Hindawi
Journal of Function Spaces
Volume 2022, Article ID 3740809, 11 pages
https://doi.org/10.1155/2022/3740809

Research Article

Estimating Fixed Points via New Iterative Scheme with an Application

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Received 25 July 2021; Revised 10 August 2021; Accepted 18 February 2022; Published 4 March 2022

Academic Editor: Andreea Fulga

In this paper, we introduce a new scheme and prove convergence results for nonexpansive mappings as well as for weak contractions in the frame of Banach spaces. Moreover, we prove analytically and numerically that the proposed scheme converges to a fixed point of a weak contraction faster than some known and leading schemes. Further, we prove that the new scheme is almost stable with respect to weak contraction. For supporting the main results, we give a couple of nontrivial numerical examples, and the visualization is shown by using the Matlab program. Finally, the solution of a nonlinear fractional differential equation is approximated by operating the main result of the paper.

1. Introduction

Throughout the paper, $\mathbb{Z}_0^+$ denotes the set of nonnegative integers. Let E be a nonempty subset of a Banach space $\mathcal{Z}$, $\mathcal{K} : E \longrightarrow E$ is a mapping and $F(\mathcal{K}) = \{w \in E : \mathcal{K}w = w\}$. A mapping $\mathcal{K} : E \longrightarrow E$ is said to be nonexpansive if for each $x, y \in E$,

$$\|\mathcal{K}x - \mathcal{K}y\| \leq \|x - y\|. \tag{1}$$

A self-map $\mathcal{K}$ on E is said to be a weak contraction [1] if for all $x, y \in E \exists$ a constant $\delta \in (0, 1)$ and some $\mathcal{L} \geq 0$ such that

$$\begin{aligned} \|\mathcal{K}x - \mathcal{K}y\| &\leq \delta\|x - y\| + \mathcal{L}\|x - \mathcal{K}y\|, \\ \|\mathcal{K}x - \mathcal{K}y\| &\leq \delta\|x - y\| + \mathcal{L}\|y - \mathcal{K}x\|. \end{aligned} \tag{2}$$

Theorem 1 (see [1]). *Let $\mathcal{K} : \mathcal{Z} \longrightarrow \mathcal{Z}$ be a weak contraction with $\delta \in (0,1)$ and some $\mathcal{L} \geq 0$ such that*

$$\|\mathcal{K}x - \mathcal{K}y\| \leq \delta\|x - y\| + \mathcal{L}\|x - \mathcal{K}x\|, \quad \forall x, y \in \mathcal{Z}. \tag{3}$$

Then, $\mathcal{K}$ has a unique fixed point, and Picard sequence converges to the fixed point.

But if we take an initial guess different from a fixed point in case of nonexpansive mapping, it is to be noted that the Picard iterative scheme fails to converge to the fixed points of such mappings; hence, we need some other iterative schemes. In the sequel, many authors gave the generalizations of nonexpansive mapping and proved existence and convergence results in linear space, e.g., see [2].

However, to find the fixed points of numerous nonlinear mappings is not an easy task. So, to overcome this kind of problem, several researchers constructed iterative schemes to approximate fixed points of mappings. A few of them are Picard-S [3], Thakur-New [4], and others [5–10].

Here, we consider some iterative schemes which are frequently used to approximate the fixed points of nonlinear mappings introduced by Picard [11], Mann [12], Ishikawa [13], Noor [14], and Agarwal et al. (S) [15], respectively, where the sequence $\{\tau_n\}$ is developed by an arbitrary point $\tau_0 \in E$ as follows:

$$\{\tau_{n+1} = \mathcal{K}\tau_0, \quad n \in \mathbb{Z}_0^+, \tag{4}$$

$$\{\tau_{n+1} = (1 - \theta_n)\tau_n + \theta_n\mathcal{K}\tau_n, \quad n \in \mathbb{Z}_0^+, \tag{5}$$

$$\begin{cases} \tau_{n+1} = (1-\theta_n)\tau_n + \theta_n \mathscr{K}\sigma_n, \\ \sigma_n = (1-\mu_n)\tau_n + \mu_n \mathscr{K}\tau_n, \quad n \in \mathbb{Z}_0^+, \end{cases} \tag{6}$$

$$\begin{cases} \tau_{n+1} = (1-\theta_n)\tau_n + \theta_n \mathscr{K}\sigma_n, \\ \sigma_n = (1-\mu_n)\tau_n + \mu_n \mathscr{K}\xi_n, \\ \xi_n = (1-\gamma_n)\tau_n + \gamma_n \mathscr{K}\tau_n, \quad n \in \mathbb{Z}_0^+, \end{cases} \tag{7}$$

$$\begin{cases} \tau_{n+1} = (1-\theta_n)\mathscr{K}\tau_n + \theta_n \mathscr{K}\sigma_n, \\ \sigma_n = (1-\mu_n)\tau_n + \mu_n \mathscr{K}\tau_n, \quad n \in \mathbb{Z}_0^+, \end{cases} \tag{8}$$

where $\{\theta_n\}$, $\{\mu_n\}$, and $\{\gamma_n\}$ are sequences in $(0, 1)$.

Motivated by the previous work, we define a new iterative scheme for finding the fixed point of a weak contraction, where the sequence $\{\tau_n\}$ is developed iteratively by $\tau_0 \in E$ and

$$\begin{cases} \tau_{n+1} = \mathscr{K}((1-\theta_n)\sigma_n + \theta_n \mathscr{K}\sigma_n), \\ \sigma_n = (1-\mu_n)\mathscr{K}\tau_n + \mu_n \mathscr{K}\xi_n, \\ \xi_n = (1-\gamma_n)\tau_n + \gamma_n \mathscr{K}\tau_n, \quad n \in \mathbb{Z}_0^+, \end{cases} \tag{9}$$

where $\{\theta_n\}$, $\{\mu_n\}$, and $\{\gamma_n\}$ are sequences in $(0, 1)$.

Our main focus is to consider those iterative schemes which save time when we approximate fixed points of mappings. Berinde [16] gave the following definitions about the rate of convergence of iterative schemes which is defined as follows:

Definition 2. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of positive numbers that converge to α and β, respectively. Assume that

$$\ell = \lim_{n\longrightarrow\infty} \frac{|\alpha_n - \alpha|}{|\beta_n - \beta|}. \tag{10}$$

(i) If $\ell = 0$, then $\{\alpha_n\}$ converges to α faster than $\{\beta_n\}$ to β

(ii) If $0 < \ell < \infty$, then $\{\alpha_n\}$ and $\{\beta_n\}$ have the same rate of convergence

Definition 3. Consider $\{\tau_n\}$ and $\{\sigma_n\}$ as two fixed point iterative schemes both converging to the same point t of a mapping with error estimates

$$\begin{aligned} |\tau_n - t| &\le \alpha_n, \\ |\sigma_n - t| &\le \beta_n. \end{aligned} \tag{11}$$

If $\lim_{n\longrightarrow\infty} \alpha_n/\beta_n = 0$, then $\{\tau_n\}$ converges faster than $\{\sigma_n\}$.

Now, we discuss another concept related to iterative schemes called stability. Let $\{\tau_n\}$ be a theoretical sequence and $\{t_n\}$ an approximate sequence which is due to rounding errors and numerical approximation of functions. We say that the approximate sequence $\{t_n\}$ converges to the fixed point of mapping $\mathscr{K}$ if and only if the given fixed point iterative scheme would be stable. Because of this fact, the concept of stability for a fixed point iterative scheme was coined by Ostrowski [17] which defined as follows.

Definition 4 (see [17]). Let $\tau_{n+1} = f(\mathscr{K}, \tau_n)$ be an iteration procedure, converging to a fixed point w, which is said to be $\mathscr{K}$ -stable or stable with respect to $\mathscr{K}$, if for $\epsilon_n = \|t_{n+1} - f(\mathscr{K}, t_n)\|$, $n \in \mathbb{Z}_0^+$, we have $\lim_{n\longrightarrow\infty} \epsilon_n = 0 \Leftrightarrow \lim_{n\longrightarrow\infty} t_n = w$, where $\{t_n\}$ is an approximate sequence in a subset E of a Banach space $\mathscr{Z}$.

In 1998, a weaker concept of stability, called almost stability, was coined by Osilike [18] which is defined as follows.

Definition 5 (see [18]). Let $\tau_{n+1} = f(\mathscr{K}, \tau_n)$ be an iteration procedure, converging to fixed point w, which is said to be almost $\mathscr{K}$ -stable or almost stable with respect to $\mathscr{K}$, if for $\epsilon_n = \|t_{n+1} - f(\mathscr{K}, t_n)\|$, $n \in \mathbb{Z}_0^+$, we have $\sum_{n=0}^{\infty} \epsilon_n < \infty \Rightarrow \lim_{n\longrightarrow\infty} t_n = w$.

Remark 6 (see [18]). It can be easily seen that any $\mathscr{K}$-stable iteration procedure is almost $\mathscr{K}$ -stable, but reverse may fail.

Lemma 7 (see [19]). *Let $0 \le \delta < 1$ and $\{\epsilon_n\}$ and $\{u_n\}$ be any two sequences of nonnegative numbers satisfying $u_{n+1} \le \delta u_n + \epsilon_n$, $n \in \mathbb{Z}_0^+$. If $\sum_{n=0}^{\infty} \epsilon_n < \infty$, then $\sum_{n=0}^{\infty} u_n < \infty$.*

2. Preliminaries

Definition 8. A self-mapping $\mathscr{K}$ on a Banach space $\mathscr{Z}$ is said to be demiclosed at y, if for any sequence $\{\tau_n\}$ which converges weakly to x, and if the sequence $\{\mathscr{K}(\tau_n)\}$ converges strongly to y, then $\mathscr{K}(x) = y$.

Definition 9. A sequence $\{\tau_n\}$ in a normed space $\mathscr{Z}$ is said to be weakly convergent (denoted by $\rightharpoonup$) if $\exists$ an element $x \in \mathscr{Z}$ such that

$$\lim_{n\longrightarrow\infty} \mathscr{K}(\tau_n) = \mathscr{K}(x), \quad \forall \mathscr{K} \in \mathscr{Z}^*. \tag{12}$$

Definition 10. A Banach space $\mathscr{Z}$ is said to satisfy Opial's property [20] if for any $\{\tau_n\} \rightharpoonup f$ in $\mathscr{Z} \Rightarrow \lim_{n\longrightarrow\infty} \inf \|\tau_n - f\| < \lim_{n\longrightarrow\infty} \inf \|\tau_n - g\|$ for all $g \in \mathscr{Z}$ with $g \ne f$.

Lemma 11 (see [21]). *Let E be a nonempty closed and convex subset of a uniformly convex Banach space $\mathscr{Z}$ and $\mathscr{K}$ a nonexpansive mapping on E. Then, $I - \mathscr{K}$ is demiclosed at zero.*

Lemma 12 (see [22]). *Let $\mathscr{Z}$ be a uniformly convex Banach space and $0 < a \le \omega_n \le b < 1$ for all $n \in \mathbb{N}$. Assume that $\{\tau_n\}$ and $\{\sigma_n\}$ are two sequences in $\mathscr{Z}$ such that $\lim_{n\longrightarrow\infty} \sup \|\tau_n\| \le \omega$, $\lim_{n\longrightarrow\infty} \sup \|\sigma_n\| \le \omega$, and $\lim_{n\longrightarrow\infty} \sup \|\omega_n \tau_n + (1-\omega_n)\sigma_n\| = \omega$ holds, for some $\omega \ge 0$. Then, $\lim_{n\longrightarrow\infty} \|\tau_n - \sigma_n\| = 0$.*

Definition 13 (see [23]). A mapping $\mathcal{K} : E \longrightarrow E$ is said to satisfy property (A), if $\exists$ a nondecreasing mapping $\psi : [0, \infty) \longrightarrow [0,\infty)$ with $\psi(0) = 0$ and $\psi(z) > 0$, $\forall z > 0$, such that $d(x, \mathcal{K}x) \geq \psi(d(x, F(\mathcal{K})))$, $\forall x \in E$.

3. Convergence Result for Weak Contractions

Throughout this section, we presume that E is a nonempty closed and convex subset of a normed linear space $\mathcal{Z}$ and $\mathcal{K} : E \longrightarrow E$ a weak contraction satisfying (3) with $F(\mathcal{K}) \neq \emptyset$.

Theorem 14. *Let $\{\tau_n\}$ be a sequence developed by new iterative scheme (9), then $\{\tau_n\}$ converges to a fixed point of $\mathcal{K}$.*

Proof. From (9), for any $w \in F(\mathcal{K})$,

$$\|\xi_n - w\| = \|(1-\gamma_n)\tau_n + \gamma_n \mathcal{K}\tau_n - w\| \leq (1-\gamma_n)\|\tau_n - w\| + \delta\gamma_n\|\tau_n - w\| = (1-(1-\delta)\gamma_n)\|\tau_n - w\|,$$

$$\begin{aligned}\|\sigma_n - w\| &= \|(1-\mu_n)\mathcal{K}\tau_n + \mu_n\mathcal{K}\xi_n - w\| \\ &\leq \delta(1-\mu_n)\|\tau_n - w\| + \delta\mu_n\|\xi_n - w\| \\ &\leq \delta(1-(1-\delta)\mu_n\gamma_n)\|\tau_n - w\|,\end{aligned}$$

$$\begin{aligned}\|\tau_{n+1} - w\| &= \|\mathcal{K}((1-\theta_n)\sigma_n + \theta_n\mathcal{K}\sigma_n) - w\| \\ &\leq \delta\|(1-\theta_n)\sigma_n + \theta_n\mathcal{K}\sigma_n - w\| \\ &\leq \delta^2(1-(1-\delta)\theta_n)(1-(1-\delta)\mu_n\gamma_n)\|\tau_n - w\|.\end{aligned} \tag{13}$$

By using the fact that $0 < (1-(1-\delta)\theta_n) \leq 1$ and $0 < (1-(1-\delta)\mu_n\gamma_n) \leq 1$, we have

$$\|\tau_{n+1} - w\| \leq \delta^2\|\tau_n - w\|. \tag{14}$$

Inductively, we get

$$\|\tau_{n+1} - w\| \leq \delta^{2(n+1)}\|\tau_0 - w\|. \tag{15}$$

Since $0 < \delta < 1$, $\{\tau_n\}$ converges to w.

Now, we prove almost stability of new iterative scheme (9) with respect to a weak contraction. □

Theorem 15. *Let $\{\tau_n\}$ be a sequence developed by iterative scheme (9), then $\{\tau_n\}$ is almost $\mathcal{K}$-stable.*

Proof. Consider $\{t_n\}$ an approximate sequence of $\{\tau_n\}$ in E. Suppose sequence defined by (9) is $\tau_{n+1} = f(\mathcal{K}, \tau_n)$ converging to a fixed point w (by Theorem 14) and $\epsilon_n = \|t_{n+1} - f(\mathcal{K}, t_n)\|$, $n \in \mathbb{Z}_0^+$. Now, we will prove that $\sum_{n=0}^{\infty}\epsilon_n < \infty \Rightarrow \lim_{n\longrightarrow\infty} t_n = w$. □

Let $\sum_{n=0}^{\infty}\epsilon_n < \infty$, then by iterative scheme (9), we have

$$\begin{aligned}\|t_{n+1} - w\| &\leq \|t_{n+1} - f(\mathcal{K}, t_n)\| + \|f(\mathcal{K}, t_n) - w\| \\ &= \epsilon_n + \|f(\mathcal{K}, t_n) - w\| \leq \epsilon_n + \delta^2(1-(1-\delta)\theta_n) \\ &\quad \cdot (1-(1-\delta)\mu_n\gamma_n)\|t_n - w\|.\end{aligned} \tag{16}$$

Since $0 < (1-(1-\delta)\theta_n) \leq 1$ and $0 < (1-(1-\delta)\mu_n\gamma_n) \leq 1$ and using (16), we get

$$\|t_{n+1} - w\| \leq \epsilon_n + \delta^2\|t_n - w\|. \tag{17}$$

Define $u_n = \|t_n - w\|$, then

$$u_{n+1} \leq \delta^2 u_n + \epsilon_n. \tag{18}$$

Since $\sum_{n=0}^{\infty}\epsilon_n < \infty$, by Lemma 7, we have $\sum_{n=0}^{\infty}u_n < \infty$. This implies $\lim_{n\longrightarrow\infty} u_n = 0$, i.e., $\lim_{n\longrightarrow\infty} t_n = w$. This shows that new iterative scheme (9) is almost $\mathcal{K}$-stable.

There is analytical comparison of the rate of convergence of iterative schemes with new iterative scheme (9) for weak contraction.

Theorem 16. *Suppose that the sequence $\{\tau_{1,n}\}$ is introduced by Picard (4), $\{\tau_{2,n}\}$ by Mann (5), $\{\tau_{3,n}\}$ by Ishikawa (6), $\{\tau_{4,n}\}$ by Noor (7), $\{\tau_{5,n}\}$ by Agrawal (8), and $\{\tau_n\}$ by (9) iterative scheme which converges to the same point w. Then iterative scheme (9) converges faster than all the schemes (4)–(8) to a fixed point of $\mathcal{K}$.*

Proof. Using equation (15) of Theorem 14, we have

$$\|\tau_{n+1} - w\| \leq \delta^{2(n+1)}\|\tau_0 - w\| = \alpha_n, \quad n \in \mathbb{Z}_0^+. \tag{19}$$

From equation (7), we get

$$\begin{aligned}\|\xi_n - w\| &= \|(1-\gamma_n)\tau_n + \gamma_n\mathcal{K}\tau_n - w\| \\ &\leq (1-(1-\delta)\gamma_n)\|\tau_n - w\|.\end{aligned} \tag{20}$$

It can be easily seen that $0 < (1-(1-\delta)\gamma_n) \leq 1$, so we get

$$\|\xi_n - w\| \leq \|\tau_n - w\|. \tag{21}$$

Using (21), we obtain that

$$\begin{aligned}\|\sigma_n - w\| &= \|(1-\mu_n)\tau_n + \mu_n\mathcal{K}\xi_n - w\| \leq (1-\mu_n)\|\tau_n - w\| \\ &\quad + \delta\mu_n\|\xi_n - w\| \leq (1-\mu_n)\|\tau_n - w\| \\ &\quad + \delta\mu_n\|\tau_n - w\| \leq (1-(1-\delta)\mu_n)\|\tau_n - w\|.\end{aligned} \tag{22}$$

Again, it can be easily seen that $0 < (1-(1-\delta)\mu_n) \leq 1$, so we get

$$\|\sigma_n - w\| \leq \|\tau_n - w\|. \tag{23}$$

Table 1: Comparison of speed of the convergence of different iterative schemes.

Iter.	Picard	Mann	Ishikawa
1	(0.200000, 0.400000)	(0.200000, 0.400000)	(0.200000, 0.400000)
2	(0.099335, 0.097355)	(0.164767, 0.294074)	(0.156965, 0.282817)
⋮			
19	(0.000001, 0.000000)	(0.006241, 0.001655)	(0.002593, 0.000796)
20	(0.000000, 0.000000)	(0.005149, 0.001221)	(0.002037, 0.000564)
⋮			
54	(0.000000, 0.000000)	(0.000007, 0.000000)	(0.000001, 0.000000)
55	(0.000000, 0.000000)	(0.000006, 0.000000)	(0.000000, 0.000000)
⋮			
67	(0.000000, 0.000000)	(0.000001, 0.000000)	(0.000000, 0.000000)
⋮			
68	(0.000000, 0.000000)	(0.000001, 0.000000)	(0.000000, 0.000000)
69	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
70	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)

Using (23), we obtain that

$$\begin{aligned}\|\tau_{n+1}-w\| = \|(1-\theta_n)\tau_n+\theta_n\mathscr{K}\sigma_n - w\| \le (1-\theta_n)\|\tau_n - w\| \\ +\delta\theta_n\|\sigma_n - w\| \le (1-\theta_n)\|\tau_n - w\| \\ +\delta\theta_n\|\tau_n - w\| \le (1-(1-\delta)\theta_n)\|\tau_n - w\|.\end{aligned} \tag{24}$$

By using the fact that $0<(1-(1-\delta)\theta_n)\le 1$, we have

$$\|\tau_{n+1}-w\| \le \|\tau_n - w\|. \tag{25}$$

Inductively, we get

$$\|\tau_{n+1}-w\| \le \|\tau_0 - w\|. \tag{26}$$

Let

$$\left\|\tau_{4,n}-w\right\| \le \left\|\tau_{4,0}-w\right\| = \alpha_{4,n}, \quad n\in\mathbb{Z}_0^+. \tag{27}$$

Then

$$\frac{\alpha_n}{\alpha_{4,n}} = \frac{\delta^{2(n+1)}\|\tau_0 - w\|}{\left\|\tau_{4,0}-w\right\|}. \tag{28}$$

Thus, $\{\tau_n\}$ converges faster than $\{\tau_{4,n}\}$ to w because $(0<\delta<1)$, then $\alpha_n/\alpha_{4,n}\longrightarrow 0$ as $n\longrightarrow\infty$. □

By applying a similar approach, we can also show that the rate of convergence of all the other leading iterative schemes to w is slower than iterative scheme (9).

We embellish the following example to support our assertion.

Example 17. Let $\mathscr{Z}=\mathbb{R}^2$ be a Banach space with respect to the norm $\|x\| = \|(x_1,x_2)\| = |x_1|+|x_2|$ and $E=\{x=(x_1,x_2) : (x_1,x_2)\in[0,1]\times[0,1]\}$ be a subset of $\mathscr{Z}$. Let $\mathscr{K}: E\longrightarrow E$ be defined by

$$\mathscr{K}(x_1,x_2) = \begin{cases}\left(\frac{1}{2}\sin(x_1), \frac{1}{4}\sin(x_2)\right), & \text{if } (x_1,x_2)\in\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right], \\ \left(\frac{1}{2}x_1, \frac{1}{4}x_2\right), & \text{if } (x_1,x_2)\in\left(\frac{1}{2},1\right]\times\left(\frac{1}{2},1\right].\end{cases} \tag{29}$$

Then $\mathscr{K}$ is a weak contraction satisfying (3) for $\delta = 1/2 = L$, but $\mathscr{K}$ is not a contraction mapping.

By Matlab 2015a, we exhibit that new iterative scheme (9) converges to a fixed point $w=(0,0)$ of the mapping $\mathscr{K}$ faster than the iterative schemes Picard, Mann, Ishikawa, Noor, and S with initial point $\tau_0=(0.20{,}0.40)$ and control sequences $\theta_n = 0.35$, $\mu_n = 0.45$, and $\gamma_n = 0.75$, $n\in\mathbb{Z}_0^+$, which can be easily seen in Tables 1 and 2 and Figure 1.

4. Convergence Results for Nonexpansive Mapping

Throughout this section, we presume that E is a nonempty, closed, and convex subset of a uniformly convex Banach space $\mathscr{Z}$ and $\mathscr{K}: E\longrightarrow E$ is a nonexpansive mapping. Now, we prove the following useful lemmas which are used to prove the next results of this section.

Lemma 18. *Let $\{\tau_n\}$ be a sequence developed by new iterative scheme (9), then $\lim_{n\longrightarrow\infty}\|\tau_n - w\|$ exists for all $w\in F(\mathscr{K})$.*

Proof. Suppose $w\in F(\mathscr{K})$ and $\{\tau_n\}\in E$. From (9), we have

$$\|\xi_n - w\| = \|(1-\gamma_n)\tau_n+\gamma_n\mathscr{K}\tau_n - w\| \le \|\tau_n - w\|, \tag{30}$$

Table 2: Comparison of speed of the convergence of different iterative schemes.

Iter.	Noor	S	New scheme
1	(0.200000, 0.400000)	(0.200000, 0.400000)	(0.200000, 0.400000)
2	(0.154063, 0.280745)	(0.091532, 0.086097)	(0.034027, 0.013435)
⋮			
8	(0.032294, 0.033698)	(0.000873, 0.000010)	(0.000001, 0.000000)
9	(0.024894, 0.023670)	(0.000402, 0.000002)	(0.000000, 0.000000)
⋮			
17	(0.003104, 0.001403)	(0.000001, 0.000000)	(0.000000, 0.000000)
18	(0.002392, 0.000985)	(0.000000, 0.000000)	(0.000000, 0.000000)
⋮			
50	(0.000001, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
51	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
⋮			
69	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
70	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)

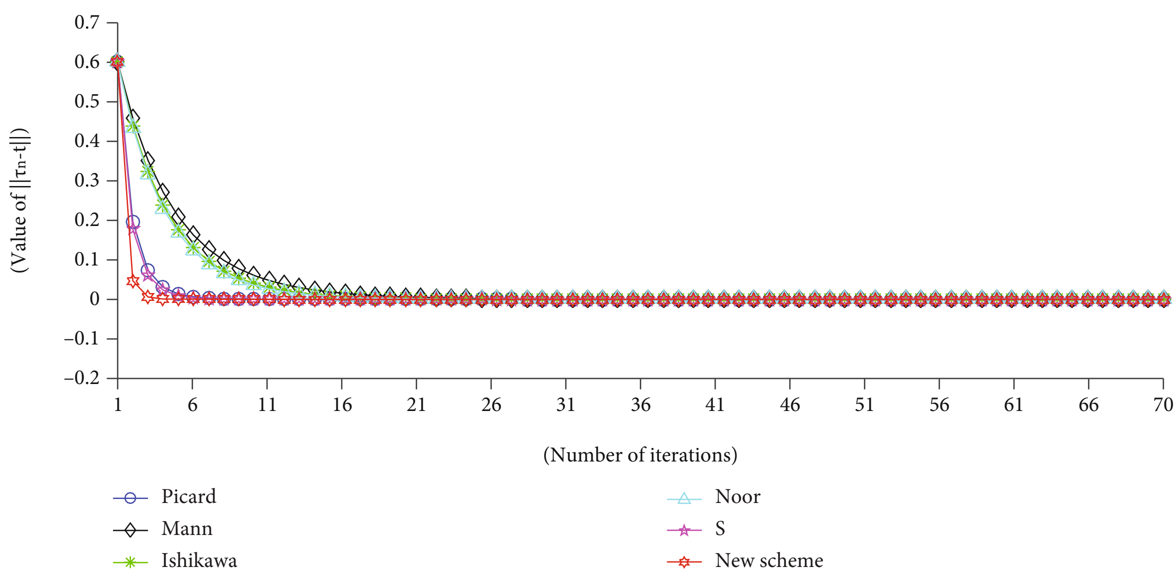

Figure 1: Graphical representation of iterative schemes.

$$\begin{aligned}\|\sigma_n - w\| &= \|(1-\mu_n)\mathscr{K}\tau_n + \mu_n\mathscr{K}\xi_n - w\| \\ &\le (1-\mu_n)\|\tau_n - w\| + \mu_n\|\xi_n - w\| \\ &\le (1-\mu_n)\|\tau_n - w\| + \mu_n\|\tau_n - w\| \le \|\tau_n - w\|.\end{aligned} \tag{31}$$

Using (31), we get

$$\begin{aligned}\|\tau_{n+1} - w\| &= \|\mathscr{K}((1-\theta_n)\sigma_n + \theta_n\mathscr{K}\sigma_n) - w\| \\ &\le \|(1-\theta_n)\sigma_n + \theta_n\mathscr{K}\sigma_n - w\| \le (1-\theta_n)\|\sigma_n - w\| \\ &\quad + \theta_n\|\sigma_n - w\| = \|\sigma_n - w\| \le \|\tau_n - w\|,\end{aligned} \tag{32}$$

which exhibit that $\{\|\tau_n - w\|\}$ is decreasing and bounded below. Therefore, $\lim_{n\longrightarrow\infty}\|\tau_n - w\|$ exists. □

Lemma 19. *Let $F(\mathscr{K}) \neq \emptyset$ and $\{\tau_n\}$ be the iterative scheme developed by equation (9). Then $\lim_{n\longrightarrow\infty}\|\tau_n - \mathscr{K}\tau_n\| = 0$.*

Proof. Since $\lim_{n\longrightarrow\infty}\|\tau_n - w\|$ exists by Lemma 18 and it is given that $F(\mathscr{K}) \neq \emptyset$ with $w \in F(\mathscr{K})$. Presume that $\lim_{n\longrightarrow\infty}\|\tau_n - w\| = c$.

By the inequalities (30) and (31), we get

$$\lim_{n\longrightarrow\infty} \sup \|\xi_n - w\| \le c, \tag{33}$$

$$\lim_{n\longrightarrow\infty} \sup \|\sigma_n - w\| \le c, \tag{34}$$

respectively. Since $\mathscr{K}$ is nonexpansive mapping, we have

$$\|\mathscr{K}\tau_n - w\| \le \|\tau_n - w\|, \|\mathscr{K}\sigma_n - w\| \le \|\sigma_n - w\|, \|\mathscr{K}\xi_n - w\| \le \|\xi_n - w\|. \tag{35}$$

Using (35), we get

$$\lim_{n\longrightarrow\infty} \sup \|\mathscr{K}\tau_n - w\| \le c, \tag{36}$$

$$\lim_{n\longrightarrow\infty} \sup \|\mathscr{K}\sigma_n - w\| \le c, \tag{37}$$

$$\lim_{n\longrightarrow\infty} \sup \|\mathscr{K}\xi_n - w\| \le c. \tag{38}$$

Since

$$\begin{aligned}\|\tau_{n+1} - w\| &= \|\mathscr{K}((1-\theta_n)\sigma_n + \theta_n\mathscr{K}\sigma_n) - w\| \\ &\le \|(1-\theta_n)\sigma_n + \theta_n\mathscr{K}\sigma_n - w\| \le (1-\theta_n)\|\sigma_n - w\| \\ &\quad + \theta_n\|\sigma_n - w\| \le \|\sigma_n - w\|.\end{aligned} \tag{39}$$

Now,

$$c = \lim_{n\longrightarrow\infty} \inf \|\tau_{n+1} - w\| \le \lim_{n\longrightarrow\infty} \inf \|\sigma_n - w\|. \tag{40}$$

So that (33) and (40) give

$$\lim_{n\longrightarrow\infty} \|\sigma_n - w\| = c, \tag{41}$$

$$\begin{aligned}c &= \lim_{n\longrightarrow\infty} \|\sigma_n - w\| = \lim_{n\longrightarrow\infty} \|(1-\mu_n)\mathscr{K}\tau_n + \mu_n\mathscr{K}\xi_n - w\| \\ &= \lim_{n\longrightarrow\infty} \|(1-\mu_n)(\mathscr{K}\tau_n - w) + \mu_n(\mathscr{K}\xi_n - w)\|\end{aligned} \tag{42}$$

by using Lemma 12 and Inequality (36) and (38), we have

$$\lim_{n\longrightarrow\infty} \|\mathscr{K}\tau_n - \mathscr{K}\xi_n\| = 0. \tag{43}$$

Now,

$$\begin{aligned}\|\sigma_n - w\| &= \|(1-\mu_n)\mathscr{K}\tau_n + \mu_n\mathscr{K}\xi_n - w\| \le \|\mathscr{K}\tau_n - w\| \\ &\quad + \mu_n\|\mathscr{K}\xi_n - \mathscr{K}\tau_n\| \le \|\mathscr{K}\tau_n - w\|,\end{aligned} \tag{44}$$

which gives

$$c \le \lim_{n\longrightarrow\infty} \inf \|\mathscr{K}\tau_n - w\|, \tag{45}$$

using (36) and (45), we get

$$\lim_{n\longrightarrow\infty} \|\mathscr{K}\tau_n - w\| = c. \tag{46}$$

On the other hand, we have

$$\begin{aligned}\|\mathscr{K}\tau_n - w\| &\le \|\mathscr{K}\tau_n - \mathscr{K}\xi_n\| + \|\mathscr{K}\xi_n - w\| \\ &\le \|\mathscr{K}\tau_n - \mathscr{K}\xi_n\| + \|\xi_n - w\|.\end{aligned} \tag{47}$$

Applying liminf on both sides, we get

$$c \le \lim_{n\longrightarrow\infty} \inf \|\xi_n - w\|, \tag{48}$$

by using (34) and (48), we have

$$\lim_{n\longrightarrow\infty} \|\xi_n - w\| = c. \tag{49}$$

So,

$$\begin{aligned}c &= \lim_{n\longrightarrow\infty} \|\xi_n - w\| = \lim_{n\longrightarrow\infty} \|(1-\gamma_n)\tau_n + \gamma_n\mathscr{K}\tau_n - w\| \\ &= \lim_{n\longrightarrow\infty} \|(1-\gamma_n)(\tau_n - w) + \gamma_n(\mathscr{K}\tau_n - w)\|.\end{aligned} \tag{50}$$

Using Lemma 12 and Inequality (50), we get

$$\lim_{n\longrightarrow\infty} \|\tau_n - \mathscr{K}\tau_n\| = 0. \tag{51}$$

Now, we prove a weak convergence result for nonexpansive mapping. □

Theorem 20. *Presume that $\mathscr{X}$ enjoys Opial's condition, then the sequence $\{\tau_n\}$ developed by iterative algorithm (9) converges weakly to a point of $F(\mathscr{K})$.*

Proof. Let $\{\tau_n\}$ be a sequence with two subsequences $\{\tau_{n_j}\}$ and $\{\tau_{n_k}\}$ and l and m are two weak subsequential limits of $\{\tau_{n_j}\}$ and $\{\tau_{n_k}\}$, respectively. From Lemmas 18 and 19, we get $\lim_{n\longrightarrow\infty} \|\tau_n - w\|$ exists and $\lim_{n\longrightarrow\infty} \|\tau_n - \mathscr{K}\tau_n\| = 0$, respectively. Now, we have to show that $\{\tau_n\}$ cannot have different weak subsequential limits in $F(\mathscr{K})$. Also, from Lemma 11, $I - \mathscr{K}$ is demiclosed at 0. This implies that $(I - \mathscr{K})l = 0$, i.e., $l = \mathscr{K}l$, similarly $m = \mathscr{K}m$. We have to show that $l = m$.

Let on contrary $l \ne m$, by Opial's condition, we have

$$\begin{aligned}\lim_{n\longrightarrow\infty} \|\tau_n - l\| &= \lim_{n_j\longrightarrow\infty} \left\|\tau_{n_j} - l\right\| < \lim_{n_j\longrightarrow\infty} \left\|\tau_{n_j} - m\right\| \\ &= \lim_{n\longrightarrow\infty} \|\tau_n - m\| = \lim_{n_k\longrightarrow\infty} \left\|\tau_{n_k} - m\right\| \\ &< \lim_{n_k\longrightarrow\infty} \left\|\tau_{n_k} - l\right\| = \lim_{n\longrightarrow\infty} \|\tau_n - l\|,\end{aligned} \tag{52}$$

which is absurd, hence $l = m$. Consequently, $\{\tau_n\} \rightharpoonup l \in F(\mathscr{K})$.

There is a strong convergence result for nonexpansive mapping. □

Theorem 21. *Let $\{\tau_n\}$ be the sequence developed by equation (9). Then $\lim_{n\longrightarrow\infty} \inf d(\tau_n, F(\mathscr{K})) = 0$ if and only if $\{\tau_n\}$*

converges to a point of $F(\mathscr{K})$*, where* $d(\tau_n, F(\mathscr{K})) = \inf\{\|\tau_n - w\| : w \in F(\mathscr{K})\}$.

Proof. If the sequence $\{\tau_n\}$ converges to a point $w \in F(\mathscr{K})$, then it is obvious that $\lim_{n\longrightarrow\infty} \inf d(\tau_n, F(\mathscr{K})) = 0$.

Now, for the first part taking $\lim_{n\longrightarrow\infty} \inf d(\tau_n, F(\mathscr{K})) = 0$ for any fixed point $w \in F(\mathscr{K})$. From Lemma 18,

$$\lim_{n\longrightarrow\infty} \|\tau_n - w\| \tag{53}$$

exists $\forall w \in F(\mathscr{K})$; therefore, $\lim_{n\longrightarrow\infty} d(\tau_n, F(\mathscr{K})) = 0$.

Now, our assertion is that $\{\tau_n\}$ is a Cauchy sequence in E. Since $\lim_{n\longrightarrow\infty} d(\tau_n, F(\mathscr{K})) = 0$, and for a given $\alpha > 0$, there exists $w_0 \in \mathbb{Z}_0^+$ such that for all $n \geq w_0$

$$\begin{aligned} d(\tau_n, F(\mathscr{K})) &< \frac{\alpha}{2}, \\ \inf\{\|\tau_n - w\| : w \in F(\mathscr{K})\} &< \frac{\alpha}{2}. \end{aligned} \tag{54}$$

Precisely, $\inf\{\|\tau_{w_0} - w\| : w \in F(\mathscr{K})\} < \alpha/2$. Therefore, there exists $w \in F(\mathscr{K})$ such that

$$\left\|\tau_{w_0} - w\right\| < \frac{\alpha}{2}. \tag{55}$$

Now, for $m, n \geq w_0$,

$$\begin{aligned} \|\tau_{n+m} - \tau_n\| \leq \|\tau_{n+m} - w\| + \|\tau_n - w\| \leq \left\|\tau_{w_0} - w\right\| \\ + \left\|\tau_{w_0} - w\right\| = 2\left\|\tau_{w_0} - w\right\| < \alpha. \end{aligned} \tag{56}$$

Thus, $\{\tau_n\}$ is a Cauchy in E. Since E is closed, $\lim_{n\longrightarrow\infty} \tau_n = q$ for some $q \in E$. Now, $\lim_{n\longrightarrow\infty} d(\tau_n, F(\mathscr{K})) = 0$ implies $d(q, F(\mathscr{K})) = 0$; hence, we get $q \in F(\mathscr{K})$.

We now prove a strong convergence result by applying property (A). □

Theorem 22. *Let* $\mathscr{K} : E \longrightarrow E$ *be a nonexpansive mapping with property* (A)*. Then* $\{\tau_n\}$ *defined by (9) converges strongly to a fixed point of* $\mathscr{K}$.

Proof. From equation (51) of Lemma 19, we have

$$\lim_{n\longrightarrow\infty} \|\tau_n - \mathscr{K}\tau_n\| = 0. \tag{57}$$

Using (57) and property (A), we get

$$\begin{aligned} 0 \leq \lim_{n\longrightarrow\infty} \psi(d(\tau_n, F(\mathscr{K}))) \leq \lim_{n\longrightarrow\infty} \|\tau_n - \mathscr{K}\tau_n\| = 0, \\ \lim_{n\longrightarrow\infty} \psi(d(\tau_n, F(\mathscr{K}))) = 0. \end{aligned} \tag{58}$$

Since ψ enjoy the conditions $\psi(z) > 0$ and $\psi(0) = 0$, $\forall z > 0$, then we obtain

$$\lim_{n\longrightarrow\infty} d(\tau_n, F(\mathscr{K})) = 0. \tag{59}$$

So by Theorem 21, we obtain the desired result. □

5. An Illuminate Numerical Example

The purpose of this section is to present a numerical example to compare the rate of convergence for nonexpansive mapping.

Example 23. Suppose $\mathscr{Z} = \mathbb{R}^3$ *a Banach space with usual norm and let* $\mathscr{K} : \mathscr{Z} \longrightarrow \mathscr{Z}$ be a mapping defined as

$$\mathscr{K}(x) = \mathscr{K}(x_1, x_2, x_3) = (0, x_1, x_2), \quad \forall x = (x_1, x_2, x_3) \in \mathscr{Z}. \tag{60}$$

Then $\mathscr{K}$ is nonexpansive, but not contraction.

Proof. Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathscr{Z}$. Then

$$\begin{aligned} \|\mathscr{K}x - \mathscr{K}y\| &= \|(0, x_1, x_2) - (0, y_1, y_2)\| \\ &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &\leq \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} \\ &= \|(x_1, x_2, x_3) - (y_1, y_2, y_3)\| = \|x - y\|. \end{aligned} \tag{61}$$

Hence, $\mathscr{K}$ is a nonexpansive mapping, but not contraction. □

Now, by taking control sequences $a_n = 0.5$, $b_n = 0.4$, and $c_n = 0.3$ with initial guess $x_1 = 0.5$, $x_2 = 0.25$, and $x_3 = 0.15$, we can show that new iterative scheme (9) converges faster than all other leading iterative schemes which is shown in Tables 3 and 4 and Figure 2.

6. Application to Nonlinear Fractional Differential Equation

In recent years, many authors pointed out that derivatives and integrals of noninteger order are very suitable for the description of properties of various real materials, e.g., polymers. It has been shown that new fractional order models are more adequate than previously used integer order models. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modelling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields. Most nonlinear fractional differential equations have no exact solution, so the approximate solution or numerical solution may be a good choice [24]. Related to this topic, we may refer the readers to [25–27] and the references therein.

Table 3: Numerical comparison of iterative schemes.

Iter.	New scheme	S
1	(0.500000, 0.250000, 0.150000)	(0.500000, 0.250000, 0.150000)
2	(0.000000, 0.000000, 0.220000)	(0.000000, 0.400000, 0.300000)
3	(0.000000, 0.000000, 0.000000)	(0.000000, 0.000000, 0.320000)
4	(0.000000, 0.000000, 0.000000)	(0.000000, 0.000000, 0.000000)

Table 4: Numerical comparison of iterative schemes.

Iter.	Ishikawa	Mann
1	(0.500000, 0.250000, 0.150000)	(0.500000, 0.250000, 0.150000)
2	(0.250000, 0.275000, 0.250000)	(0.250000, 0.375000, 0.200000)
3	(0.125000, 0.212500, 0.257500)	(0.125000, 0.312500, 0.287500)
⋮		
29	(0.000000, 0.000000, 0.000000)	(0.000000, 0.000000, 0.000001)
30	(0.000000, 0.000000, 0.000000)	(0.000000, 0.000000, 0.000000)

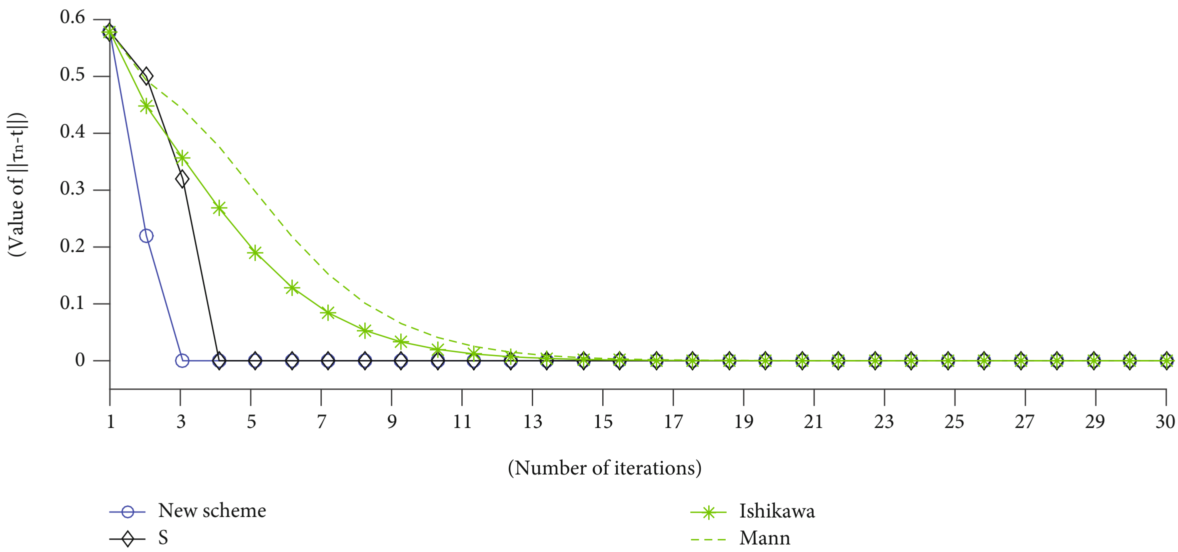

Figure 2: Graphical representation of iterative schemes.

Table 5: By using new iterative scheme observation between approximate solution and exact solution.

S.no.	t	$x(t)$	τ_1	τ_3	τ_5	τ_7	τ_{10}
1	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
2	0.1	-0.03804451	-0.03804451	-0.03804451	-0.03804451	-0.03804451	-0.03804451
3	0.2	-0.07003487	-0.07003487	-0.07003487	-0.07003487	-0.07003487	-0.07003487
4	0.3	-0.09478803	-0.09478803	-0.09478803	-0.09478803	-0.09478803	-0.09478803
5	0.4	-0.11165674	-0.11165674	-0.11165674	-0.11165674	-0.11165674	-0.11165674
6	0.5	-0.12026046	-0.12026046	-0.12026046	-0.12026046	-0.12026046	-0.12026046
7	0.6	-0.12040960	-0.12040960	-0.12040960	-0.12040960	-0.12040960	-0.12040960
8	0.7	-0.11207095	-0.11207095	-0.11207095	-0.11207095	-0.11207095	-0.11207095
9	0.8	-0.09534778	-0.09534778	-0.09534778	-0.09534778	-0.09534778	-0.09534778
10	0.9	-0.07046638	-0.07046638	-0.07046638	-0.07046638	-0.07046638	-0.07046638
11	1.0	-0.03776548	-0.03776548	-0.03776548	-0.03776548	-0.03776548	-0.03776548

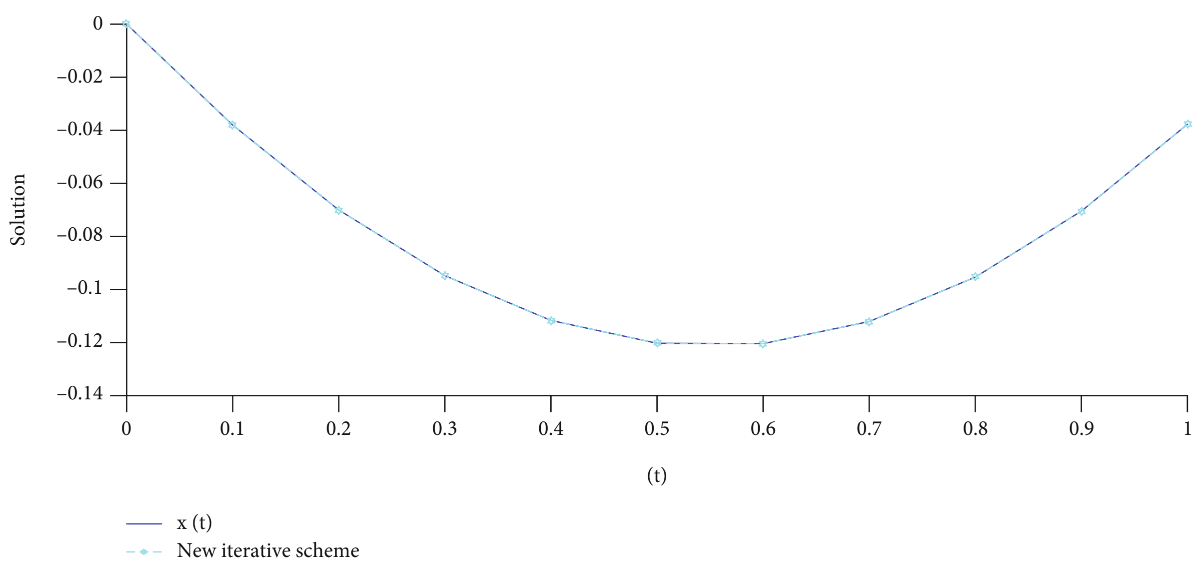

FIGURE 3: Exact solution and approximated solution of problem (68).

Consider the following fractional differential equation:

$$\begin{cases} {}^cD^{\beta} = f(t, x(t))(0,\leq t \leq 1, 1 < \beta \leq 2), \\ x(0) = 0, x(1) = \int_0^{\eta} x(s)ds(0 < \eta < 1), \end{cases} \tag{62}$$

where ${}^cD^{\beta}$ denotes the Caputo fractional derivative of order β and $f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

In this section, we approximate the solution of problem (62) via new scheme (9) with $\mathscr{Z} = C[0, 1]$ which is a Banach space of continuous function from $[0, 1]$ into $\mathbb{R}$ endowed with the maximum norm.

(C_1) Assume that

$$|f(t, a) - f(t, b)| \leq \frac{\Gamma(\beta + 1)}{5} |a - b| \tag{63}$$

for all $t \in [0, 1]$ and $a, b \in \mathbb{R}$.

Theorem 24. *Let $\mathscr{Z} = C[0, 1]$ and $\mathscr{K} : \mathscr{Z} \longrightarrow \mathscr{Z}$ be an operator defined by*

$$\begin{aligned}\mathscr{K}(x(t)) = {} & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s))ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \\ & \cdot \int_0^1 (1-s)^{\beta-1} f(s, x(s))ds + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \\ & \cdot \int_0^{\eta} \left(\int_0^s (s-m)^{\beta-1} f(m, x(m)dm \right) ds,\end{aligned} \tag{64}$$

$t \in [0, 1]$, $\forall x \in \mathscr{Z}$. Assume that the condition (C_1) is satisfied. Then the new scheme (9) converges to a solution of the problem (62), say $x^* \in \mathscr{Z}$.

Proof. Observe that $x^* \in \mathscr{Z}$ is a solution of (62) if and only if x^* is a solution of the integral equation

$$\begin{aligned}x(t) = {} & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s))ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \\ & \cdot \int_0^1 (1-s)^{\beta-1} f(s, x(s))ds + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \\ & \cdot \int_0^{\eta} \left(\int_0^s (s-m)^{\beta-1} f(m, x(m))dm \right) ds.\end{aligned} \tag{65}$$

Now, let $t \in [0, 1]$ and $x, y \in \mathscr{Z}$. Using (C_1), we get

$$\begin{aligned}& |\mathscr{K}x(t) - \mathscr{K}y(t)| \\ & = \left| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x(s))ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \right. \\ & \quad \cdot \int_0^1 (1-s)^{\beta-1} f(s, x(s))ds + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \\ & \quad \cdot \int_0^{\eta} \left(\int_0^s (s-m)^{\beta-1} f(m, x(m))dm \right) ds - \frac{1}{\Gamma(\beta)} \\ & \quad \cdot \int_0^t (t-s)^{\beta-1} f(s, y(s))ds + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \\ & \quad \cdot \int_0^1 (1-s)^{\beta-1} f(s, y(s))ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \\ & \quad \left. \cdot \int_0^{\eta} \left(\int_0^s (s-m)^{\beta-1} f(m, y(m))dm \right) ds \right| \\ & \leq \frac{\Gamma(\beta+1)}{5} \|x - y\| \sup_{t \in (0,1)} \left(\frac{1}{\Gamma(\beta)} \int_0^t |t-s|^{\beta-1} ds + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \right. \\ & \quad \left. \cdot \int_0^1 |1-s|^{\beta-1} ds + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^{\eta} \int_0^s |s-m|^{\beta-1} dm ds \right) \leq \|x - y\|.\end{aligned} \tag{66}$$

Thus, for $t \in [0, 1]$ and for each $x, y \in \mathscr{Z}$, we get

$$\|\mathscr{K}x - \mathscr{K}y\| \leq \|x - y\|. \tag{67}$$

Thus, $\mathscr{K}$ is nonexpansive mapping. Hence, iterative scheme (9) converges to the solution of (62). □

Now, for the effectiveness of Theorem 24, we present the following example.

Example 25.

$$\begin{cases} {}^{c}D^{1.25} = \sin(t) \ (0 \le t \le 1), \\ x(0) = 0, x(1) = \int_0^{0.5} x(s)ds. \end{cases} \tag{68}$$

The exact solution of problem (68) is given by

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sin(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \\ & \cdot \int_0^1 (1-s)^{\beta-1} \sin(s) ds + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \\ & \cdot \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} \sin(m) dm \right) ds. \end{aligned} \tag{69}$$

The operator $\mathscr{K} : C[0,1] \longrightarrow C[0,1]$ is defined by

$$\begin{aligned} \mathscr{K}(x(t)) = {} & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sin(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \\ & \cdot \int_0^1 (1-s)^{\beta-1} \sin(s) ds + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \\ & \cdot \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} \sin(m) dm \right) ds. \end{aligned} \tag{70}$$

Taking initial hypothesis $\tau_0(t) = t(1-t), t \in [0,1]$, $\beta = 1.25$ and $\eta = 0.5$, choose control sequences $\theta_n = 0.95$, $\mu_n = 0.65$ and $\xi_n = 0.50$, $n \in \mathbb{Z}_0^+$. It is shown in Table 5 and Figure 3 that iterative scheme (9) converges to the exact solution of problem (68) for the operator constructed in (70).

7. Conclusion

In this paper, convergence and stability results of a new three step iterative scheme has been studied. Further, the solution of a fractional differential equation is approximated by applying Theorem 24. For nonlinear mappings, we compared the rate of convergence of remarkable iterative schemes analytically and numerically. To support the main result, we gave nontrivial examples.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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