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Research Article

Infinite Product Representation for the Szegö Kernel for an Annulus

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The Szegö kernel has many applications to problems in conformal mapping and satisfies the Kerzman-Stein integral equation. The Szegö kernel for an annulus can be expressed as a bilateral series and has a unique zero. In this paper, we show how to represent the Szegö kernel for an annulus as a basic bilateral series (also known as q-bilateral series). This leads to an infinite product representation through the application of Ramanujan's sum. The infinite product clearly exhibits the unique zero of the Szegö kernel for an annulus. Its connection with the basic gamma function and modified Jacobi theta function is also presented. The results are extended to the Szegö kernel for general annulus and weighted Szegö kernel. Numerical comparisons on computing the Szegö kernel for an annulus based on the Kerzman-Stein integral equation, the bilateral series, and the infinite product are also presented.

1. Introduction

The Ahlfors map is a branching n-to-one map from an n-connected region onto the unit disk. It is intimately tied to the Szegö kernel of an n-connected region [1]. The boundary values of the Szegö kernel satisfy the Kerzman-Stein integral equation, which is a Fredholm integral equation of the second kind for a region with a smooth boundary [2]. The boundary values of the Alhfors map are completely determined from the boundary values of the Szegö kernel [1–3]. For an annulus region Ω, the Szegö kernel can be expressed as a bilateral series from which the zero can be determined analytically [4]. The Kerzman-Stein integral equation has been solved using the Adomian decomposition method in [5] to give another bilateral series form for the Szegö kernel for Ω that converges faster. There are various special functions in the form of bilateral and basic bilateral series [6–8]. For example, the bilateral basic hypergeometric series contain, as special cases, many interesting identities related to infinite products, theta functions, and Ramanujan's identities. It is therefore natural to ask if the bilateral series for the Szegö kernel for Ω can be summed as special functions or an infinite product that exhibits clearly its zero.

In this paper, we show how to express the bilateral series for the Szegö kernel for Ω as a basic bilateral series (also known as q-bilateral series). Ramanujan's sum is then applied to obtain the infinite product representation for the Szegö kernel for Ω. The product clearly exhibits the zero of the Szegö kernel for Ω, and its connection with the q-gamma function and the modified Jacobi theta function is shown. Using the symmetry of Ramanujan's sum, we show how to easily transform the bilateral series for the Szegö kernel for Ω in [4] to the bilateral series in [5].

The plan of the paper is as follows: After the presentation of some preliminaries in Section 2, we derive the basic bilateral series and infinite product representations for the Szegö kernel for Ω in Section 3. We then derive a closed form of the Szegö for Ω in terms of q-gamma function and the modified Jacobi theta function. In Section 4, we show how to extend the representations in Section 3 to the general annulus using the transformation formula for the Szegö kernel under conformal mappings. Similar q-analysis for the weighted Szegö kernel for Ω is presented in Section 5. In Section 6, we give numerical comparisons for computing the Szegö kernel for Ω using bilateral series, infinite product, and integral equation formulations.

2. Preliminaries

Let $\Omega = \{z : \rho < |z| < 1\}$ be an annulus with $0 < \rho < 1$ and a point $a \in \Omega$. The boundary Γ of Ω consists of two smooth Jordan curves with the outer curve Γ_0 oriented counterclockwise and the inner curve Γ_1 oriented clockwise. The positive direction of the contour $\Gamma = \Gamma_0 \cup \Gamma_1$ is usually that for which the region is on the left as one traces the boundary.

Let $\{\varphi_n(z)\}_{n=1}^{\infty}$ be an orthonormal basis for the Hardy spaces $H^2(\Gamma)$. Since the Szegö kernel $S(z,a)$ is the reproducing kernel for $H^2(\Gamma)$, it can be written as [4]

$$S(z,a) = \sum_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_n(a)}, a \in \Omega, \tag{1}$$

with absolute and uniform convergence on compact subsets of Ω. An orthogonal basis for $H^2(\Gamma)$ is $\{z^n\}_{n=-\infty}^{\infty}$. Thus

$$\|z^n\|^2 = \int_{\Gamma} |z|^{2n} |dz| = 2\pi\left(1 + \rho^{2n+1}\right), \tag{2}$$

where $|dz|$ is the arc length measure. Therefore, an orthonormal basis for $H^2(\Gamma)$ is [3, 4]

$$\left\{\frac{z^n}{\sqrt{2\pi(1+\rho^{2n+1})}}\right\}_{n=-\infty}^{\infty}. \tag{3}$$

Using (1) and (3), the series representation for the Szegö kernel for Ω is given by [4]

$$S(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \frac{(z\bar{a})^n}{1+\rho^{2n+1}}, a \in \Omega, z \in \Omega \cup \Gamma. \tag{4}$$

Series (4) is a bilateral series. It has a zero at $z = -\rho/\bar{a}$ [4].

Another bilateral series representation for the Szegö kernel for Ω is given by [5] (in an equivalent form)

$$S(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \frac{(-1)^n\rho^n}{\rho^{2n} - z\bar{a}}, z \in \Omega \cup \Gamma, a \in \Omega, \tag{5}$$

which is initially obtained by solving the Kerzman-Stein integral equation using the Adomian decomposition method. It is also shown in [5] how to derive (5) directly from (4) using geometric series. It is illustrated in [5] that series (5) converges faster than (4).

More generally, if Ω_1 is any doubly connected region with the smooth boundary Γ_1, and $f(z)$ is a biholomorphic map of Ω_1 onto Ω, then the Szegö kernel for Ω_1 can be obtained via the transformation formula as [1]

$$\begin{aligned} S_1(z,a) &= \sqrt{f'(z)}S(f(z), f(a))\overline{\sqrt{f'(a)}} \\ &= \frac{\sqrt{f'(z)}\overline{\sqrt{f'(a)}}}{2\pi}\sum_{n=-\infty}^{\infty}\frac{\left(f(z)\overline{f(a)}\right)^n}{1+\rho^{2n+1}}, a \in \Omega_1, z \in \Omega_1 \cup \Gamma_1, \end{aligned} \tag{6}$$

where ρ is unknown but can be computed.

The Szegö kernel $S_1(z,a)$ can also be computed without using conformal mapping. The boundary values of the Szegö kernel $S_1(z,a)$ on Γ_1 satisfy the Kerzman-Stein integral equation [2, 4],

$$S_1(z,a) + \int_{\Gamma} A(z,w) S_1(w,a) |dw| = g(z), z \in \Gamma_1, \tag{7}$$

where

$$\begin{aligned} A(z,w) &= \begin{cases} \dfrac{1}{2\pi}\left(\dfrac{T(w)}{z-w} - \dfrac{\overline{T(z)}}{\bar{z}-\bar{w}}\right), & z \neq w \in \Gamma_1, \\ 0, & z = w \in \Gamma_1, \end{cases} \\ g(z) &= -\frac{1}{2\pi i}\frac{\overline{T(z)}}{\bar{z}-\bar{a}}, z \in \Gamma_1, \\ T(z) &= \frac{z'(t)}{|z'(t)|}, z \in \Gamma_1, \end{aligned} \tag{8}$$

and $z(t)$ is a parametrization of Γ_1. The function $A(z,w)$ is known as the Kerzman-Stein kernel, and it is continuous on the boundary of Ω_1 [9, 10]. In fact, the integral equation (7) is also valid for an n-connected region.

Since bilateral series and basic bilateral series will be used throughout this paper, we recall some facts about q-series notations and results.

Let $0 < q < 1$ and $\alpha \in \mathbb{C}$. The q-shifted factorial is defined as [7]

$$(q^{\alpha}; q)_n = \begin{cases} 1, & n = 0, \\ \left(1-q^{\alpha}\right)\left(1-q^{\alpha+1}\right)\cdots\left(1-q^{\alpha+n-1}\right), & n = 1, 2, \ldots, \\ \dfrac{1}{(1-q^{\alpha-1})(1-q^{\alpha-2})\cdots(1-q^{\alpha-n})}, & n = -1, -2, \ldots. \end{cases} \tag{9}$$

This notation yields the shifted factorial as a special case through

$$\lim_{q\to 1}\frac{(q^{\alpha};q)_n}{(q;q)_n} = \alpha(\alpha+1)\cdots(\alpha+n-1), n = 1, 2, \ldots. \tag{10}$$

If α is written in place of q^{α}, then (9) becomes

$$(\alpha;q)_n = \begin{cases} 1, & n=0, \\ (1-\alpha)(1-\alpha q)\cdots\left(1-\alpha q^{n-1}\right), & n=1,2,\ldots, \\ \dfrac{1}{(1-\alpha q^{-1})(1-\alpha q^{-2})\cdots(1-\alpha q^{-n})}, & n=-1,-2,\ldots. \end{cases} \tag{11}$$

It can be shown that [7]

$$\frac{1-\alpha}{1-\alpha q^n} = \frac{(\alpha;q)_n}{(\alpha q;q)_n}, n=0,\pm1,\pm2,\ldots. \tag{12}$$

If $n \longrightarrow \infty$, it is standard to write

$$(\alpha;q)_{\infty} = \prod_{n=0}^{\infty}(1-\alpha q^n), \tag{13}$$

which is absolutely convergent for all finite values of α, real or complex, when $|q|<1$ [6]. This yields

$$(\alpha;q)_n = \frac{(\alpha;q)_{\infty}}{(\alpha q^n;q)_{\infty}}. \tag{14}$$

Observe that $(\alpha;q)_{\infty}$ would have zero as a factor if $\alpha=1$. It would be zero also if $\alpha = q^{-1}, q^{-2}, q^{-3},\ldots$, but these are all outside the circle $|z|=1$ since $|q|<1$ [8].

The bilateral basic hypergeometric series in base q with one numerator and one denominator parameters is defined by [6–8]

$${}_1\psi_1(\alpha;\beta;q;z) = \sum_{n=-\infty}^{\infty}\frac{(\alpha;q)_n}{(\beta;q)_n}z^n. \tag{15}$$

The series is convergent for $|q|<1$ and $|\beta/\alpha|<|z|<1$.

The classical Ramanujan's ${}_1\psi_1$ summation is given by [7, 8]

$${}_1\psi_1(\alpha;\beta;q;z) = \frac{(\alpha z;q)_{\infty}(q/\alpha z;q)_{\infty}(\beta/\alpha;q)_{\infty}(q;q)_{\infty}}{(z;q)_{\infty}(\beta/\alpha z;q)_{\infty}(q/\alpha;q)_{\infty}(\beta;q)_{\infty}}, |\beta/\alpha|<|z|<1. \tag{16}$$

The special case $\beta = \alpha q$ of Ramanujan's ${}_1\psi_1$ summation yields [8]

$$\sum_{n=-\infty}^{\infty}\frac{z^n}{1-\alpha q^n} = \frac{(\alpha z;q)_{\infty}((q/\alpha z);q)_{\infty}(q;q)_{\infty}^2}{(z;q)_{\infty}((q/z);q)_{\infty}(\alpha;q)_{\infty}((q/\alpha);q)_{\infty}}, \tag{17}$$

also known as Cauchy's formula. Due to symmetry in α and z on the right-hand side of (17), it implies [8]

$$\sum_{n=-\infty}^{\infty}\frac{z^n}{1-\alpha q^n} = \sum_{n=-\infty}^{\infty}\frac{\alpha^n}{1-zq^n}. \tag{18}$$

The q-gamma function is defined as [7]

$$\Gamma_q(x) = \frac{(q;q)_{\infty}}{(q^x;q)_{\infty}}(1-q)^{1-x}, 0<q<1, x=\mathbb{C}-\{0,-1,-2,\ldots\}. \tag{19}$$

Another important special function that is used in this paper is the modified Jacobi theta function defined by [7]

$$\theta(x;q) = (x;q)_{\infty}(q/x;q)_{\infty}, \tag{20}$$

where $x \neq 0$ and $|q|<1$. For a more detailed discussion on q-series and historical perspectives, see, for example, [6–8] and the references therein.

3. Szegö Kernel for an Annulus and Basic Bilateral Series

In this section, we express the bilateral series (4) as a basic bilateral series and derive the infinite product representation of the Szegö kernel for Ω. It is given in the following theorem.

Theorem 1. *Let Ω be the annulus $\{z : \rho<|z|<1\}$ bounded by Γ. For $a\in\Omega$, $z\in\Omega\cup\Gamma$, the Szegö kernel for Ω can be represented by*

$$S(z,a) = \frac{1}{2\pi(1+\rho)}{}_1\psi_1\left(-\rho;-\rho^3;\rho^2;\bar{a}z\right), \tag{21}$$

$$= \frac{1}{2\pi}\prod_{n=0}^{\infty}\frac{\left(1+\bar{a}z\rho^{2n+1}\right)\left(\bar{a}z+\rho^{2n+1}\right)\left(1-\rho^{2n+2}\right)^2}{\left(1-\bar{a}z\rho^{2n}\right)\left(\bar{a}z-\rho^{2n+2}\right)\left(1+\rho^{2n+1}\right)^2}. \tag{22}$$

The zero of $S(z,a)$ in Ω is the zero of the factor $\bar{a}z+\rho$, that is, $z=-\rho/\bar{a}$.

Proof. From (4), we have

$$S(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}z)^n}{1+\rho^{2n+1}} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}z)^n}{1-(-\rho)\rho^{2n}}. \tag{23}$$

Letting $\alpha=-\rho$ and $q=\rho^2$ yields

$$S(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}z)^n}{1-\alpha q^n}, \tag{24}$$

$$= \frac{1}{2\pi(1-\alpha)}\sum_{n=-\infty}^{\infty}\frac{1-\alpha}{1-\alpha q^n}(\bar{a}z)^n. \tag{25}$$

Applying (12) and (15) gives

$$\begin{aligned} S(z,a) &= \frac{1}{2\pi(1-\alpha)}\sum_{n=-\infty}^{\infty}\frac{(\alpha,q)_n}{(\alpha q,q)_n}(\bar{a}z)^n \\ &= \frac{1}{2\pi(1-\alpha)}{}_1\psi_1(\alpha;\alpha q;q;\bar{a}z). \end{aligned} \tag{26}$$

Note that the ${}_1\psi_1$ series above is convergent because $|q| = \rho^2 < 1$ and $|\beta/\alpha| = |\alpha q/\alpha| = |q| = \rho^2 < |\bar{a}z| < 1$. Substituting $\alpha = -\rho$ and $q = \rho^2$ into (26) gives (21).

Applying Ramanujan's sum (16) to (26), gives

$$S(z,a) = \frac{1}{2\pi(1-\alpha)} \frac{(\alpha\bar{a}z\,;q)_\infty (q/\alpha\bar{a}z\,;q)_\infty (q\,;q)_\infty^2}{(\bar{a}z\,;q)_\infty (q/\bar{a}z\,;q)_\infty (q/\alpha\,;q)_\infty (\alpha q\,;q)_\infty}. \quad (27)$$

But from (14), with $n = 1$, we have

$$(1-\alpha)(\alpha q\,;q)_\infty = (\alpha\,;q)_\infty. \quad (28)$$

Thus, (27) becomes

$$S(z,a) = \frac{1}{2\pi} \frac{(\alpha\bar{a}z\,;q)_\infty (q/\alpha\bar{a}z\,;q)_\infty (q\,;q)_\infty^2}{(\bar{a}z\,;q)_\infty (q/\bar{a}z\,;q)_\infty (q/\alpha\,;q)_\infty (\alpha\,;q)_\infty}, \quad (29)$$

$$= \frac{1}{2\pi} \prod_{n=0}^{\infty} \frac{(1-\alpha\bar{a}zq^n)\left(1-q^{n+1}/\alpha\bar{a}z\right)\left(1-q^{n+1}\right)^2}{(1-\bar{a}zq^n)(1-q^{n+1}/\bar{a}z)(1-q^{n+1}/\alpha)(1-\alpha q^n)}. \quad (30)$$

Substituting $\alpha = -\rho$ and $q = \rho^2$ into (30) gives (22).

The infinite product (22) would have poles if

$$1 - \bar{a}z\rho^{2n} = 0 \text{ or } \bar{a}z - \rho^{2n+2} = 0, \quad (31)$$

which implies

$$z = \frac{1}{\bar{a}\rho^{2n}} \text{ or } z = \frac{\rho^{2n+2}}{\bar{a}}. \quad (32)$$

But

$$\frac{1}{|a\rho^{2n}|} > 1, \left|\frac{\rho^{2n+2}}{\bar{a}}\right| < \rho^{2n+1} < \rho. \quad (33)$$

Therefore, the poles are all outside Ω.

The infinite product (22) would have zeros if

$$1 + \bar{a}z\rho^{2n+1} = 0 \text{ or } \bar{a}z + \rho^{2n+1} = 0, \quad (34)$$

which implies

$$z = -\frac{1}{\bar{a}\rho^{2n+1}} \text{ or } z = -\frac{\rho^{2n+1}}{\bar{a}}. \quad (35)$$

For the first case

$$\frac{1}{|a\rho^{2n+1}|} > \frac{1}{\rho^{2n+1}} > 1, \quad (36)$$

which is outside Ω. For the second case, observe that

$$\rho^{2n+1} < \left|\frac{\rho^{2n+1}}{\bar{a}}\right| = \frac{\rho^{2n+1}}{|a|} < \rho^{2n}, \quad (37)$$

which clearly has a zero inside Ω when $n = 0$. Thus, the infinite product (22) for $S(z,a)$ has only one zero inside Ω at $z = -\rho/\bar{a}$. This completes the proof. □

We note that the series representation (21) for $S(z,a)$ is valid only for $\rho \le |z| \le 1$, while the infinite product representation (22) for $S(z,a)$ is meaningful for all $z \in \mathbb{C}$ except for the infinitely many poles at $z = 0, \rho^{-2n}/\bar{a}, \rho^{2n+2}/\bar{a}$.

We next show that the Szegö kernel for Ω can also be expressed in terms of the basic gamma function and modified Jacobi theta function. By applying (20) to (29) and substituting $\alpha = -\rho$ and $q = \rho^2$, we have

$$\begin{aligned} S(z,a) &= \frac{1}{2\pi} \frac{\theta(\alpha\bar{a}z\,;q)_\infty (q\,;q)_\infty^2}{\theta(\bar{a}z\,;q)_\infty (q/\alpha\,;q)_\infty (\alpha\,;q)_\infty} \\ &= \frac{1}{2\pi} \frac{\theta\left(-\rho\bar{a}z\,;\rho^2\right)_\infty \left(\rho^2\,;\rho^2\right)_\infty^2}{\theta(\bar{a}z\,;\rho^2)_\infty (-\rho\,;\rho^2)_\infty^2}. \end{aligned} \quad (38)$$

Applying (19) with $q = \rho^2$, observe that

$$\frac{\left(\rho^2\,;\rho^2\right)_\infty}{\left(-\rho\,;\rho^2\right)_\infty} = \frac{\left(\rho^2\,;\rho^2\right)_\infty}{\left(\rho^{2x}\,;\rho^2\right)_\infty} = \frac{\Gamma_{\rho^2}(x)}{\left(1-\rho^2\right)^{1-x}}, \quad (39)$$

where x satisfies $\rho^{2x} = -\rho$. This equation may be written as

$$e^{(2x-1)\ \ln\,\rho} = e^{i\pi}, \quad (40)$$

which yields a solution

$$x = \frac{1}{2} + \frac{i\pi}{2\ln\rho}. \quad (41)$$

Thus, (38) becomes

$$S(z,a) = \frac{\left[\Gamma_{\rho^2}(\lambda)\right]^2}{2\pi\left(1-\rho^2\right)^{2(1-\lambda)}} \frac{\theta\left(-\rho\bar{a}z\,;\rho^2\right)_\infty}{\theta\left(\bar{a}z\,;\rho^2\right)_\infty}, \lambda = \frac{1}{2} + \frac{i\pi}{2\ln\rho}. \quad (42)$$

This can be regarded as a closed-form expression for the Szegö kernel for Ω.

In the following, we show how to easily transform series (4) to series (5) using (18). Letting $\alpha = -\rho$ and $q = \rho^2$, (4) becomes

$$S(z,a) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(\bar{a}z)^n}{1-\alpha q^n} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\alpha^n}{1-(\bar{a}z)q^n}, \quad (43)$$

where in the last step we have used (18). By replacing $\alpha = -\rho$ and $q = \rho^2$, we get

$$S(z,a) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n\rho^n}{1-(\bar{a}z)\rho^{2n}}. \quad (44)$$

Letting $n = -m$ yields

$$S(z,a) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\frac{(-1)^{-m}\rho^{-m}}{1-(\bar{a}z)\rho^{-2m}} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\frac{(-1)^{m}\rho^{m}}{\rho^{2m}-\bar{a}z}, \quad (45)$$

which is the same as (5).

4. Szegö Kernel for General Annulus

Consider the general annulus $\Omega_2 = \{z : r_2 < |z - z_0| < r_1\}$ with boundary denoted by Γ_2. The region Ω_2 reduces to Ω if $z_0 = 0$, $r_2 = \rho$, and $r_1 = 1$.

Theorem 2. *Let $z_0 \in \mathbb{C}$, $z \in \Omega_2 \cup \Gamma_2$, and $a \in \Omega_2$. The Szegö kernel for Ω_2 can be represented by the bilateral series as*

$$S_2(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}-\bar{z}_0)^n}{r_1^{2n+1}+r_2^{2n+1}}(z-z_0)^n, \quad (46)$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n r_1^{n+1} r_2^n}{r_2^{2n}r_1^2 - r_1^{2n}(z-z_0)(\bar{a}-\bar{z}_0)}. \quad (47)$$

The zero of $S_2(z,a)$ in Ω_2 is $z = z_0 - r_1 r_2/\bar{a} - \bar{z}_0$.

Proof. Observe that the function $f(z) = (z - z_0)/r_1$ maps Ω_2 onto Ω with $\rho = r_2/r_1$.

Applying the transformation formula (6) yields

$$\begin{aligned} S_2(z,a) &= \sqrt{f'(z)}S(f(z),f(a))\overline{\sqrt{f'(a)}} \\ &= \frac{1}{\sqrt{r_1}}S\left(\frac{z-z_0}{r_1},\frac{a-z_0}{r_1}\right)\frac{1}{\sqrt{r_1}} \\ &= \frac{1}{r_1}S\left(\frac{z-z_0}{r_1},\frac{a-z_0}{r_1}\right). \end{aligned} \quad (48)$$

Applying (4) to (48) with z and a replaced by $(z - z_0)/r_1$ and $(a - z_0)/r_1$, respectively, gives

$$S_2(z,a) = \frac{1}{2\pi r_1}\sum_{n=-\infty}^{\infty}\frac{\left((z-z_0)(a\bar{z}_0)/r_1^2\right)^n}{1+(r_2/r_1)^{2n+1}}, \quad (49)$$

which simplifies to (46).

Applying (5) to (48) instead of z and a replaced by $(z - z_0)/r_1$ and $(a - z_0)/r_1$, respectively, gives

$$S_2(z,a) = \frac{1}{2\pi r_1}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(r_2/r_1)^n}{(r_2/r_1)^{2n}-(z-z_0)(a\bar{z}_0)/r_1^2}, \quad (50)$$

which simplifies to (47).

Using the fact that $S(z,a)$ has a zero at $z = -\rho/\bar{a}$ for Ω, the zero of $S_2(z,a)$ for Ω_2 is $(z - z_0)/r_1 = -\rho/((a\bar{z}_0)/r_1)$ which implies $z = z_0 - (\rho r_1^2/(\bar{a} - \bar{z}_0)) = z_0 - (r_1 r_2/((\bar{a} - \bar{z}_0)))$. This completes the proof.

Similarly, the infinite product representation of $S_2(z,a)$ for Ω_2 can be obtained by applying (22) to (48) with z and a replaced by $(z - z_0)/r_1$ and $(a - z_0)/r_1$, respectively. □

5. The Weighted Szegö Kernel for an Annulus and Basic Bilateral Series

The weighted Szegö kernel is defined in [11] as

$$\widehat{K}_q^t(z,w) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{w}z)^n}{1+tq^{2n}}, \; t>0, q<|z|,|w|<1. \quad (51)$$

To adopt the notations used in this paper, we change q to ρ, w to a, and $\widehat{K}_q^t(z,w)$ to $S_\rho^t(z,a)$ in (51), which gives

$$S_\rho^t(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}z)^n}{1+t\rho^{2n}}, \; t>0, \rho<|z|,|a|<1. \quad (52)$$

Note that $S_\rho^\rho(z,a)$ is exactly the kernel $S(z,a)$ for Ω discussed in Section 1. The zeros of the kernel $S_t^\rho(z,a)$ are not discussed in [11] but have expressed interest on the effect of the weight on the location of its zeros. In the following theorem, we express the weighted Szegö kernel $S_\rho^t(z,a)$ as a basic bilateral series and derive its associated infinite product representation as well as its zeros.

Theorem 3. *Let Ω be the annulus $\{z : \rho < |z| < 1\}$ bounded by Γ. For $a \in \Omega$, $z \in \Omega \cup \Gamma$, and $t > 0$, the weighted Szegö kernel $S_\rho^t(z,a)$ for Ω can be represented by*

$$S_t^\rho(z,a) = \frac{1}{2\pi(1+t)_1}\psi_1\left(-t;-t\rho^2;\rho^2;\bar{a}z\right), \quad (53)$$

$$= \frac{1}{2\pi}\prod_{n=0}^{\infty}\frac{\left(1+t\bar{a}z\rho^{2n}\right)\left(\bar{a}z+\rho^{2n+2}/t\right)\left(1-\rho^{2n+2}\right)^2}{\left(1-\bar{a}z\rho^{2n}\right)\left(\bar{a}z-\rho^{2n+2}\right)\left(1+\rho^{2n+2}/t\right)\left(1+t\rho^{2n}\right)}. \quad (54)$$

The kernel $S_\rho^t(z,a)$ has a zero in Ω only if t takes the form $t = \rho^{\pm(2m+1)}$, $m = 0, 1, 2, \cdots$. In both cases, the zero is $z = -\rho/\bar{a}$.

Proof. Observe that

$$S_\rho^t(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}z)^n}{1-(-t)\rho^{2n}}. \quad (55)$$

Letting $\alpha = -t$ and $q = \rho^2$, the above equation becomes

$$S_\rho^t(z,a) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(\bar{a}z)^n}{1-\alpha q^n}, \quad (56)$$

which is exactly the same form as (24). Applying the result (26) with $\alpha = -t$, the above equation becomes

$$S_\rho^t(z,a) = \frac{1}{2\pi(1+t)_1}\psi_1(-t;-tq;q;\bar{a}z). \quad (57)$$

Series (57) is convergent because $|q| = \rho^2 < 1$ and $|\beta/\alpha| = |-tq/(-t)| = |q| < \rho^2 < |\bar{a}z| < 1$. Substituting $q = \rho^2$ gives (41).

Applying the result (29) with $\alpha = -t$ to (57) yields

$$S_\rho^t(z,a) = \frac{1}{2\pi}\frac{(-t\bar{a}z\,;q)_\infty (q/(-t)\bar{a}z\,;q)_\infty (q\,;q)_\infty^2}{(\bar{a}z\,;q)_\infty (q/\bar{a}z\,;q)_\infty (q/(-t)\,;q)_\infty (-t\,;q)_\infty}. \quad (58)$$

Replacing $q=\rho^2$ and applying (13) give (54).

In the proof of Theorem 1, we have shown that the factors $(1-\bar{a}z\rho^{2n})(\bar{a}z-\rho^{2n+2})$ have no zeros in Ω. The factors $(1+\rho^{2n+2}/t)(1+t\rho^{2n})$ would have zeros if

$$\rho^{2n+2}/t = -1 \text{ or } t\rho^{2n} = -1. \quad (59)$$

Since $t>0$, we conclude that the kernel $S_\rho^t(z,a)$ has no poles in Ω for any $t>0$. The factors $(1+t\bar{a}z\rho^{2n})(\bar{a}z+\rho^{2n+2}/t)$ would have zeros if

$$1+t\bar{a}z\rho^{2n} = 0 \text{ or } \bar{a}z+\rho^{2n+2}/t = 0, \quad (60)$$

which implies

$$z = -\frac{1}{t\bar{a}\rho^{2n}} \text{ or } z = -\frac{\rho^{2n+2}}{t\bar{a}}. \quad (61)$$

For the first case, observe that

$$\frac{1}{t\rho^{2n}} < \frac{1}{|t\bar{a}\rho^{2n}|} < \frac{1}{t\rho^{2n+1}}. \quad (62)$$

To have a zero in Ω, we must have the condition

$$\rho \le \frac{1}{t\rho^{2n}} < \frac{1}{|t\bar{a}\rho^{2n}|} < \frac{1}{t\rho^{2n+1}} \le 1, \quad (63)$$

which means

$$t \le \frac{1}{\rho^{2n+1}} \text{ and } t \ge \frac{1}{\rho^{2n+1}}. \quad (64)$$

Hence, we must have $t=\rho^{-(2n+1)}$. In this case, the zero of $S_\rho^t(z,a)$ in Ω is $z=-\rho/\bar{a}$.

For the second case, observe that

$$\frac{\rho^{2n+2}}{t} < \frac{\rho^{2n+2}}{|t\bar{a}|} < \frac{\rho^{2n+1}}{t}. \quad (65)$$

To have a zero in Ω, we must have the condition

$$\rho \le \frac{\rho^{2n+2}}{t} < \frac{\rho^{2n+2}}{|t\bar{a}|} < \frac{\rho^{2n+1}}{t} \le 1, \quad (66)$$

which means

$$t \le \rho^{2n+1} \text{ and } t \ge \rho^{2n+1}. \quad (67)$$

Hence, we must have $t=\rho^{2n+1}$. In this case, the zero of $S_\rho^t(z,a)$ in Ω is also $z=-\rho/\bar{a}$. This completes the proof. □

The weighted Szegö kernel can also be expressed in terms of the basic gamma function and the modified Jacobi theta function. By applying (20) to (58) with $q=\rho^2$, we have

$$S_\rho^t(z,a) = \frac{1}{2\pi}\frac{\theta(-t\bar{a}z\,;\rho^2)_\infty (\rho^2\,;\rho^2)_\infty^2}{\theta(\bar{a}z\,;\rho^2)_\infty (\rho^2/(-t)\,;\rho^2)_\infty (-t\,;\rho^2)_\infty}. \quad (68)$$

Observe that

$$\frac{(\rho^2\,;\rho^2)_\infty}{(-t\,;\rho^2)_\infty} = \frac{(\rho^2\,;\rho^2)_\infty}{(\rho^{2x}\,;\rho^2)_\infty} = \frac{\Gamma_{\rho^2}(x)}{(1-\rho^2)^{1-x}}, \quad (69)$$

where x satisfies $\rho^{2x}=-t$. This equation may be written as

$$2x\ln\rho = \ln(-t) = \ln|-t| + i\arg(-t) = \ln t + i\pi, \quad (70)$$

which yields a solution

$$x = \frac{\ln t + i\pi}{2\ln\rho}. \quad (71)$$

Observe also that

$$\frac{(\rho^2\,;\rho^2)_\infty}{(-\rho^2/t\,;\rho^2)_\infty} = \frac{(\rho^2\,;\rho^2)_\infty}{(\rho^{2y}\,;\rho^2)_\infty} = \frac{\Gamma_{\rho^2}(y)}{(1-\rho^2)^{1-y}}, \quad (72)$$

where y satisfies $\rho^{2y}=-\rho^2/t$. This equation may be written as

$$(2y-2)\ln\rho = \ln\left(-\frac{1}{t}\right) = \ln\left|-\frac{1}{t}\right| + i\arg\left(-\frac{1}{t}\right) = -\ln t + i\pi, \quad (73)$$

which yields a solution

$$y = 1 + \frac{-\ln t + i\pi}{2\ln\rho}. \quad (74)$$

Thus, (68) becomes

$$S_\rho^t(z,a) = \frac{\Gamma_{\rho^2}(\mu)\Gamma_{\rho^2}(\nu)\theta(-t\bar{a}z\,;\rho^2)_\infty}{2\pi(1-\rho^2)^{2-\mu-\nu}\theta(\bar{a}z\,;\rho^2)_\infty}, \mu = \frac{\ln t + i\pi}{2\ln\rho}, \nu = 1 + \frac{-\ln t + i\pi}{2\ln\rho}. \quad (75)$$

This can be regarded as a closed-form expression for the weighted Szegö kernel for an annulus Ω. Observe that (75) reduces to (42) when $t=\rho$.

Table 1: Error norms between S_{10} and $\tilde{S}_n$, S_{50} and $\tilde{S}_n$, and S_{100} and $\tilde{S}_n$.

n	$\|S_{10} - \tilde{S}_n\|_\infty$	$\|S_{50} - \tilde{S}_n\|_\infty$	$\|S_{100} - \tilde{S}_n\|_\infty$
16	2.4536 (-02)	2.97754 (-03)	2.97758 (-03)
32	2.75019 (-02)	1.15906 (-05)	1.16299 (-05)
64	2.75136 (-02)	3.91113 (-08)	1.88349 (-10)
128	2.75136 (-02)	3.92996 (-08)	2.28878 (-15)

Table 2: Error norms between S^*_{10} and $\tilde{S}_n$ and S^*_{50} and $\tilde{S}_n$.

n	$\|S^*_{10} - \tilde{S}_n\|_\infty$	$\|S^*_{50} - \tilde{S}_n\|_\infty$
16	2.94797 (-03)	2.97758 (-03)
32	1.78995 (-02)	1.16299 (-05)
64	1.77628 (-04)	1.88351 (-10)
128	1.77628 (-04)	1.81497 (-15)

6. Numerical Computation of the Szegö Kernel for an Annulus

In this section, we compare the speed of convergence of the three formulas for computing the Szegö kernel for Ω based on the two bilateral series (4) and (5) and the infinite product (22).

To approximate (4) numerically, we calculate

$$S(z, a) \approx S_{10}(z, a) = \frac{1}{2\pi} \sum_{k=-10}^{10} \frac{(z\bar{a})^k}{1 + \rho^{2k+1}}, \tag{76}$$

and S_{50} and S_{100}.

To approximate (5) numerically, we calculate

$$S(z, a) \approx S^*_{10}(z, a) = \frac{1}{2\pi} \sum_{k=-10}^{10} \frac{(-1)^k \rho^k}{\rho^{2k} - z\bar{a}}, \tag{77}$$

and S^*_{50}.

To approximate (22) numerically, we compute

$$S(z, a) \approx S^{**}_{15}(z, a) = \frac{1}{2\pi} \prod_{k=0}^{15} \frac{\left(1 + \bar{a}z\rho^{2k+1}\right)\left(z\bar{a} + \rho^{2k+1}\right)\left(1 - \rho^{2k+2}\right)^2}{\left(1 - z\bar{a}\rho^{2k}\right)\left(z\bar{a} - \rho^{2k+2}\right)\left(1 + \rho^{2k+1}\right)^2}, \tag{78}$$

and S^{**}_{20} and S^{**}_{25}.

The approximations are then compared with the numerical solution of the Kerzman-Stein Equation (7). To solve (7), we used the Nyström method [5] with the trapezoidal rule with n selected nodes on each boundary component Γ_0 and Γ_1. The approximate solution is represented by $\tilde{S}_n$ where n is the number of nodes. All the computations were done using MATHEMATICA 12.3. Four numerical examples are given for different values of a and ρ. The results for the error norms are presented for each example.

Table 3: Error norms between S^{**}_{15} and $\tilde{S}_n$, S^{**}_{20} and $\tilde{S}_n$, and S^{**}_{25} and $\tilde{S}_n$.

n	$\|S^{**}_{15} - \tilde{S}_n\|_\infty$	$\|S^{**}_{20} - \tilde{S}_n\|_\infty$	$\|S^{**}_{25} - \tilde{S}_n\|_\infty$
16	2.97758 (-03)	2.97758 (-03)	2.97758 (-03)
32	1.16296 (-05)	1.16299 (-05)	1.16299 (-05)
64	1.44308 (-10)	1.88038 (-10)	1.8835 (-10)
128	3.1999 (-10)	3.1275 (-13)	1.82618 (-15)

Table 4: Error norms between S_{10} and $\tilde{S}_n$, S_{50} and $\tilde{S}_n$, and S_{100} and $\tilde{S}_n$.

n	$\|S_{10} - \tilde{S}_n\|_\infty$	$\|S_{50} - \tilde{S}_n\|_\infty$	$\|S_{100} - \tilde{S}_n\|_\infty$
16	1.29695 (-02)	1.46732 (-03)	1.46732 (-03)
32	1.56432 (-02)	7.88666 (-06)	7.88666 (-06)
64	1.5646 (-02)	3.26124 (-08)	2.2539 (-10)
128	1.5646 (-02)	3.26942 (-08)	2.85127 (-15)

Table 5: Error norms between S^*_{10} and $\tilde{S}_n$ and S^*_{50} and $\tilde{S}_n$.

n	$\|S^*_{10} - \tilde{S}_n\|_\infty$	$\|S^*_{50} - \tilde{S}_n\|_\infty$
16	1.46686 (-03)	1.46732 (-03)
32	8.4009 (-06)	7.88666 (-06)
64	1.02367 (-06)	2.2539 (-10)
128	1.02367 (-06)	1.25883 (-15)

Table 6: Error norms between S^{**}_{5} and $\tilde{S}_n$, S^{**}_{10} and $\tilde{S}_n$, and S^{**}_{15} and $\tilde{S}_n$.

n	$\|S^{**}_{5} - \tilde{S}_n\|_\infty$	$\|S^{**}_{10} - \tilde{S}_n\|_\infty$	$\|S^{**}_{15} - \tilde{S}_n\|_\infty$
16	1.4675 (-03)	1.46732 (-03)	1.46732 (-03)
32	7.70793 (-06)	7.88666 (-06)	7.88666 (-06)
64	3.72977 (-07)	2.2434 (-10)	2.2539 (-10)
128	3.73107 (-07)	2.2023 (-12)	1.41308 (-15)

Table 7: Error norms between S_{10} and $\tilde{S}_n$, S_{50} and $\tilde{S}_n$, and S_{100} and $\tilde{S}_n$.

n	$\|S_{10} - \tilde{S}_n\|_\infty$	$\|S_{50} - \tilde{S}_n\|_\infty$	$\|S_{100} - \tilde{S}_n\|_\infty$
16	6.45804 (-02)	8.28061 (-03)	8.28061 (-03)
32	6.82534 (-02)	2.2673 (-04)	2.2673 (-04)
64	6.83565 (-02)	9.0045 (-06)	1.79491 (-07)
128	6.83565 (-02)	9.08614 (-06)	1.29631 (-10)

We consider an annulus Ω bounded by

$$\begin{aligned} \Gamma_0 &: z_0(t) = e^{it}, \\ \Gamma_1 &: z_1(t) = \rho e^{-it}, \end{aligned} \tag{79}$$

with $0 \le t \le 2\pi$.

Table 8: Error norms between S_{10}^* and $\tilde{S}_n$ and S_{50}^* and $\tilde{S}_n$.

n	$\|S_{10}^* - \tilde{S}_n\|_\infty$	$\|S_{50}^* - \tilde{S}_n\|_\infty$
16	8.28737 (-03)	8.28061 (-03)
32	2.33562 (-04)	2.2673 (-04)
64	1.79806 (-05)	1.79491 (-07)
128	1.78806 (-05)	1.1287 (-15)

Table 9: Error norms between S_5^{**} and $\tilde{S}_n$, S_{10}^{**} and $\tilde{S}_n$, and S_{15}^{**} and $\tilde{S}_n$.

n	$\|S_5^{**} - \tilde{S}_n\|_\infty$	$\|S_{10}^{**} - \tilde{S}_n\|_\infty$	$\|S_{15}^{**} - \tilde{S}_n\|_\infty$
16	8.27577 (-03)	8.28061 (-03)	8.28061 (-03)
32	2.2189 (-04)	2.26729 (-04)	2.2673 (-04)
64	1.13437 (-05)	1.78984 (-07)	1.79491 (-07)
128	1.14253 (-05)	1.19798 (-09)	7.90864 (-14)

Table 10: Error norms between S_{10} and $\tilde{S}_n$, S_{50} and $\tilde{S}_n$, and S_{100} and $\tilde{S}_n$.

n	$\|S_{10} - \tilde{S}_n\|_\infty$	$\|S_{50} - \tilde{S}_n\|_\infty$	$\|S_{100} - \tilde{S}_n\|_\infty$
16	3.15879 (-03)	2.61429 (-04)	2.61429 (-04)
32	3.22447 (-03)	2.08805 (-07)	2.08805 (-07)
64	3.28124 (-03)	5.91022 (-11)	1.33153 (-13)
128	3.28124 (-03)	5.91168 (-11)	1.33233 (-15)

Table 11: Error norms between S_{10}^* and $\tilde{S}_n$ and S_{50}^* and $\tilde{S}_n$.

n	$\|S_{10}^* - \tilde{S}_n\|_\infty$	$\|S_{50}^* - \tilde{S}_n\|_\infty$
16	2.61429 (-04)	2.61429 (-04)
32	2.0879 (-07)	2.08805 (-07)
64	1.68217 (-11)	1.33183 (-13)
128	1.67281 (-11)	1.16606 (-15)

Table 12: Error norms between S_5^{**} and $\tilde{S}_n$, S_{10}^{**} and $\tilde{S}_n$, and S_{15}^{**} and $\tilde{S}_n$.

n	$\|S_5^{**} - \tilde{S}_n\|_\infty$	$\|S_{10}^{**} - \tilde{S}_n\|_\infty$	$\|S_{15}^{**} - \tilde{S}_n\|_\infty$
16	2.61429 (-04)	2.61429 (-04)	2.61429 (-04)
32	2.08805 (-07)	2.08805 (-07)	2.08805 (-07)
64	6.46416 (-13)	1.3313 (-13)	1.33121 (-13)
128	6.77069 (-13)	1.49882 (-15)	1.55654 (-15)

Example 1. We consider an annulus Ω with $a = 0.7i$ and $\rho = 0.5$. The results for the error norms are presented in Tables 1–3.

Example 2. We consider an annulus Ω with $a = -0.4 - 0.6i$ and $\rho = 0.3$. The results for the error norms are presented in Tables 4–6.

Example 3. We consider an annulus Ω with $a = -0.8$ and $\rho = 0.4$. The results for the error norms are presented in Tables 7–9.

Example 4. We consider an annulus Ω with $a = -0.4 - 0.5i$ and $\rho = 0.1$. The results for the error norms are presented in Tables 10–12.

The numerical results presented in Tables 1–12 show that computations using the infinite product formula (22) converge faster than the bilateral series formulas (4) and (5).

7. Conclusion

This paper has shown that the bilateral series for the Szegö kernel for Ω is a disguised bilateral basic hypergeometric series ${}_1\psi_1$. Ramanujan's sum for ${}_1\psi_1$ is then applied to obtain the infinite product representation for the Szegö kernel for Ω. The product clearly exhibits the zero of the Szegö kernel for an Ω. The Szegö kernel can also be expressed as a closed form in terms of the q-gamma function and the modified Jacobi theta function. Similar q-analysis has also been conducted for the Szegó kernel for general Ω and for the weighted Szegö kernel for Ω. The numerical comparisons have shown that the infinite product method converges faster than the bilateral series methods for computing the Szegö kernel for Ω.

For future work, it is natural to devote further investigation on the infinite product representation for the Szegö kernel for doubly connected regions via the transformation formula (6) and Theorem 1. This however requires knowledge of conformal mapping of doubly connected regions to annulus [12–15]. For some ideas on numerical methods for computing the zero of the Szegö kernel for doubly connected regions, see [16]. Alternatively, perhaps some computational intelligence algorithms can also be considered to compute the zero, like the monarch butterfly optimization (MBO) [17], earthworm optimization algorithm (EWA) [18], elephant herding optimization (EHO) [19], moth search (MS) algorithm [20], slime mould algorithm (SMA) [21], and Harris hawks optimization (HHO) [22].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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