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Research Article

Fractional Fourier Transform and Ulam Stability of Fractional Differential Equation with Fractional Caputo-Type Derivative

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In this paper, we study the Ulam-Hyers-Mittag-Leffler stability for a linear fractional order differential equation with a fractional Caputo-type derivative using the fractional Fourier transform. Finally, we provide an enumeration of the chemical reactions of the differential equation.

1. Introduction

Fractional differential equations have more attention in the research area of mathematics, and there has been significant progress in this field. However, this idea is not new and as old as differential equations. The differential equations of fractional order have proved to be valuable tools in modeling multiple phenomena in different areas of science and engineering. Indeed, it has many uses in biology, physics, electromagnetics, mechanics, electrochemistry, etc. [1–3]. Fractional calculus was initiated from a question raised by L'Hospital to Leibniz, which related to his generalization of meaning of notation $(d^n y/dx^n)d$ for the derivative of order $n \in \mathcal{N} := 0, 1, 2, \cdots$, when $n = 1/2$?. In his reply, dated September 30, 1695, Leibniz wrote to L'Hospital [4], "This is an apparent paradox from which one-day useful consequences will be drawn." Recently, Ozaktas and Kutay [5] published on this topic, dealing with different characteristics in different ways.

A functional equation is stable if for each approximate answer there is a definite quantity about it. In 1940, the simulation and a hit theory suggested by Ulam [6] prompted the study of stability issues for numerous functional equations. He gave the University of Wisconsin Mathematical Colloquium a long form of talks, presenting a variety of unresolved questions. He raised one of the questions that were connected to the stability of the functional equation: "Give conditions for a linear function near an approximately linear function to exist." The first result concerning the stability of functional equations was presented by Hyers [7] in 1941. The stability of the form is subsequently referred to as Hyers-Ulam stability. In 1978, the generalization associated with the Hyers theorem given by Rassias [8] makes it possible for the Cauchy difference to be unbounded. In 2004, Jung [9] studied the Hyers-Ulam stability of the differential equations $\vartheta(s)p'(s) = p(s)$. Jung [10, 11] continuously published the general setting for Hyers-Ulam stability of first-order linear differential equations. In 2006, Jung [12] concentrated on the Hyers-Ulam stability of an arrangement of differential equations with coefficients through the utilization of a matrix approach. Ponmana Selvan et al. [13] have solved the different types of Ulam stability for the approximate solution of a special type of mth-order linear differential equation with initial and boundary conditions.

Zhang and Li [14] studied the Ulam stabilities of m -dimensional fractional differential systems with order $1 < \alpha < 2$ in 2011, and in the same year, Li and Zhang [15] proved the stability of fractional order derivative for differential equations. In 2013, Ibrahim [16] investigated the Ulam-Hyers stability for iterative Cauchy fractional differential equations and Lane-Emden equations. Kalvandi et al. [17], Liu *et al.* [18], and Vu et al. [19] presented and proved the different types of Hyers-Ulam stability of a linear fractional differential equations.

In 2012, Wang et al. [20] carried out pioneering work on the Hyers-Ulam stability for fractional differential equations with Caputo derivative using a fixed point approach, and in the same year, Wang and Zhou [21] proved the Hyers-Ulam stability of nonlinear impulsive problems for fractional differential equations. Wang et al. [22] investigated the Mittag-Leffler-Ulam-Hyers stability of fractional evolution equations.

In 2020, Unyong *et al.* [23] studied Ulam stabilities of linear fractional order differential equations in Lizorkin space using the fractional Fourier transform, and in the same year, Hammachukiattikul et al. [24] derived some Ulam-Hyers stability outcomes for fractional differential equations. In the next year, Ganesh *et al.* [25] derived some Mittag-Leffler-Hyers-Ulam stability, which makes sure the existence and individuation of an answer for a delay fractional differential equation by using the fractional Fourier transform. In 2022, Ganesh et al. [26] carried out pioneering in the field with the Hyers-Ulam stability for fractional order implicit differential equations with two Caputo derivatives using a fractional Fourier transform.

Motivated and inspired by the above results, in this paper, because of the help of fractional Fourier transform, we would like to investigate the Ulam-Hyers-Mittag-Leffler and Ulam-Hyers-Rassias-Mittag-Leffler stability of linear fractional order differential equations with the fractional Caputo-type derivative of the form:

$$\left({}^{\mathscr{C}}\mathscr{D}_{0+}^{\sigma}p\right)(s) + \eta\, p(s) = q(s), \tag{1}$$

where $q(s)$ is a $m-$times continuously differentiable function and ${}^{\mathscr{C}}\mathscr{D}_{0+}^{\sigma}$ is the fractional Caputo-type derivative of order $\sigma \in (m-1, m)$, $m \in N^{+}$.

2. Preliminaries

The following definitions, theorems, notations, and lemmas will be used to obtain the main objectives of this paper.

Definition 1 (see [27]). The one dimension fractional Fourier transform with rotational angle σ of function $p(s) \in \mathscr{L}'(\mathscr{R})$ is given by

$$\mathscr{F}_{\sigma}[p(s)](\omega) = \widehat{p}_{\sigma}(\omega) = \int_{\mathscr{R}} K_{\sigma}(s,\omega)p(s)ds, \omega \in \mathscr{R}, \tag{2}$$

where the kernel

$$K_{\sigma}(s,\omega) = \begin{cases} \mathscr{C}_{\sigma} e^{\left(\left(i\left(p^2+\omega^2\right)\cot\sigma\right)/2\right)-ip\omega\ \operatorname{cosec}\sigma}, & \text{if } \sigma \neq m\pi, \\ \dfrac{1}{\sqrt{2\pi}} e^{-ip\omega}, & \text{if } \sigma = \dfrac{\pi}{2}, \end{cases}$$
$$\mathscr{C}_{\sigma} = \sqrt{\frac{1-i\cot\sigma}{2\pi}}. \tag{3}$$

As such, the inversion formula of fractional Fourier transform is given by

$$p(s) = \frac{1}{2\pi}\int_{\mathscr{R}} K_{\sigma}(s,\omega)\widehat{p}_{\sigma}(\omega)d\omega, s \in \mathscr{R}, \tag{4}$$

where the kernel

$$K_{\sigma}(\bar{s},\omega) = \begin{cases} \mathscr{C}'_{\sigma} e^{\left(\left(-i\left(p^2+\omega^2\right)\cot\sigma\right)/2\right)+ip\omega\ \operatorname{cosec}\sigma}, & \text{if } \sigma \neq m\pi, \\ \dfrac{1}{\sqrt{2\pi}} e^{ip\omega}, & \text{if } \sigma = \dfrac{\pi}{2}, \end{cases}$$
$$\mathscr{C}'_{\sigma} = \sqrt{2\pi(1+i\cot\sigma)}. \tag{5}$$

Definition 2. The Mittag-Leffler function is given in the following manner:

$$\mathbb{E}_{\sigma}(s) = \sum_{m=0}^{\infty}\frac{s^m}{\Gamma(\sigma m+1)}, \quad (\sigma > 0)\ (\text{One parameter}),$$
$$\mathbb{E}_{\sigma,\mu}(s) = \sum_{m=0}^{\infty}\frac{s^m}{\Gamma(\sigma m+\mu)}, \quad (\sigma > 0, \mu > 0)\ (\text{Two parameters}). \tag{6}$$

where σ and μ are nonnegative constant.

Definition 3 (see [28]). The fractional integral operator of order $s > 0$ of a function $p \in \mathscr{L}^1(\mathscr{R}^+)$ is written as

$$I_{0+}^{\sigma}p(s) = \frac{1}{\Gamma(\sigma)}\int_0^s (s-u)^{(\sigma-1)}p(u)\,du, s > 0, \tag{7}$$

where $\Gamma(.)$ is the gamma function and $\mathscr{R}e > 0$.

Definition 4 (see [28]). The Riemann-Liouville fractional order derivative of $s > 0, m-1 < \sigma < m, m \in \mathscr{N}$, is written as

$$\left({}^{\mathscr{RL}}\mathscr{D}_{0+}^{\sigma}p\right)(s) = \frac{1}{\Gamma(m-\sigma)}\left(\frac{d}{ds}\right)^m\int_0^s (s-u)^{(m-\sigma-1)}p(u)du, \tag{8}$$

where the function $p(s)$ is a continuous derivatives upto order $(m-1)$.

Definition 5 (see [28]). The fractional Caputo-type derivative of order $s > 0, m-1 < \sigma < m, m \in \mathcal{N}$, is written as

$$\left({}^{\mathscr{C}}\mathscr{D}_{0+}^{\sigma}p\right)(s) = \frac{1}{\Gamma(m-\sigma)}\int_0^s (s-u)^{(m-\sigma-1)}p^{(n)}(u)du, \quad (9)$$

where the function $p(s)$ is a continuous derivatives up to order $(m-1)$. Then, let $s > 0, \sigma \in \mathscr{R}, m-1 < \sigma < m, m \in \mathcal{N}$. The relation between Caputo and Riemann-Liouville fractional derivative is given by

$$\left({}^{\mathscr{C}}\mathscr{D}_{0+}^{\sigma}p\right)(s) = \left(\mathscr{D}_{0+}^{\sigma}p\right)(s) - \sum_{k=0}^{m-1}\frac{(s-a)^{k-\sigma}}{\Gamma(k-\sigma+1)}p^{(k)}(0). \quad (10)$$

Definition 6. Equation (1) has Ulam-Hyers-Mittag-Leffler stability, if there exist a continuously differentiable function $p(s)$ satisfying the inequality

$$\left|\left({}^{\mathscr{C}}\mathscr{D}_{0+}^{\sigma}p\right)(s) + \eta p(s) - q(s)\right| \le \varepsilon\mathbb{E}_{\sigma}(s), \forall s > 0, \quad (11)$$

for every $\varepsilon > 0$, there exists a solution $p_{\sigma}(s)$ satisfying Equation (1) such that

$$|p(s) - p_{\sigma}(s)| \le \mathscr{H}\varepsilon\mathbb{E}_{\sigma}(s), \quad (12)$$

where $\mathscr{H}$ is a nonnegative and stability constant.

Definition 7. The considered $\phi : (0,\infty) \longrightarrow (0,\infty)$ is a function. Equation (1) has Ulam-Hyers-Rassias-Mittag-Leffler stability, if there exist a continuously differentiable function $p(s)$ satisfying the inequality

$$\left|\left({}^{\mathscr{C}}\mathscr{D}_{0+}^{\sigma}p\right)(s) + \eta p(s) - q(s)\right| \le \varepsilon\phi(s)\mathbb{E}_{\sigma}(s), \forall s > 0, \quad (13)$$

for every $\varepsilon > 0$, there exists a solution $p_{\sigma}(s)$ satisfying Equation (1) such that

$$|p(s) - p_{\sigma}(s)| \le \mathscr{H}\phi(s)\varepsilon\mathbb{E}_{\sigma}(s), \quad (14)$$

where $\mathscr{H}$ is a nonnegative and stability constant.

3. Main Results

In this section, we will investigate to help of fractional Fourier transform to study the Ulam-Hyers-Mittag-Leffler stability of (1).

Theorem 8. *If a function $p(s)$ satisfies the inequality (11) for every $\varepsilon > 0$, there exists a solution $p_{\sigma}(s)$ satisfying Equation (1) such that*

$$|p(s) - p_{\sigma}(s)| \le \mathscr{H}\varepsilon\mathbb{E}_{\sigma}(s). \quad (15)$$

Proof. Let us choose a function $y(s)$ follow as

$$y(s) = \left({}^{\mathscr{C}}\mathscr{D}_{0+}^{\sigma}p\right)(s) + \eta p(s) - q(s). \quad (16)$$

Now,

$$y(s) = (\mathscr{D}^{\sigma}p)(s) - \sum_{k=0}^{m-1}\frac{s^{k-\sigma}}{\Gamma(k-\sigma+1)}p^{(k)}(0) + \eta p(s) - q(s), \forall s > 0. \quad (17)$$

Taking $\mathscr{F}_{\sigma}$ (the fractional Fourier transform oprator) onto both sides of Equation (17), we have

$$\begin{aligned}\mathscr{F}_{\sigma}\{y(s)\} &= \mathscr{F}_{\sigma}\left\{\mathscr{D}^{\sigma}p(s) - \sum_{k=0}^{m-1}\frac{s^{k-\sigma}}{\Gamma(k-\sigma+1)}p^{(k)}(0) + \eta p(s) - q(s)\right\} \\ &= \left(i\omega^{n/\sigma}\right)^{\sigma}\mathscr{F}_{\sigma}\{p(s)\} - e^{i\omega^{n/\sigma}a}\sum_{k=0}^{m-1}a_k\frac{s^{k-\sigma}}{\Gamma(k-\sigma+1)\left(i\omega^{n/\sigma}\right)^{k-\sigma+1}} \\ &\quad + \eta\mathscr{F}_{\sigma}(p(s)) - \widehat{G}_a(\omega),\end{aligned} \quad (18)$$

where $p^{(k)}(0) = a_k$, for $k = 0, 1, \cdots, m-1$ and

$$\begin{aligned}\mathscr{F}_{\sigma}\{p(s)\} &= \frac{\mathscr{F}_{\sigma}\{y(s)\}}{\left(\left(i\omega^{n/\sigma}\right)+\eta\right)} \\ &\quad + \frac{e^{i\omega^{n/\sigma}a}}{\left(\left(i\omega^{n/\sigma}\right)+\eta\right)}\sum_{k=0}^{m-1}\frac{a_k}{\left(\left(i\omega^{n/\sigma}\right)+\eta\right)} + \frac{\widehat{G}_a(\omega)}{\left(\left(i\omega^{n/\sigma}\right)+\eta\right)}.\end{aligned} \quad (19)$$

Setting

$$p_{\sigma}(s) = \sum_{k=n}^{p-1}a_k p_k(0) + \int_0^s (s-\nu)^{\nu-1}\mathbb{E}_{\sigma}\left[\eta(s-\nu)^{\nu-1}\right]q(s)d\nu. \quad (20)$$

By using fractional Fourier transform to (20), we have

$$\mathscr{F}_{\sigma}\{p_{\sigma}(s)\} = \frac{e^{i\omega^{n/\sigma}a}}{\left(\left(i\omega^{n/\sigma}\right)+\eta\right)}\sum_{k=0}^{m-1}\frac{a_k}{\left(\left(i\omega^{n/\sigma}\right)+\eta\right)} + \frac{\widehat{G}_a(\omega)}{\left(\left(i\omega^{n/\sigma}\right)+\eta\right)}. \quad (21)$$

Hence,

$$\begin{aligned}\left({}^{\mathscr{C}}\mathscr{D}_{0+}^{\sigma}p\right)(s) + \eta p(s) &= \left(i\omega^{n/\sigma}\right)^{\sigma}\mathscr{F}_{\sigma}\{p(s)\} \\ &\quad - e^{i\omega^{n/\sigma}a}\sum_{k=0}^{m-1}a_k\frac{s^{k-\sigma}}{\Gamma(k-\sigma+1)\left(i\omega^{n/\sigma}\right)^{k-\sigma+1}} \\ &\quad + \eta\mathscr{F}_{\sigma}(p(s)) - \widehat{G}_a(\omega) \\ &= q(s).\end{aligned} \quad (22)$$

□

Since $\mathscr{F}_\sigma$ is one-to-one operator, $({}^{\mathscr{C}}\mathscr{D}^{\sigma}_{0+}p)(s) + \eta p(s) = q(s)$. Now, its follows form (19) and (21) that

$$\mathscr{F}_\sigma\{p(s)\} - \mathscr{F}_\sigma\{p_\sigma(s)\} = \frac{\mathscr{F}_\sigma\{y(s)\}}{((i\omega^{n/\sigma}) + \eta)}. \tag{23}$$

Using the convolution property, we obtain

$$\mathscr{F}_\sigma\{p(s) - p_\sigma(s)\} = \mathscr{F}_\sigma\{y(s)\} * \frac{1}{((i\omega^{n/\sigma}) + \eta)} = y(s) * y_\sigma(s), \tag{24}$$

where $y_\sigma(s) = 1/((i\omega^{n/\sigma}) + \eta)$. In view of (13), we have

$$|y(s)| \le \varepsilon \mathbb{E}_\sigma(s), \forall s > 0. \tag{25}$$

Now, applying the modules on both sides of Equation (24), we get

$$\begin{aligned} |p(s) - p_\sigma(s)| &= \left| \int_0^s (s-x)^{\sigma-1} \mathbb{E}_\sigma(\eta(s-\nu)^\sigma) * y(s) d\nu \right| \\ &\le |y(s)| \left| \int_0^s (s-\nu)^{\sigma-1} \mathbb{E}_\sigma(\eta(s-\nu)^\sigma) d\nu \right| \\ &\le \varepsilon \mathbb{E}_\sigma(s) \left| \int_0^s (s-\nu)^{\sigma-1} \mathbb{E}_\sigma(\eta(s-x)^\sigma) d\nu \right| \\ &\le \mathscr{H} \varepsilon \mathbb{E}_\sigma(s). \end{aligned} \tag{26}$$

where $\mathscr{H} = |\int_0^s (s-x)^{\sigma-1} \mathbb{E}_\sigma(\eta(s-x)^\sigma) d\nu|$. Thus Equation (1) has Ulam-Hyers-Mittag-Leffler stability.

Corollary 9. *The considered $\phi : (0,\infty) \longrightarrow (0,\infty)$ is a function. If a function $p(s)$ satisfies the inequality (13), for every $\varepsilon > 0$, there exists a solution $p_\sigma(s)$ satisfying Equation (1) such that*

$$|p(s) - p_a(s)| \le \mathscr{H} \phi(s) \varepsilon \mathbb{E}_\sigma(s), \forall s > 0. \tag{27}$$

i.e., Equation (1) has Ulam-Hyers-Rassias-Mittag-Leffler stability.

4. Applications

In this section, the standard kinetic equation in the chemical reaction that will be used to analyze this experimental data is revealed by the equation as follows: where $\mathscr{L} =$ xylan; $\mathscr{M} =$ xylose; $\mathscr{N} =$ products of decomposition; $r_1 =$ release rate of sugar; $r_2 =$ decomposition rate of sugar. The model is presented in Figure 1.

Material balance for components: "$\mathscr{L}$" and "$\mathscr{M}$" for the first-order kinetic equation, we get

$$-\frac{d\mathscr{N}_{\mathscr{L}}(s)}{ds} = r_1 \mathscr{N}_{\mathscr{L}}(s), \tag{28}$$

in which the initial concentration at $s = 0$ is presented by $\mathscr{N}_{\mathscr{L}} = \mathscr{N}_{\mathscr{L}_0}$. Also, we have the same direction for material $\mathscr{M}$:

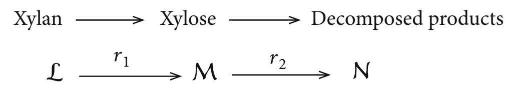

FIGURE 1: The presented model.

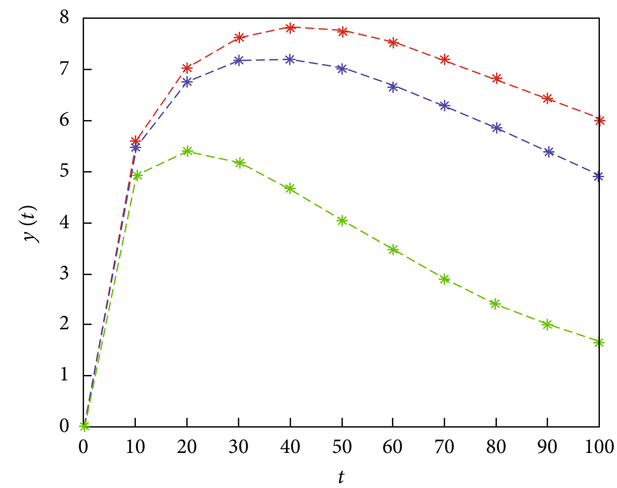

FIGURE 2: Solution of Equation (32) for different values $(r_1 = 0.012 \& r_2 = 0.005)$, $(r_1 = 0.014 \& r_2 = 0.005)$, and $(r_1 = 0.025 \& r_2 = 0.005)$ with $\sigma = 1/2$.

$$-\frac{d\mathscr{N}_{\mathscr{M}}(s)}{ds} = r_1 \mathscr{N}_{\mathscr{L}}(s) - r_2 \mathscr{N}_{\mathscr{M}}(s), \tag{29}$$

in which the initial concentration at $s = 0$ is presented by $\mathscr{N}_{\mathscr{M}} = \mathscr{N}_{\mathscr{M}_0}$. Equation (29) can be integrated and, using the provided boundary condition, yields

$$\mathscr{N}_{\mathscr{L}}(s) = \mathscr{N}_{\mathscr{L}_0} \quad \exp(-r_1 s). \tag{30}$$

Substituting (30) for (29) yields

$$\frac{d\mathscr{N}_{\mathscr{M}}(s)}{ds} + r_2 \mathscr{N}_{\mathscr{M}}(s) = r_1 \mathscr{N}_{\mathscr{L}_0} \quad \exp(-r_1 s). \tag{31}$$

Now, if we take the fractional Caputo derivative in (31) instead of the classical ones, we have

$${}^{\mathscr{C}}\mathscr{D}^{\sigma} \mathscr{N}_{\mathscr{M}}(s) + r_2 \mathscr{N}_{\mathscr{M}}(s) = r_1 \mathscr{N}_{\mathscr{L}_0} \quad \exp(-r_1 s). \tag{32}$$

Figure 2 shows the solution of Equation (32) for various r_1 and r_2.

5. Conclusions

In this paper, the objective is investigated by using the fractional Fourier transform to study the Ulam-Hyers-Mittag-Leffler stability of linear fractional differential equations. The required outcomes have been achieved by using the fractional Fourier transform. We could reach the suitable approximation value of xylose after a certain period of time, which is crucial for analyzing the kinetic equation in the chemical reaction process.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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