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Research Article

Endpoints of Generalized Contractions in $\mathcal{F}$-Metric Spaces with Application to Integral Equations

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The purpose of this article is to introduce locally α-ζ-multivalued contraction and rational Ćirić type α-ζ-multivalued contraction in the context of $\mathcal{F}$-metric spaces and prove some endpoint results. We provide a nontrivial example to show the authenticity of our main result. Our results generalize some well-known results of literature. We also present some endpoint results in both graphic $\mathcal{F}$-metric spaces and ordered $\mathcal{F}$-metric spaces. As an application of our main result, we investigate the solution of an integral equation.

1. Introduction

In 2010, Amini-Harandi [1] showed that a multivalued mapping has a unique endpoint if and only if this multivalued mapping has the approximate endpoint property. Hussain et al. [2] established some approximate endpoints of the multivalued almost I-contractions in complete metric spaces. Later on, Moradi and Khojasteh [3] proved a result for generalized weak contractive multifunctions.

On the other hand, Samet et al. [4] introduced the notion of α-admissibility and α-ζ-contraction in 2012. Asl et al. [5] extended this notion of α-admissibility to α^*-admissibility and proved some results for multivalued mappings. In 2015, Mohammadi and Rezapour [6] improved the α-admissibility concept and obtained endpoint of α-ζ-multivalued contraction. Later on, Choudhury et al. [7] used the notion of α-admissibility and proved end point results of multivalued mappings without continuity. Very recently, Isik et al. [8] proved endpoint results for α-ζ-contraction in the newly introduced space of Jleli and Samet [9] which is named as $\mathcal{F}$-metric space ($\mathcal{F}$-MS). In this artilce, we give locally α-ζ-multivalued contraction and rational Ćirić type α-ζ-multivalued contraction in the framework of $\mathcal{F}$-metric space and generalized the main result of Isik et al. [8].

2. Preliminaries

Let $\mathcal{M}=\varnothing$ and $\mathcal{T} : \mathcal{M} \longrightarrow 2^{\mathcal{M}}$ (nonempty subsets of $\mathcal{M}$) be a multivalued mapping. A point $\sigma \in \mathcal{M}$ is professed to be an endpoint (fixed point) of $\mathcal{T}$ if $\mathcal{T}\sigma = \{\sigma\}(\sigma \in \mathcal{T}\sigma)$. Now, let $(\mathcal{M}, d)$ be a metric space, then $\mathcal{T}$ is said to satisfy the approximate fixed point property if

$$\inf_{\sigma\in\mathcal{M}} \sup_{y\in\mathcal{T}\sigma} d(\sigma, y) = 0. \tag{1}$$

Let $\mathcal{CB}(\mathcal{M})$ represents the set of all nonempty, closed, and bounded subsets of $\mathcal{M}$. The Hausdorff metric $\mathcal{H}$ is defined on $\mathcal{CB}(\mathcal{M})$ as follows:

$$\mathcal{H}(A, B) = \max\left\{\sup_{\sigma\in A} d(\sigma, B), \sup_{y\in B} d(y, A)\right\}. \tag{2}$$

In 2012, Samet et al. [4] used the following set Ψ of nondecreasing functions $\zeta : [0, \infty) \longrightarrow [0, \infty)$ satisfying

$$\sum_{n=1}^{\infty} \zeta^n(t) < \infty, \text{for all } t > 0, \tag{3}$$

and introduced α-ζ-contraction. Clearly, $\zeta(t) < t$ for all $t > 0$ ([30]).

Samet et al. [4] also initiated the concept of α-admissibility of a single valued mapping in this way.

Definition 1 (see [4]). Let $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0,\infty)$ and let $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$, then $\mathcal{T}$ is said to be α-admissible if $\forall \sigma, y \in \mathcal{M}$, $\alpha(\sigma, y) \geq 1$ implies $\alpha(\mathcal{T}\sigma, \mathcal{T}y) \geq 1$.

They gave the following property of $\mathcal{M}$ that is $\mathcal{M}$ is α-regular, if for each sequence $\{\sigma_n\}$ in $\mathcal{M}$ with $\alpha(\sigma_n, \sigma_{n+1}) \geq 1$, and $\sigma_n \longrightarrow \sigma$, then $\alpha(\sigma_n, \sigma) \geq 1$, $\forall n$.

In 2013, Asl et al. [5] extended this concept to multivalued mapping and gave the notion of α^*-admissibility as follows.

Definition 2 (see [5]). Let $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0,\infty)$ and let $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$, then $\mathcal{T}$ is said to be α^*-admissible if for all $\sigma, y \in \mathcal{M}$, $\alpha(\sigma, y) \geq 1$ implies $\alpha^*(\mathcal{T}\sigma, \mathcal{T}y) \geq 1$, where $\alpha^*(A, B) = \inf \{\alpha(a, b): a \in A, b \in B\}$, for all A, $B \in \mathcal{CB}(\mathcal{M})$.

In 2015, Mohammadi and Rezapour [6] extended the above notion in this way.

Definition 3 (see [6]). Let $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0,\infty)$ and $\mathcal{T} : \mathcal{M} \longrightarrow 2^{\mathcal{M}}$, then $\mathcal{T}$ is α-admissible provided that for all $\sigma \in \mathcal{M}$ and $y \in \mathcal{T}\sigma$ with $\alpha(\sigma, y) \geq 1$, then $\alpha(y, z) \geq 1$, for all $z \in \mathcal{T}y$.

They proved endpoint results for α-ζ-multivalued contraction by using the following property.

A multivalued mapping $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ is said to satisfy the property $(\mathcal{BS})$, if for all $\sigma \in \mathcal{M}$, there exists $y \in \mathcal{T}\sigma$ such that $\mathcal{H}(\mathcal{T}\sigma, \mathcal{T}y) = \sup_{b \in \mathcal{T}y} d(y, b)$. Isik et al. [8] used the property $(\mathcal{SBS})$ of Mohammadi and Rezapour [6] to prove their results, that is, for each sequence $\{\sigma_n\}$ with

$$d(\sigma_n, \mathcal{T}\sigma_n) \leq d(\sigma_n, \sigma_{n+1}) + \zeta(d(\sigma_n, \sigma_{n+1})), \quad (4)$$

for all n and $\sigma_n \longrightarrow \sigma$, then $d(\sigma_n, \mathcal{T}\sigma_n) \leq d(\sigma_n, \sigma) + \zeta(d(\sigma_n, \sigma))$, for all $n \geq N$.

For more details in this direction, we refer the readers (see [10–14]).

Recently, Jleli and Samet [9] introduced an interesting generalization of metric space which is called $\mathcal{F}$-metric space ($\mathcal{F}$-MS) as follows.

Let $\mathcal{F}$ be the class of $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that $f(\sigma_1) < f(\sigma_2)$, for $\{\sigma_n\} \subseteq \mathbb{R}^+$, $\lim_{n\longrightarrow\infty} \sigma_n = 0 \Leftrightarrow \lim_{n\longrightarrow\infty} f(\sigma_n) = -\infty$.

Definition 4 (see [9]). Let $\mathcal{M}=\varnothing$, and let $d_{\mathcal{F}} : \mathcal{M} \times \mathcal{M} \longrightarrow [0,+\infty)$. Suppose that there exists $f \in \mathcal{F}$ and $\alpha \in [0, +\infty)$ such that

$(D_1) d_{\mathcal{F}}(\sigma, y) = 0 \Longleftarrow\Rightarrow \sigma = y$, for all $(\sigma, y) \in \mathcal{M} \times \mathcal{M}$

$(D_2) d_{\mathcal{F}}(\sigma, y) = d_{\mathcal{F}}(y, \sigma)$, for all $(\sigma, y) \in \mathcal{M} \times \mathcal{M}$

(D_3) for every $(\sigma, y) \in \mathcal{M} \times \mathcal{M}$, for every $N \in \mathbb{N}$, $N \geq 2$ and for every $(\sigma_i)_{i=1}^N \subset \mathcal{M}$ with $(u_1, u_N) = (\sigma, y)$, we have

$$d_{\mathcal{F}}(\sigma, y) > 0 \Rightarrow f(d_{\mathcal{F}}(\sigma, y)) \leq f\left(\sum_{i=1}^{N-1} d_{\mathcal{F}}(u_i, u_{i+1})\right) + \alpha \quad (5)$$

Then, $(\mathcal{M}, d_{\mathcal{F}})$ is called an $\mathcal{F}$-MS.

Theorem 5 (see[9]). *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$-MS and let $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$. Suppose that these assertions hold:*

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete

(ii) there exists $k \in (0, 1)$ such that

$$d_{\mathcal{F}}(\mathcal{T}(\sigma), \mathcal{T}(y)) \leq k d_{\mathcal{F}}(\sigma, y) \quad (6)$$

Then, there exists $\sigma^* \in \mathcal{M}$ such that $\mathcal{T}\sigma^* = \sigma^*$ which is unique.

Hussain and Kanwal [15] utilized an $\mathcal{F}$-metric space and generalized the above result by considering the notion of α-ζ-contraction to prove a fixed point theorem. Many researchers (see [16–18]) worked in this newly generalized space.

Very recently, Isik et al. [8] introduced the notion of Hausdorff metric $\mathcal{H}_{\mathcal{F}}(.,.)$ on $\mathcal{CB}(\mathcal{M})$ influence by $\mathcal{F}$-metric $d_{\mathcal{F}}$ as follows:

$$\mathcal{H}_{\mathcal{F}}(A, B) = \max \left\{\sup_{\sigma \in A} d_{\mathcal{F}}(\sigma, B), \sup_{y \in B} d_{\mathcal{F}}(y, A)\right\}, \quad (7)$$

for all $A, B \in \mathcal{CB}(\mathcal{M})$, where $d_{\mathcal{F}}(\sigma, B) = \inf_{y \in B} d_{\mathcal{F}}(\sigma, y)$ and obtained endpoint results for α-ζ-multivalued contraction in this way.

Theorem 6. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$-MS and $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ be an α-admissible mapping which satisfies the property $(\mathcal{BS})$. Suppose there exists $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0,+\infty)$ and $\zeta \in \Psi$ such that*

$$\alpha(\sigma, y) \geq 1 \Rightarrow \mathcal{H}_{\mathcal{F}}(\mathcal{T}(\sigma), \mathcal{T}(y)) \leq \zeta(d_{\mathcal{F}}(\sigma, y)). \quad (8)$$

Also, suppose that these assertions hold:

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete

(ii) $\alpha(\sigma_0, \sigma_1) \geq 1$ for an $\sigma_0 \in M$ and $\sigma_1 \in \mathcal{T}(\sigma_0)$

(iii) $\mathcal{M}$ is α-regular

Then, $\mathcal{T}$ has an endpoint.

3. Main Results

Definition 7. Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$-MS. A mapping $\mathcal{T} : \mathcal{M} \longrightarrow 2^{\mathcal{M}}$ is called a locally α-ζ-multivalued contraction if there exists $\zeta \in \Psi$ and $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ such that

$$\alpha(\sigma, y) \geq 1 \Rightarrow \mathcal{H}_{\mathcal{F}}(\mathcal{T}(\sigma), \mathcal{T}(y)) \leq \zeta(d_{\mathcal{F}}(\sigma, y)), \tag{9}$$

for $\sigma, y \in B(\bar{\sigma_0}, r)$.

Now, we state our main result regarding the existence of the endpoint of an α-ζ-multivalued contraction on the closed ball $B(\bar{\sigma_0}, r)$ which is very advantageous in the perception that it needs the contractiveness of the multivalued mapping $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ only on the closed ball instead of the whole space.

Theorem 8. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$-MS and $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ be an α-admissible, locally α - ζ -multivalued contraction such that $\mathcal{T}$ satisfies the property $(\mathcal{BS})$ and for $\sigma_0 \in \mathcal{M}$, there exists $\sigma_1 \in \mathcal{T}\sigma_0$ such that*

$$\zeta^i(d(\sigma_0, \sigma_1)) < r, \tag{10}$$

for all $n = 0, 1, 2, \cdots$ and $r > 0$. Also, suppose that the following assertions hold:

(i) $(\mathcal{M}, d_{\mathcal{F}})$ is $\mathcal{F}$-complete

(ii) $\alpha(\sigma_0, \sigma_1) \geq 1$ for an $\sigma_0 \in \mathcal{M}$ and $\sigma_1 \in \mathcal{T}(\sigma_0)$

(iii) $\mathcal{M}$ is α-regular

Then, $\mathcal{T}$ has an endpoint.

Proof. Choose $\sigma_0 \in \mathcal{M}$ and $\sigma_1 \in \mathcal{T}\sigma_0$ such that $\alpha(\sigma_0, \sigma_1) \geq 1$. It follows directly from (10); we have

$$d(\sigma_0, \sigma_1) < r, \tag{11}$$

which implies that

$$\sigma_1 \in B(\bar{\sigma_0}, r). \tag{12}$$

□

It follows from (10) that

$$\alpha(\sigma_0, \sigma_1) \geq 1 \Rightarrow H(\mathcal{T}\sigma_0, \mathcal{T}\sigma_1) \leq \zeta(d(\sigma_0, \sigma_1)). \tag{13}$$

Since $\mathcal{T}$ satisfies the property $(\mathcal{BS})$, so $\exists \sigma_2 \in \mathcal{T}\sigma_1$ such that $\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_1, \mathcal{T}\sigma_2) = \sup_{b \in \mathcal{T}\sigma_2} d_{\mathcal{F}}(\sigma_2, b)$. Now, from (13), we have

$$\begin{aligned} d(\sigma_1, \sigma_2) &\leq \sup_{b \in \mathcal{T}\sigma_1} d_{\mathcal{F}}(\sigma_1, b) = \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_0, \mathcal{T}\sigma_1) \\ &\leq \zeta(d(\sigma_0, \sigma_1)) < r. \end{aligned} \tag{14}$$

This implies that

$$\sigma_2 \in \overline{B(\sigma_0, r)}. \tag{15}$$

Since $\mathcal{T}$ is α-admissible, $\alpha(\sigma_1, \sigma_2) \geq 1$, so t follows from (9) that

$$\alpha(\sigma_1, \sigma_2) \geq 1 \Rightarrow H(\mathcal{T}\sigma_1, \mathcal{T}\sigma_2) \leq \zeta(d(\sigma_1, \sigma_2)). \tag{16}$$

Continuing this process, we obtain a sequence $\{\sigma_n\}$ in $B(\bar{\sigma_0}, r)$ such that $\sigma_{n+1} \in \mathcal{T}\sigma_n$, $\alpha(\sigma_n, \sigma_{n+1}) \geq 1$ and $\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma_{n+1}) = \sup_{b \in \mathcal{T}\sigma_{n+1}} d_{\mathcal{F}}(\sigma_{n+1}, b)$, for all n. If $\sigma_n = \sigma_{n+1}$ for some $n \in \mathbb{N}$, then we get that $\mathcal{H}_{\mathcal{F}}(\{\sigma_{n+1}\}, \mathcal{T}\sigma_{n+1}) = \sup_{b \in \mathcal{T}\sigma_{n+1}} d_{\mathcal{F}}(\sigma_{n+1}, b) = \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma_{n+1}) = 0$. It implies that σ_{n+1} is an endpoint. Hence, we suppose that $\sigma_n \neq \sigma_{n+1}$, for all $n \in \mathbb{N}$.

Now, since $\alpha(\sigma_{n-1}, \sigma_n) \geq 1$, so

$$\begin{aligned} d_{\mathcal{F}}(\sigma_n, \sigma_{n+1}) &\leq \sup_{b \in \mathcal{T}\sigma_n} d_{\mathcal{F}}(\sigma_n, b) = \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_{n-1}, \mathcal{T}\sigma_n) \\ &\leq \zeta(d_{\mathcal{F}}(\sigma_{n-1}, \sigma_n)) \leq \zeta^2(d_{\mathcal{F}}(\sigma_{n-2}, \sigma_{n-1})) \\ &\leq \cdots \leq \zeta^n(d_{\mathcal{F}}(\sigma_0, \sigma_1)), \end{aligned} \tag{17}$$

for all $n \geq 0$. Assume that $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ be such that (D_3) is satisfied and fix $\varepsilon > 0$. By (F_2),$\exists \delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\varepsilon) - \alpha. \tag{18}$$

Suppose that $N \in \mathbb{N}$ be such that $0 < \sum_{i \geq N} \zeta^{i-1}(d_{\mathcal{F}}(\sigma_1, \sigma_2)) < \delta$. Hence, by (17), (18) and $(\mathcal{F}_1)$, we have

$$\begin{aligned} f\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\sigma_i, \sigma_{i+1})\right) &\leq f\left(\sum_{i=n}^{m-1} \zeta^{i-1}(d_{\mathcal{F}}(\sigma_1, \sigma_2))\right) \\ &\leq f\left(\sum_{i \geq N} \zeta^{i-1}(d_{\mathcal{F}}(\sigma_1, \sigma_2))\right) \\ &< f(\varepsilon) - \alpha, \end{aligned} \tag{19}$$

for $m > n \geq N$. Using $(\mathfrak{D}_3)$ and (19), we obtain that $d_{\mathcal{F}}(\sigma_n, \sigma_m) > 0$ where $m > n \geq N$ which implies that

$$f(d_{\mathcal{F}}(\sigma_n, \sigma_m)) \leq f\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\sigma_i, \sigma_{i+1})\right) + \alpha < f(\varepsilon), \tag{20}$$

which implies by $(\mathcal{F}_1)$ that $d_{\mathcal{F}}(\sigma_n, \sigma_m) < \varepsilon$, for all $m > n \geq N$. This proves that $\{\sigma_n\}$ is $\mathcal{F}$-Cauchy. Because of $\mathcal{F}$-completeness of $\mathcal{M}$, there exists $\sigma^{å} \in B(\bar{\sigma_0}, r)$ such that $\sigma_n \longrightarrow \sigma^{å}$. We shall prove that σ^* is an endpoint of $\mathcal{T}$. We assume on the contrary that $\mathcal{T}\sigma^* \neq \{\sigma^*\}$. Then $\mathcal{H}_{\mathcal{F}}(\{\sigma^*\}, \mathcal{T}\sigma^*) > 0$. Since $\mathcal{M}$ is locally α-regular, so $\alpha(\sigma_n, \sigma^*) \geq 1$, for all $n \in \mathbb{N}$. Then, by (9) and $(\mathcal{F}_1)$, we have

$$\begin{aligned} f(\mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n)) &= f(\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_{n-1}, \mathcal{T}\sigma_n)) \\ &\leq f(\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_{n-1}, \mathcal{T}\sigma^*) \\ &\quad + \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma^*)) + \alpha \\ &\leq f(\zeta(d_{\mathcal{F}}(\sigma_{n-1}, \sigma^*)) \\ &\quad + \zeta(d_{\mathcal{F}}(\sigma_n, \sigma^*))) \\ &\quad + \alpha \longrightarrow -\infty, \end{aligned} \tag{21}$$

as $n \longrightarrow \infty$. Thus,

$$\lim_{n \longrightarrow \infty} \mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n) = 0. \tag{22}$$

On the other side,

$$\begin{aligned} f(\mathscr{H}_{\mathscr{F}}(\{\sigma^*\},\mathscr{T}\sigma^*)) &\le f(\mathscr{H}_{\mathscr{F}}(\{\sigma^*\},\{\sigma_n\})+\mathscr{H}_{\mathscr{F}}(\{\sigma_n\},\mathscr{T}\sigma_n) \\ &\quad +\mathscr{H}_{\mathscr{F}}(\mathscr{T}\sigma_n,\mathscr{T}\sigma^*))+\alpha \\ &\le f(d(\sigma^*,\sigma_n)+\mathscr{H}_{\mathscr{F}}(\{\sigma_n\},\mathscr{T}\sigma_n) \\ &\quad +\zeta(d_{\mathscr{F}}(\sigma_n,\sigma^*)))\longrightarrow -\infty, \end{aligned} \tag{23}$$

as $n\longrightarrow\infty$, that is a contradiction. Hence, $\{\sigma^*\}=\mathscr{T}\sigma^*$.

Definition 9. Let $(\mathscr{M},d_{\mathscr{F}})$ be an $\mathscr{F}$-MS. A mapping $\mathscr{T}:\mathscr{M}\longrightarrow 2^{\mathscr{M}}$ is called a rational Ćirić type α - ζ -multivalued contraction if there exists two functions $\alpha:\mathscr{M}\times\mathscr{M}\longrightarrow[0,+\infty)$ and $\zeta\in\Psi$ such that

$$\alpha(\sigma,y)\mathscr{H}_{\mathscr{F}}(\mathscr{T}(\sigma),\mathscr{T}(y))\le\zeta(R_{\mathscr{F}}(\sigma,y)), \tag{24}$$

for $(\sigma,y)\in\mathscr{M}\times\mathscr{M}$, where

$$R_{\mathscr{F}}(\sigma,\mathrm{y})=\max\left\{d_{\mathscr{F}}(\sigma,\mathrm{y}),\frac{d_{\mathscr{F}}(\sigma,\mathscr{T}\sigma)d_{\mathscr{F}}(y,\mathscr{T}y)}{1+d_{\mathscr{F}}(\sigma,\mathrm{y})}\right\}. \tag{25}$$

Theorem 10. *Suppose that $(\mathscr{M},d_{\mathscr{F}})$ be an $\mathscr{F}$-MS and $\mathscr{T}:\mathscr{M}\longrightarrow\mathscr{CB}(\mathscr{M})$ be an α -admissible and rational Ćirić type α - ζ -multivalued contraction such that $\mathscr{T}$ satisfies the property $(\mathscr{BS})$. Also, suppose that these conditions hold:*

(i) $(\mathscr{M},d_{\mathscr{F}})$ is $\mathscr{F}$-complete

(ii) $\alpha(\sigma_0,\sigma_1)\ge 1$ for an $\sigma_0\in\mathscr{M}$ and $\sigma_1\in\mathscr{T}(\sigma_0)$;

(iii) $\mathscr{T}$ is continuous

Then, $\mathscr{T}$ has an endpoint.

Proof. Choose $\sigma_0\in\mathscr{M}$ and $\sigma_1\in\mathscr{T}\sigma_0$ such that $\alpha(\sigma_0,\sigma_1)\ge 1$. Since $\mathscr{T}$ satisfies the property $(\mathscr{BS})$, there exists $\sigma_2\in\mathscr{T}\sigma_1$ such that

$$\mathscr{H}_{\mathscr{F}}(\mathscr{T}\sigma_1,\mathscr{T}\sigma_2)=\sup_{b\in\mathscr{T}\sigma_2}d_{\mathscr{F}}(\sigma_2,b). \tag{26}$$

□

Since $\mathscr{T}$ is α-admissible, $\alpha(\sigma_1,\sigma_2)\ge 1$. Continuing this process, we obtain a sequence $\{\sigma_n\}$ such that $\sigma_{n+1}\in\mathscr{T}\sigma_n$, $\alpha(\sigma_n,\sigma_{n+1})\ge 1$ and

$$\mathscr{H}_{\mathscr{F}}(\mathscr{T}\sigma_n,\mathscr{T}\sigma_{n+1})=\sup_{b\in\mathscr{T}\sigma_{n+1}}d_{\mathscr{F}}(\sigma_{n+1},b), \tag{27}$$

for all n. If $\sigma_n=\sigma_{n+1}$ for some $n\in\mathbb{N}$, then we get that

$$\begin{aligned} \mathscr{H}_{\mathscr{F}}(\{\sigma_{n+1}\},\mathscr{T}\sigma_{n+1}) &= \sup_{b\in\mathscr{T}\sigma_{n+1}}d_{\mathscr{F}}(\sigma_{n+1},b) \\ &= \mathscr{H}_{\mathscr{F}}(\mathscr{T}\sigma_n,\mathscr{T}\sigma_{n+1})=0. \end{aligned} \tag{28}$$

It implies that σ_{n+1} is an endpoint. Hence, we suppose that $\sigma_n\ne\sigma_{n+1}$, for all $n\in\mathbb{N}$.

Note that

$$\begin{aligned} d_{\mathscr{F}}(\sigma_n,\sigma_{n+1}) &\le \sup_{b\in\mathscr{T}\sigma_n}d_{\mathscr{F}}(\sigma_n,b)=\mathscr{H}_{\mathscr{F}}(\mathscr{T}\sigma_{n-1},\mathscr{T}\sigma_n) \\ &\le \alpha(\sigma_{n-1},\sigma_n)\mathscr{H}_{\mathscr{F}}(\mathscr{T}\sigma_{n-1},\mathscr{T}\sigma_n)\le\zeta(R_{\mathscr{F}}(\sigma_{n-1},\sigma_n)) \\ &= \zeta\left(\max\left\{d_{\mathscr{F}}(\sigma_{n-1},\sigma_n),\frac{d_{\mathscr{F}}(\sigma_{n-1},\mathscr{T}\sigma_{n-1})d_{\mathscr{F}}(\sigma_n,\mathscr{T}\sigma_n)}{1+d_{\mathscr{F}}(\sigma_{n-1},\sigma_n)}\right\}\right) \\ &\le \zeta\left(\max\left\{d_{\mathscr{F}}(\sigma_{n-1},\sigma_n),\frac{d_{\mathscr{F}}(\sigma_{n-1},\sigma_n)d_{\mathscr{F}}(\sigma_n,\sigma_{n+1})}{1+d_{\mathscr{F}}(\sigma_{n-1},\sigma_n)}\right\}\right) \\ &\le \zeta(\max\{d_{\mathscr{F}}(\sigma_{n-1},\sigma_n),d_{\mathscr{F}}(\sigma_n,\sigma_{n+1})\}), \end{aligned} \tag{29}$$

for all $n\ge 2$. If $\max\{d_{\mathscr{F}}(\sigma_{n-1},\sigma_n),d_{\mathscr{F}}(\sigma_n,\sigma_{n+1})\}=d_{\mathscr{F}}(\sigma_n,\sigma_{n+1})$, then

$$d_{\mathscr{F}}(\sigma_n,\sigma_{n+1})\le\zeta(d_{\mathscr{F}}(\sigma_n,\sigma_{n+1}))<d_{\mathscr{F}}(\sigma_n,\sigma_{n+1}), \tag{30}$$

which is a contradiction. So, we have

$$\max\{d_{\mathscr{F}}(\sigma_{n-1},\sigma_n),d_{\mathscr{F}}(\sigma_n,\sigma_{n+1})\}=d_{\mathscr{F}}(\sigma_{n-1},\sigma_n)), \tag{31}$$

which implies

$$d_{\mathscr{F}}(\sigma_n,\sigma_{n+1})\le\zeta(d_{\mathscr{F}}(\sigma_{n-1},\sigma_n)). \tag{32}$$

Continuing in this way, we obtain that

$$\begin{aligned} d_{\mathscr{F}}(\sigma_n,\sigma_{n+1}) &\le \zeta(d_{\mathscr{F}}(\sigma_{n-1},\sigma_n))\le\zeta^2(d_{\mathscr{F}}(\sigma_{n-2},\sigma_{n-1})) \\ &\le \zeta^3(d_{\mathscr{F}}(\sigma_{n-3},\sigma_{n-2}))\le\cdots\le\zeta^n(d_{\mathscr{F}}(\sigma_0,\sigma_1)), \end{aligned} \tag{33}$$

for all $n\ge 2$ which yields that

$$\sum_{i=n}^{m-1}d_{\mathscr{F}}(\sigma_i,\sigma_{i+1})\le\sum_{i=n}^{m-1}\zeta^i(d_{\mathscr{F}}(\sigma_0,\sigma_1)), \tag{34}$$

for $m>n\ge 2$. Suppose that $\varepsilon>0$ be arbitrary. Next, let $(f,\alpha)\in\mathscr{F}\times[0,+\infty)$ be such that $(d_{\mathscr{F}3})$ is satisfied. By $(\mathscr{F}_2)$, there exists $\delta>0$ such that

$$0<t<\delta\Rightarrow f(t)<f(\varepsilon)-\alpha. \tag{35}$$

Suppose that $N\in\mathbb{N}$ be such that $\sum_{i\ge N}\zeta^i(d_{\mathscr{F}}(\sigma_1,\sigma_2))<\delta$. Hence, by (24), (35) and $(\mathscr{F}_1)$, we have

$$\begin{aligned} f\left(\sum_{i=n}^{m-1}d_{\mathscr{F}}(\sigma_i,\sigma_{i+1})\right) &\le f\left(\sum_{i=n}^{m-1}\zeta^i(d_{\mathscr{F}}(\sigma_0,\sigma_1))\right) \\ &\le f\left(\sum_{i\ge N}\zeta^i(d_{\mathscr{F}}(\sigma_0,\sigma_1))\right) \\ &< f(\varepsilon)-\alpha, \end{aligned} \tag{36}$$

for $m > n \geq N$. Using $(\mathfrak{D}_3)$ and (36), we obtain that $d_{\mathcal{F}}(\sigma_n, \sigma_m) > 0$ where $m > n \geq N$ which implies that

$$f(d_{\mathcal{F}}(\sigma_n, \sigma_m)) \leq f\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(\sigma_i, \sigma_{i+1})\right) + \alpha < f(\varepsilon), \quad (37)$$

which implies by $(\mathcal{F}_1)$ that $d_{\mathcal{F}}(\sigma_n, \sigma_m) < \varepsilon$, for all $m > n \geq N$. This proves that $\{\sigma_n\}$ is $\mathcal{F}$-Cauchy. As $\mathcal{M}$ is $\mathcal{F}$-complete, so $\exists \sigma^{å} \in \mathcal{M}$ such that $\sigma_n \longrightarrow \sigma^{å}$. We shall prove that σ^* is an endpoint of $\mathcal{T}$. We assume on contrary that $\mathcal{T}\sigma^* \neq \{\sigma^*\}$. Then, $\mathcal{H}_{\mathcal{F}}(\{\sigma^*\}, \mathcal{T}\sigma^*) > 0$. Now, we have

$$\begin{aligned} &f(\mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n)) \\ &\quad = f(\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_{n-1}, \mathcal{T}\sigma_n)) \\ &\quad \leq f(\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_{n-1}, \mathcal{T}\sigma^*) + \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma^*)) + \alpha. \end{aligned} \quad (38)$$

Note that we used the property $(\mathcal{BS})$ in the above inequality. Taking the limit in both sides of the above inequality and using continuity assumption of $\mathcal{T}$, we get $\lim_{n\longrightarrow\infty} f(\mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n)) = -\infty$ which implies that $\lim_{n\longrightarrow\infty} \mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n) = 0$. Hence,

$$\begin{aligned} &f(\mathcal{H}_{\mathcal{F}}(\{\sigma^*\}, \mathcal{T}\sigma^*)) \\ &\quad \leq f(\mathcal{H}_{\mathcal{F}}(\{\sigma^*\}, \{\sigma_n\}) + \mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n) \\ &\qquad + \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma^*)) + \alpha \\ &\quad \leq f(d(\sigma^*, \sigma_n) + \mathcal{H}_{\mathcal{F}}(\{\sigma_n\}, \mathcal{T}\sigma_n) \\ &\qquad + \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma^*)) \longrightarrow -\infty, \end{aligned} \quad (39)$$

as $n \longrightarrow \infty$, that is a contradiction. Hence, $\{\sigma^*\} = \mathcal{T}\sigma^*$.

Example 1. Consider the set $\mathcal{M} = \{1, 2, 3\}$. Suppose that the mapping $d_{\mathcal{F}} : \mathcal{M} \times \mathcal{M} \longrightarrow [0, +\infty)$ be given by

$$\begin{aligned} d_{\mathcal{F}}(1,1) &= d_{\mathcal{F}}(2,2) = d_{\mathcal{F}}(3,3) = 0, \\ d_{\mathcal{F}}(1,2) &= d_{\mathcal{F}}(2,1) = \frac{1}{2}, \\ d_{\mathcal{F}}(2,3) &= d_{\mathcal{F}}(3,2) = \frac{2}{3}, \\ d_{\mathcal{F}}(1,3) &= d_{\mathcal{F}}(3,1) = \frac{4}{3}. \end{aligned} \quad (40)$$

So, $(\mathcal{M}, d_{\mathcal{F}})$ is an $\mathcal{F}$-metric on $\mathcal{M}$ with $f(t) = \ln(\sqrt{t})$ and $\alpha = \ln\sqrt{7/6}$. Now, define $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ by $\mathcal{T}(1) = \mathcal{T}(2) = \{1\}$ and $\mathcal{T}(3) = \{1, 2\}$. Taking $\zeta(t) = (3/4)t$, we have

$$\begin{aligned} \mathcal{H}_{\mathcal{F}}(\mathcal{T}(1), \mathcal{T}(2)) &= 0, \\ \mathcal{H}_{\mathcal{F}}(\mathcal{T}(1), \mathcal{T}(3)) &= d_{\mathcal{F}}(1,2) = \frac{1}{2} \leq \frac{3}{4}\frac{4}{3} = \frac{3}{4} R_{\mathcal{F}}(1,3), \end{aligned} \quad (41)$$

where

$$\begin{aligned} R_{\mathcal{F}}(1,3) &= \max\left\{d_{\mathcal{F}}(1,3), \frac{d_{\mathcal{F}}(1, \mathcal{T}(1)) d_{\mathcal{F}}(3, \mathcal{T}(3))}{1 + d_{\mathcal{F}}(1,3)}\right\} \\ &= \max\left\{d_{\mathcal{F}}(1,3), \frac{d_{\mathcal{F}}(1,1) d_{\mathcal{F}}(3, \{1,2\})}{1 + d_{\mathcal{F}}(1,3)}\right\}, \\ \mathcal{H}_{\mathcal{F}}(\mathcal{T}(2), \mathcal{T}(3)) &= d_{\mathcal{F}}(1,2) = \frac{1}{2} \leq \frac{3}{4}\frac{2}{3} = \frac{3}{4} R_{\mathcal{F}}(2,3), \end{aligned} \quad (42)$$

where

$$\begin{aligned} R_{\mathcal{F}}(2,3) &= \max\left\{d_{\mathcal{F}}(2,3), \frac{d_{\mathcal{F}}(2, \mathcal{T}(2)) d_{\mathcal{F}}(3, \mathcal{T}(3))}{1 + d_{\mathcal{F}}(1,3)}\right\} \\ &= \max\left\{d_{\mathcal{F}}(1,3), \frac{d_{\mathcal{F}}(2,1) d_{\mathcal{F}}(3, \{1,2\})}{1 + d_{\mathcal{F}}(2,3)}\right\}. \end{aligned} \quad (43)$$

Therefore,

$$\alpha(\sigma, y) \mathcal{H}_{\mathcal{F}}(\mathcal{T}(\sigma), \mathcal{T}(y)) \leq \zeta(R_{\mathcal{F}}(\sigma, y)), \quad (44)$$

where

$$R_{\mathcal{F}}(\sigma, y) = \max\left\{d_{\mathcal{F}}(\sigma, y), \frac{d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma) d_{\mathcal{F}}(y, \mathcal{T}y)}{1 + d_{\mathcal{F}}(\sigma, y)}\right\}, \quad (45)$$

for all $\sigma, y \in \mathcal{M}$. Taking $\alpha(\sigma, y) = 1$ for all $\sigma, y \in \mathcal{M}$, $\mathcal{T}$ satisfies all of the conditions of Theorem 10 and so $\mathcal{T}$ has an endpoint. Here, $\mathcal{T}(1) = \{1\}$.

4. Endpoint Theorem in Graphic $\mathcal{F}$-Metric Spaces

In the present section, we will discuss the existence of endpoints on an $\mathcal{F}$-MS equiped with a graph G, i.e, ($\mathcal{F}$-GMS).

Jachymski [19] has obtained an extension of Banach's contraction principle in metric space equiped with a graph G. Afterwars, Dinevari and Frigon [20] proved his results for multivalued mappings. Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$-MS. A set $\{(\sigma, \sigma) : \sigma \in \mathcal{M}\}$ is said to be a *diagonal* of $\mathcal{M} \times \mathcal{M}$, and represented by Γ. Let G be a graph such that the set $\mathfrak{V}(\mathfrak{G}) = \mathcal{M}$, that is, the set of its vertices and the set $\mathfrak{E}(\mathfrak{G})$ of its edges consists of all loops, i.e., $\Gamma \subseteq \mathfrak{E}(\mathfrak{G})$.

Definition 11. [21] Let $\mathcal{M}$=∅ equiped with a graph G and $\mathcal{T} : \mathcal{M} \longrightarrow 2^{\mathcal{M}}$. The mapping $\mathcal{T}$ is said to preserves edges weakly if, for all $\sigma \in \mathcal{M}$ and $y \in \mathcal{T}\sigma$ with $(\sigma, y) \in \mathfrak{E}(\mathfrak{G})$, we get $(y, z) \in \mathfrak{E}(\mathfrak{G})$, $\forall z \in \mathcal{T}y$.

We give the following definition from [21] which is required in our proof.

Definition 12. Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$ *-GMS.*

The $\mathcal{F}$-GMS $\mathcal{M}$ is called $\mathfrak{E}(\mathfrak{G})$-complete if every Cauchy sequence $\{\sigma_n\}$ in $\mathcal{M}$ with $(\sigma_n, \sigma_{n+1}) \in \mathfrak{E}(\mathfrak{G})$, for all $n \in \mathbb{N}$ converges in $\mathcal{M}$.

Definition 13. A mapping $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ is called *a* $\mathfrak{E}(\mathfrak{G})$ -continuous mapping if, for any $\sigma \in \mathcal{M}$ and any sequence $\{\sigma_n\}$ with $\lim_{n\longrightarrow\infty} d_{\mathcal{F}}(\sigma_n, \sigma) = 0$ and $(\sigma_n, \sigma_{n+1}) \in \mathfrak{E}(\mathfrak{G})$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\longrightarrow\infty} \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma) = 0. \tag{46}$$

Definition 14. A multivalued mapping $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ is called a rational Ćirić type $(\mathfrak{E}(\mathfrak{G}), \zeta)$ -contraction multivalued mapping if there exist a function $\zeta \in \Psi$ such that

$$\sigma, y \in \mathcal{M}, (\sigma, \mathfrak{y}) \in \mathfrak{E}(\mathfrak{G}) \Rightarrow \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma, \mathcal{T}y) \leq \zeta(R_{\mathcal{F}}(\sigma, y)), \tag{47}$$

where $R_{\mathcal{F}}(\sigma, y) = \max\{d_{\mathcal{F}}(\sigma, y), (d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma) d_{\mathcal{F}}(y, \mathcal{T}y))/(1 + d_{\mathcal{F}}(\sigma, y))\}$.

Theorem 15. *Suppose that $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$-GMS and $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ be a rational Ćirić type $(\mathfrak{E}(\mathfrak{G}), \zeta)$ -multivalued contraction. Suppose that the following conditions hold:*

$(S_1)(\mathcal{M}, d_{\mathcal{F}})$ is an $\mathfrak{E}(\mathfrak{G})$-complete $\mathcal{F}$-GMS

$(S_2)\mathcal{T}$ preserves edges weakly

(S_3) there exist σ_0 and $\sigma_1 \in \mathcal{T}\sigma_0$ such that $(\sigma_0, \sigma_1) \in \mathfrak{E}(\mathfrak{G})$

$(S_4)\mathcal{T}$ is an $\mathfrak{E}(\mathfrak{G})$-continuous multivalued mapping

Then, $\mathcal{T}$ has an endpoint point in $\mathcal{M}$.

Proof. This result can be obtain from Theorem 10 if we define a mapping $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ by $\alpha(\sigma, y) = 1$, if $(\sigma, \mathfrak{y}) \in \mathfrak{E}(\mathfrak{G})$ and $\alpha(\sigma, y) = 0$, otherwise. □

5. Endpoint Theorem in Ordered $\mathcal{F}$-Metric Spaces

In 2004, Ran and Reurings [22] gave the idea of ordered metric space (OMS) by combing classical metric space $(\mathcal{M}, d)$ and partial order $^\circ$ on $\mathcal{M}$. Fixed point results in OMS have many applications in integral and differential equations and other fields of mathematical analysis (see [23, 24]). In this section, we will consider ($\mathcal{F}$-OMS), i.e., $(\mathcal{M}, d_{\mathcal{F}}{}^\circ)$ where $(\mathcal{M}, d_{\mathcal{F}})$ is an $\mathcal{F}$-MS and $^\circ$ is a partial order on $\mathcal{M}$ and we will derive some new results from Theorems 8 and 10. Remember that $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{M}$ is nondecreasing if $\forall\sigma, y \in \mathcal{M}, \sigma^\circ y \Rightarrow \mathcal{T}(\sigma)^\circ \mathcal{T}(y)$.

Here, we state the following notion motivated from [25].

Definition 16. Let $\mathcal{M}$=∅ with partial order $^\circ$ *on* $\mathcal{M}$ and $\mathcal{T} : \mathcal{M} \longrightarrow 2^{\mathcal{M}}$. Then, $\mathcal{T}$ is said to be weakly increasing if, f or all $\sigma \in \mathcal{M}$ and $y \in \mathcal{T}\sigma$ with $\sigma^\circ y$, we get that $y^\circ z$, for all $z \in \mathcal{T}y$.

Definition 17. Let $(\mathcal{M}, d_{\mathcal{F}}{}^\circ)$ be an $\mathcal{F}$ *-OMS.*

The $\mathcal{F}$-OMS $\mathcal{M}$ is called $^\circ$ -complete if every Cauchy sequence $\{\sigma_n\}$ in $\mathcal{M}$ with $\sigma_n{}^\circ\sigma_{n+1}$, for all $n \in \mathbb{N}$ converges in $\mathcal{M}$.

Definition 18. A mapping $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ is said to be *a* $^\circ$ -continuous mapping if, for any $\sigma \in \mathcal{M}$ and any sequence $\{\sigma_n\}$ with $\lim_{n\longrightarrow\infty} d_{\mathcal{F}}(\sigma_n, \sigma) = 0$ and $\sigma_n{}^\circ\sigma_{n+1}$, for all $n \in \mathbb{N}$, we get

$$\lim_{n\longrightarrow\infty} \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma_n, \mathcal{T}\sigma) = 0. \tag{48}$$

Motivated from [8], we define the notion of an ordered rational C$'$iric$'$ type ζ -multivalued contraction in an $\mathcal{F}$-OMS.

Definition 19. A multivalued $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ is called an ordered rational C' *iri* c' type ζ -multivalued contraction if there exists $\zeta \in \Psi$ such that

$$\sigma, y \in \mathcal{M}, \sigma^\circ y \Rightarrow \mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma, \mathcal{T}y) \leq \zeta((R_{\mathcal{F}}(\sigma, y)), \tag{49}$$

where $R_{\mathcal{F}}(\sigma, y) = \max\{d_{\mathcal{F}}(\sigma, y), (d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma) d_{\mathcal{F}}(y, \mathcal{T}y))/(1 + d_{\mathcal{F}}(\sigma, y))\}$.

Theorem 20. *Let $(\mathcal{M}, d_{\mathcal{F}^\circ})$ be an $\mathcal{F}$-OMS $^\circ$ and $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ be an ordered rational Ćirić type ζ -multivalued contraction. Assume that these hold:*

$(S_1)(\mathcal{M}, d_{\mathcal{F}})$ is an $^\circ$-complete $\mathcal{F}$-OMS

$(S_2)\mathcal{T}$ is weakly increasing

(S_3) there exist σ_0 and $\sigma_1 \in \mathcal{T}\sigma_0$ such that $\sigma_0{}^\circ\sigma_1$

$(S_4)\mathcal{T}$ is an $^\circ$-continuous multivalued mapping

Then, $\mathcal{T}$ has an endpoint point in $\mathcal{M}$.

Proof. This result can be obtained from Theorem 10 if we define a mapping $\alpha : \mathcal{M} \times \mathcal{M} \longrightarrow [0,\infty)$ by $\alpha(\sigma, y) = 1$, if $\sigma^\circ y$, and $\alpha(\sigma, y) = 0$, otherwise. □

6. Suzuki Type Endpoint Results in $\mathcal{F}$-MS

In 2008, Suzuki [26] obtained a fixed point result as generalization of the Banach fixed point theorem. In this section, we derive endpoint results for rational Suzuki type ζ-multivalued contraction in $\mathcal{F}$-MS as consequence of our result.

Corollary 21. *Let $(\mathcal{M}, d_{\mathcal{F}})$ be a complete $\mathcal{F}$-MS, $\zeta \in \Psi$ and $\mathcal{T} : \mathcal{M} \longrightarrow \mathcal{CB}(\mathcal{M})$ such that $d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma) \leq d_{\mathcal{F}}(\sigma, y) + \zeta(d_{\mathcal{F}}(\sigma, y))$ implies*

$$\mathcal{H}_{\mathcal{F}}(\mathcal{T}\sigma, \mathcal{T}y) \leq \zeta(R_{\mathcal{F}}(\sigma, y)), \tag{50}$$

where $R_{\mathcal{F}}(\sigma, y) = \max\{d_{\mathcal{F}}(\sigma, y), (d_{\mathcal{F}}(\sigma, \mathcal{T}\sigma) d_{\mathcal{F}}(y, \mathcal{T}y))/(1 + d_{\mathcal{F}}(\sigma, y))\}$, for all $\sigma, y \in \mathcal{M}$ and $\mathcal{T}$ satisfies the property $(\mathcal{BS})$. If $\mathcal{T}$ is continuous, then $\mathcal{T}$ has an endpoint.

Proof. Define $\alpha : \mathscr{M} \times \mathscr{M} \longrightarrow [0, \infty)$ by

$$\alpha(\sigma, y) = \begin{cases} 1, & d_{\mathscr{F}}(\sigma, \mathscr{T}\sigma) \le d_{\mathscr{F}}(\sigma, y) + \zeta(d_{\mathscr{F}}(\sigma, y)), \\ 0, & \text{otherwise.} \end{cases} \quad (51)$$

□

It is easy to check that $\mathscr{T}$ is α-admissible.Also, for every $\sigma_0 \in \mathscr{M}$ and $\sigma_1 \in \mathscr{T}\sigma_0$, we have $d_{\mathscr{F}}(\sigma_0, \mathscr{T}\sigma_0) \le d_{\mathscr{F}}(\sigma_0, \sigma_1) \le d_{\mathscr{F}}(\sigma_0, \sigma_1) + \zeta(d_{\mathscr{F}}(\sigma_0, \sigma_1))$. Hence, $\alpha(\sigma_0, \sigma_1) = 1$. It is very simple to check that

$$\alpha(\sigma, y)\mathscr{H}_{\mathscr{F}}(\mathscr{T}\sigma, \mathscr{T}y) \le \zeta(R(\sigma, y)), \quad (52)$$

where $R_{\mathscr{F}}(\sigma, y) = \max\{d_{\mathscr{F}}(\sigma, y), (d_{\mathscr{F}}(\sigma, \mathscr{T}\sigma)d_{\mathscr{F}}(y, \mathscr{T}y))/(1 + d_{\mathscr{F}}(\sigma, y))\}$, for all $\sigma, y \in \mathscr{M}$. Therefore, by Theorem 10, $\mathscr{T}$ has an endpoint.

Corollary 22. *Suppose that $(\mathscr{M}, d_{\mathscr{F}})$ be a complete $\mathscr{F}$-MS, $r \in [0, 1)$ and $\mathscr{T} : \mathscr{M} \longrightarrow \mathscr{CB}(\mathscr{M})$ such that $1/(1 + r)d_{\mathscr{F}}(\sigma, \mathscr{T}\sigma) \le d_{\mathscr{F}}(\sigma, y)$ implies that*

$$\mathscr{H}_{\mathscr{F}}(\mathscr{T}\sigma, \mathscr{T}y) \le rR_{\mathscr{F}}(\sigma, y), \quad (53)$$

where $R_{\mathscr{F}}(\sigma, y) = \max\{d_{\mathscr{F}}(\sigma, y), (d_{\mathscr{F}}(\sigma, \mathscr{T}\sigma)d_{\mathscr{F}}(y, \mathscr{T}y))/(1 + d_{\mathscr{F}}(\sigma, y))\}$, for all $\sigma, y \in \mathscr{M}$ and $\mathscr{T}$ enjoys property ($\mathscr{BS}$). If $\mathscr{T}$ is continuous, then $\mathscr{T}$ has an endpoint.

7. Application to Nonlinear Integral Equations

Let CB ($\mathbb{R}$) represents the set of all nonempty closed and bounded subsets of $\mathbb{R}$ and $\mathfrak{B} = C(I, \mathbb{R})$ be the space of all real-valued continuous functions on $[0, 1]$. Clearly, $\mathfrak{B}$ equiped with the $\mathscr{F}$-metric $d_{\mathscr{F}} : \mathfrak{B} \times \mathfrak{B} \longrightarrow [0, +\infty)$ given by

$$d_{\mathscr{F}}(\sigma, y) = \begin{cases} e^{\|\sigma - y\|}, \text{if } \sigma{=}y, \\ 0, \text{otherwise,} \end{cases} \quad (54)$$

where

$$\|\sigma - y\| = \sup_{t \in I}[|\sigma(t) - y(t)|], \quad (55)$$

is a $\mathscr{F}$-complete $\mathscr{F}$-metric space (see [15]).

Now, we consider the integral equation

$$\sigma(t) {=} {}_0^t K(t, s, \sigma(s))d_{\mathscr{F}} + g(t), \quad (56)$$

$t \in I$, where $\sigma \in \mathfrak{B}$,$K : [0, 1] \times [0, 1] \times \mathbb{R} \longrightarrow CB(\mathbb{R})$ and $g : [0, 1] \longrightarrow \mathbb{R}$ is continuous.

Theorem 23. *Suppose that these conditions hold:*

(i) *for all $\sigma \in \mathfrak{B}$,$K : [0, 1] \times [0, 1] \times \mathbb{R} \longrightarrow CB(\mathbb{R})$ is such that $K(t, s, \sigma(s))$ is continuous in $[0, 1] \times [0, 1]$*

(ii) *there exists $\mathfrak{J} : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ which is continuous that satisfy the property $\inf_{t \in I}{}_0^t \mathfrak{J}(t, s)ds = \tau > 0$ such that for any $\sigma, \hbar \in \mathfrak{B}$ and each $k_\sigma(t, s) \in K_\sigma(t, s, \sigma(s))$, there exists $k_\hbar(t, s) \in K_\sigma(t, s, \hbar(s))$ such that*

$$\begin{aligned} &|k_\sigma(t, s) - k_\hbar(t, s)| \\ &\quad \le \max\{|\sigma(s) - \hbar(s)|, (|\sigma(s) - k_\sigma(t, s)| \\ &\qquad \cdot |\hbar(s) - k_\hbar(t, s)|)/(1 + |\sigma(s) - \hbar(s)|)\} - \mathfrak{J}(t, s) \end{aligned} \quad (57)$$

for all $t, s \in [0, 1]$.

Then, the integral equation (56) has at least one solution in $\mathfrak{B}$.

Proof. Suppose that multivalued mapping $\mathscr{T} : \mathfrak{B} \longrightarrow CB(\mathfrak{B})$ defined by

$$\mathscr{T}\sigma = \left\{\varpi \in \mathfrak{B} : \varpi(t) \in g(t) + \int_0^t K(t, s, \sigma(s))d_{\mathscr{F}}s, t \in [a, b]\right\}, \quad (58)$$

for all $\sigma \in \mathfrak{B}$. Evidently, each endpoint of $\mathscr{T}$ is a solution of (56). □

Next, consider the set-valued operator $K_\sigma(t, s)$: $[0, 1] \times [0, 1] \longrightarrow CB(\mathbb{R})$, defined by

$$K_\sigma(t, s) = K(t, s, \sigma(s)). \quad (59)$$

Then, by Michael's selection theorem, $\exists k_\sigma(t, s) : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ such that $k_\sigma(t, s) \in K_\sigma(t, s)$ for each $t, s \in [0, 1]$. This implies that $g(t) + \int_0^t k_\rho(t, s)d_{\mathscr{F}}s \in T\sigma$. Hence, $T\sigma{=}\varnothing$. Next, we prove that the multivalued function $\mathscr{T}$ satisfies all the conditions of Theorem 10. Let $\sigma, \mathfrak{y} \in \mathfrak{B}$ and $\rho(t) \in \mathscr{T}\sigma$. Then, $\exists k_\sigma(t, s) \in K_\sigma(t, s)$ for each $t, s \in [0, 1]$ such that

$$\rho(t) = g(t) + \int_0^t \mathrm{k}_\sigma(t, s)d_{\mathscr{F}}s, \quad (60)$$

for $t \in [0, 1]$. On the other side, by assumption (ii), $\exists$ $k_\hbar(\mathrm{t}, \mathrm{s}) \in \mathrm{K}_\sigma(\mathrm{t}, \mathrm{s})$ such that (57) holds. Now, by taking

$$\varpi(t) = g(t) + \int_0^t k_\hbar(t, s)d_{\mathscr{F}}s, \quad (61)$$

we get

$$\varpi(t) = g(t) + \int_0^t K(t, s, \hbar(s))d_{\mathscr{F}}s = \mathscr{T}\hbar, \quad (62)$$

for $t \in [0, 1]$,

$$\begin{aligned}
d_{\mathscr{F}}(\rho, \varpi) &= e^{\|\rho-\varpi\|} \le e^{\sup_{t\in[0,1]} \left|\int_0^t k_\sigma(t,s)ds - \int_0^t k_\hbar(t,s)ds\right|} \\
&\le e^{\sup_{t\in[0,1]} \int_0^t |k_\sigma(t,s)-k_\hbar(t,s)|ds} \\
&\le e^{\sup_{t\in[0,1]} \int_0^t |\max\{|\sigma(s)-y(s)|,(|\sigma(s)-k_\sigma(t,s)||\hbar(s)-k_\hbar(t,s)|)/(1+|\sigma(s)-\hbar(s)|)\}-\mathfrak{I}(t,s)|ds} \\
&= e^{\sup_{t\in[0,1]} \int_0^t |\max\{|\sigma(s)-\hbar(s)|,(|\sigma(s)-k_\sigma(t,s)||\hbar(s)-k_\hbar(t,s)|)/(1+|\sigma(s)-\hbar(s)|)\}|ds - \int_0^t \mathfrak{I}(t,s)ds} \\
&\le e^{\max\{\|\sigma(s)-\hbar(s)\|,\|(|\sigma(s)-k_\sigma(t,s)||\hbar(s)-k_\hbar(t,s)|)/(1+|\sigma(s)-\hbar(s)|)\|\} - \inf_{t\in[0,1]} \int_0^t \mathfrak{I}(t,s)ds} \\
&\le e^{\max\{\|\sigma(s)-\hbar(s)\|,\|(|\sigma(s)-k_\sigma(t,s)||\hbar(s)-k_\hbar(t,s)|)/(1+|\sigma(s)-\hbar(s)|)\|\}-\tau} \\
&\le e^{\max\{\|\sigma(s)-\hbar(s)\|,\|(|\sigma(s)-k_\sigma(t,s)||\hbar(s)-k_\hbar(t,s)|)/(1+|\sigma(s)-\hbar(s)|)\|\}} \cdot e^{-\tau} \\
&= \zeta(R_{\mathscr{F}}(\sigma, \mathrm{y})),
\end{aligned} \tag{63}$$

where $\zeta(t) = e^{-\tau} t$. By interchanging the roles of σ and y, we get that

$$\mathscr{H}_{\mathscr{F}}(\mathscr{T}\sigma, \mathscr{T}\mathscr{y}) \le \zeta(R_{\mathscr{F}}(\sigma, \mathrm{y})), \tag{64}$$

where $R_{\mathscr{F}}(\sigma, \mathrm{y}) = \max\{d_{\mathscr{F}}(\sigma, \mathrm{y}), (d_{\mathscr{F}}(\sigma, \mathscr{T}\sigma) d_{\mathscr{F}}(\mathscr{y}, \mathscr{T}\mathscr{y}))/(1 + d_{\mathscr{F}}(\sigma, \mathrm{y}))\}$, for all $\sigma, \mathfrak{y} \in \mathfrak{B}$. Taking $\alpha(\sigma, \mathrm{y}) = 1$, for all $\sigma, \mathfrak{y} \in \mathfrak{B}$, all of the conditions of Theorem 10 are satisfied, and thus, $\mathscr{T}$ has an endpoint, which is a solution of integral equation (56).

Data Availability

No such data were used for this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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