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Research Article

Some Generalized Formulas of Hadamard-Type Fractional Integral Inequalities

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This paper is aimed at establishing the generalized forms of Riemann-Liouville fractional inequalities of the Hadamard type for a class of functions known as strongly exponentially $(\alpha, h-m)$-p-convex functions. These inequalities provide some general formulas from which one can get associated inequalities for various types of exponentially convex and strongly convex functions. Refinements of well-known inequalities are also deducible from the established theorems.

1. Introduction

The notion of convexity is utilized significantly for finding solutions of essential mathematical problems in subjects of science and engineering. Leading with major developments in several branches of mathematics, convexity made its way in statistics, geometric function theory, graph theory, and economics. In recent decades, classes of functions related to convex functions are frequently used in proving new fractional integral inequalities in the form of numerous refinements and generalizations of classical inequalities.

Let I be an interval of real numbers. A function $f : I \longrightarrow \mathbb{R}$ satisfying $f(xt+(1-t)y) \le tf(x)+(1-t)f(y)$, for all $x, y \in I$ and $t \in [0,1]$, is called convex function.

A convex function satisfies the well-known Hadamard inequality:

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(\xi)d\xi \le \frac{f(x)+f(y)}{2}. \tag{1}$$

If f is concave function, then, (1) holds in a reverse order. The inequality (1) had/has been studied by many researchers and consequently obtained a lot of its variants by introducing new classes of functions. For example, in [1], it is studied for s-convex functions; in [2], it is studied for $(p-h)$-convex functions; in [3, 4], it is studied for harmonically convex functions; in [5], it is studied for strongly harmonically convex and strongly p-convex functions. Our goal in this paper is to study the inequality (1) for strongly exponentially $(\alpha, h-m)$-p-convex functions.

Definition 1 (see [6]). A function $f : (0, b] \longrightarrow \mathbb{R}$ is called strongly exponentially $(\alpha$, h-$m)$-p-convex with modulus c ≥ 0, if f is nonnegative and

$$f\left(\left(tx^p+m(1-t)y^p\right)^{1/p}\right) \le h(t^\alpha)\frac{f(x)}{e^{\eta x}} + mh(1-t^\alpha)\frac{f(y)}{e^{\eta y}} - cmh(t^\alpha)h(1-t^\alpha)\frac{|y^p-x^p|^2}{e^{\eta(x^p+y^p)}}, \tag{2}$$

holds, while $J \subseteq \mathbb{R}$ is an interval containing $(0,1)$ and $h : J \longrightarrow \mathbb{R}$ is a nonnegative function along with $x, y, m^{-1}y$,

$(tx^p + m(1-t)y^p)^{1/p} \in (0, b]$, $t \in (0,1)$, $p \in \mathbb{R} \setminus \{0\}$, and $(\alpha, m) \in [0,1]^2$.

By using (2), one can find various classes of functions closely related with the convex function and strongly convex functions already defined by different authors. Strongly convex functions provide the refinements of convex functions.

In [2], Theorem 5, if we take $I \subset (0,\infty)$, $p \in \mathbb{R} \setminus \{0\}$, and $h(t) = t$, then, we have the following theorem.

Theorem 2. *Let $f : [a,b] \subset (0,\infty) \longrightarrow \mathbb{R}$ be a positive function such that $f \in L_1[a,b]$. If f is a p-convex function on $[a,b]$, $p \in \mathbb{R} \setminus \{0\}$. Then, the following integral inequality holds:*

$$f\left(\left(\frac{a^p + b^p}{2}\right)^{1/p}\right) \le \frac{p}{b^p - a^p}\int_a^b \frac{f(t)}{t^{1-p}}dt \le \frac{f(a)+f(b)}{2}. \quad (3)$$

Our aim in this paper is the derivation of compact forms of Hadamard-type inequalities for strongly exponentially $(\alpha, h-m)$-p-convex functions via Riemann-Liouville fractional integrals involving monotone functions. The established formulas will generate Hadamard-type inequalities for fractional Riemann-Liouville integrals which have been published by various authors in the recent past (see Remarks 11 & 23). Also, Hadamard-type inequalities are deducible for some new classes of functions (see Corollaries 12–32). In the following, we give the definition of Riemann-Liouville fractional integrals:

Definition 3. Let $f \in L_1[a,b]$. Then, Riemann-Liouville fractional integral operators of order μ for a function f, where $\Re(\mu) > 0$, are given by

$$\begin{aligned} I_{a^+}^{\mu} f(x) &= \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1} f(t)dt, \quad x > a, \\ I_{b^-}^{\mu} f(x) &= \frac{1}{\Gamma(\mu)}\int_x^b (t-x)^{\mu-1} f(t)dt, \quad x < b. \end{aligned} \quad (4)$$

Next, we give Hadamard-type inequalities via Riemann-Liouville fractional integrals of convex functions as follows:

Theorem 4 (see [7]). *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then, the following fractional integral inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}\left[I_{a^+}^{\mu} f(b) + I_{b^-}^{\mu} f(a)\right] \le \frac{f(a)+f(b)}{2}, \quad (5)$$

with $\mu > 0$.

Theorem 5 (see [8]). *With the assumptions given in Theorem 4, one can have the fractional integral inequality as follows:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{2^{\mu-1}\Gamma(\mu+1)}{(b-a)^{\mu}}\left[I_{((a+b)/2)^+}^{\mu} f(b) + I_{((a+b)/2)^-}^{\mu} f(a)\right] \\ &\le \frac{f(a)+f(b)}{2}, \end{aligned} \quad (6)$$

with $\mu > 0$.

Theorem 6 (see [7]). *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then, the following fractional integral inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}\left[I_{a^+}^{\mu} f(b) + I_{b^-}^{\mu} f(a)\right]\right| \\ &\quad \le \frac{b-a}{2(\mu+1)}\left(1 - \frac{1}{2^{\mu}}\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \end{aligned} \quad (7)$$

The definition of k-fractional Riemann-Liouville integral operators is given as follows:

Definition 7 (see [9]). Let $f \in L_1[a,b]$, $k > 0$. Then, k-fractional Riemann-Liouville integrals for a function f of order μ where $\Re(\mu) > 0$ are given by

$$\begin{aligned} {}_kI_{a^+}^{\mu} f(x) &= \frac{1}{k\Gamma_k(\mu)}\int_a^x (x-t)^{(\mu/k)-1} f(t)dt, \quad x > a, \\ {}_kI_{b^-}^{\mu} f(x) &= \frac{1}{k\Gamma_k(\mu)}\int_x^b (t-x)^{(\mu/k)-1} f(t)dt, \quad x < b, \end{aligned} \quad (8)$$

where $\Gamma_k(.)$ is defined as follows:

$$\Gamma_k(\mu) = \int_0^{\infty} t^{\mu-1} e^{-(t^k/k)} dt, \quad \Re(\mu) > 0. \quad (9)$$

The generalized form of Riemann-Liouville fractional integrals is given in the following definition:

Definition 8 (see [10]). Let $f \in L_1[a,b]$. Also, let ψ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative ψ' on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function ψ on $[a,b]$ of order μ where $\Re(\mu) > 0$ are given by

$$\begin{aligned} I_{a^+}^{\mu,\psi} f(x) &= \frac{1}{\Gamma(\mu)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\mu-1} f(t)dt, \quad x > a, \\ I_{b^-}^{\mu,\psi} f(x) &= \frac{1}{\Gamma(\mu)}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{\mu-1} f(t)dt, \quad x < b. \end{aligned} \quad (10)$$

The definition of the k-analogue of the abovementioned definition is given as follows:

Definition 9 (see [11]). Let f, ψ, μ be the same as in the abovementioned definition. Then, for $k > 0$, the k-analogue of (10) and is given by

$$
\begin{aligned}
{}_kI^{\mu,\psi}_{a^+}f(x) &= \frac{1}{k\Gamma_k(\mu)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{(\mu/k)-1}f(t)dt, \quad x > a,\\
{}_kI^{\mu,\psi}_{b^-}f(x) &= \frac{1}{k\Gamma_k(\mu)}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{(\mu/k)-1}f(t)dt, \quad x < b.
\end{aligned} \tag{11}
$$

Using the fact that $\Gamma_k(\mu) = k^{(\mu/k)-1}\Gamma(\mu/k)$ in (10) after replacing μ by μ/k, we get

$$
\begin{aligned}
k^{-\mu/k}I^{\mu,\psi}_{a^+}f(x) &= {}_kI^{\mu,\psi}_{a^+}f(x),\\
k^{-\mu/k}I^{\mu,\psi}_{b^-}f(x) &= {}_kI^{\mu,\psi}_{b^-}f(x).
\end{aligned} \tag{12}
$$

For further detailed study on fractional integrals, we refer the readers to [12, 13]. In the next section, we formulate the Hadamard-type inequalities for strongly exponentially $(\alpha, h-m)$-p-convex function via integrals (10) which are compact forms of a plenty of well-known Hadamard-type inequalities holding for classes of convex, strongly convex, and exponentially convex functions. Specifically, one can have refinements of the inequalities proved in recent decades. Several special case inequalities in the form of corollaries are also given.

2. Main Results

We will use the following notations for terms which will appear frequently in the results of this section

$$
\begin{aligned}
c_{\mu,m}(\psi^p(a),\psi^p(b)) &= cm\Big[\mu(\mu+1)(\psi^p(b)-\psi^p(a))^2 + 2\\
&\quad\cdot\left(\frac{\psi^p(a)}{m} - m\psi^p(b)\right)^2 + 2\mu(\psi^p(b)-\psi^p(a))\\
&\quad\cdot\left(\frac{\psi^p(a)}{m} - m\psi^p(b)\right)\Big],
\end{aligned}
$$

$$
\begin{aligned}
R^{h,H}_{\alpha,\mu,m,\eta}(\psi^p(a),\psi^p(b)) &= cm\mu\Bigg[g_2(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{(\psi^p(b)-\psi^p(a))^2}{e^{\eta(\psi^p(a)+\psi^p(b))}}\\
&\quad + mg_3(\eta)H\left(\frac{1}{2}\right)\frac{\left(\psi^p(b)-\left(\psi^p(a)/m^2\right)\right)^2}{e^{\eta((\psi^p(a)/m^2)+\psi^p(b))}}\Bigg]\\
&\quad \int_0^1 h(t^\alpha)H(t)t^{\mu-1}dt,
\end{aligned}
$$

$$
\begin{aligned}
F_{\mu,m}(\psi^p(a),\psi^p(b)) &= cm\Big[\mu(\mu+1)(\psi^p(b)-\psi^p(a))^2 + \left(\mu^2+5\mu+8\right)\\
&\quad\cdot\left(\frac{\psi^p(a)}{m} - m\psi^p(b)\right)^2 + 2\mu(\mu+3)\times(\psi^p(b)-\psi^p(a))\\
&\quad\cdot\left(\frac{\psi^p(a)}{m} - m\psi^p(b)\right)\Big],
\end{aligned}
$$

$$
\begin{aligned}
A^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b))) &= \mu\Bigg[g_2(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(\psi(a))}{e^{\eta\psi(a)}}\\
&\quad + mg_3(\eta)H\left(\frac{1}{2}\right)\frac{f(\psi(b))}{e^{\eta\psi(b)}}\Bigg]\int_0^1 h(t^\alpha)t^{\mu-1}dt,
\end{aligned}
$$

$$
\begin{aligned}
B^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b))) &= m\mu\Bigg[g_2(\eta)h\left(\frac{1}{2^\alpha}\right)\frac{f(\psi(b))}{e^{\eta\psi(b)}}\\
&\quad + mg_3(\eta)H\left(\frac{1}{2}\right)\frac{f(\psi^p(a)/m^2)}{e^{(\eta\psi(a)/m^2)}}\Bigg]\int_0^1 H(t)t^{\mu-1}dt.
\end{aligned} \tag{13}
$$

Theorem 10. *Let $f, \psi : [a^p, mb^p] \subset (0,\infty) \longrightarrow \mathbb{R}$, range $(\psi) \subset [a^p, mb^p]$ be the positive functions such that $f \in L_1[a^p, mb^p]$, and ψ be a differentiable and strictly increasing. If f is strongly exponentially $(\alpha, h-m)$-p-convex function on $[a^p, mb^p]$ such that $p, \eta \in \mathbb{R}$ and $p \neq 0$, then, for $(\alpha, m) \in (0,1]^2$, the following fractional integral inequalities hold:*

(i) If $p > 0$,

$$
\begin{aligned}
&f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right) + \frac{c_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)h(1/2^\alpha)H(1/2)}{(\mu+1)(\mu+2)}\\
&\le \frac{\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\Bigg[g_2(\eta)h\left(\frac{1}{2^\alpha}\right)\times I^{\mu,\psi}_{\psi^{-1}(\psi^p(a))^+}(f\circ\phi)\left(\psi^{-1}(m\psi^p(b))\right)\\
&+ m^{\mu+1}g_3(\eta)H\left(\frac{1}{2}\right)I^{\mu,\psi}_{\psi^{-1}(\psi^p(b))^-}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\Bigg]\\
&\le A^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b))) + B^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))\\
&- R^{h,H}_{\alpha,\mu,m,\eta}(\psi^p(a),\psi^p(b)),
\end{aligned} \tag{14}
$$

with $\mu > 0$, $H(t) = h(1-t^\alpha)$, $\phi(t) = \psi^{1/p}(t)$ for all $t \in [a^p, mb^p]$ and

$$
\begin{aligned}
g_1(\eta) &= \begin{cases} e^{-\eta(\psi^p(a)+\psi^p(b))}, & \text{if } \eta > 0,\\ e^{-\eta(m\psi^p(b)+(\psi^p(a)/m))}, & \text{if } \eta < 0,\end{cases}\\
g_2(\eta) &= \begin{cases} e^{-\eta(m\psi^p(b))^{1/p}}, & \text{if } \eta < 0,\\ e^{-\eta\psi(a)}, & \text{if } \eta > 0,\end{cases}\\
g_3(\eta) &= \begin{cases} e^{-\eta\psi(b)}, & \text{if } \eta < 0,\\ e^{-\eta(\psi^p(a)/m)^{1/p}}, & \text{if } \eta > 0\end{cases}
\end{aligned} \tag{15}
$$

(ii) For $p < 0$, one can have

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)h(1/2^\alpha)H(1/2)}{(\mu+1)(\mu+2)}$$
$$\leq\frac{\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)h\left(\frac{1}{2^\alpha}\right)\times I^{\mu,\psi}_{\psi^{-1}(\psi^p(a))^-}(f\circ\phi)\left(\psi^{-1}(m\psi^p(b))\right)\right.$$
$$\left.+m^{\mu+1}g_3(\eta)H\left(\frac{1}{2}\right)I^{\mu,\psi}_{\psi^{-1}(\psi^p(b))^+}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]$$
$$\leq A^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))$$
$$-R^{h,H}_{\alpha,\mu,m,\eta}(\psi^p(a),\psi^p(b)),\tag{16}$$

with $\mu>0$, $H(t)=h(1-t^\alpha)$, $\phi(t)=\psi^{1/p}(t)$ *for all* $t\in[mb^p, a^p]$ *and*

$$g_1(\eta)=\begin{cases}e^{-\eta(\psi^p(a)+\psi^p(b))}, & \text{if } \eta<0,\\ e^{-\eta(m\psi^p(b)+(\psi^p(a)/m))}, & \text{if } \eta>0,\end{cases}$$
$$g_2(\eta)=\begin{cases}e^{-\eta(m\psi^p(b)1/p}, & \text{if } \eta>0,\\ e^{-\eta\psi(a)}, & \text{if } \eta<0,\end{cases}\tag{17}$$
$$g_3(\eta)=\begin{cases}e^{-\eta\psi(b)}, & \text{if } \eta>0,\\ e^{-\eta(\psi^p(a)/m)^{1/p}}, & \text{if } \eta<0\end{cases}$$

Proof. (i) The following inequality holds for a strongly exponentially $(\alpha,h-m)$-p-convex function

$$f\left(\left(\frac{\psi^p(x)+m\psi^p(y)}{2}\right)^{1/p}\right)\leq h\left(\frac{1}{2^\alpha}\right)\frac{f(\psi(x))}{e^{\eta\psi(x)}}$$
$$+mH\left(\frac{1}{2}\right)\frac{f(\psi(y))}{e^{\eta\psi(y)}}-\frac{cmh(1/2^\alpha)H(1/2)(\psi^p(y)-\psi^p(x))^2}{e^{\eta(\psi^p(x)+\psi^p(y))}}.\tag{18}$$

By setting $\psi(x)=(\psi^p(a)t+m(1-t)\psi^p(b))^{1/p}$, $\psi(y)=((\psi^p(a)/m)(1-t)+\psi^p(b)t)^{1/p}$ in (18) and then integrating on $[0,1]$ after multiplying with $t^{\mu-1}$, one can get

$$\frac{1}{\mu}f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)\leq h\left(\frac{1}{2^\alpha}\right)$$
$$\int_0^1\frac{f((\psi^p(a)t+m(1-t)\psi^p(b)))^{1/p})}{e^{\eta(\psi^p(a)t+m(1-t)\psi^p(b))^{1/p}}}t^{\mu-1}dt+mH\left(\frac{1}{2}\right)$$
$$\times\int_0^1\frac{f\left(((\psi^p(a)/m)(1-t)+\psi^p(b))t)^{1/p}\right)}{e^{\eta((\psi^p(a)/m)(1-t)+\psi^p(b))t)^{1/p}}}t^{\mu-1}dt-cmh\left(\frac{1}{2^\alpha}\right)H\left(\frac{1}{2}\right)$$
$$\int_0^1\frac{((1-t)((\psi^p(a)/m)-m\psi^p(b))+t(\psi^p(b)-\psi^p(a)))^2}{e^{\eta((1-t)((\psi^p(a)/m)+m\psi^p(b))+t(\psi^p(b)+\psi^p(a)))^{1/p}}}t^{\mu-1}dt.\tag{19}$$

Setting $\psi(u)=\psi^p(a)t+m(1-t)\psi^p(b)$ and $\psi(v)=(\psi^p(a)/m)(1-t)+\psi^p(b)t$ in (19) and multiplying by μ, after applying Definition 3, the following inequality can be obtained:

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)\leq\frac{\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}$$
$$\cdot\left[g_2(\eta)h\left(\frac{1}{2^\alpha}\right)I^\mu_{(\psi^p(a))^+}(f\circ\phi)(m\psi^p(b))+m^{\mu+1}g_3(\eta)H\right.$$
$$\left.\cdot\left(\frac{1}{2}\right)\times I^\mu_{(\psi^p(b))^-}(f\circ\phi)\left(\frac{\psi^p(a)}{m}\right)\right]$$
$$-\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)h(1/2^\alpha)H(1/2)}{(\mu+1)(\mu+2)}.\tag{20}$$

Now, by using definition of strongly exponentially $(\alpha,h-m)$-p-convex function for f and then integrating the resulting inequality on $[0,1]$ after multiplying with $t^{\mu-1}$, one can get

$$g_2(\eta)h\left(\frac{1}{2^\alpha}\right)\int_0^1 f\left((\psi^p(a)t+m(1-t)\psi^p(b))^{1/p}\right)t^{\mu-1}dt+mg_3$$
$$\cdot(\eta)H\left(\frac{1}{2}\right)\int_0^1 f\left((\psi^p(a)t+m(1-t)\psi^p(b))^{1/p}\right)t^{\mu-1}dt$$
$$\leq\frac{A^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))}{\mu}+\frac{B^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))}{\mu}$$
$$-\frac{R^{h,H}_{\alpha,\mu,m,\eta}(\psi^p(a),\psi^p(b))}{\mu}.\tag{21}$$

Again, using substitution as considered in (20) leads to the second inequality of (14)

(ii) The proof is followed on same lines as the proof of (i) □

Remark 11. The aforementioned version of the Hadamard inequalities gives (i) [4], Theorem 4 for $p=-1$, $m=\alpha=1$, $h=\psi=I$, and $c=\eta=0$; (ii) [3], Theorem 2.4 for $p=-1$, $m=\alpha=\mu=1$, $h=\psi=I$, and $c=\eta=0$; (iii) [14], Theorem 3.10 for $h=\psi=I$ and $c=\eta=0$; (iv) [15], Corollary 2.2 for $\alpha=p=1$, $\psi=I$, and $c=\eta=0$; (v) Theorem 2 for $\alpha=m=p=1$, $h=\psi=I$, and $c=\eta=0$; (vi) [16], Theorem 2.1 for $\alpha=p=1$, $h=\psi=I$, and $c=\eta=0$; (vii) [14], Theorem 2.2 for $\psi=I$ and $c=\eta=0$; (viii) Theorem 1 for $\alpha=\mu=m=1$, $h=\psi=I$, and $c=\eta=0$; (ix) [17], Theorem 2.1 for $\alpha=\mu=m=1$, $p=-1$, $h(t)=t^s$, $\psi=I$, and $c=\eta=0$; (x) [1], Theorem 2.1 for $\alpha=\mu=m=p=1$, $h(t)=t^s$, $\psi=I$, and $c=\eta=0$; and (xi) [6], Theorem 3 for $\psi=I$. Moreover, the refinements of all the deduced results will occur for c > 0.

Corollary 12. *(i) For $p>0$, one can have for the strongly $(\alpha,h-m)$-p-convex function the following fractional integral inequality:*

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))h(1/2^\alpha)H(1/2)}{(\mu+1)(\mu+2)}$$
$$\leq\frac{\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[h\left(\frac{1}{2^\alpha}\right)\times I^{\mu,\psi}_{\psi^{-1}(\psi^p(a))^+}(f\circ\phi)(\psi^{-1}(m\psi^p(b)))+m^{\mu+1}H\left(\frac{1}{2}\right)I^{\mu,\psi}_{\psi^{-1}(\psi^p(b))^-}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]$$
$$\leq A^{h,H}_{\alpha,\mu,m,0}(f(\psi(a)),f(\psi(b)))+B^{h,H}_{\alpha,\mu,m,0}(f(\psi(a)),f(\psi(b)))-R^{h,H}_{\alpha,\mu,m,0}(\psi^p(a),\psi^p(b))\tag{22}$$

Proof. The abovementioned inequality can be deduced by setting $\eta=0$ in (14).

(ii) For $p<0$, one can have for the strongly $(\alpha,h-m)$-p-convex function the following fractional integral inequality:

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))h(1/2^\alpha)H(1/2)}{(\mu+1)(\mu+2)}$$
$$\leq\frac{\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[h\left(\frac{1}{2^\alpha}\right)\times I^{\mu,\psi}_{\psi^{-1}(\psi^p(a))^-}(f\circ\phi)(\psi^{-1}(m\psi^p(b)))+m^{\mu+1}H\left(\frac{1}{2}\right)I^{\mu,\psi}_{\psi^{-1}(\psi^p(b))^+}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]$$
$$\leq A^{h,H}_{\alpha,\mu,m,0}(f(\psi(a)),f(\psi(b)))+B^{h,H}_{\alpha,\mu,m,0}(f(\psi(a)),f(\psi(b)))-R^{h,H}_{\alpha,\mu,m,0}(\psi^p(a),\psi^p(b))\tag{23}$$

□

Proof. The abovementioned inequality can be deduced by setting $\eta=0$ in (16). □

Corollary 13. (*i*) *For $p>0$, one can have for the strongly exponentially $(h-m)$-p-convex function the following fractional integral inequality:*

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)h^2(1/2)}{(\mu+1)(\mu+2)}$$
$$\leq\frac{h(1/2)\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)\times I^{\mu,\psi}_{\psi^{-1}(\psi^p(a))^+}(f\circ\phi)(\psi^{-1}(m\psi^p(b)))+m^{\mu+1}g_3(\eta)H\left(\frac{1}{2}\right)I^{\mu,\psi}_{\psi^{-1}(\psi^p(b))^-}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]$$
$$\leq A^{h,H}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{h,H}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))-R^{h,H}_{1,\mu,m,\eta}(\psi^p(a),\psi^p(b))\tag{24}$$

Proof. The abovementioned inequality can be deduced by setting $\alpha=1$ in (14).

(ii) For $p<0$, one can have for the strongly exponentially $(h-m)$-p-convex function the following fractional integral inequality:

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)h^2(1/2)}{(\mu+1)(\mu+2)}$$
$$\leq\frac{h(1/2)\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)\times I^{\mu,\psi}_{\psi^{-1}(\psi^p(a))^-}(f\circ\phi)(\psi^{-1}(m\psi^p(b)))+m^{\mu+1}g_3(\eta)H\left(\frac{1}{2}\right)I^{\mu,\psi}_{\psi^{-1}(\psi^p(b))^+}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]$$
$$\leq A^{h,H}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{h,H}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))-R^{h,H}_{1,\mu,m,\eta}(\psi^p(a),\psi^p(b))\tag{25}$$

□

Proof. The abovementioned inequality can bed deduced by setting $\alpha=1$ in (16). □

Corollary 14. (*i*) *For $p>0$, one can have for the strongly exponentially (s,m)-p-Godunova-Levin function the following fractional integral inequality:*

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)2^{2s}}{(\mu+1)(\mu+2)}$$
$$\leq\frac{\Gamma(\mu+1)}{2^s(m\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)I^{\mu,\psi}_{(\psi^p(a))^+}(f\circ\phi)(\psi^{-1}(m\psi^p(b)))+m^{\mu+1}g_3(\eta)I^{\mu,\psi}_{(\psi^p(b))^-}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]$$
$$\leq A^{t^{-s},(1-t)^{-s}}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^{-s},(1-t)^{-s}}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))-R^{t^{-s},(1-t)^{-s}}_{\alpha,\mu,m,\eta}(\psi^p(a),\psi^p(b))\tag{26}$$

Proof. The abovementioned inequality can be deduced by setting $\alpha=1$ and $h(t)=t^{-s}$ in (2.1).

(ii) For $p<0$, one can have for the strongly exponentially (s,m)-p-Godunova-Levin function the following fractional integral inequality:

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)2^{2s}}{(\mu+1)(\mu+2)}$$
$$\leq\frac{\Gamma(\mu+1)}{2^s(m\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)I^{\mu,\psi}_{(\psi^p(a))^-}(f\circ\phi)(\psi^{-1}(m\psi^p(b)))+m^{\mu+1}g_3(\eta)I^{\mu,\psi}_{(\psi^p(b))^+}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]$$
$$\leq A^{t^{-s},(1-t)^{-s}}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^{-s},(1-t)^{-s}}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))-R^{t^{-s},(1-t)^{-s}}_{\alpha,\mu,m,\eta}(\psi^p(a),\psi^p(b))\tag{27}$$

□

Proof. The abovementioned inequality can be deduced by setting $\alpha=1$ and $h(t)=t^{-s}$ in (16). □

Corollary 15. (*i*) *For $p>0$, one can have for the strongly exponentially (s,m)-p-convex function in the third sense the following fractional integral inequality:*

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)}{2^{2s}(\mu+1)(\mu+2)}$$
$$\leq\frac{\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^{\mu}}\left[g_2(\eta)I^{\mu,\psi}_{(\psi^p(a))^+}(f\circ\phi)\left(\psi^{-1}(m\psi^p(b))\right)\right.$$
$$\left.+m^{\mu+1}g_3(\eta)I^{\mu}_{(\psi^p(b))^-}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]$$
$$\leq A^{t^s,(1-t)^s}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^s,(1-t)^s}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))$$
$$-R^{t^s,(1-t)^s}_{\alpha,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \tag{28}$$

Proof. The abovementioned inequality can be deduced by setting $\alpha=1$ and $h(t)=t^s$ in (14).

(ii) For $p<0$, one can have for the strongly exponentially (s,m)-p-convex function in third sense the following fractional integral inequality:

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)}{2^{2s}(\mu+1)(\mu+2)}$$
$$\leq\frac{\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^{\mu}}\left[g_2(\eta)I^{\mu,\psi}_{(\psi^p(a))^-}(f\circ\phi)\left(\psi^{-1}(m\psi^p(b))\right)\right.$$
$$\left.+m^{\mu+1}g_3(\eta)I^{\mu}_{(\psi^p(b))^+}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]$$
$$\leq A^{t^s,(1-t)^s}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^s,(1-t)^s}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))$$
$$-R^{t^s,(1-t)^s}_{\alpha,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \tag{29}$$

□

Proof. The abovementioned inequality can be deduced by setting $\alpha=1$ and $h(t)=t^s$ in (16). □

Corollary 16. (*i*) *For $p>0$, one can have for the strongly exponentially (α,m)-p-convex function the following fractional integral inequality:*

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)(2^{\alpha}-1)}{2^{2\alpha}(\mu+1)(\mu+2)}$$
$$\leq\frac{\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^{\mu}}\left[g_2(\eta)I^{\mu,\psi}_{\psi^{-1}(\psi^p(a))^+}(f\circ\phi)\left(\psi^{-1}(m\psi^p(b))\right)\right.$$
$$\left.+m^{\mu+1}g_3(\eta)(2^{\alpha}-1)I^{\mu,\psi}_{\psi^{-1}(\psi^p(b))^-}(f\circ\phi)\left(\frac{\psi^p(a)}{m}\right)\right]$$
$$\leq A^{t^{\alpha},(1-t^{\alpha})}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^{\alpha},(1-t)^{\alpha}}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))$$
$$-R^{t^{\alpha},(1-t^{\alpha})}_{\alpha,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \tag{30}$$

Proof. The abovementioned inequality can be deduced by setting $h(t)=t$ in (14).

(ii) For $p<0$, one can have for the strongly exponentially (α,m)-p-convex function the following fractional integral inequality:

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)(2^{\alpha}-1)}{2^{2\alpha}(\mu+1)(\mu+2)}$$
$$\leq\frac{\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^{\mu}}\left[g_2(\eta)I^{\mu,\psi}_{\psi^{-1}(\psi^p(a))^-}(f\circ\phi)\left(\psi^{-1}(m\psi^p(b))\right)\right.$$
$$\left.+m^{\mu+1}g_3(\eta)(2^{\alpha}-1)I^{\mu,\psi}_{\psi^{-1}(\psi^p(b))^+}(f\circ\phi)\left(\frac{\psi^p(a)}{m}\right)\right]$$
$$\leq A^{t^{\alpha},(1-t^{\alpha})}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^{\alpha},(1-t)^{\alpha}}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))$$
$$-R^{t^{\alpha},(1-t^{\alpha})}_{\alpha,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \tag{31}$$

□

Proof. The abovementioned inequality can be deduced by setting $h(t)=t$ in (16). □

Corollary 17. (*i*) *For $p>0$, one can have for the strongly exponentially p-convex function the following fractional integral inequality:*

$$f\left(\left(\frac{\psi^p(a)+\psi^p(b)}{2}\right)^{1/p}\right)+\frac{cg_1(\eta)\left(\mu^2-\mu+2\right)(\psi^p(b)-\psi^p(a))^2}{4(\mu+1)(\mu+2)}$$
$$\leq\frac{\Gamma(\mu+1)}{(\psi^p(b)-\psi^p(a))^{\mu}}\left[g_2(\eta)I^{\mu,\psi}_{\psi^{-1}(\psi^p(a))^+}(f\circ\phi)\left(\psi^{-1}(\psi^p(b))\right)\right.$$
$$\left.+g_3(\eta)I^{\mu,\psi}_{\psi^{-1}(\psi^p(b))^-}(f\circ\phi)\left(\psi^{-1}(\psi^p(a))\right)\right]$$
$$\leq A^{t,(1-t)}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t,(1-t)}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))$$
$$-R^{t,(1-t)}_{1,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \tag{32}$$

Proof. The abovementioned inequality can be deduced by setting $\alpha=m=1$ and $h(t)=t$ in (14).

(ii) For $p<0$, one can have for strongly exponentially p-convex function the following fractional integral inequality

$$f\left(\left(\frac{\psi^p(a)+\psi^p(b)}{2}\right)^{1/p}\right)+\frac{cg_1(\eta)\left(\mu^2-\mu+2\right)(\psi^p(b)-\psi^p(a))^2}{4(\mu+1)(\mu+2)}$$
$$\leq\frac{\Gamma(\mu+1)}{(\psi^p(b)-\psi^p(a))^{\mu}}\left[g_2(\eta)I^{\mu,\psi}_{\psi^{-1}(\psi^p(a))^-}(f\circ\phi)\left(\psi^{-1}(\psi^p(b))\right)\right.$$
$$\left.+g_3(\eta)I^{\mu,\psi}_{\psi^{-1}(\psi^p(b))^+}(f\circ\phi)\left(\psi^{-1}(\psi^p(a))\right)\right]$$
$$\leq A^{t,(1-t)}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t,(1-t)}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))$$
$$-R^{t,(1-t)}_{1,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \tag{33}$$

□

Proof. The abovementioned inequality can be deduced by setting $\alpha=m=1$ and $h(t)=t$ in (16). □

Corollary 18. *For the strongly exponentially $(\alpha, h-m)$-HA-convex function, the following inequality holds:*

$$
\begin{aligned}
&f\left(\frac{2\psi(a)\psi(b)}{\psi(b)+m\psi(a)}\right)+\frac{cmh(1/2^{\alpha})H(1/2)}{(\mu+1)(\mu+2)}\left[\mu(\mu+1)\left(\frac{1}{\psi(b)}-\frac{1}{\psi(a)}\right)^{2}\right.\\
&\left.+2\left(\frac{1}{m\psi(a)}-\frac{m}{\psi(b)}\right)^{2}+2\mu\left(\frac{1}{\psi(b)}-\frac{1}{\psi(a)}\right)\times\left(\frac{1}{m\psi(a)}-\frac{m}{\psi(b)}\right)\right]\\
&\leq\frac{\Gamma(\mu+1)(\psi(a)\psi(b))^{\mu}}{(\psi(b)-m\psi(a))^{\mu}}\left[h\left(\frac{1}{2^{\alpha}}\right)I^{\mu,\psi}_{\psi^{-1}(1/\psi(a))^{-}}(f\circ\phi)\left(\psi^{-1}\left(\frac{m}{\psi(b)}\right)\right)\right.\\
&\left.+m^{\mu+1}H\left(\frac{1}{2}\right)\times I^{\mu,\psi}_{\psi^{-1}(1/\psi(b))^{+}}(f\circ\phi)\left(\psi^{-1}\left(\frac{1}{m\psi(a)}\right)\right)\right]\\
&\leq A^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))\\
&-R^{h,H}_{\alpha,\mu,m,\eta}\left(\frac{1}{\psi(a)},\frac{1}{\psi(b)}\right).
\end{aligned}\tag{34}
$$

Proof. The abovementioned inequality can be deduced by setting $p=-1$ in (16). □

Corollary 19. *For the strongly exponentially (α, m)-HA-convex function, the following inequality holds:*

$$
\begin{aligned}
&f\left(\frac{2\psi(a)\psi(b)}{\psi(b)+\psi(a)m}\right)+\frac{cmg_1(\eta)(2^{\alpha}-1)}{2^{2\alpha}(\mu+1)(\mu+2)}\left[\mu(\mu+1)\left(\frac{\psi(b)-\psi(a)}{\psi(a)\psi(b)}\right)^{2}\right.\\
&\left.+2\left(\frac{\psi(b)-\psi(a)m^{2}}{\psi(a)\psi(b)m}\right)^{2}+\frac{2\mu(\psi(a)-\psi(b))(\psi(b)-\psi(a)m^{2})}{m(\psi(a)\psi(b))^{2}}\right]\\
&\leq\frac{\Gamma(\mu+1)(\psi(a)\psi(b))^{\mu}}{2^{\alpha}(\psi(b)-m\psi(a))^{\mu}}\left[g_2(\eta)I^{\mu,\psi}_{\psi^{-1}(1/\psi(a))^{-}}(f\circ\phi)\left(\psi^{-1}\left(\frac{m}{\psi(b)}\right)\right)\right.\\
&\left.+m^{\mu+1}g_3(\eta)(2^{\alpha}-1)I^{\mu,\psi}_{\psi^{-1}(1/b)^{+}}(f\circ\phi)\left(\psi^{-1}\left(\frac{1}{m\psi(a)}\right)\right)\right]\\
&\leq A^{t^{\alpha},(1-t^{\alpha})}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^{\alpha},(1-t^{\alpha})}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))\\
&-R^{t^{\alpha},(1-t^{\alpha})}_{\alpha,\mu,m,\eta}(\psi^{p}(a),\psi^{p}(b)).
\end{aligned}\tag{35}
$$

Proof. The abovementioned inequality can be deduced by setting $p=-1$ and $h(t)=t$ in (16). □

Corollary 20. *For the strongly exponentially (s, m)-HA-convex function, the following inequality holds:*

$$
\begin{aligned}
&f\left(\frac{2\psi(a)\psi(b)}{\psi(a)m+\psi(b)}\right)+\frac{cmg_1(\eta)}{2^{2s}(\mu+1)(\mu+2)}\left[\mu(\mu+1)\left(\frac{\psi(b)-\psi(a)}{\psi(a)\psi(b)}\right)^{2}\right.\\
&\left.+2\left(\frac{\psi(b)-\psi(a)m^{2}}{\psi(a)\psi(b)m}\right)^{2}+\frac{2\mu(\psi(a)-\psi(b))(\psi(b)\psi(a)m^{2})}{m(\psi(a)\psi(b))^{2}}\right]\\
&\leq\frac{\Gamma(\mu+1)(\psi(a)\psi(b))^{\mu}}{(\psi(b)\psi(a)m)^{\mu}}\left[g_2(\eta)I^{\mu,\psi}_{(1/\psi(a))^{-}}(f\circ\phi)\left(\psi^{-1}\left(\frac{m}{b}\right)\right)\right.\\
&\left.+m^{\mu+1}g_3(\eta)I^{\mu,\psi}_{\psi^{-1}(1/\psi(b))^{+}}(f\circ\phi)\left(\psi^{-1}\left(\frac{1}{m\psi(a)}\right)\right)\right]\\
&\leq A^{t^{s},(1-t)^{s}}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^{s},(1-t)^{s}}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))\\
&-R^{t^{s},(1-t)^{s}}_{1,\mu,m,\eta}(\psi^{p}(a),\psi^{p}(b)).
\end{aligned}\tag{36}
$$

Proof. The abovementioned inequality can be deduced by setting $p=-1$, $\alpha=1$, and $h(t)=t^{s}$ in (16). □

Corollary 21. *For the Godunova-Levin type of strongly exponentially (s, m)-HA-convex function, the following inequality holds:*

$$
\begin{aligned}
&f\left(\frac{2\psi(a)\psi(b)}{\psi(b)+m\psi(a)}\right)+\frac{2^{2s}g_1(\eta)cm}{(\mu+1)(\mu+2)}\left[\mu(\mu+1)\left(\frac{1}{\psi(b)}-\frac{1}{\psi(a)}\right)^{2}\right.\\
&\left.+2\left(\frac{1}{m\psi(a)}-\frac{m}{\psi(b)}\right)^{2}+2\mu\left(\frac{1}{\psi(b)}-\frac{1}{\psi(a)}\right)\times\left(\frac{1}{m\psi(a)}-\frac{m}{\psi(b)}\right)\right]\\
&\leq\frac{2^{s}\Gamma(\mu+1)(\psi(a)\psi(b))^{\mu}}{(\psi(b)-m\psi(a))^{\mu}}\left[g_2(\eta)I^{\mu,\psi}_{\psi^{-1}(1/\psi(a))^{-}}(f\circ\phi)\left(\psi^{-1}\left(\frac{m}{\psi(b)}\right)\right)\right.\\
&\left.+m^{\mu+1}g_3(\eta)\times I^{\mu,\psi}_{\psi^{-1}(1/\psi(b))^{+}}(f\circ\phi)\left(\psi^{-1}\left(\frac{1}{m\psi(a)}\right)\right)\right]\\
&\leq A^{t^{-s},(1-t)^{-s}}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^{-s},(1-t)^{-s}}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))\\
&-R^{t^{-s},(1-t)^{-s}}_{1,\mu,m,\eta}(\psi^{p}(a),\psi^{p}(b)).
\end{aligned}\tag{37}
$$

Proof. The abovementioned inequality can be deduced by setting $p=-1$, $\alpha=1$, and $h(t)=t^{-s}$ in (2.2). □

The second variant of Hadamard inequality for strongly exponentially $(\alpha, h-m)$-p-convex function is proved as follows.

Theorem 22. *Under the assumption of Theorem 10, the following fractional integral inequalities hold:*

(i) For $p>0$, one can have

$$
\begin{aligned}
&f\left(\left(\frac{\psi^{p}(a)+m\psi^{p}(b)}{2}\right)^{1/p}\right)+\frac{F_{\mu,m}(\psi^{p}(a),\psi^{p}(b))g_1(\eta)h(1/2^{\alpha})H(1/2)}{4(\mu+1)(\mu+2)}\\
&\leq\frac{2^{\mu}\Gamma(\mu+1)}{(m\psi^{p}(b)-\psi^{p}(a))^{\mu}}\left[g_2(\eta)h\left(\frac{1}{2^{\alpha}}\right)\times I_{\psi^{-1}((\psi^{p}(a)+m\psi^{p}(b))/2)}\right.\\
&+^{\mu,\psi}(f\circ\phi)(\psi^{-1}(m\psi^{p}(b)))+m^{\mu+1}H\left(\frac{1}{2}\right)g_3(\eta)I_{\psi^{-1}((\psi^{p}(a)}\\
&\left.+m\psi^{p}(b))/2m)^{-}\mu,\psi(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^{p}(a)}{m}\right)\right)\right]\\
&\leq A^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))\\
&-R^{h,H}_{\alpha,\mu,m,\eta}(\psi^{p}(a),\psi^{p}(b)),
\end{aligned}\tag{38}
$$

with $\mu>0$, $H(t)=h(1-t^{\alpha})$, $\phi(z)=z^{1/p}$ for all $z\in[a^{p}, mb^{p}]$ and

$$g_1(\eta)=\begin{cases} e^{-\eta(\psi^p(a)+\psi^p(b))}, & \text{if } \eta>0,\\ e^{-\eta(m\psi^p(b)+(\psi^p(a)/m))}, & \text{if } \eta<0,\end{cases}$$
$$g_2(\eta)=\begin{cases} e^{-\eta(m\psi^p(b))^{1/p}}, & \text{if } \eta<0,\\ e^{-\eta\psi(a)}, & \text{if } \eta>0,\end{cases} \tag{39}$$
$$g_3(\eta)=\begin{cases} e^{-\eta\psi(b)}, & \text{if } \eta<0,\\ e^{-\eta(\psi^p(a)/m)^{1/p}}, & \text{if } \eta>0\end{cases}$$

(ii) For $p<0$, one can have

$$\begin{aligned} &f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{F_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)h(1/2^\alpha)H(1/2)}{4(\mu+1)(\mu+2)}\\ &\le \frac{2^\mu\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)h\left(\frac{1}{2^\alpha}\right)\right.\\ &\times I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2)^-}(f\circ\phi)(\psi^{-1}(m\psi^p(b)))\\ &\left.+m^{\mu+1}H\left(\frac{1}{2}\right)g_3(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2m)^+}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]\\ &\le A^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))\\ &-R^{h,H}_{\alpha,\mu,m,\eta}(\psi^p(a),\psi^p(b)), \end{aligned} \tag{40}$$

with $\mu>0$, $\phi(z)=z^{1/p}$ for all $z\in[mb^p,a^p]$ and

$$g_1(\eta)=\begin{cases} e^{-\eta(\psi^p(a)+\psi^p(b))}, & \text{if } \eta<0,\\ e^{-\eta(m\psi^p(b)+(\psi^p(a)/m))}, & \text{if } \eta>0,\end{cases}$$
$$g_2(\eta)=\begin{cases} e^{-\eta(m\psi^p(b))^{1/p}}, & \text{if } \eta>0,\\ e^{-\eta\psi(a)}, & \text{if } \eta<0,\end{cases} \tag{41}$$
$$g_3(\eta)=\begin{cases} e^{-\eta\psi(b)}, & \text{if } \eta>0,\\ e^{-\eta(\psi^p(a)/m)^{1/p}}, & \text{if } \eta<0\end{cases}$$

Proof. (i) Let $\psi(x)=(((\psi^p(a)t)/2)+m((2-t)/2)\psi^p(b))^{1/p}$, $\psi(y)=((\psi^p(a)/m)((2-t)/2)+((\psi^p(b)t)/2))^{1/p}$ in (18) and integrating the resulting inequality over the interval $[0,1]$ after multiplying with $t^{\mu-1}$, we get

$$\begin{aligned} &\frac{1}{\mu}f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)\\ &\le h\left(\frac{1}{2^\alpha}\right)\int_0^1\frac{f\left((((\psi^p(b)t)/2)+m((2-t)/2)\psi^p(b))^{1/p}\right)}{e^{\eta(((\psi^p(b)t)/2)+m((2-t)/2)\psi^p(b))^{1/p}}}t^{\mu-1}dt\\ &+mH\left(\frac{1}{2}\right)\int_0^1\frac{f\left(((\psi^p(a)/m)((2-t)/2)+((\psi^p(b)t)/2))^{1/p}\right)}{e^{\eta((\psi^p(a)/m)((2-t)/2)+((\psi^p(b)t)/2))^{1/p}}}t^{\mu-1}dt\\ &-cmh\left(\frac{1}{2^\alpha}\right)H\left(\frac{1}{2}\right)\\ &\int_0^1\frac{((t/2)(\psi^p(b)-\psi^p(a))+((2-t)/2)((\psi^p(a)/m)-m\psi^p(b)))^2}{e^{\eta(((2-t)/2)((\psi^p(a)/m)+m\psi^p(b))+(t/2)(\psi^p(b)+\psi^p(a)))^{1/p}}}t^{\mu-1}dt. \end{aligned} \tag{42}$$

Setting $\psi(u)=((\psi^p(b)t)/2)+m((2-t)/2)\psi^p(b)$ and $\psi(v)=(\psi^p(a)/m)((2-t)/2)+((\psi^p(b)t)/2)$ in (42), then, by applying Definition 3, we get the following inequality:

$$\begin{aligned} &f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)\le\frac{2^\mu\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\\ &\cdot\left[h\left(\frac{1}{2^\alpha}\right)g_2(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2)^+}(f\circ\phi)(\psi^{-1}(m\psi^p(b)))\right.\\ &\left.+m^{\mu+1}g_3(\eta)H\left(\frac{1}{2}\right)\times I^{\mu}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2m)^-}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]\\ &-\frac{F_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)h(1/2^\alpha)H(1/2)}{4(\mu+1)(\mu+2)}. \end{aligned} \tag{43}$$

Now, again, using strongly exponentially $(\alpha,h-m)$-p-convexity of f and integrating the resulting inequality over $[0,1]$ after multiplying with $t^{\mu-1}$, we get

$$\begin{aligned} &g_2(\eta)h\left(\frac{1}{2^\alpha}\right)\int_0^1 f\left(\left(\frac{\psi^p(a)t}{2}+m\left(\frac{2-t}{2}\right)\psi^p(b)\right)\right)^{1/p}t^{\mu-1}dt\\ &+mg_3(\eta)H\left(\frac{1}{2}\right)\int_0^1 f\left(\left(\frac{\psi^p(a)t}{2}+m\left(\frac{2-t}{2}\right)\psi^p(b)\right)^{1/p}\right)t^{\mu-1}dt\\ &\le\frac{A^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))}{\mu}+\frac{B^{h,H}_{\alpha,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))}{\mu}\\ &-\frac{R^{h,H}_{\alpha,\mu,m,\eta}(\psi^p(a),\psi^p(b))}{\mu}. \end{aligned} \tag{44}$$

Again, using substitution as considered in (42) leads to the second inequality of (38)

(ii) The proof is followed on the same lines as the proof of (i) □

Remark 23. The aforementioned version of the Hadamard inequalities give (i) [18], Theorem 2.4 for $c=0$ and $\psi=I$; (ii) [14], Theorem 2.4 for $c=\eta=0$ and $\psi=I$; (iii) [19], Theorem 7 for $\alpha=m=1$, $c=\eta=0$, and $h(t)=t$; (iv) [19], Theorem 7 for $\alpha=m=1$, $c=\eta=0$, and $h=\psi=I$; (v) Theorem 3 for $\alpha=m=p=1$, $c=\eta=0$, and $h=\psi=I$; (vi) [20], Theorem 2.1 for $\alpha=1=p$, $c=\eta=0$, and $h=\psi=I$; (vii) [21], Theorem 4 for $\alpha=m=1$, $p=-1$, $c=\eta=0$, and $h=\psi=I$; (viii) [3], Theorem 2.4 for $p=-1$, $\alpha=\mu=m=1$, $c=\eta=0$, and $h=\psi=I$; (ix) [22], Theorem 7 for $\alpha=\mu=m=p=1$, $c=\eta=0$, $h(t)=t^{-s}$, and $\psi=I$; (x) [23], Theorem 3.1 for $\alpha=\mu=m=1$, $p=-1$, $c=\eta=0$, $h(t)=t^{-s}$, and $\psi=I$; (xi) Theorem 1 for $\alpha=\mu=m=1$, $c=\eta=0$, and $h=\psi=I$; (xii) [24], Theorem 2.3 for $\alpha=\mu=m=1$, $\eta=0$, and $h=\psi=I$; (xiii) [25], Theorem 6 for $\alpha=\mu=m=p=1$, $\eta=0$, and $h=\psi=I$; (xiv) [17], Theorem 2.1 for $\alpha=\mu=m=1$, $p=-1$, $c=\eta=0$, $h(t)=t^s$, and $\psi=I$; (xv) [1], Theorem 2.1 for $\alpha=\mu=p=m=1$, $c=\eta=0$, $h(t)=t^s$, and $\psi=I$; and (xvi) [5], Theorem 2.1 for $\alpha=\mu=m=1$, $p=-1$, $\eta=0$, and $h=\psi=I$. Moreover, the refinements of all the deduced results will occur for $c>0$.

Corollary 24. (*i*) *For $p > 0$, one can have for the strongly $(\alpha, h-m)$-p-convex function the following fractional integral inequality:*

$$\begin{aligned} & f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))h(1/2^\alpha)H(1/2)}{(\mu+1)(\mu+2)} \\ & \leq \frac{\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[h\left(\frac{1}{2^\alpha}\right)\times I^{\mu,\psi}_{\psi^{-1}(\psi^p(a))^+}(f\circ\phi)\left(\psi^{-1}(m\psi^p(b))\right)\right. \\ & \left.+m^{\mu+1}H\left(\frac{1}{2}\right)I^{\mu,\psi}_{\psi^{-1}(\psi^p(b))^-}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right] \\ & \leq A^{h,H}_{\alpha,\mu,m,0}(f(\psi(a)),f(\psi(b)))+B^{h,H}_{\alpha,\mu,m,0}(f(\psi(a)),f(\psi(b))) \\ & -R^{h,H}_{\alpha,\mu,m,0}(\psi^p(a),\psi^p(b)) \end{aligned} \quad (45)$$

Proof. The abovementioned inequality can be deduced by setting $\eta = 0$ in (14).

(ii) For $p < 0$, one can have for the strongly $(\alpha, h-m)$-p-convex function the following fractional integral inequality:

$$\begin{aligned} & f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{c_{\mu,m}(\psi^p(a),\psi^p(b))h(1/2^\alpha)H(1/2)}{(\mu+1)(\mu+2)} \\ & \leq \frac{\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[h\left(\frac{1}{2^\alpha}\right)\times I^{\mu,\psi}_{\psi^{-1}(\psi^p(a))^-}(f\circ\phi)\left(\psi^{-1}(m\psi^p(b))\right)\right. \\ & \left.+m^{\mu+1}H\left(\frac{1}{2}\right)I^{\mu,\psi}_{\psi^{-1}(\psi^p(b))^+}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right] \\ & \leq A^{h,H}_{\alpha,\mu,m,0}(f(\psi(a)),f(\psi(b)))+B^{h,H}_{\alpha,\mu,m,0}(f(\psi(a)),f(\psi(b))) \\ & -R^{h,H}_{\alpha,\mu,m,0}(\psi^p(a),\psi^p(b)) \end{aligned} \quad (46)$$

□

Proof. The abovementioned inequality can be deduced by setting $\eta = 0$ in (14); then, one can obtain the required inequality. □

Corollary 25. (*i*) *For $p > 0$, one can have for the strongly exponentially $(h-m)$-p-convex function the following fractional integral inequality:*

$$\begin{aligned} & f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{F_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)h^2(1/2)}{4(\mu+1)(\mu+2)} \\ & \leq \frac{2^\mu h(1/2)\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)\times I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2)^+}(f\circ\phi)\left(\psi^{-1}(m\psi^p(b))\right)\right. \\ & \left.+m^{\mu+1}g_3(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2m)^-}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right] \\ & \leq A^{t,(1-t)}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t,(1-t)}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b))) \\ & -R^{t,(1-t)}_{1,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \end{aligned} \quad (47)$$

Proof. If $\alpha = 1$ in (38), then, the abovementioned inequality is obtained.

(ii) For $p < 0$, one can have for the strongly exponentially $(h-m)$-p-convex function the following fractional integral inequality:

$$\begin{aligned} & f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{F_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)h^2(1/2)}{4(\mu+1)(\mu+2)} \\ & \leq \frac{2^\mu h(1/2)\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)\times I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2)^-}(f\circ\phi)\left(\psi^{-1}(m\psi^p(b))\right)\right. \\ & \left.+m^{\mu+1}g_3(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2m)^+}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right] \\ & \leq A^{t,(1-t)}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t,(1-t)}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b))) \\ & -R^{t,(1-t)}_{1,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \end{aligned} \quad (48)$$

□

Proof. The abovementioned inequality can be deduced by setting $\alpha = 1$ in (40). □

Corollary 26. (*i*) *For $p > 0$, one can have for the strongly exponentially (s,m)-p-convex function the following fractional integral inequality:*

$$\begin{aligned} & f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{F_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)}{2^{2s+2}(\mu+1)(\mu+2)} \\ & \leq \frac{2^{\mu-s}\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[g_1(\eta)I^{\mu,\psi}_{((\psi^p(a)+m\psi^p(b))/2)^+}(f\circ\phi)\left(\psi^{-1}(mb)\right)\right. \\ & \left.+m^{\mu+1}I^{\mu}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2m)^-}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right] \\ & \leq A^{t^s,(1-t)^s}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^s,(1-t)^s}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b))) \\ & -R^{t^s,(1-t)^s}_{1,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \end{aligned} \quad (49)$$

Proof. The abovementioned inequality can be deduced by setting $\alpha = 1$ and $h(t) = t^s$ in (38).

(ii) For $p < 0$, one can have for the strongly exponentially (s,m)-p-convex function the following fractional integral inequality

$$\begin{aligned} & f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{F_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)}{2^{2s+2}(\mu+1)(\mu+2)} \\ & \leq \frac{2^{\mu-s}\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[g_1(\eta)I^{\mu,\psi}_{((\psi^p(a)+m\psi^p(b))/2)^-}(f\circ\phi)\left(\psi^{-1}(mb)\right)\right. \\ & \left.+m^{\mu+1}I^{\mu}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2m)^+}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right] \\ & \leq A^{t^s,(1-t)^s}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^s,(1-t)^s}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b))) \\ & -R^{t^s,(1-t)^s}_{1,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \end{aligned} \quad (50)$$

□

Proof. The abovementioned inequality can be deduced by setting $\alpha = 1$ and $h(t) = t^s$ in (40). □

Corollary 27. (*i*) *For $p > 0$, one can have for the strongly exponentially Godunova-Levin type of (s,m)-p-convex function the following fractional integral inequality:*

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{2^{2s}F_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)}{4(\mu+1)(\mu+2)}$$
$$\leq \frac{2^{\mu+s}\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2)^+}(f\circ\phi)(\psi^{-1}(m\psi^p(b)))\right.$$
$$\left.+m^{\mu+1}g_3(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2m)^-}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]$$
$$\leq A^{t^{-s},(1-t)^{-s}}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^{-s},(1-t)^{-s}}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))$$
$$-R^{t^{-s},(1-t)^{-s}}_{1,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \tag{51}$$

Proof. The abovementioned inequality can be deduced by setting $\alpha=1$ and $h(t)=t^{-s}$ in (38).

(ii) For $p<0$, one can have for the strongly exponentially Godunova-Levin type of (s,m)-p-convex function the following fractional integral inequality:

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{2^{2s}F_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)}{4(\mu+1)(\mu+2)}$$
$$\leq \frac{2^{\mu+s}\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2)^-}(f\circ\phi)(\psi^{-1}(m\psi^p(b)))\right.$$
$$\left.+m^{\mu+1}g_3(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2m)^+}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]$$
$$\leq A^{t^{-s},(1-t)^{-s}}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^{-s},(1-t)^{-s}}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))$$
$$-R^{t^{-s},(1-t)^{-s}}_{1,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \tag{52}$$

□

Proof. The abovementioned inequality can be deduced by setting $\alpha=1$ and $h(t)=t^{-s}$ in (40). □

Corollary 28. *(i) For $p>0$, one can have for the strongly exponentially (α,m)-p-convex function the following fractional integral inequality:*

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{(2^\alpha-1)F_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)}{2^{2\alpha+2}(\mu+1)(\mu+2)}$$
$$\leq \frac{2^{\mu-\alpha}\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2)^+}(f\circ\phi)\psi^{-1}((m\psi^p(b)))\right.$$
$$\left.+m^{\mu+1}g_3(\eta)(2^\alpha-1)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2m)^-}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]$$
$$\leq A^{t^\alpha,(1-t^\alpha)}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^\alpha,(1-t)^\alpha}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))$$
$$-R^{t^\alpha,(1-t)^\alpha}_{1,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \tag{53}$$

Proof. The abovementioned inequality can be deduced by setting $h(t)=t$ in (38).

(ii) For $p<0$, one can have for the strongly exponentially (α,m)-p-convex function the following fractional integral inequality:

$$f\left(\left(\frac{\psi^p(a)+m\psi^p(b)}{2}\right)^{1/p}\right)+\frac{(2^\alpha-1)F_{\mu,m}(\psi^p(a),\psi^p(b))g_1(\eta)}{2^{2\alpha+2}(\mu+1)(\mu+2)}$$
$$\leq \frac{2^{\mu-\alpha}\Gamma(\mu+1)}{(m\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2)^-}(f\circ\phi)\psi^{-1}((m\psi^p(b)))\right.$$
$$\left.+m^{\mu+1}g_3(\eta)(2^\alpha-1)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2m)^+}(f\circ\phi)\left(\psi^{-1}\left(\frac{\psi^p(a)}{m}\right)\right)\right]$$
$$\leq A^{t^\alpha,(1-t^\alpha)}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^\alpha,(1-t)^\alpha}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))$$
$$-R^{t^\alpha,(1-t)^\alpha}_{1,\mu,m,\eta}(\psi^p(a),\psi^p(b)) \tag{54}$$

□

Proof. The abovementioned inequality can be deduced by setting $h(t)=t$ in (40). □

Corollary 29. *(i) For $p>0$, one can have for strongly exponentially p-convex function the following fractional integral inequality:*

$$f\left(\left(\frac{\psi^p(a)+\psi^p(b)}{2}\right)^{1/p}\right)+\frac{cg_1(\eta)\left((\psi^p(b)-\psi^p(a))^2\right)}{2(\mu+1)(\mu+2)}$$
$$\leq \frac{2^{\mu-1}\Gamma(\mu+1)}{(\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2)^+}(f\circ\phi)(\psi^{-1}(\psi^p(b)))\right.$$
$$\left.+g_3(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+\psi^p(b))/2)^-}(f\circ\phi)(\psi^{-1}(\psi^p(a)))\right]$$
$$\leq A^{t^\alpha,(1-t^\alpha)}_{1,\mu,1,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^\alpha,(1-t)^\alpha}_{1,\mu,1,\eta}(f(\psi(a)),f(\psi(b)))$$
$$-R^{t^\alpha,(1-t)^\alpha}_{1,\mu,1,\eta}(\psi^p(a),\psi^p(b)) \tag{55}$$

Proof. The abovementioned inequality can be deduced by setting $\alpha=m=1$ and $h(t)=t$ in (38).

(ii) For $p<0$, one can have for the strongly exponentially p-convex function the following fractional integral inequality:

$$f\left(\left(\frac{\psi^p(a)+\psi^p(b)}{2}\right)^{1/p}\right)+\frac{cg_1(\eta)\left((\psi^p(b)-\psi^p(a))^2\right)}{2(\mu+1)(\mu+2)}$$
$$\leq \frac{2^{\mu-1}\Gamma(\mu+1)}{(\psi^p(b)-\psi^p(a))^\mu}\left[g_2(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2)^-}(f\circ\phi)(\psi^{-1}(\psi^p(b)))\right.$$
$$\left.+g_3(\eta)I^{\mu,\psi}_{\psi^{-1}((\psi^p(a)+m\psi^p(b))/2)^+}(f\circ\phi)(\psi^{-1}(\psi^p(a)))\right]$$
$$\leq A^{t^\alpha,(1-t^\alpha)}_{1,\mu,1,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^\alpha,(1-t)^\alpha}_{1,\mu,1,\eta}(f(\psi(a)),f(\psi(b)))$$
$$-R^{t^\alpha,(1-t)^\alpha}_{1,\mu,1,\eta}(\psi^p(a),\psi^p(b)) \tag{56}$$

□

Proof. The abovementioned inequality can be deduced by setting $\alpha=m=1$ and $h(t)=t$ in (40). □

Corollary 30. *For the strongly exponentially* (α, m)*-HA-convex function, the following inequality holds:*

$$f\left(\frac{2\psi(a)\psi(b)}{\psi(a)+m\psi(b)}\right)+\frac{(2^{\alpha}-1)cmg_1(\eta)}{2^{2\alpha+2}(\mu+1)(\mu+2)}\left[\mu(\mu+1)\left(\frac{\psi(b)-\psi(a)}{\psi(a)\psi(b)}\right)^2+(\mu^2+5\mu+8)\left(\frac{\psi(b)-\psi(a)m^2}{\psi(a)\psi(b)m}\right)^2+\frac{2\mu(\mu+3)(\psi(a)-\psi(b))}{m(\psi(a)\psi(b))^2}\right]\le\frac{2^{\mu-\alpha}\Gamma(\mu+1)(\psi(a)\psi(b))^{\mu}}{(\psi(b)-\psi(a)m)^{\mu}}\left[g_2(\eta)I^{\mu,\psi}_{((\psi(b)+m\psi(a))/2)^-}(f\circ\phi)\cdot\left(\psi^{-1}\left(\frac{m}{\psi(b)}\right)\right)+m^{\mu+1}g_3(\eta)\times(2^{\alpha}-1)I^{\mu}_{((\psi(a)+m\psi(b))/(2\psi(a)\psi(b)m))^+}(f\circ\phi)\cdot\left(\psi^{-1}\left(\frac{1}{\psi(a)m}\right)\right)\right]\le A^{t^{\alpha},(1-t^{\alpha})}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^{\alpha},(1-t)^{\alpha}}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))-R^{t^{\alpha},(1-t)^{\alpha}}_{1,\mu,m,\eta}\left(\frac{1}{\psi(a)},\frac{1}{\psi(b)}\right). \tag{57}$$

Proof. The abovementioned inequality can be deduced by setting $p=-1$ and $h(t)=t$ in (40). □

Corollary 31. *For the strongly exponentially* (s, m)*-HA-convex function, the following inequality holds:*

$$f\left(\frac{2\psi(a)\psi(b)}{\psi(a)+m\psi(b)}\right)+\frac{cmg_1(\eta)}{2^{2s+2}(\mu+1)(\mu+2)}\left[\mu(\mu+1)\left(\frac{\psi(b)-\psi(a)}{\psi(a)\psi(b)}\right)^2+(\mu^2+5\mu+8)\left(\frac{\psi(b)-\psi(a)m^2}{\psi(a)\psi(b)m}\right)^2+\left(\frac{2\mu(\mu+3)(\psi(a)-\psi(b))}{m(\psi(a)\psi(b))^2}\right)\right]\le\frac{2^{\mu-s}\Gamma(\mu+1)(\psi(a)\psi(b))^{\mu}}{(\psi(b)-\psi(a)m)^{\mu}}\left[g_2(\eta)I^{\mu,\psi}_{((\psi(b)+m\psi(a))/2)^-}(f\circ\phi)\left(\psi^{-1}\left(\frac{m}{\psi(b)}\right)\right)+m^{\mu+1}g_3(\eta)\times I^{\mu,\psi}_{((\psi(a)+m\psi(b))/(2\psi(a)\psi(b)m))^+}(f\circ\phi)\left(\psi^{-1}\left(\frac{1}{am}\right)\right)\right]\le A^{t^s,(1-t)^s}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^s,(1-t)^s}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))-R^{t^s,(1-t)^s}_{1,\mu,m,\eta}\left(\frac{1}{\psi(a)},\frac{1}{\psi(b)}\right). \tag{58}$$

Proof. The abovementioned inequality can be deduced by setting $p=-1$, $\alpha=1$, and $h(t)=t^s$ in (40). □

Corollary 32. *For the Godunova-Levin type of the strongly exponentially* (s, m)*-HA-convex function, the following inequality holds:*

$$f\left(\frac{2\psi(a)\psi(b)}{\psi(a)+m\psi(b)}\right)+\frac{cmg_1(\eta)}{2^{2s+2}(\mu+1)(\mu+2)}\left[\mu(\mu+1)\left(\frac{\psi(b)-\psi(a)}{\psi(a)\psi(b)}\right)^2+(\mu^2+5\mu+8)\left(\frac{\psi(b)-\psi(a)m^2}{\psi(a)\psi(b)m}\right)^2+\left(\frac{2\mu(\mu+3)(\psi(a)-\psi(b))}{m(\psi(a)\psi(b))^2}\right)\right]\le\frac{2^{\mu+s}\Gamma(\mu+1)(\psi(a)\psi(b))^{\mu}}{(\psi(b)-\psi(a)m)^{\mu}}\left[g_2(\eta)I^{\mu,\psi}_{((\psi(b)+m\psi(a))/2)^-}(f\circ\phi)\cdot\left(\psi^{-1}\left(\frac{m}{\psi(b)}\right)\right)+m^{\mu+1}g_3(\eta)I^{\mu}_{((\psi(a)+m\psi(b))/(2\psi(a)\psi(b)m))^+}(f\circ\phi)\cdot\left(\psi^{-1}\left(\frac{1}{\psi(a)m}\right)\right)\right]\le A^{t^{-s},(1-t)^{-s}}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))+B^{t^{-s},(1-t)^{-s}}_{1,\mu,m,\eta}(f(\psi(a)),f(\psi(b)))-R^{t^{-s},(1-t)^{-s}}_{1,\mu,m,\eta}\left(\frac{1}{\psi(a)},\frac{1}{\psi(b)}\right). \tag{59}$$

Proof. The abovementioned inequality can be deduced by setting $p=-1$, $\alpha=1$, and $h(t)=t^{-s}$ in (40). □

Remark 33. Using (12) with replacement of μ by μ/k in all the abovementioned inequalities, the k-fractional versions of all the abovementioned results can be obtained.

3. Conclusion

Some inequalities of the Hadamard type for the strongly exponentially $(\alpha, h-m)$-p-convex functions using generalized Riemann-Liouville fractional integrals have been proved. These inequalities give refinements of different Hadamard-type inequalities related to various types of convexities. The outcomes of this paper can also provide the k-fractional versions of established inequalities via parametric substitution along with constant multiplier.

Data Availability

There is no external data required.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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