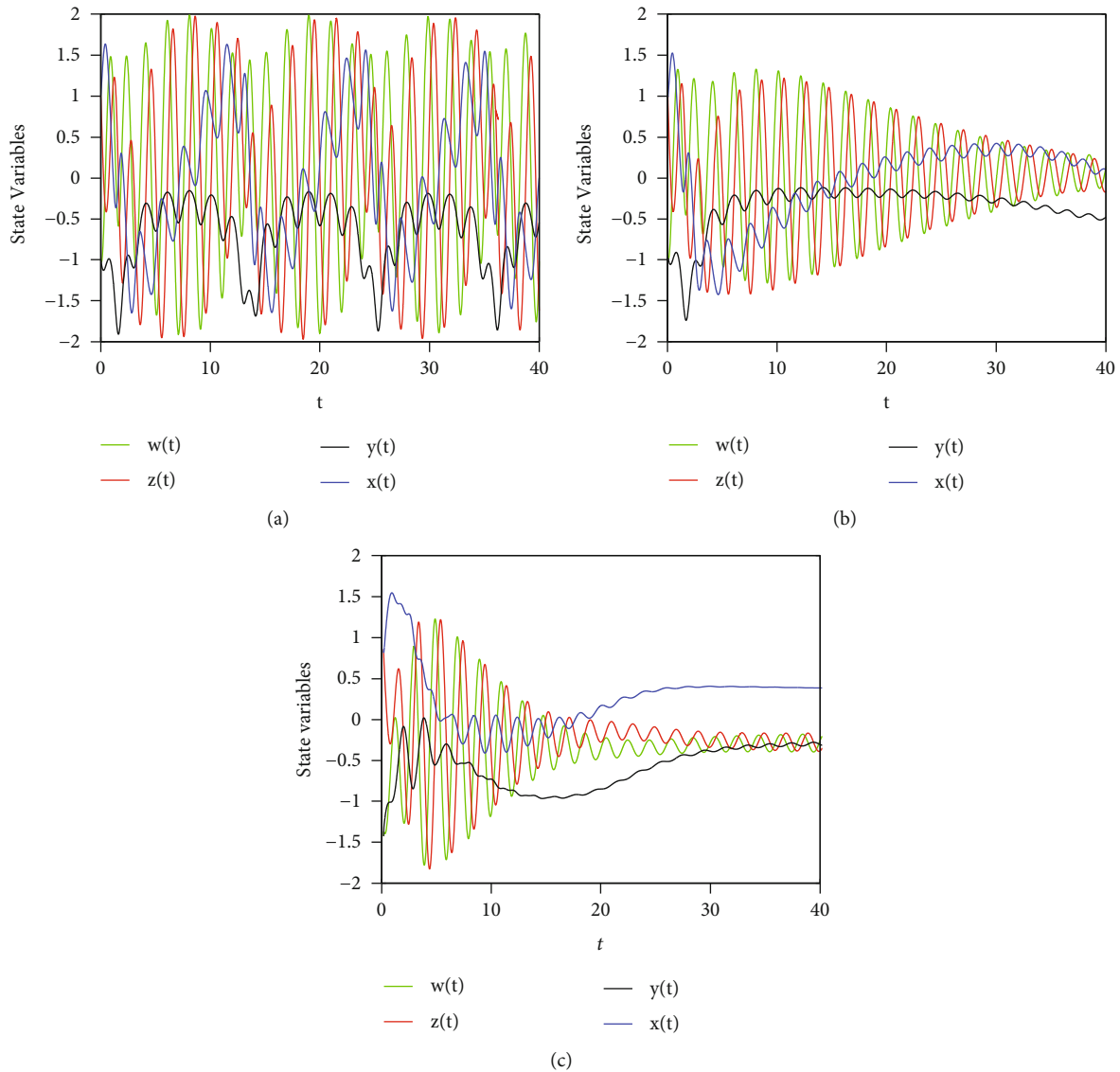


FIGURE 16: The time-series behavior of different state variables of system 1 with fractional order $\omega = 0.98$ and the parameters $c = d = 2$.

Introducing $t_n = nh$, here, h is the step size. Then, the above integral equations are rewritten as

$$\begin{aligned}
 I^\alpha \mathcal{F}_1(t_n, x) &= h^\alpha \left[k_n^\alpha \mathcal{F}_1(0) + \sum_{j=1}^n k_{n-j}^{(\alpha)} \mathcal{F}_1(t_j, x_j) \right], \\
 I^\alpha \mathcal{F}_2(t_n, y) &= h^\alpha \left[k_n^\alpha \mathcal{F}_2(0) + \sum_{j=1}^n k_{n-j}^{(\alpha)} \mathcal{F}_2(t_j, y_j) \right], \quad (31) \\
 I^\alpha \mathcal{F}_3(t_n, z) &= h^\alpha \left[k_n^\alpha \mathcal{F}_3(0) + \sum_{j=1}^n k_{n-j}^{(\alpha)} \mathcal{F}_3(t_j, z_j) \right], \\
 I^\alpha \mathcal{F}_4(t_n, w) &= h^\alpha \left[k_n^\alpha \mathcal{F}_4(0) + \sum_{j=1}^n k_{n-j}^{(\alpha)} \mathcal{F}_4(t_j, w_j) \right]. \quad (32)
 \end{aligned}$$

where

$$k_n^\alpha = \frac{(n-1)^\alpha - n^\alpha(n-\alpha-1)}{\Gamma(2+\alpha)}, \quad (33)$$

if $n = 1, 2, 3, \dots$, then parameter k can be expressed as given

$$k_0^\alpha = \frac{1}{\Gamma(2+\alpha)} \text{ and } k_n^\alpha = \frac{(n-1)^{\alpha+1} - 2n^{\alpha+1} + (n+1)^{\alpha+1}}{\Gamma(2+\alpha)}. \quad (34)$$

Now transferring the numerical approximations in the above equation, we get the following scheme, which is the implicit form of the fractional-order chaotic system

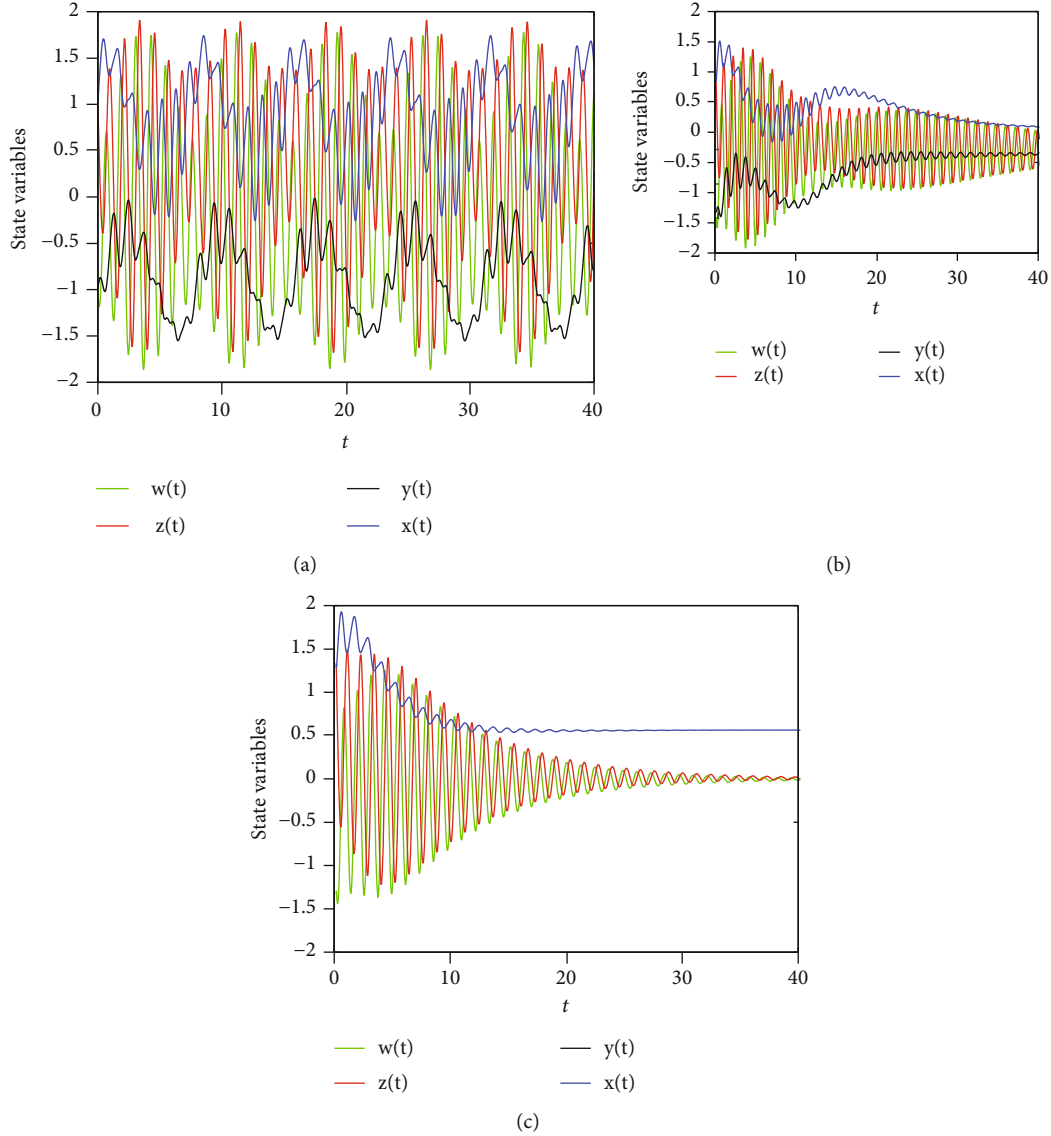


FIGURE 17: The time-series behavior of different state variables of system 1 with fractional order $\bar{\omega} = 0.98$ and the parameters $c = 2, d = 5$.

$$\begin{aligned}
 x(t_n) &= x(0) + h^\alpha \left[k_n^\alpha \mathcal{F}_1(0) + \sum_{j=1}^n k_{n-j}^{(\alpha)} \mathcal{F}_1(t_j, x_j) \right], \\
 y(t_n) &= x(0) + h^\alpha \left[k_n^\alpha \mathcal{F}_2(0) + \sum_{j=1}^n k_{n-j}^{(\alpha)} \mathcal{F}_2(t_j, y_j) \right], \quad (35) \\
 z(t_n) &= x(0) + h^\alpha \left[k_n^\alpha \mathcal{F}_3(0) + \sum_{j=1}^n k_{n-j}^{(\alpha)} \mathcal{F}_3(t_j, z_j) \right], \\
 w(t_n) &= x(0) + h^\alpha \left[k_n^\alpha \mathcal{F}_4(0) + \sum_{j=1}^n k_{n-j}^{(\alpha)} \mathcal{F}_4(t_j, w_j) \right], \quad (36)
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{F}_1(t_j, x_j) &= my_j w_j, \mathcal{F}_2(t_j, y_j) = y_j z_j, \mathcal{F}_3(t_j, z_j) \\
 &= -y_j^2 + nw_j, \mathcal{F}_4(t_j, w_j) = -mx_j y_j - nz_j. \quad (37)
 \end{aligned}$$

Following the technique presented above, we obtain the numerical solution to model (2) as

$$x(t_n) = x(0) + h^\alpha \left[k_n^\alpha \mathcal{G}_1(0) + \sum_{j=1}^n k_{n-j}^{(\alpha)} \mathcal{G}_1(t_j, x_j) \right], \quad (38)$$

$$y(t_n) = x(0) + h^\alpha \left[k_n^\alpha \mathcal{G}_2(0) + \sum_{j=1}^n k_{n-j}^{(\alpha)} \mathcal{G}_2(t_j, y_j) \right], \quad (39)$$

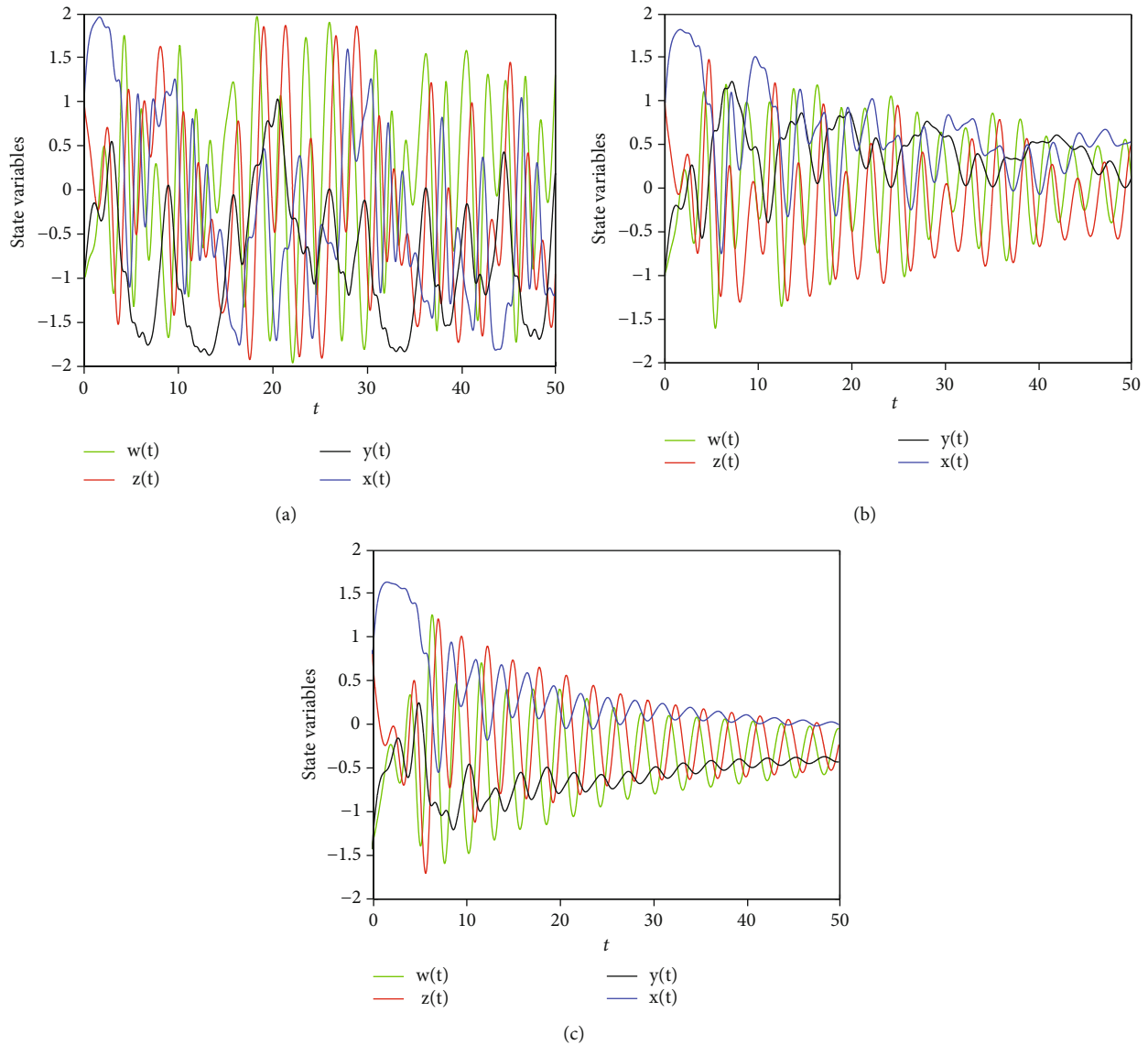


FIGURE 18: The behavior of different state variables of system 2 vs. time t with fractional order $\varpi = 0.98$ and the parameters $n = m = 2$.

$$z(t_n) = x(0) + h^\alpha \left[k_n^\alpha \mathcal{G}_3(0) + \sum_{j=1}^n k_{n-j}^{(\alpha)} \mathcal{G}_3(t_j, z_j) \right], \quad (40)$$

$$w(t_n) = x(0) + h^\alpha \left[k_n^\alpha \mathcal{G}_4(0) + \sum_{j=1}^n k_{n-j}^{(\alpha)} \mathcal{G}_4(t_j, w_j) \right], \quad (41)$$

where

$$\begin{aligned} \mathcal{G}_1(t_j, x_j) &= m y_j w_j, \mathcal{G}_2(t_j, y_j = y_j z_j, \mathcal{G}_3(t_j, z_j) \\ &= -y_j^2 + n w_j, \mathcal{G}_4(t_j, w_j = -m x_j y_j - n z_j. \end{aligned} \quad (42)$$

8. Numerical Simulation and Discussions

The numerical simulations of the numerical scheme provided above are presented in this section. We also show

phase portraits to investigate the behavior and effects of a few key parameters and fractional orders on the dynamics of models (1) and (2). We consider four different sets of parameters with fixed initial conditions throughout the sections as $[x, y, z, w] = [-1, 1, -1, 1]$. For the simulation purpose, we have considered the time $t = 30$. The step-size is considered to be $h = 0.001$ which makes the number of iterations to be 3×10^4 . The MATLAB-R13b software is used for the simulation purpose. Further, the subfigures (a), (b), (c), (d), and (e) show the state variables $x - y$, $x - z$, $x - w$, $y - z$, and $z - w$, respectively.

The dynamics of model (1) for $c = d = 2$ are shown in Figures 4–6. The attractor to which the system progress is hidden is noticed when the fractional order $\varpi = 1$. It is observed that when the fractional order is $\varpi = 0.99$, the system eventually approaches a fixed point, which is a type of attractor known as a fixed point attractor. Similarly, decreasing the order ϖ to 0.98 increases the visibility of the

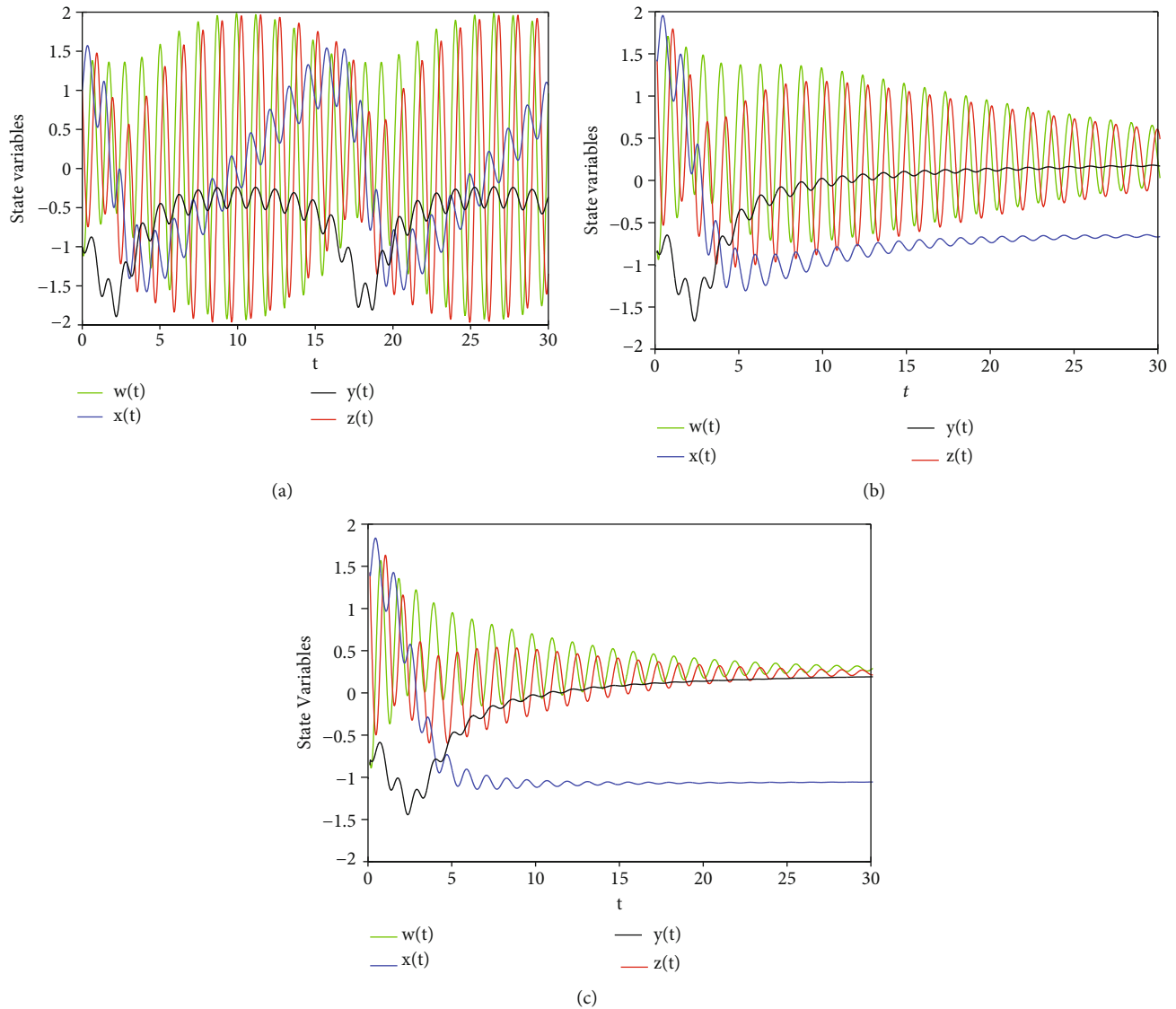


FIGURE 19: The behavior of different state variables of system 2 vs. time t with fractional order $\varpi = 0.98$ and the parameters $n = 2, m = 2$.

attractor. Model (1) is shown in Figures 7–9 with $c = 2, d = 5$. The fractional-order is set to $\varpi = 1$ in Figure 4, demonstrating the classical behavior of model (1). When the fractional order is set to $\varpi = 0.99$ as shown in Figure 8, the model (1) progresses to the fixed point attractor. The fixed point attractor is best visualized in Figure 9 by reducing the order ϖ to 0.98. It is observed from the simulations that fractional operators show significant impact on the dynamics of non-linear dynamical systems especially chaotic systems, where one can see new type of attractors that are hidden with that of integer order derivatives. In these simulations, one thing is clear that these systems dynamically evolve towards the equilibrium point E_0 , which shows that the systems become stable at lower fractional orders.

The dynamics of model (2) with $c = d = 2$ are shown in Figures 10–12. In Figure 10, the fractional-order is set to $\varpi = 1$, demonstrating the model's classical behavior (2). The fractional-order is $\varpi = 0.99$ in Figure 11, where it is noticed

that the fractional-order has a better effect on the dynamics of the model (2). The attractor with the fractional order $\varpi = 0.98$ may be seen in Figure 12. As the fractional-order decreases, the system tends to reach equilibrium points.

The parameters for Figures 13–15 are $c = 2, d = 5$, which shows the behavior of model (2). It is also noticed that changes in fractional order have a significant impact on the dynamics of the model (2). The system rapidly progresses to its attractor at $\varpi = 0.98$, as shown in the figures.

8.1. Time Series Analysis. In mathematics, a time series is a collection of data points that are indexed in time order. A time series is a collection of points taken over a period of time at uniformly spaced intervals. As a result, it is a collection of discrete-time information. Ocean tide heights, sunspot counts are a few examples of time series. Time series analysis depicts the behavior of state variables for each small value of time in the case of dynamical systems. Studying the

time series data makes it simple to study the system's stability and instability. Furthermore, oscillations that develop in a system's dynamics can be easily investigated by studying the system's time series behavior.

In Figures 16 and 17, the dynamics of different state variables of model (1) are presented with the parameters $c = d = 2$ and $c = 2, d = 5$, respectively. Similarly, in Figures 18 and 19, the dynamics of different state variables of model (2) are demonstrated with the parameters $m = n = 2$ and $m = 2, n = 5$. It is observed that both the systems oscillate rapidly when $\omega = 1$. When consider the fractional-order $\omega = 0.99$, the system oscillations decrease as compared to the integer-order. It can be seen that the system evolves towards its fixed point, which realizes the hidden attractors present in the systems. Further decreasing the fractional-order ω to 0.98, one can see that the system advances to its stable state more rapidly as compared to the $\omega = 0.99$.

9. Conclusion

We have investigated four-dimensional continuous chaotic models with Caputo's fractional derivative. The dynamical features such as Lyapunov spectra, phase portrait, and time series analysis are investigated. More than one positive exponents for each system revealed that systems are hyperchaotic. The phase portraits are presented to investigate the behavior and effect of some important parameters and fractional orders on model dynamics. It is revealed that, for fractional order, the systems approach to fixed point attractors and increase the visibility of the attractor by decreasing fractional-order which shows that change in fractional order have a significant impact on the systems. It is also revealed that, when the order of the derivative is equal to one, both systems oscillate frequently. However, as the fractional order is reduced, the system oscillations decrease as compared to the integer-order, and the system moves towards its fixed point. It shows the hidden attractors inherent in the system and enabled it to develop to a stable state more efficiently. Hence, it is concluded that fractional models are the most suitable to study such complexities in dynamical systems.

Data Availability

The data generated during the current study are available in the manuscript.

Conflicts of Interest

It is declared that all the authors have no conflict of interest regarding this manuscript.

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