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Research Article

On the Analytical Treatment for the Fractional-Order Coupled Partial Differential Equations via Fixed Point Formulation and Generalized Fractional Derivative Operators

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High-dimensional fractional equation investigation is a cutting-edge discipline with considerable pragmatic and speculative consequences in engineering, epidemiology, and other scientific disciplines. In this study, a hybrid Jafari transform mixed with the Adomian decomposition method for obtaining the analytical solution to Burgers' problem is provided. Burgers' equation is a vital mathematical expression that appears in a variety of computational modelling fields, including fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow. By considering a hybrid transform, semianalytical techniques are constructed for the Caputo and Atangana-Baleanu fractional derivative operators. Besides that, existence and uniqueness analyses are carried out with the aid of the Banach contraction-fixed point theory. To obtain the models' findings, we employed the Jafari transform on fractional-order Burger equations (BEs), supplemented by the inverse Jafari transform. The projected findings for the fractional BEs have been depicted visually. Ultimately, numerical figures are provided to validate the practicality and efficacy. The solution obtained by employing the supplied methodologies has been validated to have the appropriate rate of convergence to the precise solution. The main advantage of the suggested method is the relatively small number of computations performed. It can also be used to address fractional-order scientific issues in a multitude of fields.

1. Introduction

Fractional calculus (FC) is a novel scientific and technical area of investigation that is broadly utilized in applied mathematics, pharmacology, information theory, fluid, gas turbulence, mathematical biology, and related domains [1, 2]. The topic of FC has subsequently gained a lot of attention. Numerous researchers have made significant contributions to this topic by developing multiple fractional expressions in diverse publications. The findings of advanced calculus are frequently far more comprehensive than those of classical calculus. Furthermore, fractional differential formulations have greater characteristics than integer-order ones, including Caputo [3], Caputo and Fabrizio [4], and Atangana and Baleanu [5]. The noted Caputo order has significant difficulties as well, including the fact that their kernel is singular. This flaw has an impact on simulating major challenges. To address the aforementioned issues, the authors [5] proposed an innovative formulation that has fractional order relying on the Mittag-Leffler (ML) function. It is worth noting that their fractional integral is the fractional average of the supplied function's Riemann-Liouville fractional integral and the function themselves [6–9]. In regard to the improvements listed previously, the derivative has been shown to be particularly effective in heat-like research and structural research [10, 11]. This special generation of derivatives containing fractional orders is both a filtration derivative and a fractional derivative.

Recently, the research design of fractional derivatives [3–5] has become increasingly appropriate for simulation, permitting fractional differential equations to be progressively implemented in all disciplines of science, with increasing feasibility, effectiveness, and precision. Researchers are continuously exploring different ways of evaluating the credibility and validity of fractional differential equations (FDEs). Some prevalent numerical/analytical methodologies are the Adomian decomposition method (ADM) [12], Adams-Bashforth Moulton (ABM)[13], Hirota bilinear method (HBM) [14], Exp-function method (EFM)[15], variation iteration method (VIM) [16], Newton polynomial approach (NPA) [17], finite volume method (FVM) [18], finite difference method (FDM) [19], residue power series method (RPM) [20], and so on.

The purpose of this paper is at employing the Jafari decomposition approach to nonlinear fractional coupled BEs as follows:

$$\begin{aligned}
&\mathbf{D}_{\mathbf{t}}^{\delta_1}\mathbf{f}+\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1}+\mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2}=\frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2}+\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\},\\
&\mathbf{D}_{\mathbf{t}}^{\delta_2}\mathbf{g}+\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1}+\mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2}=\frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2}+\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}, \quad 0<\delta_1,\delta_2<1.
\end{aligned} \tag{1}$$

Model (1) of dissipation seems to be a very prominent unsteady flow system. Various scholars have contemplated investigating this notion as a framework of vibration propagation in order to gain scientific understanding. The distinctive functionality of model (1) is the simplest computational description of the rivalry between viscosity propagation and nonlinear convection. It comprises the simplest fundamental versions of the decomposition factor $\mathbf{g}(\partial \mathbf{f}/\partial \mathbf{u}_2)$ and the nonlinear convection-diffusion factor $\mathbf{f}(\partial \mathbf{f}/\partial \mathbf{u}_1)$, where $\mathscr{V}=[\mathbf{f},\mathbf{g}]$ and Re is the Reynolds number employed to imitate the practical properties of signal oscillations and hence influence the behaviour of the system. Cole [21] examined BE algebraic features. Dynamical processes are crucial in science and numerical methods. In applied sciences, the significance of attaining the exact or estimated outcomes of partial differential equations is in order to examine innovative strategies; this is currently a contentious issue for accomplishing the exact or analytical results [22–24]. For this goal, several approaches for getting the varied reported performance of diverse scientific models that represented using nonlinear PDEs have been presented. Bateman [25] created a prominent framework and established consistent conclusions that are applicable to various viscous dissipations. Burgers [26] subsequently proposed it as the most distinguished framework for addressing computational instability challenges.

During Gorge Adomian's meteoric rise in 1980, the Adomian decomposition approach established a well-known methodology. This has been increasingly incorporated into a broad range of nonlinear systems, including the Black-Scholes model [27] and the Swift-Hohenberg [28]. The ADM has been shown to be strongly related to a wide range of integral transforms [27, 28]. Jafari [29] recently proposed a hybrid integral transform known as the Jafari transform. The capacity to recapitulate numerous prior transforms is the transformation's key characteristic (see Remark 8).

Due to the aforesaid tendency, we employ the Jafari transform decomposition methodology (JTDM) to determine the required clarification of the fractional-order BEs. To generate a new algorithmic strategy, the Jafari transform incorporated the ADM in an effective way. The Jafari transform is a modification of many previous formulae; see Remark 8. Both the recommended techniques produce analytical findings in the format of a convergent series. The Atangana-Baleanu fractional derivative operator in the Caputo interpretation is used to explain the quantitative categorizations of the BEs. The proffered methodologies are well represented in modelling and compilation investigations. The obtained method is a valuable tool for evaluating the behaviour of systems that are difficult to numerically analyze, notably for fractional PDEs. Fractal-fractional phenomena can be investigated using the approximate expression.

2. Preliminaries

In this part, we revisit certain key concepts, ideas, and terminologies connected to fractional derivative formulations involving the power law and ML as a kernel, as well as the Jafari transform's specific ramifications.

Definition 1 (see [3]). The fractional derivative of Caputo (CFD) is specifically defined as follows:

$$ {}_0^c\mathbf{D}_{\mathbf{t}}^{\delta}\mathbf{f}(\mathbf{t})=\begin{cases}\dfrac{1}{\Gamma(r-\delta)}\displaystyle\int_0^{\mathbf{t}}\frac{\mathbf{f}^{(r)}(\mathbf{u}_1)}{(\mathbf{t}-\mathbf{u}_1)^{\delta+1-r}}d\mathbf{u}_1, & r-1<\delta<r,\\[2ex] \dfrac{d^r}{d\mathbf{t}^r}\mathbf{f}(\mathbf{t}), & \delta=r.\end{cases} \tag{2}$$

Definition 2 (see [5]). The ABC is specifically defined as follows:

$$ {}_{\eta_1}^{\mathrm{ABC}}\mathbf{D}_{\mathbf{t}}^{\delta}(\mathbf{f}(\mathbf{t}))=\frac{\mathbf{ABC}(\delta)}{1-\delta}\int_{\eta_1}^{\mathbf{t}}\mathbf{f}'(\mathbf{t})E_{\delta}\left[-\frac{\delta(\mathbf{t}-\mathbf{u}_1)^{\delta}}{1-\delta}\right]d\mathbf{u}_1, \tag{3}$$

where $\mathbf{f}\in\Delta^1(a_1,a_2)$(Sobolev space), $a_1<a_2,\delta\in[0,1]$, and $\mathbf{ABC}(\delta)$ indicates the normalization function as $\mathbf{ABC}(\delta)=\mathbf{ABC}(0)=\mathbf{ABC}(1)=1$.

Definition 3 (see [5]). The ABC fractional integral operator is expressed in the following form:

$$ {}_{\eta_1}^{\mathrm{ABC}}\bar{I}_{\mathbf{t}}^{\delta}(\mathbf{f}(\mathbf{t}))=\frac{1-\delta}{\mathbf{ABC}(\delta)}\mathbf{f}(\mathbf{t})+\frac{\delta}{\Gamma(\delta)\mathbf{ABC}(\delta)}\cdot\int_{\eta_1}^{\mathbf{t}}\mathbf{f}(\mathbf{u}_1)(\mathbf{t}-\mathbf{u}_1)^{\delta-1}d\mathbf{u}_1. \tag{4}$$

Definition 4 (see [29]). Consider an integrable mapping $\mathbf{f}(\mathbf{t})$ defined on a set $\mathscr{P}$, and then,

$$\mathscr{P} = \{\mathbf{f}(\mathbf{t}): \exists M > 0, \kappa > 0, |\mathbf{f}(\mathbf{t})| < M \exp(\kappa \mathbf{t}), \quad \text{if } \mathbf{t} \geq 0\}. \tag{5}$$

Definition 5 (see [29]). Assume that the functions $\Phi(\rho)$, $\Psi(\rho)$: $\mathbb{R}^+ \mapsto \mathbb{R}^+$ such that $\varphi(\rho) \neq 0 \forall \rho \in \mathbb{R}^+$. The Jafari integral transform of the function $\mathbf{f}(\mathbf{t})$ expressed by $\mathbf{Q}(\rho)$ is stated as

$$\mathbf{J}\{\mathbf{f}(\mathbf{t}), \rho\} = \mathbf{Q}(\rho) = \Phi(\rho)\int_0^\infty \mathbf{f}(\mathbf{t}) \exp(-\Psi(\rho)\mathbf{t}) d\mathbf{t}. \tag{6}$$

Theorem 6 (see [29]) (convolution property). *The following is valid for the Jafari integral transform:*

$$\mathbf{J}\{\mathbf{f}_1 * \mathbf{f}_2\} = \frac{1}{\Phi(\rho)}\mathbf{Q}_1(\rho) * \mathbf{Q}_2(\rho). \tag{7}$$

Definition 7. The following is the Jafari transform of the CFD operator:

$$\mathbf{J}\left\{ {}_0^c\mathbf{D}_\mathbf{t}^\delta(\mathbf{f}(\mathbf{t})), \rho \right\} = \Psi^\delta(\rho)\mathbf{Q}(\rho) - \Phi(\rho) \cdot \sum_{\kappa=0}^{\delta-1} \Psi^{\delta-\kappa-1}(\rho)\mathbf{f}^{(\kappa)}(0), r-1 < \delta < r, \Phi, \Psi > 0. \tag{8}$$

Remark 8. Definition 7 refers to the following resulting assumptions:

(1) Choosing $\Phi(\rho) = 1$ and $\Psi(\rho) = \rho$, then, this leads to the Laplace transform [30]

(2) Choosing $\Phi(\rho)(1/\rho)$ and $\Psi(\rho) = 1/\rho$, then, this leads to the α-Laplace transform [31]

(3) Choosing $\Phi(\rho) = 1/\rho$ and $\Psi(\rho) = 1/\rho$, then, this leads to the Sumudu transform [32]

(4) Choosing $\Phi(\rho) = 1/\rho$ and $\Psi(\rho) = 1$, then, this leads to the Aboodh transform [33]

(5) Choosing $\Phi(\rho) = \rho$ and $\Psi(\rho) = \rho^2$, then, this leads to the Pourreza transform [34, 35]

(6) Choosing $\Phi(\rho) = \rho$ and $\Psi(\rho) = 1/\rho$, then, this leads to the Elzaki transform [36]

(7) Choosing $\Phi(\rho) = \mathbf{u}_1$ and $\Psi(\rho) = \rho/\mathbf{u}_1$, then, this leads to the natural transform [37]

(8) Choosing $\Phi(\rho) = \rho^2$ and $\Psi(\rho) = \rho$, then, this leads to the Mohand transform [38]

(9) Choosing $\Phi(\rho) = 1/\rho^2$ and $\Psi(\rho) = 1/\rho$, then, this leads to the Swai transform [39]

(10) Choosing $\Phi(\rho) = 1$ and $\Psi(\rho) = 1/\rho$, then, we get the Kamal transform [40]

(11) Choosing $\Phi(\rho) = \rho^\alpha$ and $\Psi(\rho) = 1/\rho$, then, this leads to the G_transform [41, 42]

Definition 9 (see [43]). The ABC fractional derivative operator has the following Jafari transform:

$$\mathbf{J}\left\{ {}_0^{\mathrm{ABC}}\mathbf{D}_\mathbf{t}^\delta(\mathbf{f}(\mathbf{t})), \rho \right\}(\delta) = \frac{\mathbf{ABC}(\delta)\Psi^\delta(\rho)}{\delta + (1-\delta)\Psi^\delta(\rho)} \cdot \left(\mathbf{Q}(\rho) - \frac{\Phi(\rho)}{\Psi(\rho)}\mathbf{f}(0)\right). \tag{9}$$

Remark 10. Definition 9 leads to the following conclusions:

(1) Choosing $\Phi(\rho) = 1$ and $\Psi(\rho) = \rho$, then, this leads to the Laplace transform of the ABC fractional derivative operator [44]

(2) Choosing $\Phi(\rho) = \rho$ and $\Psi(\rho) = 1/\rho$, then, this leads to the Elzaki transform of the ABC fractional derivative operator [45]

(3) Choosing $\Phi(\rho) = \Psi(\rho) = 1/\rho$, then, we get the Sumudu transform of the ABC fractional derivative operator [46]

(4) Choosing $\Phi(\rho) = 1$ and $\Psi(\rho) = \rho/\mathbf{u}_1$, then, we get the Shehu transform of the ABC fractional derivative operator [46]

Definition 11 (see [47]). The ML function for a single parameter is defined as

$$E_\delta(z) = \sum_{\kappa=0}^{\infty} \frac{z_1^\kappa}{\Gamma(\kappa\delta + 1)}, \quad \delta, z_1 \in \mathbb{C}, \mathfrak{R}(\delta) \geq 0. \tag{10}$$

3. Analysis of Semianalytical Techniques

In this part, we illustrate an explanation of the generic methodology for the subsequent system via the Jafari transform.

$$\mathbf{D}_\mathbf{t}^\delta\mathbf{f}(\mathbf{u}_1, \mathbf{t}) + \mathfrak{L}\mathbf{f}(\mathbf{u}_1, \mathbf{t}) + \tilde{N}\mathbf{f}(\mathbf{u}_1, \mathbf{t}) = \mathscr{F}(\mathbf{u}_1, \mathbf{t}), \quad \mathbf{t} > 0, 0 < \delta \leq 1, \tag{11}$$

with ICs

$$\mathbf{f}(\mathbf{u}_1, 0) = \mathscr{G}(\mathbf{u}_1), \tag{12}$$

where $\mathscr{L}$ denotes being linear and $\mathscr{N}$ is nonlinear, whilst $\hbar(\mathbf{u}_1, \tau)$ represents the source terms.

Considering the Jafari transform to (11), we obtain

$$\mathbf{J}\left[\mathbf{D}_\mathbf{t}^\delta\mathbf{f}(\mathbf{u}_1, \mathbf{t}) + \mathfrak{L}\mathbf{f}(\mathbf{u}_1, \mathbf{t}) + \tilde{N}\mathbf{f}(\mathbf{u}_1, \mathbf{t})\right] = \mathbf{J}[\mathscr{F}(\mathbf{u}_1, \mathbf{t})]. \tag{13}$$

Now, implement the differentiation property of Jafari transform in regard of CFD and then utilize the ABC fractional derivative operator as follows:

$$\Psi^{\delta}(\rho)\mathcal{U}(\mathbf{u}_1,\rho)=\Phi(\rho)\sum_{\kappa=0}^{\mathbf{r}-1}\Psi^{\delta-1-\kappa}(\rho)\mathbf{f}^{(\kappa)}(0)+\mathbf{J}\left[\mathfrak{L}\mathbf{f}(\mathbf{u}_1,\mathbf{t})+\tilde{N}\mathbf{f}(\mathbf{u}_1,\mathbf{t})\right]+\mathbf{J}[\mathcal{F}(\mathbf{u}_1,\mathbf{t})], \tag{14}$$

and

$$\frac{\Psi^{\delta}(\rho)\mathbf{ABC}(\delta)}{\delta+(1-\delta)\Psi^{\delta}(\rho)}\mathcal{U}(\mathbf{u}_1,\rho)=\frac{\Phi(\rho)}{\Psi(\rho)}\frac{\Psi^{\delta}(\rho)\mathbf{ABC}(\delta)}{\delta+(1-\delta)\Psi^{\delta}(\rho)}\mathbf{f}(0)+\mathbf{J}\left[\mathfrak{L}\mathbf{f}(\mathbf{u}_1,\mathbf{t})+\tilde{N}\mathbf{f}(\mathbf{u}_1,\mathbf{t})\right]+\mathbf{J}[\mathcal{F}(\mathbf{u}_1,\mathbf{t})]. \tag{15}$$

The inverse Jafari transform of (14) and (15) gives

$$\mathbf{f}(\mathbf{u}_1,\mathbf{t})=\mathbf{J}^{-1}\left[\Phi(\rho)\sum_{\kappa=0}^{\mathbf{r}-1}\Psi(\rho)^{\delta-\kappa-1}\mathbf{f}^{(\kappa)}(0)+\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}[\mathcal{F}(\mathbf{u}_1,\mathbf{t})]\right]-\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\mathfrak{L}\mathbf{f}(\mathbf{u}_1,\mathbf{t})+\tilde{N}\mathbf{f}(\mathbf{u}_1,\mathbf{t})\right]\right], \tag{16}$$

and

$$\mathbf{f}(\mathbf{u}_1,\mathbf{t})=\mathbf{J}^{-1}\left[\frac{\Phi(\rho)}{\Psi(\rho)}\mathbf{f}(0)+\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\Psi^{\delta}(\rho)\mathbf{ABC}(\delta)}\mathbf{J}[\mathcal{F}(\mathbf{u}_1,\mathbf{t})]\right]-\mathbf{J}^{-1}\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\Psi^{\delta}(\rho)\mathbf{ABC}(\delta)}\mathbf{J}\left[\mathfrak{L}\mathbf{f}(\mathbf{u}_1,\mathbf{t})+\tilde{N}\mathbf{f}(\mathbf{u}_1,\mathbf{t})\right]\right]. \tag{17}$$

The Jafari decomposition method solution $\mathbf{f}(\mathbf{u}_1,\mathbf{t})$ is described by the subsequent infinite series

$$\mathbf{f}(\mathbf{u}_1,\mathbf{t})=\sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t}). \tag{18}$$

Consequently, the nonlinear component $\tilde{N}(\mathbf{u}_1,\mathbf{t})$ can be analyzed utilizing the Adomian decomposition approach

$$\tilde{N}\mathbf{f}(\mathbf{u}_1,\mathbf{t})=\sum_{\mathbf{r}=0}^{\infty}\tilde{A}_{\mathbf{r}}(\mathbf{f}_0,\mathbf{f}_1,\cdots,),\quad \mathbf{r}=0,1,\cdots, \tag{19}$$

where

$$\tilde{A}_{\mathbf{r}}(\mathbf{f}_0,\mathbf{f}_1,\cdots,)=\frac{1}{\mathbf{r}!}\left[\frac{d^{\mathbf{r}}}{d\delta^{\mathbf{r}}}\tilde{N}\left(\sum_{j=0}^{\infty}\delta^j\mathbf{f}_j\right)\right]_{\delta=0},\quad \mathbf{r}>0. \tag{20}$$

Plugging (17) and (18) into (15) and (16), respectively, we get

$$\sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t})=\mathcal{G}(\mathbf{u}_1)+\tilde{\mathcal{G}}(\mathbf{u}_1)-\mathbf{J}^{-1}\cdot\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\mathfrak{L}\mathbf{f}(\mathbf{u}_1,\mathbf{t})+\sum_{\mathbf{r}=0}^{\infty}\tilde{A}_{\mathbf{r}}\right]\right], \tag{21}$$

and

$$\sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t})=\mathcal{G}(\mathbf{u}_1)+\tilde{\mathcal{G}}(\mathbf{u}_1)-\mathbf{J}^{-1}\cdot\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\mathbf{J}\left[\mathfrak{L}\mathbf{f}(\mathbf{u}_1,\mathbf{t})+\sum_{\mathbf{r}=0}^{\infty}\tilde{A}_{\mathbf{r}}\right]\right]. \tag{22}$$

As a result, the iterative approach for (19) and (21) is constructed as

$$\begin{aligned}
\mathbf{f}_0(\mathbf{u}_1,\mathbf{t})&=\mathcal{G}(\mathbf{u}_1)+\tilde{\mathcal{G}}(\mathbf{u}_1),\quad \mathbf{r}=0,\\
\mathbf{f}_{\mathbf{r}+1}(\mathbf{u}_1,\mathbf{t})&=-\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\mathfrak{L}(\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t}))+\sum_{\mathbf{r}=0}^{\infty}\tilde{A}_{\mathbf{r}}\right]\right],\quad \mathbf{r}\geq 1,\\
\mathbf{f}_{\mathbf{r}+1}(\mathbf{u}_1,\mathbf{t})&=-\mathbf{J}^{-1}\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\mathbf{J}\left[\mathfrak{L}(\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t}))+\sum_{\mathbf{r}=0}^{\infty}\tilde{A}_{\mathbf{r}}\right]\right],\quad \mathbf{r}\geq 1.
\end{aligned} \tag{23}$$

4. Mathematical Formulation of the Jafari Transform Decomposition Approach

The following classifications will reveal how the essential prerequisites ensure the development of a unique solution. In the case of JTDM, we predict the existence of solutions that are accompanied by [48].

4.1. Uniqueness Results. In this part, we will present the uniqueness analysis for the JTDM_C and $\mathrm{JTDM}_{\mathrm{ABC}}$ fractional operator.

Theorem 12. *The JTDM_C solution of (23) is unique when $0<\varepsilon<1$, where $\varepsilon=((Y_1+Y_2+Y_3)\mathbf{t}^{\delta})/(\Gamma(\delta+1))$.*

Proof. Given that all continuous mappings on the Banach space are represented by $\Omega=(\mathbb{C}[\bar{I}],\|.\|)$. Further, surmise that $\bar{I}=[0,\bar{T}]$ have the norm $\|.\|$. Presently, we specify a mapping $\mathcal{U}:\Omega\mapsto\mho$ as

$$\mathbf{f}_{\mathbf{r}+1}(\mathbf{u}_1,\mathbf{t})=\mathbf{f}(\mathbf{u}_1,\mathbf{t})+\mathbf{J}^{-1}\cdot\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\mathfrak{L}[\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t})]+\bar{P}[\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t})]+\tilde{N}[\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t})]\right]\right],\quad \mathbf{r}\geq 0, \tag{24}$$

where $\mathfrak{L}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})] \equiv (\partial^3\mathbf{f}(\mathbf{u}_1,\mathbf{t}))/(\partial\mathbf{u}_1^2)$ and $\bar{P}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})] \equiv (\partial\mathbf{f}(\mathbf{u}_1,\mathbf{t}))/(\partial\mathbf{u}_1)$. Either, surmise that $\mathfrak{L}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]$ and $\mathfrak{M}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]$ are also Lipschitzian with $|\bar{P}\mathbf{f}-\bar{P}\mathbf{f\!\!\!-}| < Y_1|\mathbf{f}-\mathbf{f\!\!\!-}|$ and $|\mathfrak{L}\mathbf{f}-\mathfrak{L}\mathbf{f\!\!\!-}| < Y_2|\mathbf{f}-\mathbf{f\!\!\!-}|$, where Y_1 and Y_2 are Lipschitz constants and $\mathbf{f},\mathbf{f\!\!\!-}$ are distinguishable functional variables.

$$
\begin{aligned}
&\|\mathscr{U}\mathbf{f}-\mathscr{U}\mathbf{f\!\!\!-}\| \\
&\quad= \max_{\mathbf{t}\in\bar{I}}\left|\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\mathfrak{L}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]+\bar{P}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]+\tilde{N}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]\right]\right]\right. \\
&\qquad\left.-\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\mathfrak{L}[\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})]+\bar{P}[\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})]+\tilde{N}[\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})]\right]\right]\right| \\
&\quad\le \max_{\mathbf{t}\in\bar{I}}\left|\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}[\mathfrak{L}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]-\mathfrak{L}[\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})]]\right]\right. \\
&\qquad+\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\bar{P}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]-\bar{P}[\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})]\right]\right] \\
&\qquad\left.+\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\tilde{N}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]-\tilde{N}[\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})]\right]\right]\right| \\
&\quad\le \max_{\mathbf{t}\in\bar{I}}\left[Y_1\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}|\mathbf{f}(\mathbf{u}_1,\mathbf{t})-\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})|\right]\right. \\
&\qquad+Y_2\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}|\mathbf{f}(\mathbf{u}_1,\mathbf{t})-\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})|\right] \\
&\qquad\left.+Y_3\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}|\mathbf{f}(\mathbf{u}_1,\mathbf{t})-\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})|\right]\right] \\
&\quad\le \max_{\mathbf{t}\in\bar{I}}(Y_1+Y_2+Y_3)\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}|\mathbf{f}(\mathbf{u}_1,\mathbf{t})-\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})|\right] \\
&\quad\le (Y_1+Y_2+Y_3)\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\|\mathbf{f}(\mathbf{u}_1,\mathbf{t})-\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})\|\right] \\
&\quad= (Y_1+Y_2+Y_3)\mathbf{J}^{-1}\left[\frac{\Phi(\rho)}{\Psi^{\delta+1}(\rho)}\|\mathbf{f}(\mathbf{u}_1,\mathbf{t})-\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})\|\right] \\
&\quad= \frac{((Y_1+Y_2+Y_3))\mathbf{t}^{(\delta)}}{\Gamma(\delta+1)}\|\mathbf{f}(\mathbf{u}_1,\mathbf{t})-\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})\|.
\end{aligned}
\tag{25}
$$

As $0<\varepsilon<1$, so, $\mathscr{U}$ is contraction. As a conclusion of the Banach contraction fixed point theorem, (11) is unique. This yields the intended outcome. □

Theorem 13. *The JTDM$_{ABC}$ solution of (23) is unique when $0<\varepsilon<1$, where $\varepsilon=(Y_1+Y_2+Y_3)\{(\delta\mathbf{t}^{\delta})/(\Gamma(\delta+1))+(1-\delta)\}$.*

Proof. Given that all continuous mappings on the Banach space are represented by $\mho=(\mathbb{C}[\bar{I}],\|.\|)$, further, surmise that $\bar{I}=[0,\bar{T}]$ have the norm $\|.\|$. Presently, we specify a mapping $\mathscr{U}:\mho\mapsto\mho$ as

$$
\begin{aligned}
\mathbf{f}_{\mathbf{r}+1}(\mathbf{u}_1,\mathbf{t}) = \mathbf{f}(\mathbf{u}_1,\mathbf{t}) + \mathbf{J}^{-1}&\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\mathbf{J}[\mathfrak{L}[\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t})]\right. \\
&\left.+\bar{P}[\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t})]+\tilde{N}[\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t})]]\right], \quad \mathbf{r}\ge 0,
\end{aligned}
\tag{26}
$$

where $\mathfrak{L}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})] \equiv (\partial^3\mathbf{f}(\mathbf{u}_1,\mathbf{t}))/(\partial\mathbf{u}_1^2)$ and $\bar{P}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})] \equiv (\partial\mathbf{f}(\mathbf{u}_1,\mathbf{t}))/(\partial\mathbf{u}_1)$. Either, surmise that $\mathfrak{L}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]$ and $\mathfrak{M}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]$ are also Lipschitzian with $|\bar{P}\mathbf{f}-\bar{P}\mathbf{f\!\!\!-}| < Y_1|\mathbf{f}-\mathbf{f\!\!\!-}|$ and $|\mathfrak{L}\mathbf{f}-\mathfrak{L}\mathbf{f\!\!\!-}| < Y_2|\mathbf{f}-\mathbf{f\!\!\!-}|$, where Y_1 and Y_2 are the Lipschitz constant, respectively, and $\mathbf{f},\mathbf{f\!\!\!-}$ are distinguishable functional variables.

$$
\begin{aligned}
&\|\mathscr{U}\mathbf{f}-\mathscr{U}\mathbf{f\!\!\!-}\| \\
&\quad= \max_{\mathbf{t}\in\bar{I}}\left|\mathbf{J}^{-1}\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\mathbf{J}[\mathfrak{L}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]\right.\right. \\
&\qquad\left.+\bar{P}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]+\tilde{N}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]]\right] \\
&\qquad-\mathbf{J}^{-1}\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\mathbf{J}[\mathfrak{L}[\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})]\right. \\
&\qquad\left.\left.+\bar{P}[\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})]+\tilde{N}[\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})]]\right]\right| \\
&\quad\le \max_{\mathbf{t}\in\bar{I}}\left|\mathbf{J}^{-1}\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\mathbf{J}[\mathfrak{L}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]-\mathfrak{L}[\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})]]\right]\right. \\
&\qquad+\mathbf{J}^{-1}\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\mathbf{J}\left[\bar{P}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]-\bar{P}[\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})]\right]\right] \\
&\qquad\left.+\mathbf{J}^{-1}\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\mathbf{J}\left[\tilde{N}[\mathbf{f}(\mathbf{u}_1,\mathbf{t})]-\tilde{N}[\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})]\right]\right]\right| \\
&\quad\le \max_{\mathbf{t}\in\bar{I}}\left[Y_1\mathbf{J}^{-1}\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\mathbf{J}|\mathbf{f}(\mathbf{u}_1,\mathbf{t})-\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})|\right]\right. \\
&\qquad+Y_2\mathbf{J}^{-1}\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\mathbf{J}|\mathbf{f}(\mathbf{u}_1,\mathbf{t})-\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})|\right] \\
&\qquad\left.+Y_3\mathbf{J}^{-1}\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\mathbf{J}|\mathbf{f}(\mathbf{u}_1,\mathbf{t})-\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})|\right]\right] \\
&\quad\le \max_{\mathbf{t}\in\bar{I}}(Y_1+Y_2+Y_3)\mathbf{J}^{-1} \\
&\qquad\cdot\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\mathbf{J}|\mathbf{f}(\mathbf{u}_1,\mathbf{t})-\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})|\right] \\
&\quad\le (Y_1+Y_2+Y_3)\mathbf{J}^{-1} \\
&\qquad\cdot\left[\frac{\delta+(1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\mathbf{J}\|\mathbf{f}(\mathbf{u}_1,\mathbf{t})-\mathbf{f\!\!\!-}(\mathbf{u}_1,\mathbf{t})\|\right]
\end{aligned}
$$

$$= (Y_1 + Y_2 + Y_3)\mathbf{J}^{-1} \cdot \left[\frac{\Phi(\rho)}{\Psi(\rho)}\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)}\|\mathbf{f}(\mathbf{u}_1,\mathbf{t}) - \mathbf{f}(\mathbf{u}_1,\mathbf{t})\|\right]$$
$$= (Y_1 + Y_2 + Y_3)\left\{\frac{\delta\mathbf{t}^{\delta}}{\Gamma(\delta+1)} + (1-\delta)\right\}\|\mathbf{f}(\mathbf{u}_1,\mathbf{t}) - \mathbf{f}(\mathbf{u}_1,\mathbf{t})\|. \tag{27}$$

As $0 < \varepsilon < 1$, so, $\mathscr{U}$ is contraction. As a conclusion of the Banach contraction fixed point theorem, (11) is unique. This yields the intended outcome. □

4.2. Convergence Analysis. This section consists of the convergence analysis based on JTDM_C and JTDM_{ABC}.

Theorem 14. *The JTDM_C solution of (11) is convergent.*

Proof. Surmise that $\widehat{Q}_{\mathbf{r}} = \sum_{m=0}^{\mathbf{r}}\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t})$. Furthermore, in order to show that $\{\widehat{Q}_{\mathbf{r}}\}$ is Cauchy sequence in U, by analyzing a model consisting of Adomian polynomials, we obtain

$$\bar{R}(\widehat{Q}_{\mathbf{r}}) = \tilde{\Delta}_{\mathbf{r}} + \sum_{p=0}^{\mathbf{r}-1}\tilde{\Delta}_p,$$
$$\tilde{N}(\widehat{Q}_{\mathbf{r}}) = \tilde{\Delta}_{\mathbf{r}} + \sum_{c=0}^{\mathbf{r}-1}\tilde{\Delta}_c. \tag{28}$$

Now,

$$\begin{aligned}
\|\widehat{Q}_{\mathbf{r}} - \widehat{Q}_q\| &= \max_{\mathbf{t}\in\bar{I}}|\widehat{Q}_{\mathbf{r}} - \widehat{Q}_q| \\
&= \max_{\mathbf{t}\in\bar{I}}\left|\sum_{m=q+1}^{\mathbf{r}}\mathbf{f}(\mathbf{u}_1,\mathbf{t})\right|, (m = 1,2,3,\cdots,) \\
&\le \max_{\mathbf{t}\in\bar{I}}\left|\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\sum_{m=q+1}^{\mathbf{r}}\mathfrak{L}[\mathbf{f}_{\mathbf{r}-1}(\mathbf{u}_1,\mathbf{t})]\right]\right]\right. \\
&\quad + \mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\sum_{m=q+1}^{\mathbf{r}}\bar{P}[\mathbf{f}_{\mathbf{r}-1}(\mathbf{u}_1,\mathbf{t})]\right]\right] \\
&\quad \left.+ \mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\sum_{m=q+1}^{\mathbf{r}}\tilde{\Delta}_{\mathbf{r}-1}(\mathbf{u}_1,\mathbf{t})\right]\right]\right| \\
&= \max_{\mathbf{t}\in\bar{I}}\left|\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\sum_{m=q}^{\mathbf{r}-1}\mathfrak{L}[\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t})]\right]\right]\right. \\
&\quad + \mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\sum_{m=q}^{\mathbf{r}-1}\bar{P}[\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t})]\right]\right] \\
&\quad \left.+ \mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\sum_{m=q}^{\mathbf{r}-1}\tilde{\Delta}_{\mathbf{r}}(\mathbf{u}_1,\mathbf{t})\right]\right]\right| \\
&\le \max_{\mathbf{t}\in\bar{I}}\left|\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\sum_{m=q}^{\mathbf{r}-1}\mathfrak{L}(\widehat{Q}_{\mathbf{r}-1}) - \mathfrak{L}(\widehat{Q}_{q-1})\right]\right]\right. \\
&\quad + \mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\sum_{m=q}^{\mathbf{r}-1}\bar{P}(\widehat{Q}_{\mathbf{r}-1}) - \bar{P}(\widehat{Q}_{q-1})\right]\right] \\
&\quad \left.+ \mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}\left[\sum_{m=q}^{\mathbf{r}-1}\tilde{N}(\widehat{Q}_{\mathbf{r}-1}) - \tilde{N}(\widehat{Q}_{q-1})\right]\right]\right| \\
&\le \max_{\mathbf{t}\in\bar{I}}\left|\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}[\mathfrak{L}(\widehat{Q}_{\mathbf{r}-1}) - \mathfrak{L}(\widehat{Q}_{q-1})]\right]\right. \\
&\quad + \mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}[\bar{P}(\widehat{Q}_{\mathbf{r}-1}) - \bar{P}(\widehat{Q}_{q-1})]\right] \\
&\quad \left.+ \mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}[\tilde{N}(\widehat{Q}_{\mathbf{r}-1}) - \tilde{N}(\widehat{Q}_{q-1})]\right]\right| \\
&\le Y_1\max_{\mathbf{t}\in\bar{I}}\mathbf{J}^{-1}\left\|\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}[(\widehat{Q}_{\mathbf{r}-1}) - (\widehat{Q}_{q-1})]\right]\right\| \\
&\quad + Y_2\max_{\mathbf{t}\in\bar{I}}\left|\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}[(\widehat{Q}_{\mathbf{r}-1}) - (\widehat{Q}_{q-1})]\right]\right| \\
&\quad + Y_3\max_{\mathbf{t}\in\bar{I}}\left|\mathbf{J}^{-1}\left[\frac{1}{\Psi^{\delta}(\rho)}\mathbf{J}[(\widehat{Q}_{\mathbf{r}-1}) - (\widehat{Q}_{q-1})]\right]\right| \\
&= (Y_1 + Y_2 + Y_3)\mathbf{J}^{-1}\left[\frac{\Phi(\rho)}{\Psi^{\delta+1}(\rho)}\|\widehat{Q}_{\mathbf{r}-1} - \widehat{Q}_{q-1}\|.\right] \\
&= \frac{(Y_1 + Y_2 + Y_3)\mathbf{t}^{\delta}}{\Gamma(\delta+1)}\|\widehat{Q}_{\mathbf{r}-1} - \widehat{Q}_{q-1}\|.
\end{aligned} \tag{29}$$

Take $n = q + 1$; then,

$$\begin{aligned}
\|\widehat{Q}_{q+1} - \widehat{Q}_q\| &\le \varepsilon\|\widehat{Q}_q - \widehat{Q}_{q-1}\| \le \varepsilon^2\|\widehat{Q}_{q-1} - \widehat{Q}_{q-2}\| \\
&\le \cdots \le \varepsilon^q\|\widehat{Q}_1 - \widehat{Q}_0\|,
\end{aligned} \tag{30}$$

where $\varepsilon = ((Y_1 + Y_2 + Y_3)\mathbf{t}^{\delta})/(\Gamma(\delta+1))$. Analogously, we have

$$\begin{aligned}
\|\widehat{Q}_{\mathbf{r}} - \widehat{Q}_q\| &\le \|\widehat{Q}_{q+1} - \widehat{Q}_q\| + \|\widehat{Q}_{q+2} - \widehat{Q}_{q+1}\| + \cdots + \|\widehat{Q}_{\mathbf{r}} - \widehat{Q}_{\mathbf{r}-1}\| \\
&\le [\varepsilon^q + \varepsilon^{q+1} + \cdots + \varepsilon^{\mathbf{r}-1}]\|\widehat{Q}_1 - \widehat{Q}_0\| \\
&\le \varepsilon^q\left(\frac{1 - \varepsilon^{\mathbf{r}-q}}{\varepsilon}\right)\|\mathbf{f}_1\|.
\end{aligned} \tag{31}$$

As $0 < \varepsilon < 1$, we have $(1 - \varepsilon^{\mathbf{r}-q}) < 1$. Accordingly,

$$\|\widehat{Q}_{\mathbf{r}} - \widehat{Q}_q\| \le \frac{\varepsilon^q}{1-\varepsilon}\max_{\mathbf{t}\in\bar{I}}\|\mathbf{f}_1\|. \tag{32}$$

Therefore, $\|\mathbf{f}_1\| < \infty$ (since $\mathbf{f}(\mathbf{u}_1,\mathbf{t})$ is bounded). Additionally, as $q \mapsto \infty$, then, $\|\widehat{Q}_{\mathbf{r}} - \widehat{Q}_q\| \mapsto 0$. As a result, $\{\widehat{Q}_1\}$ is a Cauchy sequence in K. Thus, the series $\sum_{n=0}^{\infty}\mathbf{f}_{\mathbf{r}}$ is convergent. □

Theorem 15. *The JTDM_{ABC} solution of (11) is convergent.*

Proof. Surmise that $\widehat{Q}_{\mathbf{r}} = \sum_{m=0}^{\mathbf{r}} \mathbf{f}_{\mathbf{r}}(\mathbf{u}_1, \mathbf{t})$. Furthermore, in order to show that $\{\widehat{Q}_{\mathbf{r}}\}$ is a Cauchy sequence in U, by analyzing a model consisting of Adomian polynomials, we obtain

$$\begin{aligned} \bar{R}(\widehat{Q}_{\mathbf{r}}) &= \tilde{\Delta}_{\mathbf{r}} + \sum_{p=0}^{\mathbf{r}-1} \tilde{\Delta}_p, \\ \tilde{N}(\widehat{Q}_{\mathbf{r}}) &= \tilde{\Delta}_{\mathbf{r}} + \sum_{c=0}^{\mathbf{r}-1} \tilde{\Delta}_c. \end{aligned} \tag{33}$$

Now,

$$\begin{aligned} \|\widehat{Q}_{\mathbf{r}} - \widehat{Q}_q\| &= \max_{\mathbf{t}\in \bar{I}} |\widehat{Q}_{\mathbf{r}} - \widehat{Q}_q| \\ &= \max_{\mathbf{t}\in \bar{I}} \left| \sum_{m=q+1}^{\mathbf{r}} \mathbf{f}(\mathbf{u}_1, \mathbf{t}) \right|, (m = 1, 2, 3, \cdots,) \\ &\le \max_{\mathbf{t}\in \bar{I}} \left| \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}\left[\sum_{m=q+1}^{\mathbf{r}} \mathfrak{L}[\mathbf{f}_{\mathbf{r}-1}(\mathbf{u}_1, \mathbf{t})]\right]\right] \right. \\ &\quad + \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}\left[\sum_{m=q+1}^{\mathbf{r}} \bar{P}[\mathbf{f}_{\mathbf{r}-1}(\mathbf{u}_1, \mathbf{t})]\right]\right] \\ &\quad \left. + \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}\left[\sum_{m=q+1}^{\mathbf{r}} \tilde{\Delta}_{\mathbf{r}-1}(\mathbf{u}_1, \mathbf{t})\right]\right] \right| \\ &= \max_{\mathbf{t}\in \bar{I}} \left| \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}\left[\sum_{m=q}^{\mathbf{r}-1} \mathfrak{L}[\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1, \mathbf{t})]\right]\right] \right. \\ &\quad + \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}\left[\sum_{m=q}^{\mathbf{r}-1} \bar{P}[\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1, \mathbf{t})]\right]\right] \\ &\quad \left. + \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}\left[\sum_{m=q}^{\mathbf{r}-1} \tilde{\Delta}_{\mathbf{r}}(\mathbf{u}_1, \mathbf{t})\right]\right] \right| \\ &\le \max_{\mathbf{t}\in \bar{I}} \left| \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}\left[\sum_{m=q}^{\mathbf{r}-1} \mathfrak{L}(\widehat{Q}_{\mathbf{r}-1}) - \mathfrak{L}(\widehat{Q}_{q-1})\right]\right] \right. \\ &\quad + \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}\left[\sum_{m=q}^{\mathbf{r}-1} \bar{P}(\widehat{Q}_{\mathbf{r}-1}) - \bar{P}(\widehat{Q}_{q-1})\right]\right] \\ &\quad \left. + \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}\left[\sum_{m=q}^{\mathbf{r}-1} \tilde{N}(\widehat{Q}_{\mathbf{r}-1}) - \tilde{N}(\widehat{Q}_{q-1})\right]\right] \right| \\ &\le \max_{\mathbf{t}\in \bar{I}} \left| \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}[\mathfrak{L}(\widehat{Q}_{\mathbf{r}-1}) - \mathfrak{L}(\widehat{Q}_{q-1})]\right] \right. \\ &\quad + \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}[\bar{P}(\widehat{Q}_{\mathbf{r}-1}) - \bar{P}(\widehat{Q}_{q-1})]\right] \\ &\quad \left. + \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}[\tilde{N}(\widehat{Q}_{\mathbf{r}-1}) - \tilde{N}(\widehat{Q}_{q-1})]\right] \right| \\ &\le Y_1 \max_{\mathbf{t}\in \bar{I}} \mathbf{J}^{-1} \left\| \left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}[(\widehat{Q}_{\mathbf{r}-1}) - (\widehat{Q}_{q-1})]\right] \right\| \\ &\quad + Y_2 \max_{\mathbf{t}\in \bar{I}} \left| \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}[(\widehat{Q}_{\mathbf{r}-1}) - (\widehat{Q}_{q-1})]\right] \right| \\ &\quad + Y_3 \max_{\mathbf{t}\in \bar{I}} \left| \mathbf{J}^{-1}\left[\frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \mathbf{J}[(\widehat{Q}_{\mathbf{r}-1}) - (\widehat{Q}_{q-1})]\right] \right| \\ &= (Y_1 + Y_2 + Y_3)\mathbf{J}^{-1} \\ &\quad \cdot \left[\frac{\Phi(\rho)}{\Psi^{\delta+1}(\rho)} \frac{\delta + (1-\delta)\Psi^{\delta}(\rho)}{\mathbf{ABC}(\delta)\Psi^{\delta}(\rho)} \|\widehat{Q}_{\mathbf{r}-1} - \widehat{Q}_{q-1}\|\right] \\ &= (Y_1 + Y_2 + Y_3)\left\{\frac{\delta \mathbf{t}^{\delta}}{\Gamma(\delta+1)} + (1-\delta)\right\} \|\widehat{Q}_{\mathbf{r}-1} - \widehat{Q}_{q-1}\|. \end{aligned} \tag{34}$$

Take $n = q + 1$; then,

$$\begin{aligned} \|\widehat{Q}_{q+1} - \widehat{Q}_q\| &\le \varepsilon \|\widehat{Q}_q - \widehat{Q}_{q-1}\| \le \varepsilon^2 \|\widehat{Q}_{q-1} - \widehat{Q}_{q-2}\| \\ &\le \cdots \le \varepsilon^q \|\widehat{Q}_1 - \widehat{Q}_0\|, \end{aligned} \tag{35}$$

where $\varepsilon = (Y_1 + Y_2 + Y_3)\{(\delta \mathbf{t}^{\delta}/\Gamma(\delta+1)) + (1-\delta)\}$. Analogously, we have

$$\begin{aligned} \|\widehat{Q}_{\mathbf{r}} - \widehat{Q}_q\| &\le \|\widehat{Q}_{q+1} - \widehat{Q}_q\| + \|\widehat{Q}_{q+2} - \widehat{Q}_{q+1}\| + \cdots + \|\widehat{Q}_{\mathbf{r}} - \widehat{Q}_{\mathbf{r}-1}\| \\ &\le [\varepsilon^q + \varepsilon^{q+1} + \cdots + \varepsilon^{\mathbf{r}-1}] \|\widehat{Q}_1 - \widehat{Q}_0\| \\ &\le \varepsilon^q \left(\frac{1 - \varepsilon^{\mathbf{r}-q}}{\varepsilon}\right) \|\mathbf{f}_1\|. \end{aligned} \tag{36}$$

As $0 < \varepsilon < 1$, we have $(1 - \varepsilon^{\mathbf{r}-q}) < 1$. Accordingly,

$$\|\widehat{Q}_{\mathbf{r}} - \widehat{Q}_q\| \le \frac{\varepsilon^q}{1-\varepsilon} \max_{\mathbf{t}\in \bar{I}} \|\mathbf{f}_1\|. \tag{37}$$

Therefore, $\|\mathbf{f}_1\| < \infty$ (since $\mathbf{f}(\mathbf{u}_1, \mathbf{t})$ is bounded). Additionally, as $q \mapsto \infty$, then, $\|\widehat{Q}_{\mathbf{r}} - \widehat{Q}_q\| \mapsto 0$. As a result, $\{\widehat{Q}_1\}$ is a Cauchy sequence in K. Thus, the series $\sum_{n=0}^{\infty} \mathbf{f}_{\mathbf{r}}$ is convergent. □

5. Solutions of Fractional-Order Burgers Equation

In this part, we demonstrate how to use the aforementioned strategies to derive analytical findings for the Burgers equations under various initial guesses.

Example 16. Let us surmise the generic form of BEs (1) as follows:

$$\begin{aligned} \mathbf{D}_{\mathbf{t}}^{\delta_1}\mathbf{f} + \mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1} + \mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2} &= \frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\}, \\ \mathbf{D}_{\mathbf{t}}^{\delta_2}\mathbf{g} + \mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1} + \mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2} &= \frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}, \quad 0 < \delta_1, \delta_2 < 1, \end{aligned} \tag{38}$$

having ICs

$$\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, 0) = \frac{3}{4} - \frac{1}{4(1 + \exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))},$$
$$\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, 0) = \frac{3}{4} + \frac{1}{4(1 + \exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))}, \tag{39}$$

where Re denotes the Reynolds number.

In the exact solution of (38) when $\delta_1 = \delta_2 = 1$, then,

$$\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) = \frac{3}{4} - \frac{1}{4(1 + \exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2 - \mathbf{t}))},$$
$$\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) = \frac{3}{4} + \frac{1}{4(1 + \exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2 - \mathbf{t}))}. \tag{40}$$

Case 17 (Caputo fractional operator). Applying the Jafari transform on (38), we have

$$\mathbf{J}\left[\mathbf{D}_{\mathbf{t}}^{\delta_1}\mathbf{f}\right] + \mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1}\right] + \mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2}\right] = \frac{1}{\mathrm{Re}}\mathbf{J}\left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\},$$
$$\mathbf{J}\left[\mathbf{D}_{\mathbf{t}}^{\delta_2}\mathbf{g}\right] + \mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1}\right] + \mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2}\right] = \frac{1}{\mathrm{Re}}\mathbf{J}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}. \tag{41}$$

It follows that

$$\Psi^{\delta_1}(\rho)\mathbf{J}[\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \rho)] - \Phi(\rho)\sum_{\kappa=1}^{m_1-1}\Psi^{\delta_1-\kappa-1}(\rho)\mathbf{f}^{(\kappa)}(\mathbf{u}_1, \mathbf{u}_2, 0)$$
$$= -\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1}\right] - \mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2}\right] + \frac{1}{\mathrm{Re}}\mathbf{J}\left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\},$$
$$\Psi^{\delta_2}(\rho)\mathbf{J}[\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \rho)] - \Phi(\rho)\sum_{\kappa=1}^{m_1-1}\Psi^{\delta_2-\kappa-1}(\rho)\mathbf{g}^{(\kappa)}(\mathbf{u}_1, \mathbf{u}_2, 0)$$
$$= -\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1}\right] - \mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2}\right] + \frac{1}{\mathrm{Re}}\mathbf{J}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}. \tag{42}$$

The aforementioned equation can be written as

$$\mathbf{J}[\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \rho)] = \frac{\Phi(\rho)}{\Psi(\rho)}\left[\frac{3}{4} - \frac{1}{4(1 + \exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))}\right] - \frac{1}{\Psi^{\delta_1}(\rho)}\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1}\right] - \frac{1}{\Psi^{\delta_1}(\rho)}\mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2}\right] + \frac{1}{\mathrm{Re}}\frac{1}{\Psi^{\delta_1}(\rho)}\mathbf{J}\left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\}, \tag{43}$$

$$\mathbf{J}[\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \rho)] = \frac{\Phi(\rho)}{\Psi(\rho)}\left[\frac{3}{4} + \frac{1}{4(1 + \exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))}\right] - \frac{1}{\Psi^{\delta_2}(\rho)}\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1}\right] - \frac{1}{\Psi^{\delta_2}(\rho)}\mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2}\right] + \frac{1}{\mathrm{Re}}\frac{1}{\Psi^{\delta_2}(\rho)}\mathbf{J}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}. \tag{44}$$

Further, implementing the inverse Jafari transform on (43), then, it diminishes to

$$\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \rho) = \mathbf{J}^{-1}\left[\frac{\Phi(\rho)}{\Psi(\rho)}\left[\frac{3}{4} - \frac{1}{4(1 + \exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))}\right]\right] - \mathbf{J}^{-1}\left\{\frac{1}{\Psi^{\delta_1}(\rho)}\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1} + \mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2} - \frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\}\right]\right\}, \tag{45}$$

$$\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \rho) = \mathbf{J}^{-1}\left[\frac{\Phi(\rho)}{\Psi(\rho)}\left[\frac{3}{4} + \frac{1}{4(1 + \exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))}\right]\right] - \mathbf{J}^{-1}\left\{\frac{1}{\Psi^{\delta_2}(\rho)}\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1} + \mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2} - \frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}\right]\right\}. \tag{46}$$

In this case, we hypothesize that the undefined functions $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ and $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ can be described as an infinite series of the mode as follows

$$\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) = \sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}),$$
$$\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) = \sum_{\mathbf{r}=0}^{\infty}\mathbf{g}_{\mathbf{r}}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}). \tag{47}$$

It is clear that $\mathbf{f}\mathbf{f}_{\mathbf{u}_1} = \sum_{\mathbf{r}=0}^{\infty}\mathscr{A}_{\mathbf{r}}$, $\mathbf{f}\mathbf{g}_{\mathbf{u}_1} = \sum_{\mathbf{r}=0}^{\infty}\mathscr{B}_{\mathbf{r}}$, $\mathbf{g}\mathbf{f}_{\mathbf{u}_2} = \sum_{\mathbf{r}=0}^{\infty}\mathscr{C}_{\mathbf{r}}$, $\mathbf{g}\mathbf{g}_{\mathbf{u}_2} = \sum_{\mathbf{r}=0}^{\infty}\mathscr{D}_{\mathbf{r}}$ indicate the Adomian polynomials and are referred to as the nonlinear factors.

In view of the Adomian polynomials, (45) can be described as

$$\sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}+1}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) = \frac{3}{4} - \frac{1}{4(1 + \exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))} - \mathbf{J}^{-1}\left\{\frac{1}{\Psi^{\delta_1}(\rho)}\mathbf{J}\left[\sum_{\mathbf{r}=0}^{\infty}\mathscr{A}_{\mathbf{r}} + \sum_{\mathbf{r}=0}^{\infty}\mathscr{B}_{\mathbf{r}} - \frac{1}{\mathrm{Re}}\left\{\sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}\mathbf{u}_1\mathbf{u}_1} + \sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}\mathbf{u}_2\mathbf{u}_2}\right\}\right]\right\},$$

$$\begin{aligned}&\sum_{\mathbf{r}=0}^{\infty}\mathbf{g}_{\mathbf{r}+1}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})\\&\quad=\frac{3}{4}+\frac{1}{4(1+\exp(((0.03125)\,\mathrm{Re})-4\mathbf{u}_1+4\mathbf{u}_2))}\\&\qquad-\mathbf{J}^{-1}\left\{\frac{1}{\Psi^{\delta_2}(\rho)}\mathbf{J}\left[\sum_{\mathbf{r}=0}^{\infty}\mathscr{C}_{\mathbf{r}}+\sum_{\mathbf{r}=0}^{\infty}\mathscr{D}_{\mathbf{r}}\right.\right.\\&\qquad\left.\left.+\frac{1}{\mathrm{Re}}\left\{\sum_{\mathbf{r}=0}^{\infty}\mathbf{g}_{\mathbf{r}\mathbf{u}_1\mathbf{u}_1}+\sum_{\mathbf{r}=0}^{\infty}\mathbf{g}_{\mathbf{r}\mathbf{u}_2\mathbf{u}_2}\right\}\right]\right\}.\end{aligned}\tag{48}$$

Analyzing term by term (48), we may immediately get the iterative terms indicated as follows:

$$\begin{aligned}&\mathbf{f}_0(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})\\&\quad=\frac{3}{4}-\frac{1}{4(1+\exp(((0.03125)\,\mathrm{Re})-4\mathbf{u}_1+4\mathbf{u}_2))},\\&\mathbf{g}_0(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})\\&\quad=\frac{3}{4}+\frac{1}{4(1+\exp(((0.03125)\,\mathrm{Re})-4\mathbf{u}_1+4\mathbf{u}_2))},\\&\mathbf{f}_1(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})\\&\quad=-\frac{\mathrm{Re}.\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{128[1+\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^2}\cdot\frac{\mathbf{t}^{\delta_1}}{\Gamma(\delta_1+1)},\\&\mathbf{g}_1(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})\\&\quad=\frac{\mathrm{Re}.\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{128[1+\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^2}\cdot\frac{\mathbf{t}^{\delta_2}}{\Gamma(\delta_2+1)},\\&\mathbf{f}_2(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})\\&\quad=-\frac{\mathrm{Re}.\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{4096[1+\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^4}\cdot\frac{\mathbf{t}^{2\delta_1}}{\Gamma(2\delta_1+1)}\\&\qquad\times\{(\mathrm{Re})^2(-\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2)))\\&\qquad+(\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))-1)\\&\qquad+\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))\},\\&\mathbf{g}_2(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})\\&\quad=-\frac{\mathrm{Re}.\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{4096[1+\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^4}\cdot\frac{\mathbf{t}^{2\delta_2}}{\Gamma(2\delta_2+1)}\\&\qquad\times\left\{(\mathrm{Re})^2\left(\exp\left(\left(\frac{\mathrm{Re}}{4}\right)(\mathbf{u}_1-\mathbf{u}_2)\right)\right)\right.\\&\qquad+(\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))-1)\\&\qquad\left.+\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))\right\}.\end{aligned}\tag{49}$$

Proceeding in the analogous manner, the additional factors of $\mathbf{f}_\mathbf{r}$ and $\mathbf{g}_\mathbf{r}, (\mathbf{r}\geq 3)$ of the JTDM solution can be attained effortlessly. As a result, we arrive to the mathematical formulation as

$$\begin{aligned}&\mathbf{f}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})\\&\quad=\sum_{\mathbf{r}=0}^{\infty}\mathbf{f}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})\\&\quad=\mathbf{f}_0(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})+\mathbf{f}_1(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})+\mathbf{f}_2(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})+,\cdots,\\&\mathbf{f}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})\\&\quad=\frac{3}{4}-\frac{1}{4(1+\exp(((0.03125)\,\mathrm{Re})-4\mathbf{u}_1+4\mathbf{u}_2))}\\&\qquad-\frac{\mathrm{Re}.\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{128[1+\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^2}\cdot\frac{\mathbf{t}^{\delta_1}}{\Gamma(\delta_1+1)}\\&\qquad-\frac{\mathrm{Re}.\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{4096[1+\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^4}\cdot\frac{\mathbf{t}^{2\delta_1}}{\Gamma(2\delta_1+1)}\\&\qquad\times\{(\mathrm{Re})^2(-\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2)))\\&\qquad+(\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))-1)\\&\qquad+\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))\}+,\cdots,\end{aligned}\tag{50}$$

and

$$\begin{aligned}&\mathbf{g}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})\\&\quad=\sum_{\mathbf{r}=0}^{\infty}\mathbf{g}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})\\&\quad=\mathbf{g}_0(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})+\mathbf{g}_1(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})+\mathbf{g}_2(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})+,\cdots,\\&\mathbf{g}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})\\&\quad=\frac{3}{4}+\frac{1}{4(1+\exp(((0.03125)\,\mathrm{Re})-4\mathbf{u}_1+4\mathbf{u}_2))}\\&\qquad-\frac{\mathrm{Re}.\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{128[1+\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^2}\cdot\frac{\mathbf{t}^{\delta_2}}{\Gamma(\delta_2+1)}\\&\qquad-\frac{\mathrm{Re}.\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{4096[1+\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^4}\cdot\frac{\mathbf{t}^{2\delta_2}}{\Gamma(2\delta_2+1)}\\&\qquad\times\left\{(\mathrm{Re})^2\left(-\exp\left(\left(\frac{\mathrm{Re}}{4}\right)(\mathbf{u}_1-\mathbf{u}_2)\right)\right)\right.\\&\qquad+(\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))-1)\\&\qquad\left.+\exp(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))\right\}+\cdots+.\end{aligned}\tag{51}$$

Case 18 (ABC fractional operator). Applying the Jafari transform on (38) with respect to ABC-fractional derivative operator sense as

$$\begin{aligned}&\frac{\Psi^{\delta_1}(\rho)\mathbf{ABC}(\rho)}{\delta_1+(1-\delta_1)\Psi^{\delta_1}(\rho)}\mathbf{J}[\mathbf{f}(\mathbf{u}_1,\mathbf{u}_2,\rho)]\\&\quad-\Phi(\rho)\sum_{\kappa=1}^{m_1-1}\Psi^{\delta-\kappa-1}(\rho)\mathbf{f}^{(\kappa)}(\mathbf{u}_1,\mathbf{u}_2,0)\end{aligned}$$

$$= -\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1}\right] - \mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2}\right] + \frac{1}{\mathrm{Re}}\mathbf{J} \cdot \left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\}, \frac{\Psi^{\delta_2}(\rho)\mathbf{ABC}(\rho)}{\delta_2 + (1-\delta_2)\Psi^{\delta_2}(\rho)}\mathbf{J}[\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \rho)] - \Phi(\rho)\sum_{\kappa=1}^{m_1-1}\Psi^{\delta-\kappa-1}(\rho)\mathbf{g}^{(\kappa)}(\mathbf{u}_1, \mathbf{u}_2, 0) = -\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1}\right] - \mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2}\right] + \frac{1}{\mathrm{Re}}\mathbf{J}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}. \tag{52}$$

The aforementioned equation can be written as

$$\mathbf{J}[\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \rho)] = \frac{\Phi(\rho)}{\Psi(\rho)}\left[\frac{3}{4} - \frac{1}{4(1+\exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))}\right] - \frac{\delta_1 + (1-\delta_1)\Psi^{\delta_1}(\rho)}{\Psi^{\delta_1}(\rho)\mathbf{ABC}(\rho)}\mathbf{J} \cdot \left[\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1} + \mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2} - \frac{1}{\mathrm{Re}}\mathbf{J}\left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\}\right], \tag{53}$$

$$\mathbf{J}[\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \rho)] = \frac{\Phi(\rho)}{\Psi(\rho)}\left[\frac{3}{4} + \frac{1}{4(1+\exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))}\right] - \frac{\delta_2 + (1-\delta_2)\Psi^{\delta_2}(\rho)}{\Psi^{\delta_2}(\rho)\mathbf{ABC}(\rho)}\mathbf{J} \cdot \left[\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1} + \mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2} - \frac{1}{\mathrm{Re}}\mathbf{J}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}\right]. \tag{54}$$

Further, implementing the inverse Jafari transform on (53), then, it diminishes to

$$\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \rho) = \mathbf{J}^{-1}\left[\frac{\Phi(\rho)}{\Psi(\rho)}\left[\frac{3}{4} - \frac{1}{4(1+\exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))}\right]\right] - \mathbf{J}^{-1}\left\{\frac{\delta_1 + (1-\delta_1)\Psi^{\delta_1}(\rho)}{\Psi^{\delta_1}(\rho)\mathbf{ABC}(\rho)}\mathbf{J} \cdot \left[\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1} + \mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2} - \frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\}\right]\right\}, \tag{55}$$

$$\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \rho) = \mathbf{J}^{-1}\left[\frac{\Phi(\rho)}{\Psi(\rho)}\left[\frac{3}{4} + \frac{1}{4(1+\exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))}\right]\right] - \mathbf{J}^{-1}\left\{\frac{\delta_2 + (1-\delta_2)\Psi^{\delta_2}(\rho)}{\Psi^{\delta_2}(\rho)\mathbf{ABC}(\rho)}\mathbf{J} \cdot \left[\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1} + \mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2} - \frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}\right]\right\}. \tag{56}$$

In this case, we hypothesize that the undefined functions $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ and $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ can be described as an infinite series of the mode as follows:

$$\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) = \sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}), \quad \mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) = \sum_{\mathbf{r}=0}^{\infty}\mathbf{g}_{\mathbf{r}}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}). \tag{57}$$

It is clear that $\mathbf{f}\mathbf{f}_{\mathbf{u}_1} = \sum_{\mathbf{r}=0}^{\infty}\mathscr{A}_{\mathbf{r}}$, $\mathbf{f}\mathbf{g}_{\mathbf{u}_1} = \sum_{\mathbf{r}=0}^{\infty}\mathscr{B}_{\mathbf{r}}$, $\mathbf{g}\mathbf{f}_{\mathbf{u}_2} = \sum_{\mathbf{r}=0}^{\infty}\mathscr{C}_{\mathbf{r}}$, $\mathbf{g}\mathbf{g}_{\mathbf{u}_2} = \sum_{\mathbf{r}=0}^{\infty}\mathscr{D}_{\mathbf{r}}$ indicate the Adomian polynomials and were referred to as the nonlinear factors.

In view of the Adomian polynomials, (55) can be described as

$$\sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}+1}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) = \frac{3}{4} - \frac{1}{4(1+\exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))} - \mathbf{J}^{-1}\left\{\frac{\delta_1 + (1-\delta_1)\Psi^{\delta_1}(\rho)}{\Psi^{\delta_1}(\rho)\mathbf{ABC}(\rho)}\mathbf{J} \cdot \left[\sum_{\mathbf{r}=0}^{\infty}\mathscr{A}_{\mathbf{r}} + \sum_{\mathbf{r}=0}^{\infty}\mathscr{B}_{\mathbf{r}} - \frac{1}{\mathrm{Re}}\left\{\sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}\mathbf{u}_1\mathbf{u}_1} + \sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}\mathbf{u}_2\mathbf{u}_2}\right\}\right]\right\}, \tag{58}$$

$$\sum_{\mathbf{r}=0}^{\infty}\mathbf{g}_{\mathbf{r}+1}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) = \frac{3}{4} + \frac{1}{4(1+\exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))} - \mathbf{J}^{-1}\left\{\frac{\delta_2 + (1-\delta_2)\Psi^{\delta_2}(\rho)}{\Psi^{\delta_2}(\rho)\mathbf{ABC}(\rho)}\mathbf{J} \cdot \left[\sum_{\mathbf{r}=0}^{\infty}\mathscr{C}_{\mathbf{r}} + \sum_{\mathbf{r}=0}^{\infty}\mathscr{D}_{\mathbf{r}} + \frac{1}{\mathrm{Re}}\left\{\sum_{\mathbf{r}=0}^{\infty}\mathbf{g}_{\mathbf{r}\mathbf{u}_1\mathbf{u}_1} + \sum_{\mathbf{r}=0}^{\infty}\mathbf{g}_{\mathbf{r}\mathbf{u}_2\mathbf{u}_2}\right\}\right]\right\}. \tag{59}$$

Analyzing term by term (58), we may immediately get the iterative terms indicated below

$$\mathbf{f}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) = \frac{3}{4} - \frac{1}{4(1+\exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))},$$

$$\mathbf{g}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) = \frac{3}{4} + \frac{1}{4(1+\exp(((0.03125)\,\mathrm{Re}) - 4\mathbf{u}_1 + 4\mathbf{u}_2))},$$

$$\mathbf{f}_1(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t}) = -\frac{\mathrm{Re.exp}\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{128[1+\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^2} \cdot\left\{\frac{\mathbf{t}^{\delta_1}}{\Gamma(\delta_1+1)}+(1-\delta_1)\right\},$$

$$\mathbf{g}_1(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t}) = \frac{\mathrm{Re.exp}\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{128[1+\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^2} \cdot\left\{\frac{\mathbf{t}^{\delta_2}}{\Gamma(\delta_2+1)}+(1-\delta_2)\right\},$$

$$\begin{aligned}\mathbf{f}_2(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t}) = {}& -\frac{\mathrm{Re.exp}\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{4096[1+\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^4} \cdot\left\{\frac{\delta_1\mathbf{t}^{\delta_1}}{\Gamma(2\delta_1+1)}+2\delta_1(1-\delta_1)\frac{\mathbf{t}^{\delta_1}}{\Gamma(\delta_1+1)} + (1-\delta_1)^2\right\} \\ & \times\left\{(\mathrm{Re})^2(-\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))) + (\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))-1) + \exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))\right\},\end{aligned}$$

$$\begin{aligned}\mathbf{g}_2(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t}) = {}& -\frac{\mathrm{Re.exp}\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{4096[1+\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^4} \cdot\left\{\frac{\delta_2\mathbf{t}^{\delta_2}}{\Gamma(2\delta_2+1)}+2\delta_2(1-\delta_2)\frac{\mathbf{t}^{\delta_2}}{\Gamma(\delta_2+1)} + (1-\delta_2)^2\right\} \\ & \times\left\{(\mathrm{Re})^2\left(\exp\left(\left(\frac{\mathrm{Re}}{4}\right)(\mathbf{u}_1-\mathbf{u}_2)\right)\right) + (\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))-1) + \exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))\right\}.\end{aligned} \tag{60}$$

Proceeding in the analogous manner, the additional factors of $\mathbf{f}_\mathbf{r}$ and $\mathbf{g}_\mathbf{r}$, $(\mathbf{r}\geq 3)$ of the JTDM solution can be attained effortlessly. As a result, we arrive to the mathematical formulation as

$$\begin{aligned}\mathbf{f}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t}) &= \sum_{\mathbf{r}=0}^{\infty}\mathbf{f}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t}) \\ &= \mathbf{f}_0(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})+\mathbf{f}_1(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})+\mathbf{f}_2(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})+,\cdots, \\ \mathbf{f}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t}) &= \frac{3}{4}-\frac{1}{4(1+\exp\,(((0.03125)\,\mathrm{Re})-4\mathbf{u}_1+4\mathbf{u}_2))} \\ &\quad -\frac{\mathrm{Re.exp}\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{128[1+\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^2}\times\left\{\frac{\delta_1\mathbf{t}^{\delta_1}}{\Gamma(\delta_1+1)}+(1-\delta_1)\right\} \\ &\quad -\frac{\mathrm{Re.exp}\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{4096[1+\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^4} \cdot\left\{\frac{\delta_1\mathbf{t}^{\delta_1}}{\Gamma(2\delta_1+1)}+2\delta_1(1-\delta_1)\frac{\mathbf{t}^{\delta_1}}{\Gamma(\delta_1+1)}+(1-\delta_1)^2\right\} \\ &\quad \times\left\{(\mathrm{Re})^2(-\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))) + (\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))-1) + \exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))\right\}+,\cdots,\end{aligned} \tag{61}$$

and

$$\begin{aligned}\mathbf{g}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t}) &= \sum_{\mathbf{r}=0}^{\infty}\mathbf{g}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t}) \\ &= \mathbf{g}_0(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})+\mathbf{g}_1(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})+\mathbf{g}_2(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t})+,\cdots, \\ \mathbf{g}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{t}) &= \frac{3}{4}+\frac{1}{4(1+\exp\,(((0.03125)\,\mathrm{Re})-4\mathbf{u}_1+4\mathbf{u}_2))} \\ &\quad -\frac{\mathrm{Re.exp}\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{128[1+\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^2}\times\left\{\frac{\delta_2\mathbf{t}^{\delta_2}}{\Gamma(\delta_2+1)}+(1-\delta_2)\right\} \\ &\quad -\frac{\mathrm{Re.exp}\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))}{4096[1+\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))]^4} \times\left\{\frac{\delta_2\mathbf{t}^{\delta_2}}{\Gamma(2\delta_2+1)}+2\delta_2(1-\delta_2)\frac{\mathbf{t}^{\delta_2}}{\Gamma(\delta_2+1)}+(1-\delta_2)^2\right\} \\ &\quad \times\left\{(\mathrm{Re})^2\left(-\exp\left(\left(\frac{\mathrm{Re}}{4}\right)(\mathbf{u}_1-\mathbf{u}_2)\right)\right) + (\exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))-1) + \exp\,(((0.125)\,\mathrm{Re})(\mathbf{u}_1-\mathbf{u}_2))\right\}+\cdots+.\end{aligned} \tag{62}$$

Example 19. Let us surmise the generic form of BEs (1) as

$$\begin{aligned}&\mathbf{D}_\mathbf{t}^{\delta_1}\mathbf{f}-2\mathbf{f}\frac{\partial\mathbf{f}}{\partial\mathbf{u}_1}-2\mathbf{g}\frac{\partial\mathbf{f}}{\partial\mathbf{u}_2}=\frac{\partial^2\mathbf{f}}{\partial\mathbf{u}_1^2}+\frac{\partial^2\mathbf{f}}{\partial\mathbf{u}_2^2}, \\ &\mathbf{D}_\mathbf{t}^{\delta_2}\mathbf{g}-2\mathbf{f}\frac{\partial\mathbf{g}}{\partial\mathbf{u}_1}-2\mathbf{g}\frac{\partial\mathbf{g}}{\partial\mathbf{u}_2}=\frac{\partial^2\mathbf{g}}{\partial\mathbf{u}_1^2}+\frac{\partial^2\mathbf{g}}{\partial\mathbf{u}_2^2}, \quad 0<\delta_1,\delta_2<1,\end{aligned} \tag{63}$$

having ICs

$$\begin{aligned}\mathbf{f}(\mathbf{u}_1,\mathbf{u}_2,0) &= 1-\tan h(1-\mathbf{u}_1+2\mathbf{u}_2), \\ \mathbf{g}(\mathbf{u}_1,\mathbf{u}_2,0) &= 1+2\tan h(1-\mathbf{u}_1+2\mathbf{u}_2),\end{aligned} \tag{64}$$

where Re denotes the Reynolds number.

For the exact solution of (63) when $\delta_1 = \delta_2 = 1$, then,

$$\begin{aligned}\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= 1 - \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{t}),\\ \mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= 1 + 2\tanh(1 - \mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{t}).\end{aligned} \tag{65}$$

Case 20 (Caputo fractional operator). Applying the Jafari transform on (18), we have

$$\begin{aligned}\mathbf{J}\left[\mathbf{D}_{\mathbf{t}}^{\delta_1}\mathbf{f}\right] - 2\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1}\right] - 2\mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2}\right] &= \mathbf{J}\left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\},\\ \mathbf{J}\left[\mathbf{D}_{\mathbf{t}}^{\delta_2}\mathbf{g}\right] - 2\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1}\right] - 2\mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2}\right] &= \mathbf{J}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}.\end{aligned} \tag{66}$$

It follows that

$$\begin{aligned}&\Psi^{\delta_1}(\rho)\mathbf{J}[\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \rho)] - \Phi(\rho)\sum_{\kappa=1}^{m_1-1}\Psi^{\delta_1-\kappa-1}(\rho)\mathbf{f}^{(\kappa)}(\mathbf{u}_1, \mathbf{u}_2, 0)\\ &= 2\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1}\right] + 2\mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2}\right] + \mathbf{J}\left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\},\\ &\Psi^{\delta_2}(\rho)\mathbf{J}[\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \rho)] - \Phi(\rho)\sum_{\kappa=1}^{m_1-1}\Psi^{\delta_2-\kappa-1}(\rho)\mathbf{g}^{(\kappa)}(\mathbf{u}_1, \mathbf{u}_2, 0)\\ &= 2\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1}\right] + 2\mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2}\right] + \mathbf{J}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}.\end{aligned} \tag{67}$$

The aforementioned equation can be written as

$$\begin{aligned}\mathbf{J}[\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \rho)] &= \frac{\Phi(\rho)}{\Psi(\rho)}(1 - \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2))\\ &\quad + \frac{2}{\Psi^{\delta_1}(\rho)}\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1}\right] + \frac{2}{\Psi^{\delta_1}(\rho)}\mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2}\right]\\ &\quad + \frac{1}{\Psi^{\delta_1}(\rho)}\mathbf{J}\left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\},\\ \mathbf{J}[\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \rho)] &= \frac{\Phi(\rho)}{\Psi(\rho)}(1 + 2\tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2))\\ &\quad + \frac{2}{\Psi^{\delta_2}(\rho)}\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1}\right] + \frac{2}{\Psi^{\delta_2}(\rho)}\mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2}\right]\\ &\quad + \frac{1}{\Psi^{\delta_2}(\rho)}\mathbf{J}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}.\end{aligned} \tag{68}$$

Further, implementing the inverse Jafari transform on (68), then, it diminishes to

$$\begin{aligned}&\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \rho)\\ &= \mathbf{J}^{-1}\left[\frac{\Phi(\rho)}{\Psi(\rho)}(1 - \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2))\right]\\ &\quad + \mathbf{J}^{-1}\left\{\frac{1}{\Psi^{\delta_1}(\rho)}\mathbf{J}\left[2\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1} + 2\mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2} + \left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\}\right]\right\},\\ &\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \rho)\\ &= \mathbf{J}^{-1}\left[\frac{\Phi(\rho)}{\Psi(\rho)}(1 - 2\tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2))\right]\\ &\quad + \mathbf{J}^{-1}\left\{\frac{1}{\Psi^{\delta_2}(\rho)}\mathbf{J}\left[2\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1} + 2\mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2} + \left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}\right]\right\}.\end{aligned} \tag{69}$$

In this case, we hypothesize that the undefined functions $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ and $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ can be described as an infinite series of the mode as follows

$$\begin{aligned}\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= \sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}),\\ \mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= \sum_{\mathbf{r}=0}^{\infty}\mathbf{g}_{\mathbf{r}}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}).\end{aligned} \tag{70}$$

It is clear that $\mathbf{ff}_{\mathbf{u}_1} = \sum_{\mathbf{r}=0}^{\infty}\mathscr{A}_{\mathbf{r}}$, $\mathbf{fg}_{\mathbf{u}_1} = \sum_{\mathbf{r}=0}^{\infty}\mathscr{B}_{\mathbf{r}}$, $\mathbf{gf}_{\mathbf{u}_2} = \sum_{\mathbf{r}=0}^{\infty}\mathscr{C}_{\mathbf{r}}$, $\mathbf{gg}_{\mathbf{u}_2} = \sum_{\mathbf{r}=0}^{\infty}\mathscr{D}_{\mathbf{r}}$ indicate the Adomian polynomials and were referred to as the nonlinear factors.

In view of the Adomian polynomials, (69) can be described as

$$\begin{aligned}&\sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}+1}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})\\ &= (1 - \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2))\\ &\quad + \mathbf{J}^{-1}\left\{\frac{1}{\Psi^{\delta_1}(\rho)}\mathbf{J}\left[2\sum_{\mathbf{r}=0}^{\infty}\mathscr{A}_{\mathbf{r}} + 2\sum_{\mathbf{r}=0}^{\infty}\mathscr{B}_{\mathbf{r}} + \sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}\mathbf{u}_1\mathbf{u}_1} + \sum_{\mathbf{r}=0}^{\infty}\mathbf{f}_{\mathbf{r}\mathbf{u}_2\mathbf{u}_2}\right]\right\},\\ &\sum_{\mathbf{r}=0}^{\infty}\mathbf{g}_{\mathbf{r}+1}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})\\ &= (1 + 2\tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2))\\ &\quad + \mathbf{J}^{-1}\left\{\frac{1}{\Psi^{\delta_2}(\rho)}\mathbf{J}\left[2\sum_{\mathbf{r}=0}^{\infty}\mathscr{C}_{\mathbf{r}} + 2\sum_{\mathbf{r}=0}^{\infty}\mathscr{D}_{\mathbf{r}} + \sum_{\mathbf{r}=0}^{\infty}\mathbf{g}_{\mathbf{r}\mathbf{u}_1\mathbf{u}_1} + \sum_{\mathbf{r}=0}^{\infty}\mathbf{g}_{\mathbf{r}\mathbf{u}_2\mathbf{u}_2}\right]\right\}.\end{aligned} \tag{71}$$

Analyzing term by term (71), we may immediately get the iterative terms indicated as follows:

$$\begin{aligned}\mathbf{f}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= (1 - \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)),\\ \mathbf{g}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= (1 + 2\tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)),\end{aligned}$$

Table 1: The exact, $JTDM_C$ and $JTDM_{ABC}$ solutions of $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ of Example 16 for multiple fractional orders considering different values of $\mathbf{u}_1$ and $\mathbf{t}$ when $\mathbf{u}_2 = 0.5$ and Re = 75.

$\mathbf{u}_1$	$\mathbf{t}$	VIM [16]	$\delta_1 = 0.8$	$\delta_1 = 0.9$	$\delta_1 = 1(JTDM_C)$	$\delta_1 = 1(JTDM_{ABC})$	Exact
	0.1	0.7288940367	0.7304717694	0.7317182942	0.7326950774	0.7334569903	0.7323601010
	0.2	0.7244905939	0.7264485371	0.7281195526	0.7295324719	0.7295324719	0.7281090401
0.2	0.3	0.7172263899	0.7193479362	0.7213252216	0.7231448924	0.7231448924	0.7167589400
	0.4	0.6956692898	0.6956692898	0.6576549272	0.6380690943	0.6380690943	0.6378819374
	0.5	0.7139883762	0.7160468984	0.7180328004	0.7199199188	0.7199199188	0.7094049908
	0.1	0.6523781568	0.6587998081	0.6637696860	0.6637696860	0.6637696860	0.6672200651
	0.2	0.6340771846	0.6424628939	0.6494205219	0.6551698837	0.6551698837	0.6537719639
0.4	0.3	0.6176799777	0.6271117813	0.6352922301	0.6423444246	0.6423444246	0.6423444246
	0.4	0.6022648786	0.6122191242	0.6211622134	0.6291282309	0.6291282309	0.6250000000
	0.5	0.5874586328	0.5975666615	0.6069361144	0.6155213025	0.6155213025	0.6104182514
	0.1	0.5411354332	0.5484360252	0.5538986153	0.5580038915	0.5591290591	0.5591290591
	0.2	0.5196457108	0.5299324114	0.5381249783	0.5446586051	0.5446586051	0.5492065509
0.6	0.3	0.4993253071	0.5116288663	0.5218489771	0.5303155430	0.5303155430	0.5405950091
	0.4	0.4794182423	0.4931204368	0.5049063674	0.5149747051	0.5149747051	0.5332410600
	0.5	0.4596433818	0.4742623671	0.4872406024	0.4986360914	0.4986360914	0.5270472386
	0.1	0.5061774126	0.5082262729	0.5082262729	0.5108585978	0.5108585978	0.5113381451
	0.2	1.119192701	1.077177686	1.038080351	1.002540819	1.002540819	1.002518917
0.8	0.3	0.4941613229	0.4978031514	0.5007813010	0.5032110583	0.5032110583	0.5072177268
	0.4	1.267592092	1.233366109	1.198325407	1.162637693	1.162637693	1.162296670
	0.5	0.4823885868	0.4868650609	0.4907853335	0.4941788224	0.4941788224	0.5045660696
	0.1	0.5009207999	0.5012717327	0.5015297207	0.5017208223	0.5017208223	0.5018081909
	0.2	0.4998710565	0.5003839800	0.5007847127	0.5010987130	0.5010987130	0.5014325616
1.0	0.3	0.4988531468	0.4994832916	0.4999970993	0.5004150813	0.5004150813	0.5011346079
	0.4	0.4978378804	0.4985549681	0.4991611613	0.4996699272	0.4996699272	0.5008984007
	0.5	0.4968152514	0.4975945837	0.4982754230	0.4988632509	0.4988632509	0.5007112274

$$\begin{aligned}
\mathbf{f}_1(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= -2 \sec h(1 - \mathbf{u}_1 + 2\mathbf{u}_2) . \frac{\mathbf{t}^{\delta_1}}{\Gamma(\delta_1 + 1)}, \\
\mathbf{g}_1(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= 4 \sec h(1 - \mathbf{u}_1 + 2\mathbf{u}_2) . \frac{\mathbf{t}^{\delta_2}}{\Gamma(\delta_2 + 1)}, \\
\mathbf{f}_2(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= -8 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2) \\
&\quad \cdot \left(-2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right. \\
&\quad + \left(2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right. \\
&\quad \left.\left. + \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right)\right) . \frac{\mathbf{t}^{2\delta_1}}{\Gamma(2\delta_1 + 1)}, \\
\mathbf{g}_2(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= -8 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2) \\
&\quad \cdot \left(-2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right. \\
&\quad + \left(2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right. \\
&\quad \left.\left. + 2 \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right)\right) . \frac{\mathbf{t}^{2\delta_1}}{\Gamma(2\delta_1 + 1)}.
\end{aligned} \tag{72}$$

Proceeding in the analogous manner, the additional factors of $\mathbf{f}_\mathbf{r}$ and $\mathbf{g}_\mathbf{r}$, $(\mathbf{r} \geq 3)$ of the JTDM solution can be attained effortlessly. As a result, we arrive to the mathematical formulation as

$$\begin{aligned}
\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= \sum_{\mathbf{r}=0}^{\infty} \mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) \\
&= \mathbf{f}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) + \mathbf{f}_1(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) + \mathbf{f}_2(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) +, \cdots, \\
\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= (1 - \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)) \\
&\quad - 2 \sec h(1 - \mathbf{u}_1 + 2\mathbf{u}_2) . \frac{\mathbf{t}^{\delta_1}}{\Gamma(\delta_1 + 1)} \\
&\quad + 8 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2) \\
&\quad \times \left(-2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right. \\
&\quad + \left(2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right. \\
&\quad \left.\left. + \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right)\right) . \frac{\mathbf{t}^{2\delta_1}}{\Gamma(2\delta_1 + 1)} -, \cdots,
\end{aligned} \tag{73}$$

Table 2: The exact $JTDM_C$ and $JTDM_{ABC}$ solutions of $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ of Example 16 for multiple fractional orders considering different values of $\mathbf{u}_1$ and $\mathbf{t}$ when $\mathbf{u}_2 = 0.5$ and $Re = 75$.

$\mathbf{u}_1$	$\mathbf{t}$	VIM [16]	$\delta_2 = 0.8$	$\delta_2 = 0.9$	$\delta_2 = 1(JTDM_C)$	$\delta_2 = 1(JTDM_{ABC})$	Exact
	0.1	0.7711725919	0.7695647581	0.7683013593	0.7673153173	0.7673153173	0.7676398990
	0.2	0.7756852399	0.7736621934	0.7719488849	0.7705091069	0.7705091069	0.7718909599
0.2	0.3	0.7796201166	0.7774192502	0.7754645427	0.7737444753	0.7737444753	0.7770472386
	0.4	0.7832376385	0.7809877359	0.7789130917	0.7770214225	0.7770214225	0.7832410600
	0.5	0.7281682904	0.7121411619	0.6954314260	0.6777823827	0.6777823827	0.6773315511
	0.1	0.8488741222	0.8418867225	0.8365997002	0.8325907590	0.8325907590	0.8327799349
	0.2	0.8692275994	0.8596182777	0.8518657558	0.8456115855	0.8456115855	0.8462280361
0.4	0.3	0.8881500532	0.8768697846	0.8673764652	0.8594138813	0.8594138813	0.8604182514
	0.4	0.9064564989	0.8940898084	0.8833168655	0.8739976461	0.8739976461	0.8750000000
	0.5	0.9244608963	0.9114492971	0.8997569768	0.8893628803	0.8893628803	0.8895817486
	0.1	0.8893628803	0.9533170921	0.9470446456	0.9424949964	0.9424949964	0.9408709409
	0.2	0.9887933510	0.9753820465	0.9651596468	0.9651596468	0.9651596468	0.9651596468
0.6	0.3	1.015562198	0.9985384184	0.9849657751	0.9849657751	0.9849657751	0.9594049909
	0.4	1.037816628	1.009794763	0.9811058630	0.9518872337	0.9518872337	0.9516080283
	0.5	1.086299440	1.062389774	1.037461777	1.011132498	1.011132498	1.010459936
	0.1	0.9949320528	0.9923819637	0.9905903603	0.9893144893	0.9893144893	0.9886618549
	0.2	1.002863153	0.9988021771	0.9957649707	0.9934844331	0.9934844331	0.9909449860
0.8	0.3	1.011003834	1.005724345	1.001583048	0.9983467251	0.9983467251	0.9927822732
	0.4	1.019436900	1.013169065	1.008046795	1.003901365	1.003901365	0.9942556575
	0.5	1.028171604	1.021122691	1.015144457	1.010148353	1.010148353	0.9954339304
	0.1	0.9992763753	0.9988363637	0.9985284403	0.9983099389	0.9983099389	0.9981918091
	0.2	1.000649292	0.9999437068	0.9994178157	0.9990243317	0.9990243317	0.9985674384
1.0	0.3	1.002064809	1.001143618	1.000423095	0.9998617693	0.9998617693	0.9988653921
	0.4	1.003535327	1.002438392	1.001544083	1.000822252	1.000822252	0.9991015993
	0.5	1.005061514	1.003825006	1.002778424	1.001905779	1.001905779	0.9992887726

and

$$\begin{aligned}\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= \sum_{\mathbf{r}=0}^{\infty} \mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) \\ &= \mathbf{g}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) + \mathbf{g}_1(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) + \mathbf{g}_2(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) +, \cdots, \\ \mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= (1 + 2\tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)) \\ &\quad + 4 \sec h(1 - \mathbf{u}_1 + 2\mathbf{u}_2) \cdot \frac{\mathbf{t}^{\delta_1}}{\Gamma(\delta_1 + 1)} \\ &\quad - 8 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2) \\ &\quad \times \left(-\sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right. \\ &\quad + \left(2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right. \\ &\quad \left.\left. + 2\tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right)\right) \\ &\quad \cdot \frac{\mathbf{t}^{2\delta_1}}{\Gamma(2\delta_1 + 1)} - \cdots - .\end{aligned} \tag{74}$$

Case 21 (ABC fractional operator). Applying the Jafari transform on (63) with respect to the ABC-fractional derivative operator sense as

$$\begin{aligned}&\frac{\Psi^{\delta_1}(\rho)\mathbf{ABC}(\rho)}{\delta_1 + (1-\delta_1)\Psi^{\delta_1}(\rho)} \mathbf{J}[\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \rho)] \\ &\quad - \Phi(\rho) \sum_{\kappa=1}^{m_1 - 1} \Psi^{\delta-\kappa-1}(\rho)\mathbf{f}^{(\kappa)}(\mathbf{u}_1, \mathbf{u}_2, 0) \\ &\quad = 2\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1}\right] + 2\mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2}\right] + \mathbf{J}\left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\}, \\ &\frac{\Psi^{\delta_2}(\rho)\mathbf{ABC}(\rho)}{\delta_2 + (1-\delta_2)\Psi^{\delta_2}(\rho)} \mathbf{J}[\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \rho)] \\ &\quad - \Phi(\rho) \sum_{\kappa=1}^{m_1 - 1} \Psi^{\delta-\kappa-1}(\rho)\mathbf{g}^{(\kappa)}(\mathbf{u}_1, \mathbf{u}_2, 0) \\ &\quad = 2\mathbf{J}\left[\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1}\right] + 2\mathbf{J}\left[\mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2}\right] + \mathbf{J}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}.\end{aligned} \tag{75}$$

Table 3: The comparison study among VIM [16], JTDM_C, and JTDM_{ABC} of Example 16 for estimated solutions of $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ and absolute error $E_{\text{abs}} = \|E^{\text{exact}} - E^{\text{approx}}\|$ at $\delta_1 = 1$ considering multiple values of $\mathbf{u}_1$ and $\mathbf{t}$.

$\mathbf{u}_1$	$\mathbf{t}$	$\|\text{Exact} - \text{VIM}\|$	$\|\text{Exact} - \text{JTDM}_C\|$	$\|\text{Exact} - \text{JTDM}_{\text{ABC}}\|$
	0.1	0.000334976	0.0003349764	0.00034446772
	0.2	0.0014234318	0.0014234318	0.0014234318
0.2	0.3	0.0033963154	0.0033963154	0.0033963154
	0.4	0.0063859524	0.0063859524	0.0063859524
	0.5	0.0105149280	0.0105149280	0.0105149280
	0.1	0.0003845432	0.0003845432	0.0003845432
	0.2	0.0013979198	0.0013979198	0.0013979198
0.4	0.3	0.0027626761	0.0027626760	0.0027626760
	0.4	0.0041282309	0.0041282309	0.0041282309
	0.5	0.0051030511	0.0051030511	0.0051030511
	0.1	-0.0011251676	-0.0011251676	-0.0011251676
	0.2	-0.0045479458	-0.0045479458	-0.0045479458
0.6	0.3	-0.0102794660	-0.0102794661	-0.0102794661
	0.4	-0.0182663549	-0.0182663549	-0.0182663549
	0.5	-0.0284111473	-0.0284111472	-0.0284111472
	0.1	-0.0004795473	-0.0004795473	-0.0004795473
	0.2	-0.0018470989	-0.0018470989	-0.0018470989
0.8	0.3	-0.0040066685	-0.0040066685	-0.0040066685
	0.4	-0.0068763152	-0.0068763152	-0.0068763152
	0.5	-0.0103872472	-0.0103872472	-0.0103872472
	0.1	0.5018081909	-0.0000873686	-0.0000873686
	0.2	-0.0003338486	-0.0003338486	-0.0003338486
1.0	0.3	-0.0007195266	-0.0007195266	-0.0007195266
	0.4	-0.0012284735	-0.0012284735	-0.0012284735
	0.5	-0.0018479765	-0.0018479765	-0.0018479765

The aforementioned equation can be written as

$$\begin{aligned}
\mathbf{J}[\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \rho)] &= \frac{\Phi(\rho)}{\Psi(\rho)}(1 - \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)) \\
&+ \frac{\delta_1 + (1-\delta_1)\Psi^{\delta_1}(\rho)}{\Psi^{\delta_1}(\rho)\mathbf{ABC}(\rho)}\mathbf{J} \\
&\cdot \left[2\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1} + 2\mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2} + \mathbf{J}\left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\}\right], \\
\mathbf{J}[\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \rho)] &= \frac{\Phi(\rho)}{\Psi(\rho)}(1 + 2\tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)) \\
&+ \frac{\delta_2 + (1-\delta_2)\Psi^{\delta_2}(\rho)}{\Psi^{\delta_2}(\rho)\mathbf{ABC}(\rho)}\mathbf{J} \\
&\cdot \left[2\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1} + 2\mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2} + \mathbf{J}\left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}\right].
\end{aligned} \tag{76}$$

Further, implementing the inverse Jafari transform on (76), then, it diminishes to

$$\begin{aligned}
\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \rho) &= \mathbf{J}^{-1}\left[\frac{\Phi(\rho)}{\Psi(\rho)}(1 - \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2))\right] \\
&+ \mathbf{J}^{-1}\left\{\frac{\delta_1 + (1-\delta_1)\Psi^{\delta_1}(\rho)}{\Psi^{\delta_1}(\rho)\mathbf{ABC}(\rho)}\mathbf{J}\right. \\
&\left.\cdot \left[2\mathbf{f}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_1} + 2\mathbf{g}\frac{\partial \mathbf{f}}{\partial \mathbf{u}_2} + \left\{\frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{f}}{\partial \mathbf{u}_2^2}\right\}\right]\right\}, \\
\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \rho) &= \mathbf{J}^{-1}\left[\frac{\Phi(\rho)}{\Psi(\rho)}(1 + 2\tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2))\right] \\
&+ \mathbf{J}^{-1}\left\{\frac{\delta_2 + (1-\delta_2)\Psi^{\delta_2}(\rho)}{\Psi^{\delta_2}(\rho)\mathbf{ABC}(\rho)}\mathbf{J}\right. \\
&\left.\cdot \left[2\mathbf{f}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_1} + 2\mathbf{g}\frac{\partial \mathbf{g}}{\partial \mathbf{u}_2} + \left\{\frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_1^2} + \frac{\partial^2 \mathbf{g}}{\partial \mathbf{u}_2^2}\right\}\right]\right\}.
\end{aligned} \tag{77}$$

Table 4: The comparison study among VIM [16], JTDM_C, and JTDM_{ABC} of Example 16 for estimated solutions of $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ and absolute error $E_{abs} = \|E^{exact} - E^{approx}\|$ at $\delta_2 = 1$ considering multiple values of $\mathbf{u}_1$ and $\mathbf{t}$.

$\mathbf{u}_1$	$\mathbf{t}$	$\|\text{Exact} - \text{VIM}\|$	$\|\text{Exact} - \text{JTDM}_C\|$	$\|\text{Exact} - \text{JTDM}_{ABC}\|$
	0.1	−0.0003349764	−0.0003245817	−0.0003245817
	0.2	−0.0014234318	−0.0013818530	−0.0013818530
0.2	0.3	−0.0033963154	−0.0033027633	−0.0033027633
	0.4	−0.0063859524	−0.0062196375	−0.0062196375
	0.5	−0.0105149280	−0.0102550608	−0.0102550608
	0.1	−0.0003845432	−0.0001891759	−0.0001891759
	0.2	−0.0013979198	−0.0006164506	−0.0006164506
0.4	0.3	−0.0027626761	−0.0010043701	−0.0010043701
	0.4	−0.0041282309	−0.0010023539	−0.0010023539
	0.5	−0.0051030511	−0.0002188683	−0.0002188683
	0.1	0.0011251676	0.0016240555	0.0016240555
	0.2	0.0045479458	0.0065434974	0.0065434974
0.6	0.3	0.0102794660	0.0147694573	0.0147694573
	0.4	0.0182663549	0.0262485615	0.0262485615
	0.5	0.0284111476	0.0408833446	0.0408833446
	0.1	0.0004795473	0.0006526344	0.0006526344
	0.2	0.0018470989	0.0025394471	0.0025394471
0.8	0.3	0.0040066685	0.0055644519	0.0055644519
	0.4	0.0068763155	0.0096457075	0.0096457075
	0.5	0.0103872466	0.0147144226	0.0147144226
	0.1	0.0000873686	0.0001181298	0.0001181298
	0.2	0.0003338486	0.0004568933	0.0004568933
1.0	0.3	0.0007195266	0.0009963772	0.0009963772
	0.4	0.0012284737	0.0017206527	0.0017206527
	0.5	0.0018479764	0.0026170064	0.0026170064

In this case, we hypothesize that the undefined functions $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ and $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ can be described as an infinite series of the mode as follows:

$$\begin{aligned} \mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= \sum_{\mathbf{r}=0}^{\infty} \mathbf{f}_{\mathbf{r}}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}), \\ \mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= \sum_{\mathbf{r}=0}^{\infty} \mathbf{g}_{\mathbf{r}}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}). \end{aligned} \tag{78}$$

It is clear that $\mathbf{ff}_{\mathbf{u}_1} = \sum_{\mathbf{r}=0}^{\infty} \mathscr{A}_{\mathbf{r}}, \mathbf{fg}_{\mathbf{u}_1} = \sum_{\mathbf{r}=0}^{\infty} \mathscr{B}_{\mathbf{r}}, \mathbf{gf}_{\mathbf{u}_2} = \sum_{\mathbf{r}=0}^{\infty} \mathscr{C}_{\mathbf{r}}, \mathbf{gg}_{\mathbf{u}_2} = \sum_{\mathbf{r}=0}^{\infty} \mathscr{D}_{\mathbf{r}}$ indicate the Adomian polynomials and were referred to as the nonlinear factors.

In view of the Adomian polynomials, (77) can be described as

$$\begin{aligned} &\sum_{\mathbf{r}=0}^{\infty} \mathbf{f}_{\mathbf{r}+1}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) \\ &\quad = (1 - \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)) + \mathbf{J}^{-1} \left\{ \frac{\delta_1 + (1-\delta_1)\Psi^{\delta_1}(\rho)}{\Psi^{\delta_1}(\rho)\mathbf{ABC}(\rho)} \mathbf{J} \right. \\ &\quad \left. \cdot \left[2\sum_{\mathbf{r}=0}^{\infty} \mathscr{A}_{\mathbf{r}} + 2\sum_{\mathbf{r}=0}^{\infty} \mathscr{B}_{\mathbf{r}} + \sum_{\mathbf{r}=0}^{\infty} \mathbf{f}_{\mathbf{r}\mathbf{u}_1\mathbf{u}_1} + \sum_{\mathbf{r}=0}^{\infty} \mathbf{f}_{\mathbf{r}\mathbf{u}_2\mathbf{u}_2} \right] \right\}, \\ &\sum_{\mathbf{r}=0}^{\infty} \mathbf{g}_{\mathbf{r}+1}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) \\ &\quad = (1 + 2\tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)) \\ &\qquad + \mathbf{J}^{-1} \left\{ \frac{\delta_2 + (1-\delta_2)\Psi^{\delta_2}(\rho)}{\Psi^{\delta_2}(\rho)\mathbf{ABC}(\rho)} \mathbf{J} \right. \\ &\qquad \left. \cdot \left[2\sum_{\mathbf{r}=0}^{\infty} \mathscr{C}_{\mathbf{r}} + 2\sum_{\mathbf{r}=0}^{\infty} \mathscr{D}_{\mathbf{r}} + \sum_{\mathbf{r}=0}^{\infty} \mathbf{g}_{\mathbf{r}\mathbf{u}_1\mathbf{u}_1} + \sum_{\mathbf{r}=0}^{\infty} \mathbf{g}_{\mathbf{r}\mathbf{u}_2\mathbf{u}_2} \right] \right\}. \end{aligned} \tag{79}$$

Analyzing term by term (79), we may immediately get the iterative terms indicated as follows:

$$\begin{aligned} \mathbf{f}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= (1 - \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)), \\ \mathbf{g}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= (1 + 2\tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)), \\ \mathbf{f}_1(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= -2\sec h(1 - \mathbf{u}_1 + 2\mathbf{u}_2) . \left\{ \frac{\mathbf{t}^{\delta_1}}{\Gamma(\delta_1 + 1)} + (1 - \delta_1) \right\}, \end{aligned}$$

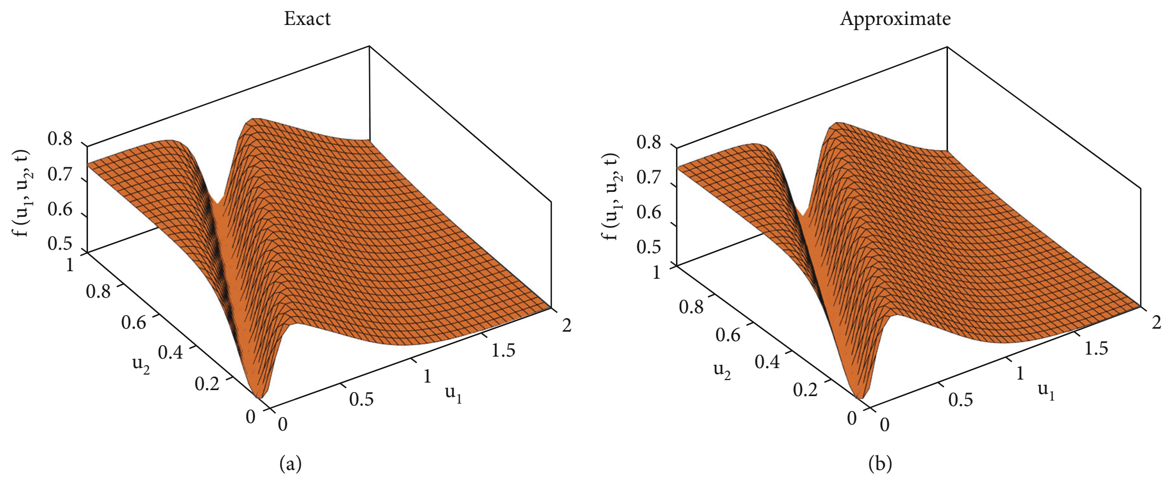


FIGURE 1: 3D view of the exact and approximate solution via JTDM of Example 16 for $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ when $\delta_1 = 1$.

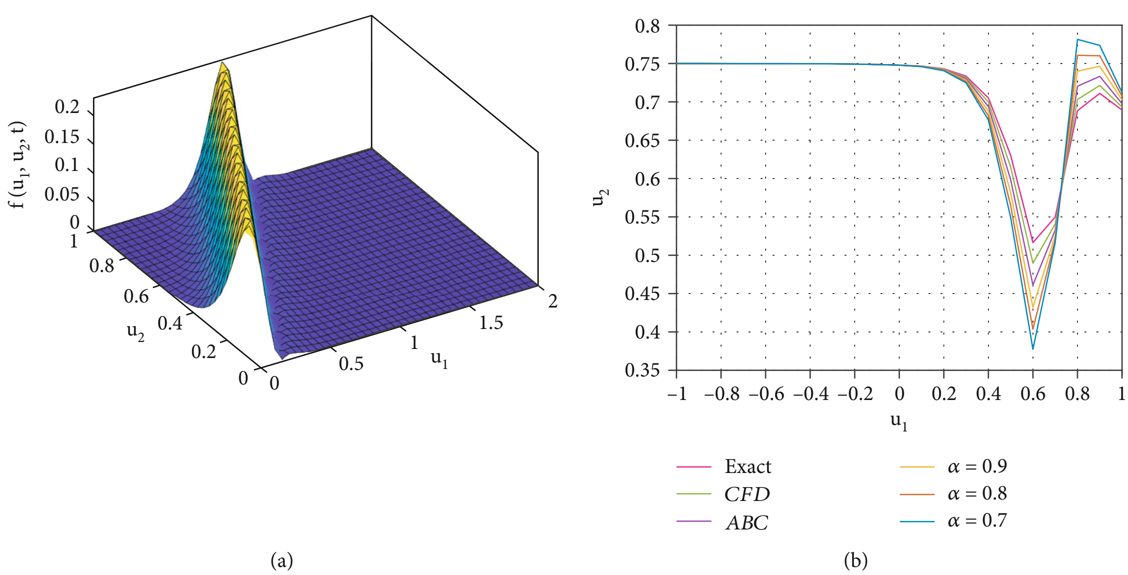
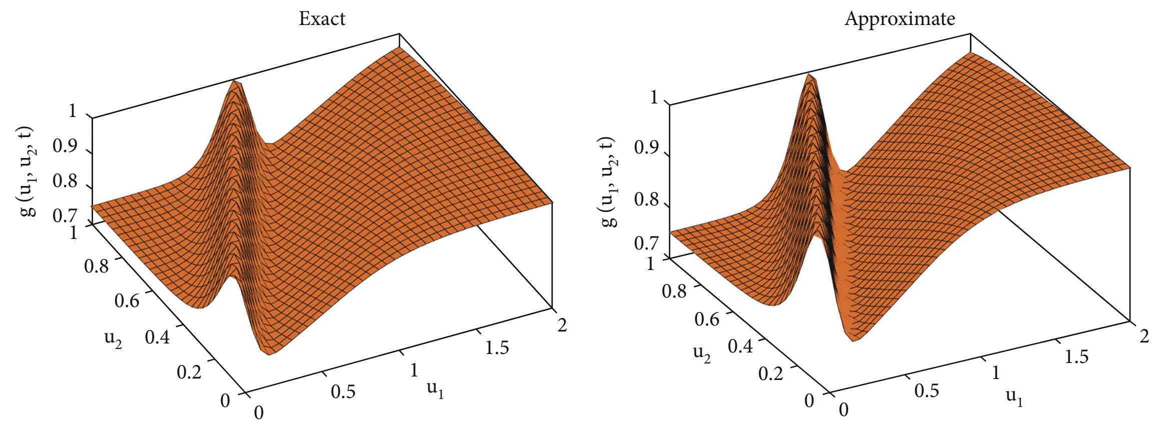


FIGURE 2: (a) 3D view of absolute error for $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$. (b) 2D view of multiple fractional orders via JTDM of Example 16 for $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ when $\mathbf{t} = 0.7$.

$$
\begin{aligned}
&\mathbf{g}_1(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) \\
&\quad = 4 \sec h(1 - \mathbf{u}_1 + 2\mathbf{u}_2) . \left\{ \frac{\mathbf{t}^{\delta_2}}{\Gamma(\delta_2 + 1)} + (1 - \delta_2) \right\}, \\
&\mathbf{f}_2(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) \\
&\quad = -8 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\left(-2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right. \\
&\qquad \left. + \left(2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2) + \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right)\right) \\
&\qquad . \left\{ \frac{\delta_1 \mathbf{t}^{\delta_1}}{\Gamma(2\delta_1 + 1)} + 2\delta_1(1 - \delta_1) \frac{\mathbf{t}^{\delta_1}}{\Gamma(\delta_1 + 1)} + (1 - \delta_1)^2 \right\}, \\
&\mathbf{g}_2(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) \\
&\quad = -8 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\left(-2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right. \\
&\qquad \left. + \left(2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2) + 2 \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\right)\right) \\
&\qquad . \left\{ \frac{\delta_2 \mathbf{t}^{\delta_2}}{\Gamma(2\delta_2 + 1)} + 2\delta_2(1 - \delta_2) \frac{\mathbf{t}^{\delta_2}}{\Gamma(\delta_2 + 1)} + (1 - \delta_2)^2 \right\}.
\end{aligned} \tag{80}
$$

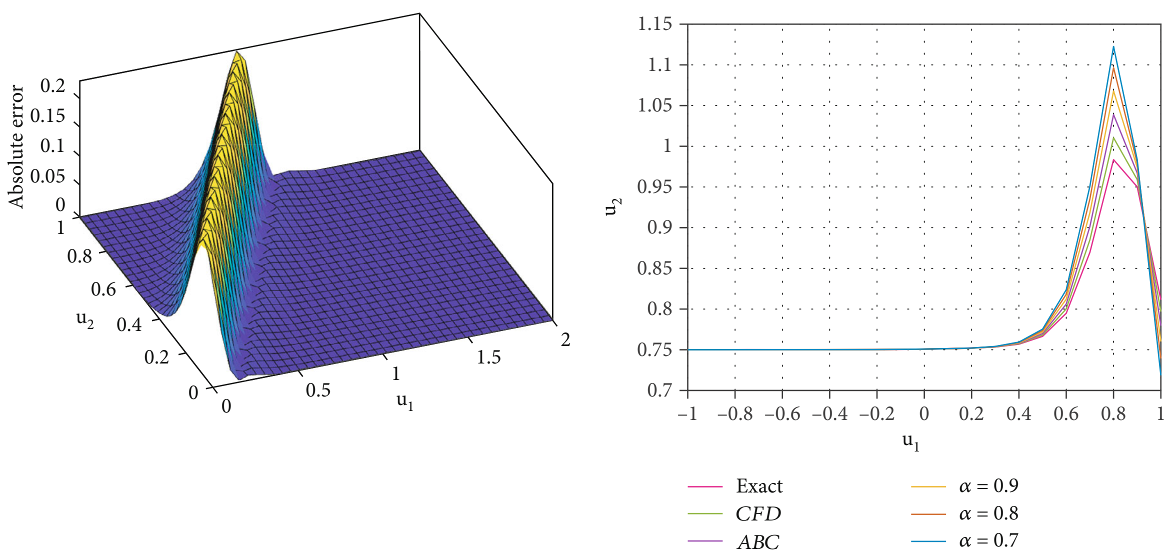
Proceeding in the analogous manner, the additional factors of $\mathbf{f}_\mathbf{r}$ and $\mathbf{g}_\mathbf{r}, (\mathbf{r} \geq 3)$ of the JTDM solution can be attained effortlessly. As a result, we arrive to the mathematical formulation as

$$
\begin{aligned}
\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= \sum_{\mathbf{r}=0}^{\infty} \mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) \\
&= \mathbf{f}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) + \mathbf{f}_1(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) + \mathbf{f}_2(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) +, \cdots,
\end{aligned}
$$



(a) (b)

Figure 3: 3D view of the exact and approximate solution via JTDM of Example 16 for $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ when $\delta_2 = 1$.



(a) (b)

Figure 4: (a) 3D view of absolute error for $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$. (b) 2D view of multiple fractional orders via JTDM of Example 16 for $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ when $\mathbf{t} = 0.7$.

$$\begin{aligned}
\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= (1 - \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)) - 2 \sec h(1 - \mathbf{u}_1 + 2\mathbf{u}_2) \\
&\quad \cdot \left\{ \frac{\mathbf{t}^{\delta_1}}{\Gamma(\delta_1 + 1)} + (1 - \delta_1) \right\} \\
&\quad - 8 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\big(-2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2) \\
&\quad + \big(2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2) + \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)\big)\big) \\
&\quad \cdot \left\{ \frac{\delta_1 \mathbf{t}^{\delta_1}}{\Gamma(2\delta_1 + 1)} + 2\delta_1(1 - \delta_1) \frac{\mathbf{t}^{\delta_1}}{\Gamma(\delta_1 + 1)} \right. \\
&\quad \left. + (1 - \delta_1)^2 \right\} +, \cdots ,
\end{aligned} \tag{81}$$

and

$$\begin{aligned}
\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= \sum_{\mathbf{r}=0}^{\infty} \mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) \\
&= \mathbf{g}_0(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) + \mathbf{g}_1(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) + \mathbf{g}_2(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) +, \cdots , \\
\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t}) &= (1 + 2 \tan h(1 - \mathbf{u}_1 + 2\mathbf{u}_2)) \\
&\quad + 4 \sec h(1 - \mathbf{u}_1 + 2\mathbf{u}_2) \\
&\quad \cdot \left\{ \frac{\mathbf{t}^{\delta_2}}{\Gamma(\delta_2 + 1)} + (1 - \delta_2) \right\} \\
&\quad - 8 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2) \\
&\quad \cdot \big(-2 \sec h^2(1 - \mathbf{u}_1 + 2\mathbf{u}_2)
\end{aligned}$$

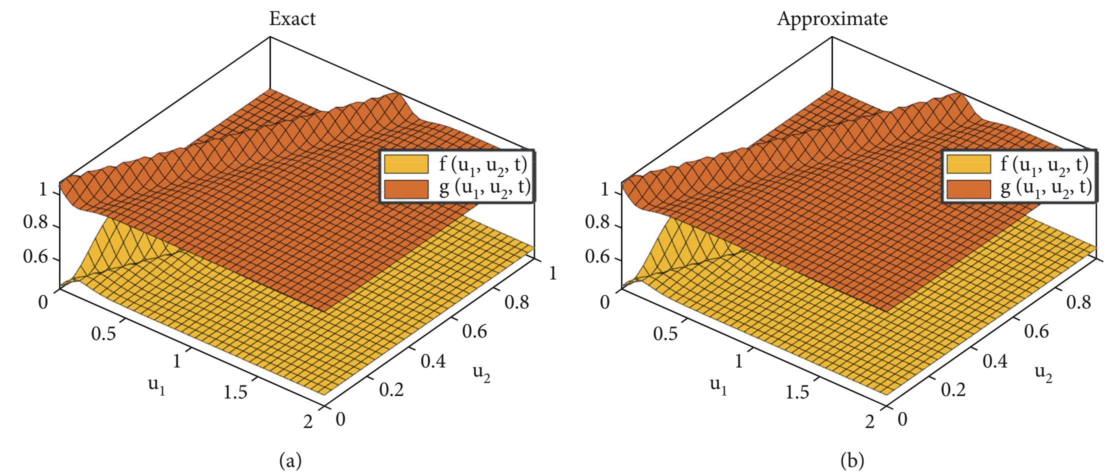


Figure 5: 3D comparison view of the exact and approximate solution of Example 16 when $\delta_1 = \delta_2 = 1$.

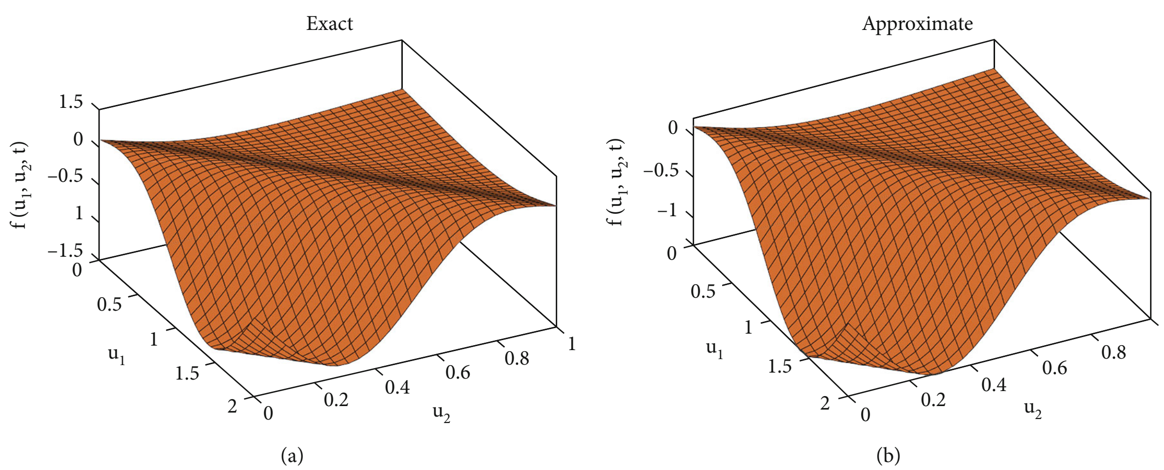


Figure 6: 3D view of the exact and approximate solution via JTDM of Example 19 for $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ when $\delta_1 = 1$.

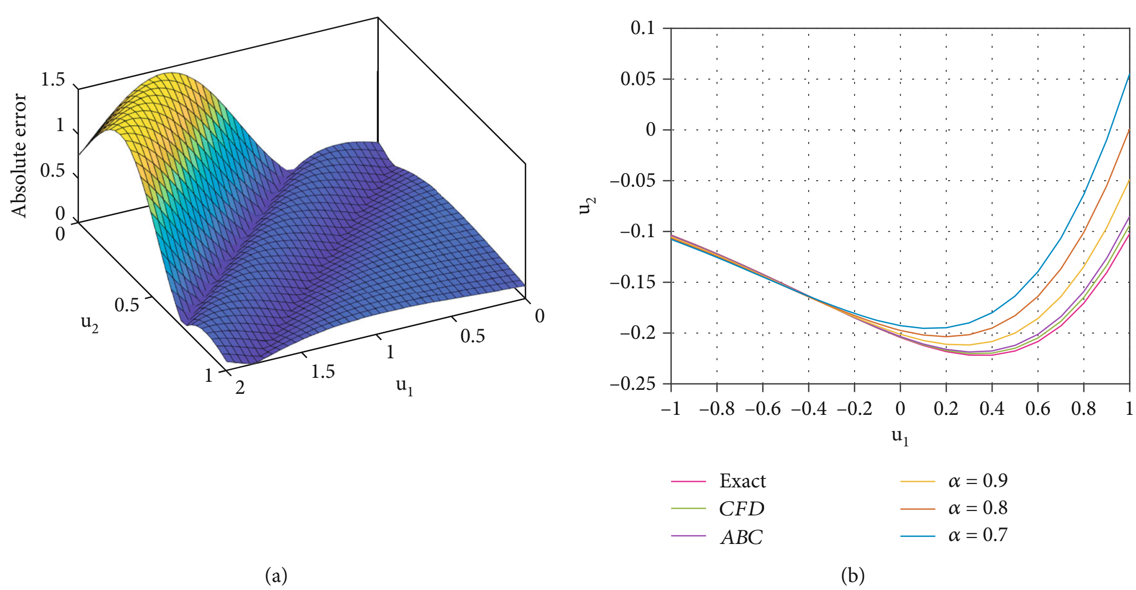


Figure 7: (a) 3D view of absolute error for $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$. (b) 2D view of multiple fractional orders via JTDM of Example 19 for $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ when $\mathbf{t} = 0.7$.

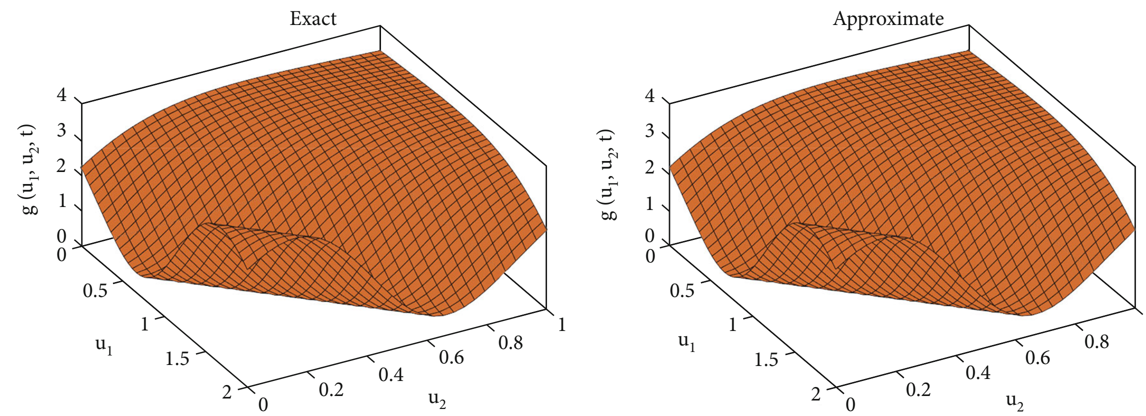


Figure 8: 3D view of the exact and approximate solution via JTDM of Example 19 for $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ when $\delta_2 = 1$.

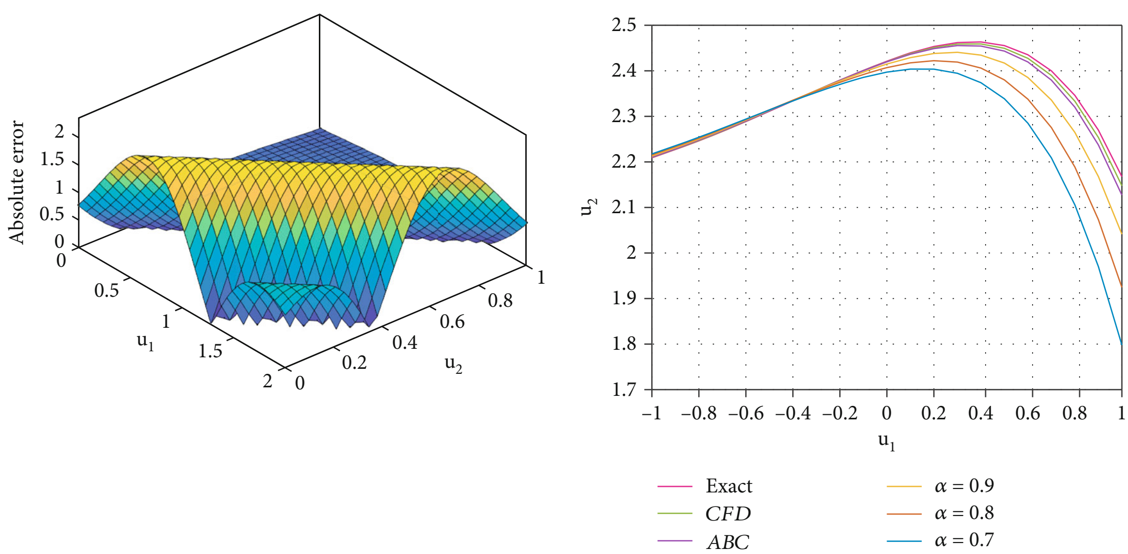


Figure 9: (a) 3D view of absolute error for $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$. (b) 2D view of multiple fractional orders via JTDM of Example 19 for $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ when $\mathbf{t} = 0.7$.

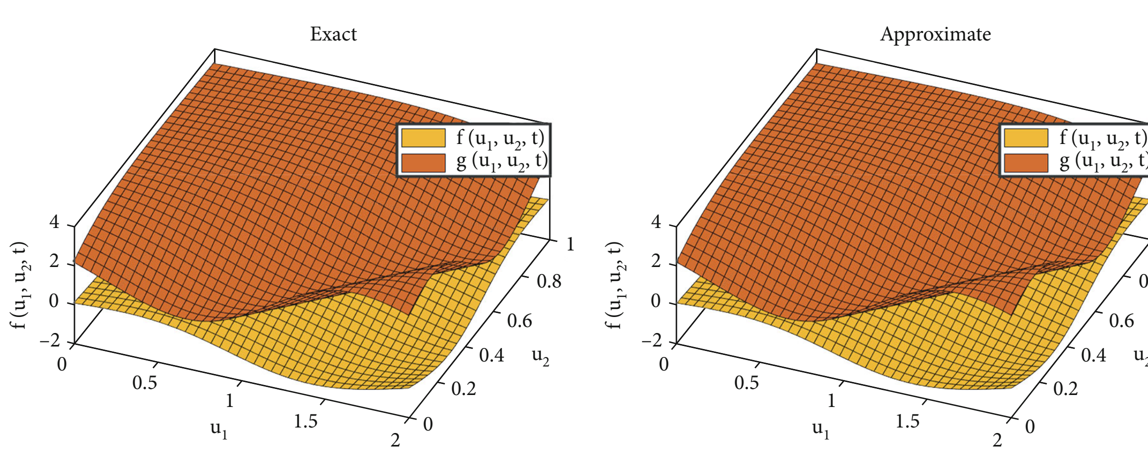


Figure 10: 3D comparison view of the exact and approximate solution of Example 19 when $\delta_1 = \delta_2 = 1$.

$$+\left(2\sec h^2(1-\mathbf{u}_1+2\mathbf{u}_2)+2\tan h(1-\mathbf{u}_1+2\mathbf{u}_2)\right)\right) \cdot\left\{\frac{\delta_2\mathbf{t}^{\delta_2}}{\Gamma(2\delta_2+1)}+2\delta_2(1-\delta_2)\frac{\mathbf{t}^{\delta_2}}{\Gamma(\delta_2+1)}+(1-\delta_2)^2\right\}+\cdots+. \tag{82}$$

6. Numerical Results and Discussion

We employed two distinctive techniques to analyze the approximate findings of fractional-order paired BEs in this research. With the aid of MATLAB 2021, one may acquire computational information for the framework of BEs in any configuration for varying parameters of spatial and temporal factors. We attempted graphical studies for multiple fractional processes in Tables 1 and 2 for the model in Example 16 assuming varied components of $\mathbf{u}_1$ and $\mathbf{t}$. Tables 1–4 illustrate a mathematical evaluation of the VIM [16] and the JTDM in perspective of absolute error for model (38). The findings of a simulation work for the interacting mechanism addressed in Example 16 are included in Tables 1–4. With relevant facts in the given data, we can deduce that the findings acquired by the JTDM are trustworthy. Figure 1 describes the performance of the JTDM result via $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ and exact outcome for Example 16, whereas Figure 2 depicts the structure of the absolute error and varied fractional parameters of δ_1. Also, Figure 3 represents the comparison analysis of the both integer and fractional order for Example 16.

In the analogous fashion, Figure 4 displays the obtained outcome $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ for Example 16. Figure 5 demonstrates the effect of collected information for Example 16 considering various classical and fractional orders $\delta_2 = 1,0.9,0.8, 0.7$. Figure 6 exhibits the performance of the JTDM outcome and exact solution from $\mathbf{f}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ for Example 19, whilst Figure 7 indicates the structure of the absolute error as well as multiple fractional orders. The various fractional orders $\delta_1 = 1,0.9,0.8,0.7$ are included in plot 7. Analogously, Figure 8 symbolizes the acquired solution $\mathbf{g}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{t})$ for (39). Figure 9 represent the strength of collected information for Example 19 considering various fractional orders and absolute errors. Furthermore, the comparison of the exact and approximate solution for both compartments is compared in Figure 10. Based on modelling, we discovered that fractional-order solution trajectories incorporate integer-order solution trajectories. In a nutshell, the JTDM allows a framework for performing productively in a unified manner. Therefore, the JTDM is a well-known system that produces a reasonable approach avoiding any linearization assumptions [27].

7. Conclusion

The objective of this study is at putting into practice the Jafari transform decomposition approach to address Burgers' equation by incorporating the Caputo and Antagana-Baleanu fractional derivative operators. Moreover, we present extensive conceptual evidence for the proposed strategy's existence and uniqueness. Furthermore, we provide two mathematical formulations to demonstrate that suggested Burgers' equation approach is viable and effective. The results for fractional problems are determined, and they are intimately associated with their realistic values. The proposed approach yields a series of outcomes of a recurrence connection with extreme precision and the fewest computations. Several numerical findings are evaluated using well-known analytical approaches, and the exact results are obtained when $\delta_1 = \delta_2 = 1$. The plots indicate that the precise and analytical findings have a clear association. The appropriateness of the specified procedures was validated by the generated figures. To better monitor the mechanisms of the provided challenges, results in various fractional orders are generated and displayed using graphs. The effectiveness of the proposed strategy has been validated by the convergence process. It was then extended by the investigators to address various scenarios using fractional partial differential equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors read and approved the final manuscript.

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