Hindawi

Retraction

Retracted: Toward the Nash Equilibrium Solutions for Large-Scale Pentagonal Fuzzy Continuous Static Games

Journal of Function Spaces

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope

(2) Discrepancies in the description of the research reported

(3) Discrepancies between the availability of data and the research described

(4) Inappropriate citations

(5) Incoherent, meaningless and/or irrelevant content included in the article

(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] H. A. E.-W. Khalifa, S. A. Edalatpanah, and A. Alburaikan, "Toward the Nash Equilibrium Solutions for Large-Scale Pentagonal Fuzzy Continuous Static Games," *Journal of Function Spaces*, vol. 2022, Article ID 3709186, 11 pages, 2022.

Research Article

Toward the Nash Equilibrium Solutions for Large-Scale Pentagonal Fuzzy Continuous Static Games

Hamiden Abd El-Wahed Khalifa[1,2] **S. A. Edalatpanah**[3] **and Alhanouf Alburaikan**[2]

^{1}Department of Operations Research, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza 12613, Egypt
^{2}Department of Mathematics, College of Science and Arts, Qassim University, Al-Badaya 51951, Saudi Arabia
^{3}Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran

Correspondence should be addressed to S. A. Edalatpanah; saedalatpanah@gmail.com

This study aims to characterize and clarify a pentagonal fuzzy continuous static game (PF-CSG) that constraints and cost functions are fuzzy numbers. Pentagonal fuzzy numbers characterize their fuzzy parameters. The α − Pareto optimal solution concept has specified. The decomposition approach has applied to decompose the problem into subproblems each of them having smaller and independent subproblems. In addition, the Nash equilibrium solution concept was used to obtain the solutions of these subproblems. The advantages of this study are the players independently without collaboration with any of the others and that each player seeks to minimize the cost function. Also, the information available to each player consists of the cost function and constraints. An illustrated numerical example has discussed for proper understanding and interpretation of the proposed concept.

1. Introduction

Game theory plays a vital role in economics, engineering, biology, and other computational cum mathematical sciences with wide range of applications in real-world problems. Differential games, continuous static games, and matrix games are three major types of games. Matrix games derive their name from a discrete relationship between a finite/countable number of possible decisions and the corresponding costs. The relationship is conveniently represented in terms of a matrix (or two-player games) in which the decision of one player relates to the choice of a row and the decision of other player is corresponding to the choice of a column, with the corresponding entries denoting the costs. It is vivid that decision probabilities are not mandatory in the cooperative games. In addition, there is no time in the relationship between costs and decisions in static games. Differential games are categorized by varying costs along with a dynamic system administrated by ODE. For continuous static games, there are several solution concepts. How a player uses these concepts depends not only on

information concerning the nature of the other players, but also on his/her own personality. A given player may or may not play rationally, cheat, cooperate, and bargain. A player in making the ultimate choice of his/her control vector must consider all of these factors. The three basic solution concepts for these games (Vincent and Grantham [1]) are

(1) Nash equilibrium solution

(2) Min-Max solutions

(3) Pareto minimal solutions

Early, several researchers worked in fuzzy set theory; Zadeh [2] introduced the notion of a fuzzy set in an attempt to develop the ideology of fuzzy set and mathematical framework in which to treat systems or phenomena, which is due to intrinsic indefiniteness as distinguished from a mere statistical variation, cannot themselves be characterized precisely. Dubois and Prade [3] developed the view of using algebraic operations on fuzzy numbers using a fuzzification principle. Decisions in a fuzzy situation were first proposed

by Bellman and Zadeh [4], which is a great help in managers' decision-making problems. Kaufmann and Gupta [5] deliberated numerous fuzzy mathematical prototypes that have significant applications in science and engineering. Lasdon [6] introduced an optimization theory for large-scale system. Osman et al. [7] developed the Nash equilibrium solution for large-scale CSG, where players are able to minimize the cost function independently and without cooperating with any of the other players. In addition, the information available to each player consists of the cost functions and constraints. Elshafei [8] familiarized an interactive model for solving Nash CSG and resulted a stability set accordingly. Hosseinzadeh Lotfi et al. [9] applied Nash bargaining theory, for performance assessment, and suggested a model of data envelopment analysis. The idea of equilibrium for a fuzzy noncooperative game has presented by Kacher and Larbain [10]. Cruz and Simaan [11] described a theory in which players could rank the order of their choice against the selection of other players instead of the payoff function. Navidi et al. [12] considered a multiresponse optimization problem and offered an attitude based on games theory. Corley [13] defined a dual to the Nash equilibrium for n – person in strategic procedure, where the strategy of each player maximizes his/her own expected payoff for the other n – 1 player's strategies. Also, the comparison between the dual and the related to the mixed Nash equilibrium and both topological and algebraic conditions is given. Farooqui and Niazi [14] introduced a comprehensive multidisciplinary state-of-the-art review and taxonomy of the game theory models of complex interactions between agents. Sasikala and Kumaraghuru [15] developed an interactive approach based on the cooperation programming and the method of concession weights for solving Nash continuous cooperative static games (NCCSTGs). Awaya and Krishna [16] deliberated the character of communication in repeated games with private monitoring and compared the set of equilibria under two regimes. Silbermayr [17] introduced a review on the use of noncooperative game theory in the inventory management. Shuler [18] investigated cooperation games in which poor agents do not benefit from cooperation with wealthy agents. Badri and Yarmohamadi [19], based on game theory, suggested a method for modest market of dental tourism issues. Khalifa and Kumar [20] studied the cooperative continuous static games in crisp environment, defined, and strongminded the stability set without differentiability. Wang and Garg [21] constructed several novel interactive operational rules for Pythagorean fuzzy numbers in the light of Archimedean t-conorm and t-norm, based on which, some novel interactive AOs are explored, they are Pythagorean fuzzy interactive weighted averaging operator and Archimedean based Pythagorean fuzzy interactive weighted geometric operator. In addition, they have discussed their properties, such as their idempotency, monotonicity boundedness, and shift invariance. Recently, there are enormous papers introduced to deal with the Nash equilibrium for solving the CSG (for instance, [22], [23], [24], [25, 26], [27], and [28]).

In this paper, a Nash equilibrium solution for solving large-scale CSG with pentagonal fuzzy information is introduced. In this type of games, each player tries to minimize his/her cost functions independently.

1.1. Research Gap and Motivation. The phrase, "pentagonal fuzzy number", is actually meant for dispensing the fuzzy value to each attribute/subattribute in the domain of single argument/multiargument approximate function.

(1) Many researchers discussed the fuzzy set-like structures under fuzzy set environment with fuzzy set-like settings.

(2) Along these lines another construction requests its place in writing for tending to such obstacle, so fuzzy set is conceptualized to handle such situations.

The rest of the paper is outlined as follows in Figure 1:

2. Preliminaries

In this section, some essential definitions and terminologies are recalled from fuzzy-like literature for proper understanding of the proposed work ([29], [30], and [31]) .

Definition 1 (see [2]). A fuzzy set $\tilde{P}$ defined on the set of real numbers $\Re$ is said to be fuzzy numbers if its membership function:

$\mu_{\tilde{P}}(x): \Re \longrightarrow [0, 1]$, have the following properties:

(1) $\mu_{\tilde{P}}(x)$ is an upper semi- continuous membership function;

(2) $\tilde{P}$ is convex fuzzy set, i.e., $\mu_{\tilde{P}}(\delta x + (1 - \delta) y) \geq \min \{\mu_{\tilde{P}}(x), \mu_{\tilde{P}}(y) \}$ for all x, y $\in \Re$; $0 \leq \delta \leq 1$;

(3) $\tilde{P}$ is normal, i.e., $\exists x_0 \in \Re$ for which $\mu_{\tilde{P}}(x_0) = 1$;

(4) Supp $(\tilde{P}) = \{x \in \Re : \mu_{\tilde{P}}(x) > 0 \}$ is the support of $\tilde{P}$, and the closure cl(Supp($\tilde{P}$)) is compact set.

Definition 2 (see [29]). The membership function of A linear PFN $\tilde{A}_{PFN} = (a_1, a_2, a_3, a_4, a_5), a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$, is defined as (Figure 2)

$$\pi_{\tilde{A}_{PFN}}(x) = \begin{cases} 0, x < a_1, \\ w_1\left(\dfrac{x - a_1}{a_2 - a_1}\right), \text{for } a_1 \leq x \leq a_2, \\ 1 - (1 - w_1)\left(\dfrac{x - a_2}{a_3 - a_2}\right), \text{for } a_2 \leq x \leq a_3, \\ 1, \text{for } x = a_3, \\ 1 - (1 - w_2)\left(\dfrac{a_4 - x}{a_4 - a_3}\right), \text{for } a_3 \leq x \leq a_4, \\ w_2\left(\dfrac{a_5 - x}{a_5 - a_4}\right), \text{for } a_4 \leq x \leq a_5, \\ 0, \text{for } x > a_5. \end{cases} \quad (1)$$
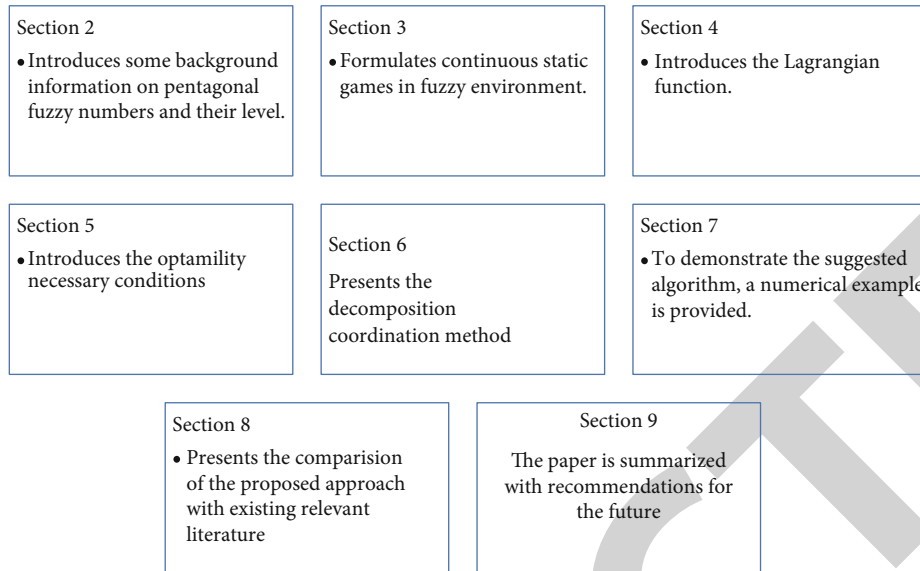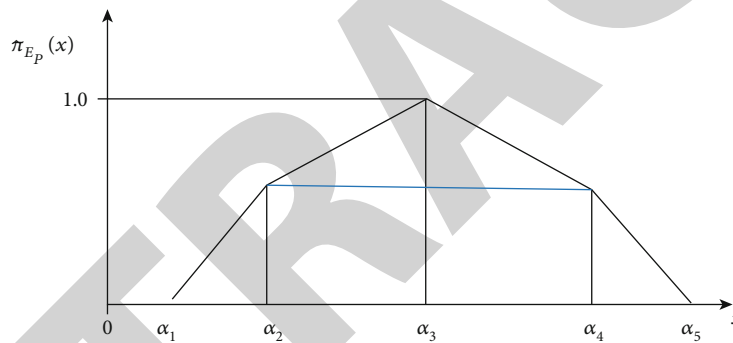
Figure 1: Layout of remaining paper.

Figure 2: Depiction of a PFN [29].

Definition 3. For two PFNs $Y = (y_1, y_2, y_3, y_4, y_5)$, and $F = (f_1, f_2, f_3, f_4, f_5)$, we have

$$
\begin{aligned}
Y \oplus F &= (y_1 + f_1, y_2 + f_2, y_3 + f_3, y_4 + f_4, y_5 + f_5) \\
Y \ominus F &= (y_1 - f_5, y_2 - f_4, y_3 - f_3, y_4 - f_2, y_5 - f_1), \\
k\, Y &= \begin{cases} (ky_1, ky_2, ky_3, ky_4, ky_5), k > 0 \\ (ky_5, ky_4, ky_3, ky_2, ky_1), k < 0 \end{cases}.
\end{aligned}
\tag{2}
$$

The interval of confidence at level α for the pentagonal fuzzy number is defined as

$$
(Y)_\alpha = [a_1 + 2(a_2 - a_1)\alpha, a_5 - 2(a_5 - a_4)\alpha]; \forall \alpha \in [0, 1]. \tag{3}
$$

Definition 4. An interval $[\tilde{E}_P] = [E_\alpha^-, E_\alpha^+]$ of a pentagonal fuzzy $\tilde{E}_P$ is called closed an inexact rough interval if

$$
E_\alpha^- = \inf\left\{ x \in \Re : \mu_{\tilde{E}_P} \geq 0.75 \right\}, \text{ and } E_\alpha^+ = \sup\left\{ x \in \Re : \mu_{\tilde{E}_P} \geq 0.75 \right\}.
\tag{4}
$$

Definition 5. Let $[\tilde{E}_P] = [E_\alpha^-, E_\alpha^+]$ *and* $[\tilde{F}_P] = [F_\alpha^-, F_\alpha^+]$ be two inexact rough intervals of pentagonal fuzzy numbers $\tilde{E}_P$ and $\tilde{F}_P$. Then, the arithmetic operations are

(1) Addition: $[\tilde{E}_P] \oplus [\tilde{F}_P] = [E_\alpha^- + F_\alpha^-, E_\alpha^+ + F_\alpha^+]$,

(2) Subtraction: $[\tilde{E}_P] \ominus [\tilde{F}_P] = [E_\alpha^- - F_\alpha^+, E_\alpha^+ - F_\alpha^-]$,

(3) Scalar multiplication: $k.[\tilde{E}_P] = \begin{cases} [k\, E_\alpha^-, k\, E_\alpha^+], k > 0, \\ [k \quad E_\alpha^+, k\, E_\alpha^-], k < 0. \end{cases}$

3. Problem Formulation and Solution Concepts

The large-scale CSG with pentagonal fuzzy numbers in both the cost functions and constraints can be formulated as follows:

$$(\text{PF} - \text{CSGs}) \min \tilde{G}_l^P = \sum_{k=1}^{r} f_{lk}\left(x_{lk}, u_{lk}, \tilde{a}_{lk}^P\right), l = \overline{1, r}$$

Subject to

$$x \in \tilde{\Phi}^P = \left\{ \begin{array}{l} (x, u) \in \mathfrak{R}^{n \times s}: \quad x, u \text{ satisfies} \\[4pt] \displaystyle\sum_{i=1}^{p_1} h_{it}\left(x_{it}, u_{it}, \tilde{a}_{it}^P\right) = 0, i = \overline{1, n}, \\[4pt] \cdot \\ \cdot \\ \cdot \\[4pt] \displaystyle\sum_{m=p_n+1}^{r} h_{im}\left(x_{im}, u_{im}, \tilde{a}_{im}^P\right) = 0, i = \overline{1, n}, \\[4pt] \displaystyle\sum_{t'=1}^{p_1'} g_{kt'}\left(x_{kt'}, u_{kt'}, \tilde{a}_{kt'}^P\right) \geq 0, k = \overline{1, q}, \\[4pt] \cdot \\ \cdot \\ \cdot \\[4pt] \displaystyle\sum_{m'=p_n'+1}^{r} g_{km'}\left(x_{km'}, u_{km'}, \widetilde{a}_{km'}^P\right) \geq 0, ik = \overline{1, q}, \\[4pt] \displaystyle\sum_{c=1}^{r} \varphi_{jc}\left(x_{jc}, u_{jc}, \widetilde{a}_{jc}^P\right) \geq 0, j = \overline{1, n}, \\[4pt] \cdot \\ \cdot \\ \cdot \\[4pt] \displaystyle\sum_{s=1}^{r} \psi_{ks}\left(x_{ks}, u_{ks}, \widetilde{a}_{ks}^P\right) \geq 0, k = \overline{1, q}; \end{array} \right.$$

(5)

where the objective functions and the constraints are assumed to have an additively separable form, $\tilde{a}_{lk}^{P}$; $(\tilde{a}_{jt}^{P}, \cdots, \tilde{a}_{jm}^{P})$ and $(\tilde{a}_{kt}^{P}, \cdots, \tilde{a}_{km}^{P})$; $\tilde{a}_{jb}^{P}, \tilde{a}_{ks}^{P}$ are vectors of fuzzy parameters in the cost functions, in equality and inequality constraints and in common constraints; , respectively. Pentagonal fuzzy numbers represents these fuzzy parameters.

Definition 6 (see [3]) (**α – level set**). The α – level set of the fuzzy numbers $\tilde{a}_{lk}^{P}$ are defined as the ordinary set $L_{\alpha}(\tilde{a}_{lk}^{P})$ in

which the degree of their membership functions exceeds the level α :

$$L_{\alpha}\left(\tilde{a}_{lk}^{P}\right) = \left\{ a_{lk} : \mu_{\tilde{a}_{lk}^{P}}(a_{lk}) \geq \alpha, l = \overline{1, r} ; k = \overline{1, q} \right\}. \quad (6)$$

For a certain degree of α, the (PF-CSGs) can be converted into large-scale nonfuzzy continues static games as

$$(\alpha - \text{CSGs}) \min G_l = \sum_{k=1}^{r} f_{lk}(x_{lk}, u_{lk}, a_{lk}), l = \overline{1, r}$$

Subject to

$$(x, u) \in \mathfrak{R}^{n \times s} : x, u \text{ satisfies}$$

$$\sum_{i=1}^{p_1} h_{it}(x_{it}, u_{it}, a_{lt}) = 0, i = \overline{1, n},$$

$$\vdots$$

$$\sum_{m=p_n+1}^{r} h_{im}(x_{im}, u_{im}, a_{im}) = 0, i = \overline{1, n},$$

$$\sum_{t'=1}^{p_1'} g_{kt'}(x_{kt'}, u_{kt'}, a_{kt'}) \geq 0, k = \overline{1, q},$$

$$\mathrm{x} \in \tilde{\Phi}_{\alpha} = \left\{ \qquad\qquad \vdots \right\}. \quad (7)$$

$$\sum_{m'=p_n'+1}^{r} g_{km'}(x_{km'}, u_{km'}, a_{km'}) \geq 0, ik = \overline{1, q},$$

$$\sum_{c=1}^{r} \varphi_{jc}(x_{jc}, u_{jc}, a_{jc}) \geq 0, j = \overline{1, n},$$

$$\vdots$$

$$\sum_{s=1}^{r} \psi_{ks}(x_{ks}, u_{ks}, a_{ks}) \geq 0, k = \overline{1, q},$$

$$\left(a_{lk}, a_{lt}, a_{im}, a_{kt'}, a_{km'}, a_{jc}, a_{ks} \right) \in L_{\alpha}\left(\tilde{a}_{lk}^{P}, \tilde{a}_{it}^{P}, \tilde{a}_{im}^{P}, \tilde{a}_{kt'}^{P}, \tilde{a}_{km'}^{P}, \tilde{a}_{jc}^{P}, \tilde{a}_{ks}^{P} \right)$$

It is noted that, the (α-CSGs) problem can be transformed into the following problem using the concept of inexact rough interval of pentagonal fuzzy numbers as

$$(\gamma - \text{CSGs}) \min G_l = \sum_{k=1}^{r} f_{lk}(x_{lk}, u_{lk}, a_{lk}), l = 1, r$$

Subject to

$$X \in \Phi^{(-,+)} = \begin{cases} (x, u) \in \mathfrak{R}^{n \times s} : \quad x, u \text{ satisfies} \\ \sum_{i=1}^{p_1} h_{it}(x_{it}, u_{it}, a_{it}) = 0, i = 1, n, \\ \cdot \\ \cdot \\ \cdot \\ \sum_{m=p_n+1}^{r} h_{im}(x_{im}, u_{im}, a_{im}) = 0, i = 1, n, \\ \sum_{t'=1}^{p_1'} g_{kt'}(x_{kt'}, u_{kt'}, a_{kt'}) \geq 0, k = 1, q, \\ \cdot \\ \cdot \\ \cdot \\ \sum_{m'=p_n'+1}^{r} g_{km'}(x_{km'}, u_{km'}, a_{km'}) \geq 0, ik = 1, q, \\ \sum_{c=1}^{r} \varphi_{jc}(x_{jc}, u_{jc}, a_{jc}) \geq 0, j = 1, n, \\ \cdot \\ \cdot \\ \cdot \\ \sum_{s=1}^{r} \psi_{ks}(x_{ks}, u_{ks}, a_{ks}) \geq 0, k = 1, q, \\ a_{lk} \in [(a_{lk})_\alpha^-, (a_{lk})_\alpha^+], a_{lt} \in [(a_{lt})_\alpha^-, (a_{lt})_\alpha^+], \\ a_{im} \in [(a_{im})_\alpha^-, (a_{im})_\alpha^+], a_{kt'} \in [(a_{kt'})_\alpha^-, (a_{kt'})_\alpha^+], \\ a_{km'} \in [(a_{km'})_\alpha^-, (a_{km'})_\alpha^+], a_{jc} \in [(a_{jc})_\alpha^-, (a_{jc})_\alpha^+], \\ a_{ks} \in [(a_{ks})_\alpha^-, (a_{ks})_\alpha^+] \end{cases}$$

(8)

Definition 7 (**α** – Nash equilibrium solution). A point $u^* \in \Psi$ is an α – Nash equilibrium solution to the (γ-CSGs) problem if and only if for each $l = 1, r$, we have $G_l(\zeta(u^*), u^*, a_{lk}) \leq G_l(\zeta(u^l, v^*), u°, v^*, a_{lk})$, for all $u^l \in u$, where $u^* = (u^{*l}, v^*) \in \Psi, u = \{u^l \in \mathfrak{R}^l : g(\zeta(u^l, v), u^l, v) \geq 0\}, x \in \zeta(u)$ is the solution to the system

$$\sum_{i=1}^{p_1} h_{it}(x_{it}, u_{it}, a_{lt}) = 0, i = 1, n \; ; \sum_{m=p_n+1}^{r} h_{im}(x_{im}, u_{im}, a_{im})$$
$$= 0, i = 1, n,$$

$$A_{lk} \in [(a_{lk})_\alpha^-, (a_{lk})_\alpha^+], a_{lt} \in [(a_{lt})_\alpha^-, (a_{lt})_\alpha^+], a_{im} \in$$
$$\cdot [(a_{im})_\alpha^-, (a_{im})_\alpha^+], a_{kt'} \in [(a_{kt'})_\alpha^-, (a_{kt'})_\alpha^+],$$

$$a_{km'} \in [(a_{km'})_\alpha^-, (a_{km'})_\alpha^+], a_{jc} \in$$
$$\cdot [(a_{jc})_\alpha^-, (a_{jc})_\alpha^+], a_{ks} \in [(a_{ks})_\alpha^-, (a_{ks})_\alpha^+].$$

(9)

4. Lagrangian Function

The Lagrangian function corresponding to the (γ-CSGs) problem is represented by

$$L_N = G_l - \left(\sum_{t=1}^{p_1} V_{jt}^t(l) h_{jt}(x_{jt}, u_{jt}, a_{jt}) + \cdots + \right.$$
$$\left. \sum_{m=p_n+1}^{r} V_{jm}^t(l) h_{jm}(x_{jm}, u_{jm}, a_{jm}) \right)$$
$$- \left(\sum_{t'=1}^{p_1'} Y_{kt'}^t(l) g_{kt'}(x_{kt'}, u_{kt'}, a_{kt'}) + \cdots + \right.$$
$$\left. \sum_{m'=p_n'+1}^{r} Y_{kt'}^t(l) g_{km'}(x_{km'}, u_{km'}, a_{km'}) \right)$$
$$- \left(\sum_{c=1}^{r} N_{jc}^t \varphi_{jc}(x_{jc}, u_{jc}, a_{jc}) - F_{ks}^t(l) \left[\sum_{s=1}^{r} \psi_{ks}(x_{ks}, u_{ks}, a_{ks}) - J \right] \right)$$
$$- Z_{lk}^t(l) \left(a_{lk} - (a_{lk})_\alpha^- \right) - Z_{lk}^t(l) \left((a_{lk})_\alpha^+ - a_{lk} \right),$$

(10)

where $V_{jt}^t(l), \cdots, V_{jm}^t(l)$ and N_{jc}^t are the Lagrangian multipliers, and
$Y_{kt'}^t(l), \cdots, Y_{kt'}^t(l) \geq 0, X_{ks}(l) \geq 0$, and $Z_{lk}^t(l) \geq 0$ are the Kuhn-Tucker multipliers.

5. Optimality Necessary Conditions

If $u^* \in \Psi$ is completely regular α–Nash equilibrium solution to the (γ-CSGs) problem, and $x^* \in \zeta(u^*)$ is the solution to the system $\sum_{i=1}^{p_1} h_{it}(x_{it}, u_{it}, a_{lt}) = 0, i = 1, n \; ; \sum_{m=p_n+1}^{r} h_{im}(x_{im}, u_{im}, a_{im}) = 0, i = 1, n$, then for each $l = 1, r$, there exists a vector $V(l) \in \mathfrak{R}^n, Y(l) \in \mathfrak{R}^q, N(l) \in \mathfrak{R}^n, X(l) \in \mathfrak{R}^q, Z(l) \in \mathfrak{R}^{2r}$ such that:

$$\frac{\partial}{\partial x} L_N(x^*, u^*, a^*, V(l), Y(l), N(l), X(l), Z(l)) = 0,$$

$$\frac{\partial}{\partial u} L_N(x^*, u^*, a^*, V(l), Y(l), N(l), X(l), Z(l)) = 0,$$

$$\frac{\partial}{\partial a} L_N(x^*, u^*, a^*, V(l), Y(l), N(l), X(l), Z(l)) = 0,$$

$$\sum_{t=1}^{p_1} h_{jt}(x_{jt}, u_{jt}, a_{jt}) = 0, j = 1, n \; ; \sum_{m=p_n+1}^{r} h_{jm}(x_{jm}, u_{jm}, a_{jm})$$
$$= 0, j = 1, n \; ;$$

$$\sum_{t'=1}^{p_1'} g_{kt'}(x_{kt'}, u_{kt'}, a_{kt'}) \geq 0, k$$

$$= 1, \bar{q}; \sum_{m'=p_n'+1}^{r} g_{km'}(x_{km'}, u_{km'}, a_{km'})$$

$$\geq 0, k = 1, \bar{q};$$

$$\sum_{c=1}^{r} \varphi_{jc}(x_{jc}, u_{jc}, a_{jc}) = 0, j = 1, \bar{n}; \sum_{s=1}^{r} \psi_{ks}(x_{ks}, u_{ks}, a_{ks}) - J$$

$$\geq 0, k = 1, \bar{q};$$

$$\sum_{t'=1}^{p'} Y_{kt'}^{t}(l) g_{kt'}(x_{kt'}, u_{kt'}, a_{kt'}) = 0, k = 1, \bar{q};$$

$$\vdots$$

$$\sum_{m'=p_n'+1}^{r} Y_{kt'}^{t}(l) g_{km'}(x_{km'}, u_{km'}, a_{km'}) \geq 0, k = 1, \bar{q};$$

$$Y_{kt'}(l) \geq 0, \cdots, Y_{km'}(l) \geq 0;$$

$$X_{ks}^{t}(l) \left[\sum_{s=1}^{r} \psi_{ks}(x_{ks}, u_{ks}, a_{ks}) - J \right] = 0, Z_{ks}(l) \geq 0;$$

$$(a_{lk} - (a_{lk})_\alpha^-) \geq 0; (a_{lk})_\alpha^+ - a_{lk} \geq 0; Z_{lk}^{t}(l)(a_{lk} - (a_{lk})_\alpha^-) = 0;$$

$$Z_{lk}^{t}(l)((a_{lk})_\alpha^+ - a_{lk}) = 0, Z_{lk}^{t}(l) \geq 0. \tag{11}$$

6. Decomposition Coordination Method

In this section, the decomposition coordination method is used to solve the (γ-CSGs) problem, where the solution of the problem can be found by solving (r) subproblems P_β^r, where β is a given coordination parameter through the sequence $(\beta^0, \beta^1, \cdots, \widehat{\beta})$. So the kth subproblem is defined as

$$\min G_l = f_{lk} - N_{jc}^{t} \varphi_{jc}(x_{jc}, u_{jc}, a_{jc}) - X_{ks}^{t}(l) \sum_{s=1}^{r} \psi_{ks}(x_{ks}, u_{ks}, a_{ks})$$

$$\text{Subject to } \sum_{I=1}^{p_1} h_{it}(x_{it}, u_{it}, a_{it}) = 0, i = 1, \bar{n};$$

$$\vdots$$

$$\sum_{m=p_n+1}^{r} h_{im}(x_{im}, u_{im}, a_{im}) = 0, i = 1, \bar{n};$$

$$\sum_{t'=1}^{p_1'} g_{kt'}(x_{kt'}, u_{kt'}, a_{kt'}) \geq 0, k = 1, \bar{q};$$

$$\vdots$$

$$\sum_{m'=p_n'+1}^{r} g_{km'}(x_{km'}, u_{km'}, a_{km'}) \geq 0, ik = 1, \bar{q};$$

$$(a_{lk} - (a_{lk})_\alpha^-) \geq 0; (a_{lk})_\alpha^+ - a_{lk} \geq 0. \tag{12}$$

By applying the Kuhn-Tucker conditions for the kth subproblem, we have the necessary conditions for optimality, for $k = 1, \bar{n}$ are

$$\frac{\partial}{\partial x} f_l - \left[\left(\frac{\partial}{\partial x} h_{lt} \right)^t V_{lt}(l) + \cdots + \left(\frac{\partial}{\partial x} h_{jm} \right)^t V_{jm}(l) \right]$$

$$- \left[\begin{array}{c} \left(\frac{\partial}{\partial x} g_{kt'} \right)^t Y_{kt'}(l) + \cdots \\ + \left(\frac{\partial}{\partial x} h_{km'} \right)^t Y_{km'}(l) \end{array} \right] = 0,$$

$$\frac{\partial}{\partial u} f_l - \left[\left(\frac{\partial}{\partial u} h_{lt} \right)^t V_{lt}(l) + \cdots + \left(\frac{\partial}{\partial x} h_{jm} \right)^t V_{jm}(l) \right]$$

$$- \left[\begin{array}{c} \left(\frac{\partial}{\partial x} g_{kt'} \right)^t Y_{kt'}(l) + \cdots \\ + \left(\frac{\partial}{\partial x} h_{km'} \right)^t Y_{km'}(l) \end{array} \right] = 0,$$

$$\frac{\partial}{\partial a} f_l - \left[\left(\frac{\partial}{\partial x} h_{lt} \right)^t V_{lt}(l) + \cdots + \left(\frac{\partial}{\partial x} h_{jm} \right)^t V_{jm}(l) \right]$$

$$- \left[\left(\frac{\partial}{\partial a} g_{kt'} \right)^t Y_{kt'}(l) + \cdots + \left(\frac{\partial}{\partial a} h_{km'} \right)^t Y_{km'}(l) \right]$$

$$- \left(\frac{\partial}{\partial a} \varphi_{lc} \right)^t N_{kt'}(l) - \left(\frac{\partial}{\partial a} \psi_{ks} \right)^t X_{kS}(l) = 0,$$

$$\sum_{I=1}^{p_1} h_{it}(x_{it}, u_{it}, a_{it}) = 0, i = 1, \bar{n};$$

$$\vdots$$

$$\sum_{m=p_n+1}^{r} h_{im}(x_{im}, u_{im}, a_{im}) = 0, i = 1, \bar{n};$$

$$\sum_{t'=1}^{p_1'} g_{kt'}(x_{kt'}, u_{kt'}, a_{kt'}) \geq 0, k = 1, \bar{q};$$

$$\vdots$$

$$\sum_{m'=p_n'+1}^{r} g_{km'}(x_{km'}, u_{km'}, a_{km'}) \geq 0, ik = 1, \bar{q};$$

$$(a_{lk} - (a_{lk})_\alpha^-) \geq 0; (a_{lk})_\alpha^+ - a_{lk} \geq 0.$$

$$\sum_{T'=1}^{p_1'} Y_{kt'}^{t}(l) g_{kt'}(x_{kt'}, u_{kt'}, a_{kt'}) = 0, k = 1, \bar{q};$$

$$\vdots$$

$$\sum_{m'=p_n'+1}^{r} Y_{kt'}^t(l) g_{km'}(x_{km'}, u_{km'}, a_{km'}) = 0, k = 1, \bar{q};$$

$$Y_{kt'}(l) \geq 0, \cdots, Y_{km'}(l) \geq 0;$$

$$Z_{lk}^t(l)\left(a_{lk} - (a_{lk})_\alpha^-\right) = 0; Z_{lk}^t(l)\left((a_{lk})_\alpha^+ - a_{lk}\right)$$
$$= 0, \quad Z_{lk}^t(l) \geq 0. \tag{13}$$

By comparing these conditions with the corresponding conditions of (γ-CSGs) problem, it is obvious that after solving the r subproblems, all optimality necessary conditions in (γ-CSGs) problem are satisfied except

$$\sum_{C=1}^{r} \varphi_{jc}(x_{jc}, u_{jc}, a_{jc}) \geq 0, j = 1, \bar{n};$$

$$F_{ks}^t(l)\left[\sum_{s=1}^{r} \psi_{ks}(x_{ks}, u_{ks}, a_{ks}) - J\right] = 0, k = 1, \bar{q}; \tag{14}$$

$$F_{ks}(l) \geq 0, \quad \sum_{s=1}^{r} \psi_{ks}(x_{ks}, u_{ks}, a_{ks}) \geq J, k = 1, \bar{q}$$

It is clear that these conditions can be satisfied equivalently by solving the dual Lagrangian problem ([32]).

7. Numerical Example

Consider the two players problem.

$$\min \tilde{G}_1^P = \left((x_{11} - 2)^2 + (u_{11} - 2)^2\right) + \left((\tilde{a}_{12}^P)^2 + v_{12}^2\right),$$

$$\min \tilde{G}_2^P = \left(x_{21}^2 - u_{21}^2\right) + \left((\tilde{a}_{22}^P - 2)^2 + v_{22}^2\right)$$

Subject to

$$3u_{11} - x_{11}^2 \geq 0, 4 - (v_{12} - 2)^2 \geq 0, u_{21}^2 \geq 0, 3v_{22}^2 \geq 0,$$

$$x_{11} + u_{11} + 8\tilde{a}_{12}^P + 2v_{12} - 12 \geq 0, a_{21}^2 - u_{21} - v_{22} \geq 0,$$

$$\tag{15}$$

where player I selects $u \in \Re^l$ and player II selectedv $\in \Re^l$. The membership function for $\tilde{a}_{12}^P$ and $\tilde{a}_{22}^P$ is illustrated in Figures 3 and 4.

According to problem (8), problem (15) takes the form

$$\min G_1 = \left((x_{11} - 2)^2 + (u_{11} - 2)^2\right) + \left(a_{12}^2 + v_{12}^2\right)$$

$$\min G_2 = \left(x_{21}^2 - u_{21}^2\right) + \left((a_{22} - 2)^2 + v_{22}^2\right)$$

Subject to

$$3u_{11} - x_{11}^2 \geq 0, 4 - (v_{12} - 2)^2 \geq 0, u_{21}^2 \geq 0, 3v_{22}^2 \geq 0,$$

$$x_{11} + u_{11} + 8a_{12} + 2v_{12} - 12 \geq 0, x_{21}^2 - u_{21} - v_{22} \geq 0,$$

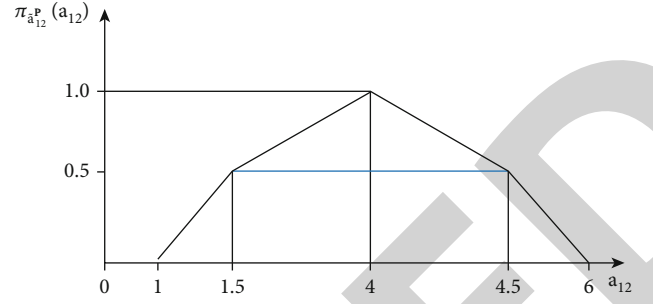$$2 \leq a_{12} \leq 6, 2 \leq a_{22} \leq 5.$$

$$\tag{16}$$

FIGURE 3: Graphical representation of pentagonal fuzzy number $\tilde{a}_{12}^P$.

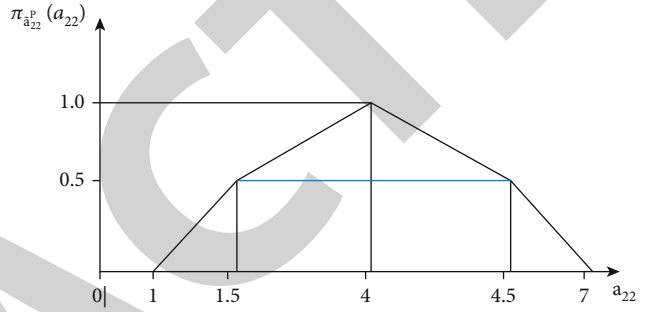FIGURE 4: Graphical representation of pentagonal fuzzy number $\tilde{a}_{22}^P$.

The problem for player 1 is given by

$$\min G_1 = \left((x_{11} - 2)^2 + (u_{11} - 2)^2\right) + \left(a_{12}^2 + v_{12}^2\right)$$

Subject to

$$3u_{11} - x_{11}^2 \geq 0, 4 - (v_{12} - 2)^2 \geq 0,$$

$$x_{11} + u_{11} + 8a_{12} + 2v_{12} - 12 \geq 0, 2 \leq a_{12} \leq 6. \tag{17}$$

Based on the additively separable structure of the functions in problem (17), it can be decomposed into two subproblems with $F_{11}(I)$ as the coordinating parameters. The two subproblems are

Subproblem 1 $\min f_1 = (x_{11} - 2)^2 + (u_{11} - 2)^2 - F_{11}(1)[x_{11} + u_{11}]$

Subject to

$$3u_{11} - x_{11}^2 \geq 0.$$

$$\tag{18}$$

Subproblem 2 $\min f_2 = a_{12}^2 + v_{12}^2 - F_{11}(1)[8a_{12} + 2v_{12}]$

Subject to

$$4 - (v_{12} - 2)^2 \geq 0, 2 \leq a_{12} \leq 6.$$

$$\tag{19}$$

Using the necessary conditions to subproblems (18) and (19), we have

TABLE 1: Comparative study.

Author's name	Kuhn-Tucher conditions	Decomposition technique	α – equilibrium solution	Fuzzy optimal solution	Environment
Elshafei [8]	No	No	No	No	Crisp
Sasikala and Kumaraghuru [17]	Yes	No	No	No	Fuzzy
Khalifa and Kumar [20]	Yes	No	No	No	Crisp
Ganzfried [23]	Yes	No	No	No	Crisp
Megahed [24]	Yes	No	No	Yes	Crisp
Our investigation	Yes	Yes	Yes	Yes	Fuzzy

$$L_1 = (x_{11} - 2)^2 + (u_{11} - 2)^2 - F_{11}(1)(x_{11} + u_{11})$$
$$- Y_{11}(1)(3u_{11} - x_{11}^2),$$

$$L_2 = \min f_2 = a_{12}^2 + v_{12}^2 - F_{11}(1)[8a_{12} + 2v_{12}]$$
$$- Y_{12}(1)(4 - (v_{12} - 2)^2) - Z_{11}(1)(a_{12} - 2) \quad (20)$$
$$- Z_{12}(1)(6 - a_{12}).$$

Hence,

$$x_{11} = \frac{4 + X_{11}(1)}{2 + 2Y_{11}(1)},$$

$$u_{11} = \frac{X_{11}(1) + 3Y_{11}(1)}{2} + 2,$$

$$a_{12} = \frac{8X_{11}(1) + Z_{11}(1) - Z_{12}(l)}{2}, \quad (21)$$

$$v_{12} = \frac{X_{11}(1) + 2Y_{12}(1)}{1 + Y_{12}(1)}.$$

The dual problem for player 1 is

$$\max \Lambda(F_{11}) = \min \left((x_{11} - 2)^2 + (u_{11} - 2)^2 + a_{12}^2 + v_{12}^2\right.$$
$$\left. - Z_{11}(1)(x_{11} + u_{11} + 8a_{12} + 2v_{12} - 12)\right). \quad (22)$$

By using the gradient algorithm (Bazaraa [32]), the optimal solution is

$$x_{11} = 2116, a_{12} = 2.928, u_{11} = 2116, v_{12} = 0.232, F_{11}(1) = 0.232. \quad (23)$$

The problem for player II is

$$\min \tilde{G}_2^P = (x_{21}^2 - u_{21}^2) + \left((\tilde{a}_{22}^P - 2)^2 + v_{22}^2\right)$$
$$\text{Subject to} \quad (24)$$

$$u_{21}^2 \geq 0, 3v_{22}^2 \geq 0, x_{21}^2 - u_{21} - v_{22} \geq 0, 2 \leq a_{22} \leq 5.$$

Due to the additively separable structure of function in problem (24), it can be decomposed into the following two subproblems with $F_{22}(2)$ as the coordinating parameter.

The two subproblems are

Subproblem 3 $\min f_1 = (x_{11} - 2)^2 + (u_{11} - 2)^2 - F_{22}(2)[x_{11} + u_{11}]$
Subject to
$$3u_{11} - x_{11}^2 \geq 0. \quad (25)$$

Subproblem 4 $\min f_2 = a_{12}^2 + v_{12}^2 - F_{22}(2)[8a_{12} + 2v_{12}]$
Subject to
$$4 - (v_{12} - 2)^2 \geq 0, 2 \leq a_{12} \leq 6. \quad (26)$$

By applying the necessary conditions to the two subproblems 3 and 4, we obtain

$$L_3 = x_{21}^2 - u_{21}^2 - F_{22}(2)(x_{21}^2 - u_{21}) - Y_{21}(2)(u_{21}^2),$$

$$L_4 = \min f_2 = (a_{22}^2 - 2)^2 + v_{22}^2 + F_{22}(2)(v_{22})$$
$$- Y_{22}(2)(3v_{22}^2) - Z_{21}(2)(a_{22} - 2) - Z_{22}(2)(5 - a_{22}).$$

$$x_{21} = 0,$$

$$u_{21} = \frac{X_{22}(2)}{2 + 2Y_{21}(2)},$$

$$a_{22} = \frac{Z_{21}(2) - Z_{22}(2)}{2} + 2,$$

$$v_{22} = \frac{X_{22}(2)}{6Y_{22}(2) - 1}. \quad (27)$$

The dual problem for player 2 takes the form

$$\max \Delta(F_{22}) = \min \left(x_{21}^2 - u_{21}^2 + (a_{22}^2 - 2)^2 + v_{22}^2 - Z_{22}(2)(x_{21}^2 - u_{21} - v_{22})\right). \quad (28)$$

By using the gradient algorithm introduced by Bazaraa [32], the optimal solution is

$$x_{21} = 0, a_{22} = 2, u_{21} = 0, v_{22} = 0, F_{22}(2) = 0. \quad (29)$$

8. Comparative Study

This section introduces the comparison between the proposed approach with some existing literature to illustrate the advantages of the proposed approach as shown in Table 1.

9. Conclusion and Future Works

In this study, pentagonal fuzzy CGS with n players having fuzzy factors both in the cost functions and the right-hand side of constraints is studied. The optimal solution concept and the inexact intervals for the pentagonal fuzzy number are specified. The decomposition approach is applied to decompose the problem into subproblems each of them having smaller and independent subproblem. The key features of this work can be summarized as follows:

   (i) The fundamental theory of fuzzy set is developed and its decision constructed. A real world problem is discussed with the support of proposed algorithm and decision support of fuzzy set

   (ii) The rudiments of f fuzzy set are characterized

   (iii) The proposed model and its decision-making based system are developed. A real-life problem is studied with the help of proposed algorithm, and decision system of fuzzy set is compared professionally via strategy with some existing relevant models keeping in view important evaluating features

   (iv) The particular cases of proposed models of fuzzy set are discussed with the generalization of these structures

   (v) As the proposed model is inadequate, the situation in the domain of pentagonal fuzzy numbers is mandatory. Therefore, future work may include the addressing of this limitation and the determination

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Acknowledgments

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