Hindawi
Journal of Function Spaces
Volume 2022, Article ID 3787529, 24 pages
https://doi.org/10.1155/2022/3787529

Research Article

L_p-Curvature Measures and $L_{p,q}$-Mixed Volumes

Tongyi Ma

College of Mathematics and Statistics, Hexi University, Zhangye, Gansu 734000, China

Correspondence should be addressed to Tongyi Ma; matongyi@126.com

Received 8 March 2022; Revised 1 May 2022; Accepted 5 July 2022; Published 18 August 2022

Academic Editor: Raúl E. Curto

Motivated by Lutwak et al.'s L_p-dual curvature measures, we introduce the concept of L_p-curvature measures. This new L_p-curvature measure is an extension of the classical surface area measure, L_p-surface area measure, and curvature measure. In this paper, we first prove some properties of the L_p-curvature measure. Next, using the L_p-curvature measure, we define the $L_{p,q}$-mixed volume which includes L_p-mixed volume as the special cases. Further, the Minkowski-type inequality related $L_{p,q}$-mixed volume and the uniqueness of the solution for the $L_{p,q}$- Minkowski problem are obtained. Finally, we propose several problems that need to be studied further.

1. Introduction

Surface area measure and integral curvature measure are two important measures in classical Brunn-Minkowski theory. Minkowski problem describing surface area measure and Aleksandrov problem describing integral curvature are two famous problems. As a generalization, L_p-surface area measure and L_p-integral curvature are defined in [1, 2], respectively. At the same time, the hyperbolic measure as the curvature measure of dual Fiedler is constructed in [3]. Lutwak et al. introduce L_p-dual curvature measure in [4], which is a generalization of the dual curvature, L_p-surface area measure and L_p-integral curvature. L_p-dual mixed volume (also known as (p, q)-dual mixed volume) is defined by [4] and Minkowski inequality is established. Furthermore, they study the L_p-dual Minkowski problem of L_p-dual curvature measure by reference to [5].

Inspired by Lutwak et al.'s L_p-dual curvature measure, a new concept of L_p-curvature measure is introduced in this paper. It includes classical surface area measure, L_p-surface area measure and curvature measure. In this paper, we first prove some properties of L_p-curvature measure. Next, based on L_p-curvature measure, we define $L_{p,q}$-mixed volume, which includes L_p-mixed volume as a special case. Furthermore, the Minkowski inequality for $L_{p,q}$-mixed volume and the uniqueness of the solution for $L_{p,q}$-Minkowski problem are obtained. Finally, some problems which need further study are put forward.

Let $\mathscr{K}^n$ represent the set of convex bodies in n-dimensional Euclidean (compact convex subsets with nonempty embedding) space $\mathbb{R}^n$, for convex bodies containing the origin inside in $\mathbb{R}^n$, we write $\mathscr{K}_o^n$. Set B said centered on the origin of the unit sphere, B surface written as S^{n-1}, in $\mathbb{R}^n$. $V(K)$ represents the n dimensional volume of the body K and writes $V(B) = \omega_n$.

For $K \in K^n$, its support function, $h_K = h(K, \cdot)$: $\mathbb{R}^n \longrightarrow (-\infty, +\infty)$, is defined by (see [6])

$$h(K, x) = \max \{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n, \tag{1}$$

where $x \cdot y$ denotes the standard inner product of x and y.

For $K, L \in \mathscr{K}^n$ and $s, t \geq 0$ (not both zero), the Minkowski combination, $sK + tL \in \mathscr{K}^n$, of K and L is defined by the following:

$$h(sK + tL, \cdot) = sh(K, \cdot) + th(L, \cdot), \tag{2}$$

i.e., $sK + tL = \{sx + ty : x \in K, y \in L\}$.

The surface area measure $S(K,\cdot)$ of $K \in \mathscr{K}^n$ can be defined by the following:

$$\frac{d}{dt}V(K+tL)\bigg|_{t=0^+} = \int_{S^{n-1}} h_L(u)dS(K,u), \qquad (3)$$

for any $L \in \mathscr{K}^n$. From Equation (3), the Minkowski's first mixed volume of K and L is given as follows:

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(u)dS(K,u). \qquad (4)$$

The mixed volume $V_1(K,L)$ generalizes the concepts of volume, surface area, and mean width.

We say that $K \in \mathscr{K}^n$ has a positive continuous curvature function $f(K,\cdot) = f_K(\cdot)$: $S^{n-1} \longrightarrow \mathbb{R}$, if for all $L \in \mathscr{K}^n$,

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(u)f_K(u)dS(u), \qquad (5)$$

where S is spherical Lebesgue measure. Clearly, Equations (4) and (5) imply the following:

$$f(K,u) = \frac{dS(K,u)}{dS}. \qquad (6)$$

Let $p \geq 1$. Using the L_p-Minkowski conbinations (see Equation (60)), Lutwak [2] defined the L_p-surface area measure $S_p(K,\cdot)$ of a convex body $K \in \mathscr{K}_o^n$, namely, for each $L \in \mathscr{K}_o^n$,

$$\frac{d}{dt}V\left(K +_p t\cdot L\right)\bigg|_{t=0^+} = \frac{1}{p}\int_{S^{n-1}} h_L(u)^p dS_p(K,u). \qquad (7)$$

For $K, L \in \mathscr{K}_o^n$, the L_p-mixed volume $V_p(K,L)$ is given by the following (see [4]):

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(u)^p dS_p(K,u). \qquad (8)$$

We say that $K \in \mathscr{K}_o^n$ has a positive continuous L_p-curvature function $f_p(K,\cdot)$: $S^{n-1} \longrightarrow \mathbb{R}$, if the integral representation

$$V_p(K,Q) = \frac{1}{n}\int_{S^{n-1}} h_Q(u)^p f_p(K,u)dS(u), \qquad (9)$$

for all $Q \in \mathscr{K}_o^n$. For $K \in \mathscr{K}_o^n$ with a positive continuous curvature functions, it follows from Equation (8) and Equation (9) that

$$f_p(K,u) = \frac{dS_p(K,u)}{dS}. \qquad (10)$$

The L_p-Minkowski inequality of the L_p-mixed volume is (see [2, 7]) that for $p \geq 1$,

$$V_p(K,L)^n \geq V(K)^{n-p}V(L)^p, \qquad (11)$$

with equality for $p > 1$ if and only if K and L are dilates, for $p = 1$ and if and only if K and L are homothetic.

According to Equation (10), the curvature function of L_p is the Radon-Nikodym derivative of L_p-surface area measure with respect to the spherical Lebesgue measure. The integral of L_p-curvature function (raised to an appropriate power) over the unit sphere is the L_p-affine surface area, which is an important research point of affine geometry and valuation theory, see, e.g., [8–24]. The L_p-Minkowski problem (see [2]) is a necessary and sufficient condition to find a given measure such that it is only the L_p-surface area measure of a convex body. Solving the L_p-Minkowski problem requires solving a degenerate singular Monge-Ampère-type equation on the unit sphere. The L_p-Minkowski problem has been solved for $p \geq 1$, see [2, 25, 26], but critical cases for $p < 1$ remain open, see, e.g., [25, 27–31]. For its applications, see [5, 7, 27, 32–35].

A star body $Q \subset \mathbb{R}^n$ is a compact star-shaped set about the origin whose radial function $\rho_Q : S^{n-1} \longrightarrow (0,\infty)$ is defined by the following:

$$\rho_Q(u) = \max\{\lambda \geq 0 : \lambda u \in Q\}, \qquad (12)$$

for $u \in S^{n-1}$. If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Denote the set of star bodies in $\mathbb{R}^n$ by $\mathscr{S}_o^n$. Obviously, $\mathscr{K}_o^n \subset \mathscr{S}_o^n$.

The dual Brunn-Minkowski theory is the theory of dual mixed volumes of star bodies. For $q \in \mathbb{R}$, the q-th dual mixed volume, $\tilde{V}_q(K,Q)$, of $K, Q \in \mathscr{S}_o^n$ is defined by the following:

$$\tilde{V}_q(K,Q) = \frac{1}{n}\int_{S^{n-1}} \rho_K^q(u)\rho_Q^{n-q}(u)du, \qquad (13)$$

where the integration is with respect to spherical Lebesgue measure. For $q \neq 0$, the q-th dual volume $\tilde{V}_q(K)$ of $K \in \mathscr{S}_o^n$ is defined by $\tilde{V}_q(K) = \tilde{V}_q(K,B)$. The q-th dual volume is important in geometric tomography, one of the reasons that is that for integers $q = 1, 2, \cdots, n-1$ and each $K \in \mathscr{S}_o^n$,

$$\tilde{V}_q(K) = c_{n,q}\int_{G(n,q)} \mathrm{vol}_q(K \cap \xi)d\xi, \qquad (14)$$

where vol_q denotes volume in $\mathbb{R}^q$, $G(n,q)(q = 1, 2, \cdots, n-1)$ denote the Grassmann manifold of q-dimensional subspaces of $\mathbb{R}^n$, the integration is with respect to the rotation invariant probability measure on $G(n,q)$ and constant $c_{n,q}$ is trivially determined by taking K to be B.

For the real $q \neq 0$, the q-th dual curvature $\tilde{C}_q(K,\cdot)$ of $K \in \mathscr{K}_o^n$ is a Borel measure on the unit sphere, which can be defined in [3] by using the variational formula:

$$\frac{d}{dt}\tilde{V}_q(K+tL)\bigg|_{t=0^+} = q\int_{S^{n-1}} h_L(v)h_K^{-1}(v)d\tilde{C}_q(K,v), \quad (15)$$

for every $L \in \mathcal{K}_o^n$. Similar to the critical role as L_p-surface area measures playing in the L_p Brunn-Minkowski theory, dual curvature measures is a central concept within the dual Brunn-Minkowski theory.

The singularity case $q=0$ of dual volume leads to dual entropy of star body. For $K \in \mathcal{S}_o^n$, the dual entropy $\tilde{E}(K)$ can be defined as follows:

$$\tilde{E}(K) = \frac{1}{n}\int_{S^{n-1}} \log \rho_K(u)du. \quad (16)$$

The L_p-integral curvature, $J_p(K,\cdot)$, of $K \in \mathcal{K}_o^n$ (see [1]) can be defined by a variational formula:

$$\frac{d}{dt}\tilde{E}\left(K +_p t\cdot L\right)\bigg|_{t=0^+} = \frac{1}{np}\int_{S^{n-1}} h_L^p(v)dJ_p(K^*,v), \quad (17)$$

for all $L \in \mathcal{K}_o^n$, where K^* is the polar body of K is given by $K^* = \{x \in \mathbb{R}^n : x\cdot y \le 1 \text{ for all } y \in K\}$.

In [4], Lutwak et al. introduced L_p-dual curvature measures, which are a generalization of dual curvatures, L_p-surface area measure and L_p-integral curvatures. For $p, q \in \mathbb{R}$, $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$, the L_p-dual curvature measure, $\tilde{C}_{p,q}$, is the Borel measure on S^{n-1} defined by the following:

$$\int_{S^{n-1}} g(v)d\tilde{C}_{p,q}(K,Q,v) = \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))h_K(\alpha_K(u))^{-p}\rho_K(u)^q\rho_Q(u)^{n-q}du, \quad (18)$$

for each continuous $g: S^{n-1} \longrightarrow \mathbb{R}$, where α_K is the radial Gauss map (see Section 2 for details).

L_p-dual mixed volume (also known as (p,q)-dual mixed volume) is defined by Lutwak et al. [4] using the L_p-dual curvature:

For $p, q \in \mathbb{R}$, $K, L \in \mathcal{K}_o^n$, and $Q \in \mathcal{S}_o^n$, the L_p-dual mixed volume $\tilde{V}_{p,q}(K,L,Q)$ is defined by the following:

$$\tilde{V}_{p,q}(K,L,Q) = \int_{S^{n-1}} h_L^p(v)d\tilde{C}_{p,q}(K,Q,v). \quad (19)$$

By Equation (18), the L_p-dual mixed volume has the following integral formula:

$$\tilde{V}_{p,q}(K,L,Q) = \frac{1}{n}\int_{S^{n-1}} h_L(\alpha_K(u))^p h_K(\alpha_K(u))^{-p}\rho_K(u)^q\rho_Q(u)^{n-q}du. \quad (20)$$

Specifically, $\tilde{C}_{p,n}(K,B) = (1/n)S_p(K,\cdot)$, namely,

$$dS_p(K,u) = n\tilde{C}_{p,n}(K,B) = h_K(\alpha_K(u))^{-p}\rho_K(u)^n du, \quad (21)$$

and for $K, L \in \mathcal{K}_o^n$,

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^p(\alpha_K(u))\rho_K^n(u)du. \quad (22)$$

For the (p,q)-dual mixed volumes, the related Minkowski inequality is given in [4]. Suppose $1 \le q/n \le p$, if K, $L \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$, then

$$\tilde{V}_{p,q}(K,L,Q)^n \ge V(K)^{q-p}V(L)^pV(Q)^{n-q}, \quad (23)$$

with equality when $q > n$ if and only if K, L and Q are dilates; while when $q=n$ and $p>1$, with equality if and only if K and L are dilates; while when $q=n$ and $p=1$, with equality if and only if K and L are homothetic.

In [4], the authors studied the L_p-dual Minkowski problems for L_p-dual curvature measures. The results of L_p-dual Minkowski problem caught many attentions, for example, see [3, 27, 36–42]. In addition, based on the (p,q)-dual mixed volumes, Ma et al. studied (p,q)-John ellipsoids in [43], which contain the classical John ellipsoid and the L_p-John ellipsoids. They also solved two involving optimization problem about the (p,q)-dual mixed volumes for all $0 < p \le q$. A different extension of the L_p-John ellipsoid was considered by Li et al. in [44].

In this paper, motivated by Lutwak et al.'s works in [4], we introduce the following L_p-curvature measures which is a new curvature measure.

Definition 1. For $p, q \in \mathbb{R}$ and $K, Q \in \mathcal{K}_o^n$, we define the L_p-curvature measure $C_{p,q}(K,Q,\cdot)$ by the following:

$$\int_{S^{n-1}} g(v)dC_{p,q}(K,Q,v) = \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))h_Q^{q-p}(\alpha_K(u))h_K^{-q}(\alpha_K(u))\rho_K^n(u)du, \quad (24)$$

for each continuous $g: S^{n-1} \longrightarrow \mathbb{R}$.

According to Definition 1, the L_p-curvature measure $C_{p,q}(K,Q,\cdot)$ has the following integral expression.

Property 2. Suppose $p, q \in \mathbb{R}$. If $K, Q \in \mathcal{K}_o^n$, then

$$C_{p,q}(K,Q,\eta) = \frac{1}{n}\int_{\alpha_K^*(\eta)} h_Q^{q-p}(\alpha_K(u))h_K^{-q}(\alpha_K(u))\rho_K^n(u)du, \quad (25)$$

for each Borel set $\eta \subseteq S^{n-1}$. Here,

$$\alpha_K^*(\eta) = \left\{\frac{x}{|x|} = \bar{x} \in S^{n-1} \text{ where } x \in H_K(v) \text{ for some } v \in \eta\right\}, \quad (26)$$

and $H_K(v)$ is the supporting hyperplane to K with outer normal vector $v \in \mathbb{R}^n \setminus \{0\}$.

Property 3. Suppose $p, q \in \mathbb{R}$. If $K, Q \in \mathscr{K}_o^n$, then for each Borel set $\eta \subseteq S^{n-1}$,

$$C_{p,q}(K, Q, \eta) = \frac{1}{n} \int_{x \in \mathbf{x}_K(\eta)} (x \cdot \nu_K(x))^{1-q} \|\nu_K(x)\|_{Q^*}^{q-p} d\mathscr{H}^{n-1}(x). \tag{27}$$

Among them, $\mathscr{H}^{n-1}(\cdot)$ represents the $(n-1)$-dimensional Hausdorff measure, and $\nu_K(x)$ represents the regular radial vector of $x \in \partial K$, as well as $\mathbf{x}_K(\eta)$ represents the reverse spherical image of $\eta \subset S^{n-1}$.

The L_p-curvature measures unify the surface area measures, L_p-surface area measures and curvature measures, as well as other measures. In particular, for $p, q \in \mathbb{R}$ and $K, Q \in \mathscr{K}_o^n$ the L_p-surface area measures and the q-th curvature measures (see Section 3 for its definition) are special cases of the L_p-curvature measures:

$$C_{q,q}(K, Q, \cdot) = \frac{1}{n} S_q(K, \cdot), \tag{28}$$

$$C_{p,q}(K, K, \cdot) = \frac{1}{n} S_p(K, \cdot), \tag{29}$$

$$C_{p,q}(K, B, \cdot) = \frac{1}{n} S_q(K, \cdot), \tag{30}$$

$$C_{p,0}(K, K, \cdot) = \frac{1}{n} S_p(K, \cdot), \tag{31}$$

$$C_{0,q}(K, B, \cdot) = \frac{1}{n} S_q(K, \cdot). \tag{32}$$

According to the L_p-curvature measures, we now define the notion of the $L_{p,q}$-mixed volumes which unifies L_p-mixed volumes and dual-mixed volumes.

Definition 4. For $p, q \in \mathbb{R}$ and $K, L, Q \in \mathscr{K}_o^n$, the $L_{p,q}$-mixed volume, $V_{p,q}(K, L, Q)$, of K and L (with respect to Q) is defined by the following:

$$V_{p,q}(K, L, Q) = \int_{S^{n-1}} h_L(u)^p dC_{p,q}(K, Q, u). \tag{33}$$

The following variational formula is an extension of Equations (3) and (7).

Theorem 5. *If reals $p, q \neq 0$ and $K, L, Q \in \mathscr{K}_o^n$, then the $L_{p,q}$-mixed volume $V_{p,q}(K, L, Q)$ via the variational formula of K and L (with respect to Q) by the following:*

$$V_{p,q}(K, L, Q) = \frac{p}{q} \lim_{t \longrightarrow 0} \frac{V_q(K, Q +_p t \cdot L) - V_q(K, Q)}{t}. \tag{34}$$

Using Equation (24), the $L_{p,q}$-mixed volume can be written by the following integral formula:

$$V_{p,q}(K, L, Q) = \frac{1}{n} \int_{S^{n-1}} h_L(\alpha_K(u))^p h_Q(\alpha_K(u))^{q-p} h_K(\alpha_K(u))^{-q} \rho_K(u)^n du. \tag{35}$$

It will be shown that the L_p-mixed volume (Equation (8)) is the special case of the $L_{p,q}$-mixed volumes of convex bodies, i.e.,

$$\begin{aligned} V_{p,q}(K, L, K) &= V_p(K, L), \\ V_{p,p}(K, L, Q) &= V_p(K, L), \\ V_{p,q}(K, L, L) &= V_q(K, L), \\ V_{p,p}(K, L, B) &= V_p(K, L), \\ V_{0,q}(K, L, Q) &= V_q(K, Q). \end{aligned} \tag{36}$$

The Minkowski-type inequality for $L_{p,q}$-mixed volume is as follows:

Theorem 6. *Let $K, L, Q \in \mathscr{K}_o^n$ and $q \geq 1, p < 0$. Then,*

$$V_{p,q}(K, L, Q)^n \geq V(K)^{n-q} V(L)^p V(Q)^{q-p}, \tag{37}$$

with equality if and only if K, L, Q are dilates when $q > 1$ and K, Q are homothetic when $q = 1$.

For $Q, K \in \mathscr{K}_o^n$, we say that the convex body Q with respect to K has a positive continuous (p, q)-curvature function $f_{p,q}(K, Q, \cdot)$: $S^{n-1} \longrightarrow \mathbb{R}$, if

$$V_{p,q}(K, L, Q) = \frac{1}{n} \int_{S^{n-1}} h_L(u)^p f_{p,q}(K, Q, u) dS(u), \tag{38}$$

for all $L \in \mathscr{K}_o^n$. From Equations (33) and (38), we get that for $K \in \mathscr{K}_o^n$ with a positive continuous curvature functions and a fixed $Q \in \mathscr{K}_o^n$,

$$\frac{1}{n} f_{p,q}(K, Q, u) = \frac{dC_{p,q}(K, Q, u)}{dS}. \tag{39}$$

For $q \in \mathbb{R}$ and $t \in (0, \infty)$, the normalized power function $t^{\bar{q}}$ can be defined by the following:

$$t^{\bar{q}} = \begin{cases} \frac{1}{q} t^q, & \text{if } q \neq 0, \\ \log t, & \text{if } q = 0. \end{cases} \tag{40}$$

For $q \in \mathbb{R}$ and $K, Q \in \mathscr{K}_o^n$, the normalized L_p-mixed volume $V_{\bar{q}}(K, Q)$ is defined by the following:

$$V_{\bar{q}}(K, Q) = \frac{1}{n} \int_{S^{n-1}} \left(\frac{h_Q}{h_K} \right)^{\bar{q}} (\alpha_K(u)) \rho_K^n(u) du. \tag{41}$$

Note that for $q \neq 0$, we have $qV_{\bar{q}}(K, Q) = V_q(K, Q)$, while for $p=0$ the normalized L_p-mixed volume $V_{\bar{p}}(K,Q)$ is not just $V_p(K,Q)$ multiplied by a constant but it can be considered from the mixed entropy (see Section 2 for details).

Another aim of this paper is to show that for $p,q\in\mathbb{R}$ and $K,Q\in\mathscr{K}_o^n$, there exists a variational formula that defines the L_p-curvature measure $C_{p,q}(K,Q,\cdot)$ by the following:

$$\frac{d}{dt}V_{\bar{q}}\left(K,Q+_p t\cdot L\right)\bigg|_{t=0^+}=\int_{S^{n-1}}h_L(v)^{\bar{p}}\,dC_{p,q}(K,Q,v), \quad (42)$$

for every $L\in\mathscr{K}_o^n$. This plays a key role to solve the associated Minkowski-type problems using a variational method.

Associated with L_p-curvature measures, (p,q)-Minkowski problem related to L_p-curvature measure asks: For a given Borel measure ϕ on a sphere, what are the necessary and sufficient conditions for the existence of a K convex body whose L_p-curvature measure is ϕ? The uniqueness of the problem is to ask to what extent is a convex body uniquely determined by its L_p-curvature measure?

The new (p,q)-Minkowski problem is equivalent to a degenerate singular Monge-Ampère equation on S^{n-1}: For fixed $p,q\in\mathbb{R}$,

$$h^{1-p}\left\|\nu\circ\left(\bar{\nabla}h+h\ell\right)\right\|_{Q^*}^{q-p}\det\left(\bar{\nabla}^2h+hI\right)=f, \quad (43)$$

where $f:S^{n-1}\longrightarrow[0,\infty)$ is the given "data" function, $h:S^{n-1}\longrightarrow(0,\infty)$ is the unknown function, and $\ell:S^{n-1}\longrightarrow S^{n-1}$ is the identity map. Here, $\bar{\nabla}h$ and $\bar{\nabla}^2h$ denote the gradient vector and the Hessian matrix of h, respectively, with respect to an orthonormal frame on S^{n-1}, and I is the identity matrix. If we assume that the range of the gradient function $\bar{\nabla}h$ is D, then $\nu:D\longrightarrow S^{n-1}$ is also an unknown function related to h.

Finally, we propose some problems that need further study, i.e., $L_{p,q}$-affine surface area problem, $L_{p,q}$-geominimal surface area problem and $L_{p,q}$-John ellipsoid problem.

2. Preliminaries

2.1. Basics in Convex Geometry. We work in the n-dimensional Euclidean space $\mathbb{R}^n$. For $x,y\in\mathbb{R}^n$, we use $x\cdot y$ to denote the standard inner product of x and y, and $|x|=\sqrt{x\cdot x}$ to denote the Euclidean norm of x. For $x\in\mathbb{R}^n\setminus\{0\}$, we will use both $\bar{x}$ and $\langle x\rangle$ to abbreviate $x/|x|$.

We denote by $C(S^{n-1})$ the family of continuous functions defined on S^{n-1} as endowed with the topology induced by the max-norm: $\|f\|_\infty=\max_{v\in S^{n-1}}|f(v)|$, for $f\in C(S^{n-1})$.

For the support function, we know that for $\lambda>0$ and $x\in\mathbb{R}^n$,

$$h_{\lambda K}(x)=\lambda h_K(x). \quad (44)$$

Generally, for $\phi\in GL(n)$, the image $\phi K=\{\phi x: x\in K\}$ satisfies that for $x\in\mathbb{R}^n$,

$$h_{\phi K}(x)=h_K\left(\phi^t x\right), \quad (45)$$

where ϕ^t denotes the transpose of ϕ.

Since the support function is positive homogeneous of degree 1, we can restricted it on the unit sphere. For convex bodies $K,L\in\mathscr{K}^n$, their Hausdorff metric is given by the following:

$$\delta_H(K,L)\coloneqq\|h_K-h_L\|_\infty=\max_{u\in S^{n-1}}|h_K(u)-h_L(u)|. \quad (46)$$

At the point $v\in S^{n-1}$ where h_K is differentiable, the gradient of h_K in $\mathbb{R}^n$ is as follows:

$$\nabla h_K(v)=\bar{\nabla}h_K(v)+h_K(v)v, \quad (47)$$

where $\bar{\nabla}h_K$ denotes the gradient of h_K on S^{n-1} with respect to the standard metric of S^{n-1}.

For the radial function, we see that for $K\in\mathcal{S}_o^n$, $\phi\in GL(n)$ and $x\in\mathbb{R}^n\setminus\{0\}$,

$$\rho_{\phi K}(x)=\rho_K\left(\phi^{-1}x\right). \quad (48)$$

Using the radial function, the volume of $K\in\mathcal{S}_o^n$ can be expressed as follows:

$$V(K)=\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^n\,dS(u). \quad (49)$$

For $K\in\mathscr{K}_o^n$, the polar body K^* of K is defined by the following:

$$K^*=\{x\in\mathbb{R}^n: x\cdot y\le 1 \text{ for all } y\in K\}. \quad (50)$$

From this definition, we get that for $u\in S^{n-1}$,

$$\begin{aligned}\rho_K(u)&=\frac{1}{h_{K^*}(u)},\\ h_K(u)&=\frac{1}{\rho_{K^*}(u)},\end{aligned} \quad (51)$$

and for $K\in\mathscr{K}_o^n$,

$$(K^*)^*=K^{**}=K. \quad (52)$$

For $K\in\mathscr{K}_o^n$, the Minkowski function of K is defined by the following:

$$\|x\|_K=\min\{a\ge 0: x\in aK\}. \quad (53)$$

Obviously, it is a continuous function on $\mathbb{R}^n$, and

$$\|x\|_{K^*}=\rho_{K^*}(x)^{-1}=h_K(x). \quad (54)$$

In the whole process, $\Omega\subset S^{n-1}$ will represent a closed set that cannot be contained in any of the closed hemispheres of

S^{n-1}. Wulff shape $[h] \in \mathscr{K}_o^n$, a continuous function $h : \Omega \longrightarrow (0,\infty)$, also known as h of the Aleksandrov body, is defined by the following:

$$[h] = \bigcap_{v \in \Omega} \{x \in \mathbb{R}^n : x \cdot v \le h(v)\}. \tag{55}$$

If $K \in \mathscr{K}_o^n$, then it is easily seen that

$$[h_K] = K. \tag{56}$$

Assume that the function $\rho : \Omega \longrightarrow (0,\infty)$ is continuous. Since $\Omega \subset S^{n-1}$ is assumed to be closed, and ρ is continuous, we have $\{\rho(u)u : u \in \Omega\}$ is a compact set in $\mathbb{R}^n$. The convex hull $\langle \rho \rangle$ generated by ρ,

$$\langle \rho \rangle = \mathrm{conv}\{\rho(u)u : u \in \Omega\}, \tag{57}$$

is compact as well (see Schneider [40], Theorem 1.1.11). Since Ω is not contained in any closed hemisphere of S^{n-1}, we get that $\langle \rho \rangle$ contains the origin in its interior; namely, $\langle \rho \rangle \in \mathscr{K}_o^n$. Obviously, if $K \in \mathscr{K}_o^n$,

$$\langle \rho_K \rangle = K. \tag{58}$$

The following lemma will be required.

Lemma 7 (see [3]). *Let $\Omega \subset S^{n-1}$ be a closed set that is not contained in any closed hemisphere of S^{n-1}. Let $h : \Omega \longrightarrow (0,\infty)$ be continuous. Then, the Wulff shape $[h]$ determined by h and the convex hull $\langle 1/h \rangle$ generated by the function $1/h$ are polar reciprocals of each other; namely,*

$$[h]^* = \left\langle \frac{1}{h} \right\rangle. \tag{59}$$

Let $K, L \in \mathscr{K}_o^n$ and $p \ge 1$. The L_p-Minkowski combination $s \cdot K +_p t \cdot L$ is the convex body whose support function is given by the following (see [2]):

$$h(s \cdot K +_p t \cdot L, \cdot)^p = s h(K, \cdot)^p + t h(L, \cdot)^p. \tag{60}$$

From Equation (53), we can extend the L_p-Minkowski combinations to the cases of $p < 1$.

Let $p \ne 0$. For $K, L \in \mathscr{K}_o^n$, and $s, t \in \mathbb{R}$ such that $s h_K^p + t h_L^p$ is a strictly positive function on S^{n-1}, Lutwak et al. [4] defined the L_p-Minkowski combination $s \cdot K +_p t \cdot L \in \mathscr{K}_o^n$ by the following:

$$s \cdot K +_p t \cdot L = \left[\left(s h_K^p + t h_L^p \right)^{1/p} \right]. \tag{61}$$

When $p = 0$, define $s \cdot K +_0 t \cdot L$ by the following:

$$s \cdot K +_0 t \cdot L = \left[h_K^s h_L^t \right]. \tag{62}$$

Note that $s \cdot K +_0 t \cdot L$ is defined for all $s, t \in \mathbb{R}$, since h_K, h_L are strictly positive functions on S^{n-1}.

Given $\phi \in SL(n)$ and $p \ne 0$ (see [4]), we obtain that for $s, t \in \mathbb{R}$,

$$s \cdot \phi K +_p t \cdot \phi L = \phi\left(s \cdot K +_p t \cdot L\right). \tag{63}$$

If $s + t = 1$, then Equation (63) holds for $p = 0$ as well.

For $p \in \mathbb{R} \setminus \{0\}$ and $K, L \in \mathscr{K}_o^n$, the L_p-mixed volume $V_p(K, L)$ is defined by the following:

$$\begin{aligned} V_p(K, L) &= \frac{1}{n} \int_{S^{n-1}} h_L^p(u) dS_p(K, u) \\ &= \frac{p}{n} \lim_{t \longrightarrow 0^+} \frac{V\left(K +_p t \cdot L\right) - V(K)}{t}. \end{aligned} \tag{64}$$

From Equations (64) and (63), we get that for $\phi \in SL(n)$ (see [45]),

$$V_p(\phi K, \phi L) = V_p(K, L). \tag{65}$$

The L_p-surface area $S_p(K)$ of $K \in \mathscr{K}_o^n$ is given by $S_p(K) = n V_p(K, B)$.

The following definition will be required.

Definition 8 (see [4]). Let $p \in \mathbb{R}$. If μ is a Borel measure on S^{n-1} and $\phi \in SL(n)$, then $\phi_p \dashv \mu$, the L_p image of μ under ϕ, is a Borel measure such that

$$\int_{S^{n-1}} f(u) d\phi_p \dashv \mu(u) = \int_{S^{n-1}} \left|\phi^{-1} u\right|^p f\left(\left\langle \phi^{-1} u \right\rangle\right) d\mu(u), \tag{66}$$

for each Borel $f : S^{n-1} \longrightarrow \mathbb{R}$.

Recall that the L_p-mixed volume has a dual integral formulation (see [4]): If $K, L \in \mathscr{K}_o^n$, then

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) \rho_K^n(u) du, \tag{67}$$

where α_K is the radial Gauss map of K.

For $K, L \in \mathscr{K}_o^n$ and $p > 0$, we define the volume-normalized L_p-mixed volume by the following:

$$\bar{V}_p(K, L) = \left(\frac{V_p(K, L)}{V(K)} \right)^{1/p} = \left(\int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p (\alpha_K(u)) d\tilde{V}_K(u) \right)^{1/p}. \tag{68}$$

Note that $d\tilde{V}_K = (1/n)\rho_K^n du / V(K)$ is the normalized dual conical measure of K, it is a probability measure on supp $S(\cdot)$. Let $p \longrightarrow 0$. Then,

$$\begin{aligned} \bar{V}_0(K, L) &= \exp\left(\int_{S^{n-1}} \log \left(\frac{h_L}{h_K} \right) (\alpha_K(u)) d\tilde{V}_K(u) \right) \\ &= \exp\left(\frac{1}{n V(K)} \int_{S^{n-1}} \log \left(\frac{h_L}{h_K} \right) (\alpha_K(u)) \rho_K^n(u) du \right). \end{aligned} \tag{69}$$

The mixed entropy $E(K,L)$ of $K,L\in\mathscr{K}_o^n$ is defined by the following:

$$E(K,L)=\frac{1}{n}\int_{S^{n-1}}\log\left(\frac{h_L}{h_K}\right)(\alpha_K(u))\rho_K^n(u)du. \tag{70}$$

Note that $E(K,L)=V_{\bar{0}}(K,L)$. As the case in Equation (63), for the dual mixed entropy, we have that for $\phi\in SL(n)$,

$$E(\phi K,\phi L)=E(K,L). \tag{71}$$

2.2. The Radial Gauss Map. The following results come from the articles [3, 4].

Suppose K is a convex body in $\mathbb{R}^n$. For each $v\in\mathbb{R}^n\setminus\{0\}$, the hyperplane

$$H_K(v)=\{x\in\mathbb{R}^n : x\cdot v=h_K(v)\} \tag{72}$$

is called the supporting hyperplane to K with outer normal vector v.

The spherical image of $\sigma\subset\partial K$ is defined by the following:

$$\mathbf{v}_K(\sigma)=\left\{v\in S^{n-1} : x\in H_K(v)\ \text{for some}\ x\in\sigma\right\}\subset S^{n-1}. \tag{73}$$

The reverse spherical image of $\eta\subset S^{n-1}$ is defined by the following:

$$\mathbf{x}_K(\eta)=\{x\in\partial K : x\in H_K(v)\ \text{for some}\ v\in\eta\}\subset\partial K. \tag{74}$$

Suppose $\sigma_K\subset\partial K$ is a set consisting of all $x\in\partial K$, for which the set $\mathbf{v}_K(\{x\})$, which we frequently abbreviate as $\mathbf{v}_K(x)$, contains more than a single element. It is a well-known fact that $\mathscr{H}^{n-1}(\sigma_K)=0$ (see Schneider [46], p. 84). The function on the set of regular radial vectors of ∂K is precisely defined by the following:

$$\nu_K : \partial K\setminus\sigma_K\longrightarrow S^{n-1}, \tag{75}$$

by making $\nu_K(x)$ be the unique element in $\mathbf{v}_K(x)$ for each $x\in\partial K\setminus\sigma_K$, The function ν_K is called the spherical image map of K and is known to be continuous (see Schneider [40], Lemma 2.2.12). It will be very convenient to abbreviate $\partial K\setminus\sigma_K$ by $\partial' K$. Since $\mathscr{H}^{n-1}(\sigma_K)=0$, when the integration is about $\mathscr{H}^{n-1}$, it does not matter if the domain is over subsets of $\partial' K$ or ∂K.

The set $\eta_K\subset S^{n-1}$ consisting of all $v\in S^{n-1}$, for which the set $\mathbf{x}_K(v)$ contains more than a single element, is of $\mathscr{H}^{n-1}$–measure 0 (see Schneider [40], Theorem 2.2.11). The function is precisely defined on the set of regular unit normal vectors of K:

$$x_K : S^{n-1}\backslash\eta_K\longrightarrow\partial K, \tag{76}$$

by making $x_K(v)$ be the unique element in $\mathbf{x}_K(v)$, for each $v\in S^{n-1}\setminus\eta_K$. The function x_K is called the reverse spherical image map and is well known to be continuous (see Schneider [40], Lemma 2.2.12). By extending x_K to be a homogeneous function of degree 0 in $\mathbb{R}^n\setminus\{0\}$, we get a natural definition of x_K on the set of all regular normal vectors on ∂K.

For $\omega\subset S^{n-1}$, the radial Gauss image of ω is defined by the following:

$$\boldsymbol{\alpha}_K(\omega)=\left\{v\in S^{n-1} : \rho_K(u)u\in H_K(v)\ \text{for some}\ u\in\omega\right\}. \tag{77}$$

For a subset $\eta\subset S^{n-1}$, the reverse radial Gauss image of η is defined by the following:

$$\boldsymbol{\alpha}_K^*(\eta)=\left\{u\in S^{n-1} : \rho_K(u)u\in H_K(v)\ \text{for some}\ v\in\eta\right\}. \tag{78}$$

Thus,

$$\boldsymbol{\alpha}_K^*(\eta)=\{\bar{x} : x\in\partial K\ \text{where}\ x\in H_K(v)\ \text{for some}\ v\in\eta\}. \tag{79}$$

In particular, we can see that if η contains only a single vector $v\in S^{n-1}$,

$$\boldsymbol{\alpha}_K^*(v)=\{\bar{x} : x\in\partial K\ \text{where}\ x\in H_K(v)\}. \tag{80}$$

Note that Equation (78), and hence for $u\in S^{n-1}$ and $\eta\subset S^{n-1}$, we see from Equation (77) that

$$u\in\boldsymbol{\alpha}_K^*(\eta)\Leftrightarrow\alpha_K(u)\cap\eta\neq\varnothing. \tag{81}$$

Thus, for $\eta_1,\eta_2\subseteq S^{n-1}$,

$$\eta_1\subseteq\eta_2\Rightarrow\boldsymbol{\alpha}_K^*(\eta_1)\subseteq\boldsymbol{\alpha}_K^*(\eta_2). \tag{82}$$

We shall need to make use of the fact that for $u,v\in S^{n-1}$,

$$u\in\boldsymbol{\alpha}_{K^*}(v)\Leftrightarrow v\in\boldsymbol{\alpha}_K(u). \tag{83}$$

If $u\in\omega_K$, then $\boldsymbol{\alpha}_K(u)=\{\alpha_K(u)\}$, and Equation (77) becomes

$$u\in\boldsymbol{\alpha}_{K^*}(\eta)\Leftrightarrow\boldsymbol{\alpha}_K(u)\in\eta, \tag{84}$$

and hence Equation (84) holds for almost all $u\in S^{n-1}$, with respect to spherical Lebesgue measure.

The following lemma will be used.

Lemma 9 (see [4]). *If $K\in\mathscr{K}_o^n$, then*

$$\boldsymbol{\alpha}_K^*(\eta)=\boldsymbol{\alpha}_{K^*}(\eta), \tag{85}$$

for each $\eta\subseteq S^{n-1}$.

Since $\boldsymbol{\alpha}_K^*(v)=\{\alpha_K^*(v)\}$ for almost all $v\in S^{n-1}$ with respect to spherical Lebesgue measure, and $\boldsymbol{\alpha}_{K^*}(v)=\{\alpha_{K^*}(v)\}$ for almost all $v\in S^{n-1}$ with respect to spherical Lebesgue measure, Lemma 9 implies that if $K\in\mathscr{K}_o^n$, then

$$\boldsymbol{\alpha}_K^*=\boldsymbol{\alpha}_{K^*}, \tag{86}$$

almost everywhere with respect to spherical Lebesgue measure.

For $K \in \mathcal{K}_o^n$, the radial map of K is defined by the following:

$$r_K : S^{n-1} \longrightarrow \partial K \text{ by } r_K(u) = \rho_K(u)u \in \partial K, \tag{87}$$

for $u \in S^{n-1}$. Note that $r_K^{-1} : \partial K \longrightarrow S^{n-1}$ is just the restriction to ∂K of the map $\mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1}$.

The radial Gauss map of the convex body $K \in \mathcal{K}_o^n$ is defined by the following:

$$\alpha_K : S^{n-1} \setminus \omega_K \longrightarrow S^{n-1} \text{ by } \alpha_K = \nu_K \circ r_K, \tag{88}$$

where $\omega_K = \bar{\sigma_K} = r_K^{-1}(\sigma_K)$. Since $r_K^{-1} = \bar{\cdot}$ is a bi-Lipschitz map between the spaces ∂K and S^{n-1}, so it follows that ω_K has spherical Lebesgue measure 0. We observed that if $u \in S^{n-1}/\omega_K$, then $\boldsymbol{\alpha}_K(u)$ contains only the element $\alpha_K(u)$. Since both ν_K and r_K are continuous, α_K is continuous. Notice that for $x \in \partial' K$,

$$\alpha_K(\bar{x}) = \nu_K(x), \tag{89}$$

and hence for $x \in \partial' K$,

$$h_K(\alpha_K(\bar{x})) = h_K(\nu_K(x)) = x \cdot \nu_K(x). \tag{90}$$

If $u \in S^{n-1}/\omega_K$, we see that $x = \rho_K(u)u \in \partial K/\omega_K$ with $\bar{x} = u$ from the definition of ω_K. Hence from Equation (89) we have $\alpha_K(u) = \alpha_K(\bar{x}) = \nu_K(x)$ and we get the following (see [4]):

$$\alpha_K(u) = -\frac{\nabla \rho_K(u)}{|\nabla \rho_K(u)|} = \frac{\nabla h_{K^*}(u)}{|\nabla h_{K^*}(u)|}, \quad u \in S^{n-1} \setminus \omega_K. \tag{91}$$

Combining with Equations (86) and (91), we have the following:

$$\alpha_K^*(\nu) = \frac{\nabla h_K(\nu)}{|\nabla h_K(\nu)|}, \tag{92}$$

for almost all ν with respect to spherical Lebesgue measure.

The surface area measure $S(K, \cdot)$ of a convex body K can be defined, for Borel $\eta \subseteq S^{n-1}$, by the following:

$$S(K, \eta) = \mathcal{H}^{n-1}(\mathbf{x}_K(\eta)), \tag{93}$$

where $\mathbf{x}_K(\eta)$ is the reverse spherical image of $\eta \subset S^{n-1}$.

If the boundary of a convex body K, denoted by ∂K, is smooth with positive Gauss curvature, the surface area measure of K is absolutely continuous with respect to spherical Lebesgue measure. The density can be regarded as the reciprocal of Gauss curvature and expressed in terms of the support function and its Hessian matrix on S^{n-1}:

$$\frac{dS(K, \cdot)}{dS} = \det\left(\bar{\nabla}^2 h_K + h_K I\right), \tag{94}$$

where $\bar{\nabla}^2 h_K$ denotes the Hessian matrix of h_K and I is the identity matrix with respect to an orthonormal frame on S^{n-1}. See Schneider [46].

For $p \in \mathbb{R}$ and $K \in \mathcal{K}_o^n$, its L_p-surface area measure $S_p(K, \cdot)$ introduced in [2] is defined by the following:

$$dS_p(K, \cdot) = h_K^{1-p} dS(K, \cdot), \tag{95}$$

or equivalently by the following:

$$S_p(K, \eta) = \int_{\mathbf{x}_K(\eta)} (x \cdot \nu_K(x))^{1-p} d\mathcal{H}^{n-1}(x), \tag{96}$$

for each Borel $\eta \subseteq S^{n-1}$, where ν_K is the spherical image function of $\sigma \subset \partial K$.

For $\lambda > 0$, we easily see $h_{\lambda K} = \lambda h_K$ and $S(\lambda K, \cdot) = \lambda^{n-1} S(K, \cdot)$. Then, Equation (91) implies the following:

$$S_p(\lambda K, \cdot) = \lambda^{n-q} S_p(K, \cdot). \tag{97}$$

The following integral identity is established in [3].

Lemma 10. *If $q \in \mathbb{R}$ and $K \in \mathcal{K}_o^n$, while $f : S^{n-1} \longrightarrow \mathbb{R}$ is bounded and Lebesgue integrable, then*

$$\int_{S^{n-1}} f(u) \rho_K(u)^q du = \int_{\partial' K} f(\bar{x}) |x|^{q-n} (x \cdot \nu_K(x)) d\mathcal{H}^{n-1}(x). \tag{98}$$

In [3], we see that

Lemma 11. *If $K \in \mathcal{K}_o^n$ is strictly convex, and $f : S^{n-1} \longrightarrow \mathbb{R}$ and $F : \partial K \longrightarrow \mathbb{R}$ are both continuous, then*

$$\begin{aligned} &\int_{S^{n-1}} f(u) F(\nabla h_K(u)) h_K(u) dS(K, u) \\ &\quad = \int_{\partial' K} (x \cdot \nu_K(x)) f(\nu_K(x)) F(x) d\mathcal{H}^{n-1}(x), \end{aligned} \tag{99}$$

where ∇h_K is the gradient of h_K in $\mathbb{R}^n$, and ν_K is defined only on $\partial K \setminus \sigma_K$, the set σ_K has $\mathcal{H}^{n-1}$ measure 0.

We will require a slight extension of Equation (97). To be specific, if $p \in \mathbb{R}$, while $K \in \mathcal{K}_o^n$ is strictly convex, and $f : S^{n-1} \longrightarrow \mathbb{R}$ and $F : \partial K \longrightarrow \mathbb{R}$ are both continuous, then (see [4])

$$\begin{aligned} &\int_{S^{n-1}} f(u) F(\nabla h_K(u)) dS_p(K, u) \\ &\quad = \int_{\partial' K} (x \cdot \nu_K(x))^{1-p} f(\nu_K(x)) F(x) d\mathcal{H}^{n-1}(x). \end{aligned} \tag{100}$$

The following lemma will be used.

Lemma 12 (see [4]). *For each $p \in \mathbb{R}$, the set*

$$\left\{ch_K^{\bar{p}} - ch_B^{\bar{p}} : K \in \mathcal{K}_o^n, \quad c > 0\right\} \tag{101}$$

is dense in $C(S^{n-1})$.

3. L_p-Curvature Measures

For a star body $Q \in \mathcal{S}_o^n$, define $\|\cdot\|_Q : \mathbb{R}^n \longrightarrow [0, \infty)$ by letting (see [4])

$$\|x\|_Q = \begin{cases} \dfrac{1}{\rho_Q(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{102}$$

Note that $\|\cdot\|_Q$ is continuous and positively homogeneous of degree 1. If Q is an origin-symmetric convex body in $\mathbb{R}^n$, then $\|\cdot\|_Q$ is just an ordinary norm in $\mathbb{R}^n$, and $(\mathbb{R}^n, \|\cdot\|_Q)$ is the n-dimensional Banach space whose unit ball is Q.

Note that the definition (Equation (102)) is an extension of Minkowski functional (Equation (53)) of convex body $K \in \mathcal{K}_o^n$.

Definition 13. Suppose $q \in \mathbb{R}$. For $K, Q \in \mathcal{K}_o^n$, the q-th area measure $S_q(K, Q, \cdot)$ is defined by the following:

$$S_q(K, Q, \omega) = \frac{1}{n}\int_\omega \left(\frac{h_Q}{h_K}\right)^q (\alpha_K(u))\rho_K^n(u)du, \tag{103}$$

for each Lebesgue measurable $\omega \subseteq S^{n-1}$, and the q-th curvature measure $C_q(K, Q, \cdot)$ is defined by the following:

$$C_q(K, Q, \eta) = \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} \left(\frac{h_Q}{h_K}\right)^q (\alpha_K(u))\rho_K^n(u)du, \tag{104}$$

for each Borel $\eta \subseteq S^{n-1}$. Moreover, for each $p \in \mathbb{R}$, the L_p-curvature measure $C_{p,q}(K, Q, \cdot)$ is defined by the following:

$$dC_{p,q}(K, Q, \cdot) = h_Q^{-p} dC_q(K, Q, \cdot). \tag{105}$$

Observe that

$$C_{0,q}(K, Q, \cdot) = C_q(K, Q, \cdot). \tag{106}$$

Note that from definition (Equation (104)) and the fact that Equation (84) holds off of the set ω_K of spherical Lebesgue measure 0, so for each Borel $\eta \subseteq S^{n-1}$, we get the following:

$$\begin{aligned}
&\int_{S^{n-1}} 1_\eta(v)dC_q(K, Q, v) \\
&\quad = C_q(K, Q, \eta) = \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} \left(\frac{h_Q}{h_K}\right)^q (\alpha_K(v))\rho_K^n(v)dv \\
&\quad = \frac{1}{n}\int_{S^{n-1}} 1_{\boldsymbol{\alpha}_K^*(\eta)}(v)\left(\frac{h_Q}{h_K}\right)^q (\alpha_K(v))\rho_K^n(v)dv \\
&\quad = \frac{1}{n}\int_{S^{n-1}} 1_\eta(\alpha_K(u))h_Q^q(\alpha_K(u))h_K^{-q}(\alpha_K(u))\rho_K^n(u)du.
\end{aligned} \tag{107}$$

That is,

$$\begin{aligned}
&\int_{S^{n-1}} 1_\eta(v)dC_q(K, Q, v) \\
&\quad = \frac{1}{n}\int_{S^{n-1}} 1_\eta(\alpha_K(u))h_Q^q(\alpha_K(u))h_K^{-q}(\alpha_K(u))\rho_K^n(u)du.
\end{aligned} \tag{108}$$

We observed that $C_q(K, Q, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure. Then, from Equation (108), we deduce that

Lemma 14. *Let $K \in \mathcal{K}_o^n$ and $q \in \mathbb{R}$. If each function $f : S^{n-1} \longrightarrow \mathbb{R}$ is bounded and Borel, then*

$$\begin{aligned}
&\int_{S^{n-1}} f(v)dC_q(K, Q, v) \\
&\quad = \frac{1}{n}\int_{S^{n-1}} f(\alpha_K(u))h_Q^q(\alpha_K(u))h_K^{-q}(\alpha_K(u))\rho_K^n(u)du.
\end{aligned} \tag{109}$$

Proof. Because Equation (109) is shown by Equation (108) as an indicator function of the Borel set, we see that Equation (109) holds for a linear combination of the indicator functions of the Borel set, namely, simple functions $\phi : S^{n-1} \longrightarrow \mathbb{R}$, is given by the following:

$$\phi = \sum_{i=1}^m c_i 1_{\eta_i}, \tag{110}$$

where $c_i \in \mathbb{R}$ and Borel $\eta_i \subset S^{n-1}$. Now let us choose a sequence of simple functions $\phi_k : S^{n-1} \longrightarrow \mathbb{R}$ converging to the bounded Borel function $f : S^{n-1} \longrightarrow \mathbb{R}$. Note that f is bounded, ϕ_k can be selected as uniformly bounded. Then, $\phi_k \circ \alpha_K$ converges pointwise to $f \circ \alpha_K$ on $S^{n-1} \setminus \omega_K$. Since $f : S^{n-1} \longrightarrow \mathbb{R}$ is a Borel function and the radial Gauss map $\alpha_K : S^{n-1} \setminus \omega_K \longrightarrow S^{n-1}$ is continuous; thus, $f \circ \alpha_K$ is a Borel function on $S^{n-1} \setminus \omega_K$. Because f is bounded, and ω_K has spherical Lebesgue measure 0, we can infer that f is $C_q(K, q, \cdot)$ integrable, and $f \circ \alpha_K$ is spherical Lebesgue integrable in S^{n-1}. Since $C_q(K, q, \cdot)$ is a finite measure, by taking the limit $k \longrightarrow \infty$, we obtain Equation (109).□ □

Proposition 15. *Let $p, q \in \mathbb{R}$. If $K, Q \in \mathscr{K}_o^n$, then*

$$C_{p,q}(K, Q, \eta) = \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} h_Q^{q-p}(\alpha_K(u))h_K^{-q}(\alpha_K(u))\rho_K^n(u)du, \tag{111}$$

for each Borel set $\eta \subseteq S^{n-1}$.

Proof. From Equations (105), (109), and (84), we have for each Borel $\eta \subseteq S^{n-1}$,

$$\begin{aligned} &C_{p,q}(K, Q, \eta) \\ &= \int_{S^{n-1}} 1_\eta(u)dC_{p,q}(K, Q, u) \\ &= \int_{S^{n-1}} 1_\eta(u)h_Q(u)^{-p}dC_q(K, Q, u) \\ &= \frac{1}{n}\int_{S^{n-1}} 1_\eta(\alpha_K(u))h_Q^{-p}(\alpha_K(u))h_Q^q(\alpha_K(u))h_K^{-q}(\alpha_K(u))\rho_K^n(u)du \\ &= \frac{1}{n}\int_{S^{n-1}} 1_{\boldsymbol{\alpha}_K^*(\eta)}(u)h_Q^{q-p}(\alpha_K(u))h_K^{-q}(\alpha_K(u))\rho_K^n(u)du \\ &= \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} h_Q^{q-p}(\alpha_K(u))h_K^{-q}(\alpha_K(u))\rho_K^n(u)du. \end{aligned} \tag{112}$$

□ □

Obviously, the total measures of the q-th curvature measure and the q-th area measure are the q-th mixed volume, i.e.,

$$V_q(Q, K) = S_q\left(K, Q, S^{n-1}\right) = C_q\left(K, Q, S^{n-1}\right). \tag{113}$$

It follows immediately from Equations (103) and (104) that

$$C_q(K, Q, \eta) = S_q(K, Q, \boldsymbol{\alpha}_K^*(\eta)). \tag{114}$$

The L_p-curvature measures have the following properties.

Property 16. Let $p, q \in \mathbb{R}$. If $K, Q \in \mathscr{K}_o^n$. Then, for each Borel set $\eta \subseteq S^{n-1}$ and each bounded Borel function $g : S^{n-1} \longrightarrow \mathbb{R}$, we have the following:

$$\begin{aligned} &\int_{S^{n-1}} g(v)dC_{p,q}(K, Q, v) \\ &\quad = \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))h_Q^{q-p}(\alpha_K(u))h_K^{-q}(\alpha_K(u))\rho_K^n(u)du, \end{aligned} \tag{115}$$

$$\begin{aligned} &\int_{S^{n-1}} g(v)dC_{p,q}(K, Q, v) \\ &\quad = \frac{1}{n}\int_{\partial' K} g(\nu_K(x))(x \cdot \nu_K(x))^{1-q}\|\nu_K(x)\|_{Q^*}^{q-p}d\mathscr{H}^{n-1}(x), \end{aligned} \tag{116}$$

$$C_{p,q}(K, Q, \eta) = \frac{1}{n}\int_{x \in \mathbf{x}_K(\eta)} (x \cdot \nu_K(x))^{1-q}\|\nu_K(x)\|_{Q^*}^{q-p}d\mathscr{H}^{n-1}(x). \tag{117}$$

Proof. Because $h_Q^{-p} : S^{n-1} \longrightarrow \mathbb{R}$ is a bounded Borel function, from Equation (109) with $f = gh_Q^{-p}$, we have the following:

$$\begin{aligned} &\int_{S^{n-1}} g(v)h_Q^{-p}(v)dC_q(K, Q, v) \\ &\quad = \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))h_Q^{q-p}(\alpha_K(u))h_K^{-q}(\alpha_K(u))\rho_K^n(u)du. \end{aligned} \tag{118}$$

Thus, in light of Equation (105) is the desired result (Equation (115)).

By Equations (115), (89), and (90), and letting $f = (g \circ \alpha_K)(h_Q^{q-p} \circ \alpha_K)(h_K^{-q} \circ \alpha_K)$ and $q = n$ in Equation (98), we have the following:

$$\begin{aligned} &\int_{S^{n-1}} g(v)dC_{p,q}(K, Q, v) \\ &= \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))h_Q^{q-p}(\alpha_K(u))h_K^{-q}(\alpha_K(u))\rho_K^n(u)du \\ &= \frac{1}{n}\int_{\partial' K} g(\alpha_K(\bar{x}))h_Q^{q-p}(\alpha_K(\bar{x}))h_K^{-q}(\alpha_K(\bar{x}))(x \cdot \nu_K(x))d\mathscr{H}^{n-1}(x) \\ &= \frac{1}{n}\int_{\partial' K} g(\nu_K(x))(x \cdot \nu_K(x))^{1-q}h_Q^{q-p}(\nu_K(x))d\mathscr{H}^{n-1}(x) \\ &= \frac{1}{n}\int_{\partial' K} g(\nu_K(x))(x \cdot \nu_K(x))^{1-q}\|\nu_K(x)\|_{Q^*}^{q-p}d\mathscr{H}^{n-1}(x). \end{aligned} \tag{119}$$

□ □

This yields Equation (116).

Take $g = 1_\eta$ in Equation (116). Notice that $\nu_K(x) \in \eta \Leftrightarrow x \in \mathbf{x}_K(\eta)$ for almost all x with respect to spherical Lebesgue measure. So, we immediately obtain Equation (117).

Remark 17. Equation (115) tells us the rationality for Definition 1 of the L_p-curvature measure $C_{p,q}(K, Q, \cdot)$.

Example 18 (L_p-curvature measures of polytopes). Suppose $P \in \mathscr{K}_o^n$ be a polytope with outer unit normal vectors $v_1, v_2, \cdots, v_m$. If Δ_i is a cone consisting of all rays emanating from the origin and passing through the face of P whose outer normal is v_i. Remember that we abbreviate $\boldsymbol{\alpha}_P^*(\{v_i\})$ by $\boldsymbol{\alpha}_P^*(v_i)$, and from Equation (80), we get the following:

$$\boldsymbol{\alpha}_P^*(v_i) = S^{n-1} \cap \Delta_i, \alpha_P(u) = v_i, \text{for almost all } u \in \Delta_i \cap S^{n-1}. \tag{120}$$

If $\eta \subset S^{n-1}$ is a Borel set such that $\{v_1, v_2, \cdots, v_m\} \cap \eta = \varnothing$, then $\boldsymbol{\alpha}_P^*(\eta)$ has spherical Lebesgue measure 0. So, the L_p-curvature measure $C_{p,q}(P, Q, \cdot)$ is discrete and concentrated on $\{v_1, v_2, \cdots, v_m\}$. From Proposition 15 and Equation (120), we have the following:

$$C_{p,q}(P, Q \cdot) = \sum_{i=1}^{m} d_i \delta_{v_i}, \tag{121}$$

where δ_{v_i} represents the delta measure centered on v_i, and

$$d_i = \frac{1}{n} h_Q^{q-p}(v_i) h_P^{-q}(v_i) \int_{S^{n-1} \cap \Delta_i} \rho_P^n(u) du. \tag{122}$$

Example 19 (L_p-curvature measures of strictly convex bodies). Let $K, Q \in \mathscr{K}_o^n$ are strictly convex. Suppose $g : S^{n-1} \longrightarrow \mathbb{R}$ is continuous, then we start with Equations (116) and (100)(taking $F(x) = \|\nu_K(x)\|_{Q^*}^{q-p}$) and combine the fact that $\partial K / \partial' K$ has measure 0, it follows that

$$\begin{aligned} &\int_{S^{n-1}} g(v) dC_{p,q}(K, Q, v) \\ &\quad = \frac{1}{n} \int_{\partial' K} (x \cdot \nu_K(x))^{1-q} g(\nu_K(x)) \|\nu_K(x)\|_{Q*}^{q-p} d\mathscr{H}^{n-1}(x) \\ &\quad = \frac{1}{n} \int_{S^{n-1}} g(v) \|\nu_K(\nabla h_K(v))\|_{Q^*}^{q-p} dS_q(K, v). \end{aligned} \tag{123}$$

Using Equation (95), this shows that

$$\begin{aligned} dC_{p,q}(K, Q, \cdot) &= \frac{1}{n} \|\nu_K(\nabla h_K)\|_{Q^*}^{q-p} dS_q(K, \cdot) \\ &= \frac{1}{n} h_K^{1-q} \|\nu_K \circ \nabla h_K\|_{Q^*}^{q-p} dS(K, \cdot). \end{aligned} \tag{124}$$

Example 20 (L_p-curvature measures of smooth convex bodies). Let $Q \in \mathscr{K}_o^n$ has a C^2 boundary with everywhere positive Gauss curvature. Because in this case, $S(Q, \cdot)$ is absolutely continuous for the spherical Lebesgue measure; therefore, $C_{p,q}(K, Q, \cdot)$ is absolutely continuous for the spherical Lebesgue measure, and from Equations (124), (94), and (47), we get the following:

$$\begin{aligned} \frac{dC_{p,q}(K, Q, v)}{dv} &= \frac{1}{n} h_K^{1-q}(v) \left\| \nu_K\left(\bar{\nabla} h_K(v) + h_K(v) v\right) \right\|_{Q^*}^{q-p} \\ &\quad \cdot \det\left(\bar{\nabla}^2 h_K(v) + h_K(v) I\right), \end{aligned} \tag{125}$$

where $\bar{\nabla} h_K(v)$ represents the gradient of h_K on S^{n-1} at v and $\bar{\nabla}^2 h_K$ represents the Hessian matrix of h_K with respect to an orthonormal frame on S^{n-1}. We write Equation (125) as $1/nf_{p,q}(K, Q, v)$, that is,

$$\frac{1}{n} f_{p,q}(K, Q, v) = \frac{dC_{p,q}(K, Q, v)}{dv}. \tag{126}$$

We say convex body Q with respect to a fixed convex body K as a parameter have a positive continuous (p, q)-curvature function $f_{p,q}(K, Q, \cdot)$.

The weak convergence of L_p-curvature measure is an important property contained in the following propositions.

Proposition 21. *Let $p, q \in \mathbb{R}$ and $Q \in \mathscr{K}_o^n$. If $K_i \in \mathscr{K}_o^n$ with $K_i \longrightarrow K_0 \in \mathscr{K}_o^n$, then $C_{p,q}(K_i, Q, \cdot) \longrightarrow C_{p,q}(K_0, Q, \cdot)$, weakly.*

Proof. Let $g : S^{n-1} \longrightarrow \mathbb{R}$ is continuous. From Equation (115) we know that

$$\begin{aligned} &\int_{S^{n-1}} g(v) dC_{p,q}(K_i, Q, v) \\ &\quad = \frac{1}{n} \int_{S^{n-1}} g\left(\alpha_{K_i}(u)\right) h_Q^{q-p}\left(\alpha_{K_i}(u)\right) h_{K_i}^{-q}\left(\alpha_{K_i}(u)\right) \rho_{K_i}^n(u) du, \end{aligned} \tag{127}$$

for all i. Since $K_i \longrightarrow K_0$, with respect to the Hausdorff metric, we have that $h_{K_i} \longrightarrow h_{K_0}$, uniformly on S^{n-1}, and the surface area measure has the following property (see [2, 7, 23]):

$$K_i \longrightarrow K_0 \Rightarrow S_{K_i} \longrightarrow S_{K_0} \text{ weakly}. \tag{128}$$

Thus,

$$\begin{aligned} &\int_{S^{n-1}} g\left(\alpha_{K_i}(u)\right) h_Q^{q-p}\left(\alpha_{K_i}(u)\right) h_{K_i}^{-q}\left(\alpha_{K_i}(u)\right) \rho_{K_i}^n(u) du, \\ &\quad \longrightarrow \int_{S^{n-1}} g\left(\alpha_{K_0}(u)\right) h_Q^{q-p}\left(\alpha_{K_0}(u)\right) h_{K_0}^{-q}\left(\alpha_{K_0}(u)\right) \rho_{K_0}^n(u) du. \end{aligned} \tag{129}$$

It follows that $C_{p,q}(K_i, Q, \cdot) \longrightarrow C_{p,q}(K_0, Q, \cdot)$, weakly. □

□

The following statement contains the absolute continuity of L_p-curvature measure with respect to surface area measure.

Proposition 22. *Let $p, q \in \mathbb{R}$. If $K, Q \in \mathscr{K}_o^n$, then L_p-curvature measure $C_{p,q}(K, Q, \cdot)$ is absolutely continuous with respect to the surface area measure $S(K, \cdot)$.*

Proof. Let $\eta \subset S^{n-1}$ be such that $S(K, \eta) = 0$, or equivalently by definition (Equation (96)), $\mathscr{H}^{n-1}(\mathbf{x}_K(\eta)) = 0$. Then, Equation (117) states that

$$C_{p,q}(K,Q,\eta)=\frac{1}{n}\int_{x\in\mathbf{x}_K(\eta)}(x\cdot\nu_K(x))^{1-q}\cdot\|\nu_K(x)\|_{Q*}^{q-p}d\mathscr{H}^{n-1}(x)=0. \tag{130}$$

Thus, the integration is over a set of measure 0.□ □

The following proposition shows that the L_p-curvature measure including the classical surface area measures and the L_p-surface area measures. Therefore, the classical surface area measures and the L_p-surface area measures are special cases of the L_p-curvature measures.

Proposition 23. *Suppose $K,Q\in\mathscr{K}_o^n$ and $p,q\in\mathbb{R}$. Then,*

$$C_{q,q}(K,Q,\cdot)=\frac{1}{n}S_q(K,\cdot), \tag{131}$$

$$C_{p,q}(K,K,\cdot)=\frac{1}{n}S_p(K,\cdot), \tag{132}$$

$$C_{p,q}(K,B,\cdot)=\frac{1}{n}S_q(K,\cdot), \tag{133}$$

$$C_{p,0}(K,K,\cdot)=\frac{1}{n}S_p(K,\cdot), \tag{134}$$

$$C_{0,q}(K,B,\cdot)=\frac{1}{n}S_q(K,\cdot). \tag{135}$$

Proof. Let $\eta\subset S^{n-1}$ be a Borel set. From Equations (117) and (96), we have the following:

$$\begin{aligned}C_{q,q}(K,Q,\eta)&=\int_{x\in\mathbf{x}_K(\eta)}(x\cdot\nu_K(x))^{1-q}d\mathscr{H}^{n-1}(x)\\&=C_{p,q}(K,B,\eta)=\frac{1}{n}S_q(K,\eta).\end{aligned} \tag{136}$$

Therefore, we get Equations (131) and (133).

From Equations (117), (54), (90), and (96), we have the following:

$$\begin{aligned}C_{p,q}(K,K,\eta)&=\frac{1}{n}\int_{x\in\mathbf{x}_K(\eta)}(x\cdot\nu_K(x))^{1-q}\|\nu_K(x)\|_{K^*}^{q-p}d\mathscr{H}^{n-1}(x)\\&=\frac{1}{n}\int_{x\in\mathbf{x}_K(\eta)}(x\cdot\nu_K(x))^{1-q}h_K(\nu_K(x))^{q-p}d\mathscr{H}^{n-1}(x)\\&=\frac{1}{n}\int_{x\in\mathbf{x}_K(\eta)}(x\cdot\nu_K(x))^{1-p}d\mathscr{H}^{n-1}(x)\\&=\frac{1}{n}S_p(K,\eta).\end{aligned} \tag{137}$$

Therefore, we get Equation (132). Similarly, we can get the rest.□ □

Recall that the concept of the valuation. A function Φ defined on the space $\mathscr{K}^n$ of convex bodies and taking values in an abelian semigroup is called a valuation if

$$\Phi(K\cup L)+\Phi(K\cap L)=\Phi K+\Phi L, \tag{138}$$

whenever $K,L,K\cap L,K\cup L\in\mathscr{K}^n$.

The set of Borel measures on S^{n-1} is represented by $\mathscr{M}(S^{n-1})$. We are going to prove that now, for fixed indices $p,q\in\mathbb{R}$, and a fixed convex body $K\in\mathscr{K}_o^n$, the functional $\mathscr{K}_o^n\longrightarrow\mathscr{M}(S^{n-1})$, defined by $Q\mapsto C_{p,q}(K,Q,\cdot)$ is a valuation; namely, if $K,L\in\mathscr{K}_o^n$, are such that $K\cup L\in\mathscr{K}_o^n$ then

$$\begin{aligned}&C_{p,q}(K\cup L,Q,\cdot)+C_{p,q}(K\cap L,Q,\cdot)\\&=C_{p,q}(K,Q,\cdot)+C_{p,q}(L,Q,\cdot).\end{aligned} \tag{139}$$

To prove the valuation of L_p-curvature measure, we shall employ Weil's approximation lemma (see [4]):

Lemma 24. *If $K,L\in\mathscr{K}_o^n$ are such that $K\cup L$ is convex, then K and L may be approximated by sequences of bodies $K_i,L_i\in\mathscr{K}_o^n$ that are both strictly convex and smooth and such that $K_i\cup L_i\in\mathscr{K}_o^n$.*

We appeal to Proposition 21 together with Weil's approximation lemma in order to complete our proof.

Theorem 25. *Suppose $p,q\in\mathbb{R}$ and $Q\in\mathscr{K}_o^n$. Then, the functional*

$$C_{p,q}(\cdot,Q,\cdot):\mathscr{K}_o^n\longrightarrow\mathscr{M}(S^{n-1}), \tag{140}$$

defined by $K\mapsto C_{p,q}(K,Q,\cdot)$, is a valuation.

Proof. We will use the fact that if $K,L\in\mathscr{K}_o^n$ are such that $K\cup L\in\mathscr{K}_o^n$, then $h_{K\cup L}=\max\{h_K,h_L\}$ and $h_{K\cap L}=\min\{h_K,h_L\}$. We will also take advantage of the fact that ν_K and ν_L are defined $\mathscr{H}^{n-1}$ almost everywhere on the boundaries of K and L, respectively.

First of all, let us assume that K and L are both strictly convex. For a fixed $\theta\subset S^{n-1}$, write θ as the union of three disjoint pieces $\theta=\theta_0\cup\theta_K\cup\theta_L$, where

$$\theta_K=\{u\in\theta: h_K(u)>h_L(u),\theta_L=\{u\in\theta: h_K(u)<h_L(u)\}, \tag{141}$$

while

$$\theta_0=\{u\in\theta: h_K(u)=h_L(u). \tag{142}$$

In this case, we have the following:

$$\begin{aligned}&\int_{x\in\mathbf{x}_{K\cup L}(\theta_K)}(x\cdot\nu_{K\cup L}(x))^{1-q}h_Q^{q-p}(\nu_{K\cup L}(x))d\mathscr{H}^{n-1}(x)\\&=\int_{x\in\mathbf{x}_K(\theta_K)}(x\cdot\nu_K(x))^{1-q}h_Q^{q-p}(\nu_K(x))d\mathscr{H}^{n-1}(x),\end{aligned} \tag{143}$$

while

$$\int_{x\in\mathbf{x}_{K\cap L}(\theta_K)}(x\cdot\nu_{K\cap L}(x))^{1-q}h_Q^{q-p}(\nu_{K\cap L}(x))d\mathscr{H}^{n-1}(x) = \int_{x\in\mathbf{x}_L(\theta_K)}(x\cdot\nu_L(x))^{1-q}h_Q^{q-p}(\nu_L(x))d\mathscr{H}^{n-1}(x). \quad (144)$$

Alternatively, using Equation (117), this has

$$C_{p,q}(K\cup L,Q,\theta_K)=C_{p,q}(K,Q,\theta_K), C_{p,q}(K\cap L,\theta_K) = C_{p,q}(L,Q,\theta_K). \quad (145)$$

Similarly,

$$C_{p,q}(K\cup L,Q,\theta_L)=C_{p,q}(L,Q,\theta_L), C_{p,q}(K\cap L,Q,\theta_L) = C_{p,q}(K,Q,\theta_L). \quad (146)$$

It is also the case that

$$C_{p,q}(K\cup L,Q,\theta_0)=C_{p,q}(K,Q,\theta_0), C_{p,q}(K\cap L,Q,\theta_0) = C_{p,q}(L,Q,\theta_0). \quad (147)$$

In order to see the fact that the last one, we observe that the strict convexity of K and L forces $\mathbf{x}_{K\cup L}(\theta_0)=\mathbf{x}_{K\cap L}(\theta_0)$.

Using the fact that $C_{p,q}(K,\cdot,\cdot)$ is a measure in the third argument on S^{n-1}, combined with the fact that the union $\theta=\theta_0\cup\theta_K\cup\theta_L$ is disjoint, by adding Equations (145), (146), and (147) we obtain that

$$C_{p,q}(K\cup L,Q,\theta)+C_{p,q}(K\cap L,Q,\theta) = C_{p,q}(K,Q,\theta)+C_{p,q}(L,Q,\theta), \quad (148)$$

which is the desired result.

For any $K,L\in\mathscr{K}_o^n$, we resort to Proposition 21 in order to use the weak continuity of $C_{p,q}(\cdot,Q,\cdot)$ in the first argument.□ □

4. Variational Formulas for $\mathbf{L_{p,q}}$-Mixed Volumes

Suppose Ω is a closed subset of S^{n-1} that is not contained in any closed hemisphere. Let $h_0:\Omega\longrightarrow(0,\infty)$ and $f:\Omega\longrightarrow\mathbb{R}$ be consecutive, and $\delta>0$. Let $h_t:\Omega\longrightarrow(0,\infty)$ be a positive continuous function defined as follows:

$$\log h_t(v)=\log h_0(v)+tf(v)+o(t,v), \quad (149)$$

for each $t\in(-\delta,\delta)$, where $o(t,\cdot)$: $\Omega\longrightarrow\mathbb{R}$ is continuous and $\lim_{t\longrightarrow 0} o(t,\cdot)/t=0$, uniformly on Ω. And denote by

$$[h_t]=\{x\in\mathbb{R}^n : x\cdot v\le h_t(v) \text{ for all } v\in\Omega\}, \quad (150)$$

Wulff shape determined by h_t. We call $[h_t]$ the logarithmic Wulff shape family generated by (h_0,f). If h_0 is the support function h_K of convex body K, we also put $[h_t]$ written $[K,f,t]$.

Let $\rho_0:\Omega\longrightarrow(0,\infty)$ and $g:\Omega\longrightarrow\mathbb{R}$ be continuous, and $\delta>0$. Let $\rho_t:\Omega\longrightarrow(0,\infty)$ be a positive continuous function defined by the following:

$$\log\rho_t(u)=\log\rho_0(u)+tg(u)+o(t,u), \quad (151)$$

for each $t\in(-\delta,\delta)$, where again $o(t,\cdot)$: $\Omega\longrightarrow\mathbb{R}$ is continuous and $\lim_{t\longrightarrow 0} o(t,\cdot)/t=0$ uniformly on Ω. And denote by

$$\langle\rho_t\rangle=\mathrm{conv}\{\rho_t(u)u : u\in S^{n-1}\}, \quad (152)$$

the convex hull generated by ρ_t. We call $\langle\rho_t\rangle$ the logarithmic family of convex hull generated by (ρ_0,g). If ρ_0 is the radial function ρ_K of convex body K, we also put $\langle\rho_t\rangle$ as $\langle K,g,t\rangle$.

The following lemma shows that the support functions of a logarithmic family of the polar of convex hulls are differentiable with respect to the variational variable.

Lemma 26. *Suppose $\Omega\subset S^{n-1}$ be a closed set that is not contained in any closed hemisphere of S^{n-1}. Let $\rho_0:\Omega\longrightarrow(0,\infty)$ and $g:\Omega\longrightarrow\mathbb{R}$ be continuous. If $\langle\rho_t\rangle$ is a logarithmic family of convex hulls of (ρ_0,g) and $q\in\mathbb{R}$, then*

$$\lim_{t\longrightarrow 0}\frac{h^q_{\langle\rho_t\rangle^*}(v)-h^q_{\langle\rho_0\rangle^*}(v)}{t}=-q\rho_0^{-q}(v)g(v), \quad (153)$$

for all $v\in S^{n-1}/\eta_{\langle\rho_0\rangle^*}$; namely, for all regular normals v of $\langle\rho_0\rangle^*$, where Equation (153) holds a.e. with respect to spherical Lebesgue measure. Moreover, there exist $\delta>0$ and $M>0$ so that

$$\left|\log h^q_{\langle\rho_t\rangle^*}(v)-\log h^q_{\langle\rho_0\rangle^*}(v)\right|\le M|t|, \quad (154)$$

for all $v\in S^{n-1}$ and all $t\in(-\delta,\delta)$.

Proof. Obviously,

$$\lim_{t\longrightarrow 0}\frac{\rho_t^{-q}(v)-\rho_0^{-q}(v)}{t} = -p\rho_0^{-q}(v)\lim_{t\longrightarrow 0}\frac{\log\rho_t(v)-\log\rho_0(v)}{t} = -q\rho_0^{-q}(v)g(v). \quad (155)$$

Therefore,

$$\lim_{t\longrightarrow 0}\frac{h^q_{\langle\rho_t\rangle^*}(v)-h^q_{\langle\rho_0\rangle^*}(v)}{t}=\lim_{t\longrightarrow 0}\frac{\rho_t^{-q}(v)-\rho_0^{-q}(v)}{t} = -q\rho_0^{-q}(v)g(v). \quad (156)$$

Since $\langle\rho_0\rangle$ and $\langle\rho_0\rangle^*$ are two convex bodies in $\mathscr{K}_o^n$, and $\langle\rho_t\rangle^*\longrightarrow\langle\rho_0\rangle^*$ as $t\longrightarrow 0$, there exist $m_0,m_1\in(0,\infty)$ and

$\delta_0 > 0$ such that

$$0 < m_0 < h_{\langle\rho_t\rangle^*} < m_1 \; on \; S^{n-1}, \tag{157}$$

for each $t \in (-\delta_0, \delta_0)$. From this, it follows that there exists $M_1 > 1$ so that

$$0 < \frac{h^{-q}_{\langle\rho_t\rangle^*}}{h^{-q}_{\langle\rho_0\rangle^*}} < M_1 \text{ on } S^{n-1}. \tag{158}$$

It is easily seen that $s - 1 \geq \log s$ whenever $s \in (0, 1)$, whereas $s - 1 \leq M_1 \log s$ whenever $s \in [1, M_1]$. Thus,

$$|s-1| \leq M_1 |\log s| \text{when}\, s \in (0, M_1). \tag{159}$$

It follows that

$$\left|\frac{h^{-q}_{\langle\rho_t\rangle^*}}{h^{-q}_{\langle\rho_0\rangle^*}} - 1\right| \leq M_1 \left|\log \frac{h^{-q}_{\langle\rho_t\rangle^*}}{h^{-q}_{\langle\rho_0\rangle^*}}\right| \text{when}\, s \in (0, M_1), \tag{160}$$

that is

$$\begin{aligned}\left|h^{-q}_{\langle\rho_t\rangle^*} - h^{-q}_{\langle\rho_0\rangle^*}\right| &\leq h^{-q}_{\langle\rho_0\rangle^*} M_1 \left|\log h_{\langle\rho_t\rangle^*} - \log h_{\langle\rho_0\rangle^*}\right| \\ &\leq \frac{M_1}{\min\left\{m_0^q, m_1^q\right\}} \left|\log h_{\langle\rho_t\rangle^*} - \log h_{\langle\rho_0\rangle^*}\right| \\ &= \frac{M_1}{\min\left\{m_0^q, m_1^q\right\}} |\log \rho_t - \log \rho_0|\end{aligned} \tag{161}$$

on S^{n-1}, whenever $t \in (-\delta_0, \delta_0)$.

Let $M_0 = \max_{u\in\Omega} |g(u)|$. Since $o(t, \cdot)/t \longrightarrow 0$ as $t \longrightarrow 0$ uniformly on Ω, we may choose $\delta_1 > 0$ so that for all $t \in (-\delta_1, \delta_1)$, we have $|o(t, \cdot)| \leq |t|$ on Ω. From Equation (151) and the definition of M_0, we immediately see that

$$|\log \rho_t - \log \rho_0| \leq (M_0 + 1)|t| \tag{162}$$

on S^{n-1}, whenever $t \in (-\delta_1, \delta_1)$. Let $(-\delta, \delta) = (-\delta_0, \delta_0) \cap (-\delta_1, \delta_1)$. Together with Equations (161) and (162), we give Equation (154).□ □

The following theorem gives variational formulas for the L_p-mixed volume and L_p-mixed entropy for a family of logarithmic convex hulls.

Theorem 27. *Let $\Omega \subset S^{n-1}$ is a closed set not contained in any closed hemisphere of S^{n-1}. If $\rho_0 : \Omega \longrightarrow (0,\infty)$ and $g : \Omega \longrightarrow \mathbb{R}$ are continuous, and $\langle\rho_t\rangle$ is a logarithmic family of convex hulls of (ρ_0, g), then for $K \in \mathcal{K}^n_o$ and $q \neq 0$,*

$$\lim_{t\longrightarrow 0} \frac{V_q(K, \langle\rho_t\rangle^*) - V_q(K, \langle\rho_0\rangle^*)}{t} = -q\int_\Omega g(u) dC_q(K, \langle\rho_0\rangle^*, u), \tag{163}$$

for $q = 0$,

$$\begin{aligned}&\lim_{t\longrightarrow 0} \frac{E(K, \langle\rho_t\rangle^*) - E(K, \langle\rho_0\rangle^*)}{t} = -\int_\Omega g(u) dC_0(K, \langle\rho_0\rangle^*, u), \\ &\lim_{t\longrightarrow 0} \frac{\log \bar{V}_0(K, \langle\rho_t\rangle^*) - \log \bar{V}_0(K, \langle\rho_0\rangle^*)}{t} \\ &\qquad = -\frac{1}{V(K)} \int_\Omega g(u) dC_0(K, \langle\rho_0\rangle^*, u).\end{aligned} \tag{164}$$

Proof. Abbreviate η_{K^*} by η_0. Recall that η_0 is the set of spherical Lebesgue measure zero that consists of the complement, in S^{n-1}, of the regular normal vectors of the convex body K^*. Note that the continuous function

$$\alpha^*_{K^*} : S^{n-1} \setminus \eta_0 \longrightarrow S^{n-1} \tag{165}$$

is well defined by $\alpha^*_{K^*}(v) \in \boldsymbol{\alpha}^*_{K^*}(v) = \{\alpha^*_{K^*}(v)\}$ for all $v \in S^{n-1} \setminus \eta_0$.

Let $v \in S^{n-1}/\eta_0$. To see that $\boldsymbol{\alpha}^*_{K^*}(v) \subset \Omega$, let

$$h_{K^*}(v) = \max_{u\in\Omega} \rho_{K^*}(u) u \cdot v = \rho_{K^*}(u_0) u_0 \cdot v, \tag{166}$$

for some $u_0 \in \Omega$. This means that

$$\rho_{K^*}(u_0) u_0 \in H_{K^*}(v), \tag{167}$$

and hence $\rho_{K^*}(u_0) u_0 \in \partial K^*$. Because in addition to $\rho_{K^*}(u_0) u_0$ obviously belonging to K^*, it also belongs to $H_{K^*}(v)$. But v is a regular normal vector of K^*, and therefore, $\alpha^*_{K^*}(v) = u_0 \in \Omega$. Then,

$$\alpha^*_{K^*}\left(S^{n-1} \setminus \eta_0\right) \subset \Omega. \tag{168}$$

From this, Equation (168), Equation (52), and Lemma 9 yield the following facts:

$$\alpha_K\left(S^{n-1} \setminus \eta_0\right) = \alpha^*_{K^*}\left(S^{n-1} \setminus \eta_0\right) \subset \Omega. \tag{169}$$

As Ω is closed, by using the Tietze extension theorem, extend the continuous function $g : \Omega \longrightarrow \mathbb{R}$ to a continuous function $\widehat{g} : S^{n-1} \longrightarrow \mathbb{R}$. Therefore, using Equation (169) we see that

$$g(\alpha_K(v)) = (\widehat{g} 1_K)(\alpha_K(v)), \tag{170}$$

for $v \in S^{n-1} \setminus \eta_0$.

Using Equation (22), the fact that η_0 has measure zero, Equation (51), Equation (154), the dominated convergence theorem, Lemma 26, Equation (86), Equation (170), Lemma

14, and again Equation (170), we have the following:

$$\begin{aligned}
&\lim_{t\longrightarrow 0}\frac{V_q(K,\langle\rho_t\rangle^*)-V_q(K,\langle\rho_0\rangle^*)}{t}\\
&=\lim_{t\longrightarrow 0}\frac{1}{n}\int_{S^{n-1}}\frac{\left[h^q_{\langle\rho_t\rangle^*}(\alpha_K(v))-h^q_{\langle\rho_0\rangle^*}(\alpha_K(v))\right]h_K^{-q}(\alpha_K(v))\rho_K^n(v)dv}{t}\\
&=\lim_{t\longrightarrow 0}\frac{1}{n}\int_{S^{n-1}/\eta_0}\frac{\left[h^q_{\langle\rho_t\rangle^*}(\alpha_K(v))-h^q_{\langle\rho_0\rangle^*}(\alpha_K(v))\right]h_K^{-q}(\alpha_K(v))\rho_K^n(v)\mathrm{d}v}{t}\\
&=-\frac{q}{n}\int_{S^{n-1}/\eta_0}g(\alpha_K(v))h^q_{\langle\rho_{00}\rangle^*}(\alpha_K(v))h_K^{-q}(\alpha_K(v))\rho_K^n(v)\mathrm{d}v\\
&=-\frac{q}{n}\int_{S^{n-1}}(\widehat{g}1_\Omega)(\alpha_K(v))h^q_{\langle\rho_0\rangle^*}(\alpha_K(v))h_K^{-q}(\alpha_K(v))\rho_K^n(v)dv\\
&=-q\int_{S^{n-1}}(\widehat{g}1_\Omega)(u)dC_q(K,\langle\rho_0\rangle^*,u)\\
&=-q\int_\Omega g(u)dC_q(K,\langle\rho_0\rangle^*,u).
\end{aligned}\tag{171}$$

□ □

According to Equations (70) and (51), the fact that η_0 has measure zero, the dominated convergence theorem, Equation (151), together with Equations (170) and (86), Lemma 14, and again Equation (170), we have the following:

$$\begin{aligned}
&\lim_{t\longrightarrow 0}\frac{E(K,\langle\rho_t\rangle^*)-E(K,\langle\rho_0\rangle^*)}{t}\\
&=\lim_{t\longrightarrow 0}\frac{1}{n}\int_{S^{n-1}}\frac{\log\left(\left(h_{\langle\rho_t\rangle^*}/h_K\right)(\alpha_K(v))\right)-\log\left(\left(h_{\langle\rho_0\rangle^*}/h_K\right)(\alpha_K(v))\right)}{t}\rho_K^n(v)dv\\
&=\lim_{t\longrightarrow 0}\frac{1}{n}\int_{S^{n-1}}\frac{\log h_{\langle\rho_t\rangle^*}(\alpha_K(v))-\log h_{\langle\rho_0\rangle^*}(\alpha_K(v))}{t}\rho_K^n(v)dv\\
&=-\lim_{t\longrightarrow 0}\frac{1}{n}\int_{S^{n-1}}\frac{\log\rho_t(\alpha_K(v))-\log\rho_0(\alpha_K(v))}{t}\rho_K^n(v)dv\\
&=-\lim_{t\longrightarrow 0}\frac{1}{n}\int_{S^{n-1}/\eta_0}\frac{\log\rho_t(\alpha_K(v))-\log\rho_0(\alpha_K(v))}{t}\rho_K^n(v)dv\\
&=-\frac{1}{n}\int_{S^{n-1}/\eta_0}g(\alpha_K(v))\rho_K^n(v)dv=-\frac{1}{n}\int_{S^{n-1}}(\widehat{g}1_\Omega)(\alpha_K(v))\rho_K^n(v)dv\\
&=-\int_{S^{n-1}}(\widehat{g}1_\Omega)(u)dC_0(K,\langle\rho_0\rangle^*,u)=-\int_\Omega g(u)dC_0(K,\langle\rho_0\rangle^*,u).
\end{aligned}\tag{172}$$

Using the same argument as in the second part of the proof, we get that

$$\begin{aligned}
&\lim_{t\longrightarrow 0}\frac{\log\bar{V}_0(K,\langle\rho_t\rangle^*)-\log\bar{V}_0(K,\langle\rho_0\rangle^*)}{t}\\
&=\frac{1}{nV(K)}\lim_{t\longrightarrow 0}\int_{S^{n-1}}\frac{\log h_{\langle\rho_t\rangle^*}(\alpha_K(v))-\log h_{\langle\rho_0\rangle^*}(\alpha_K(v))}{t}\\
&\quad\cdot\rho_K^n(v)dv=-\frac{1}{V(K)}\int_\Omega g(u)dC_0(K,\langle\rho_0\rangle^*,u).
\end{aligned}\tag{173}$$

The following theorem gives the variational formulas for the L_p-mixed volumes and mixed entropy of the logarithmic family of Wulff shapes.

Theorem 28. *Suppose $\Omega\subset S^{n-1}$ is a closed set not contained in any closed hemisphere of S^{n-1}. Let $h_0:\Omega\longrightarrow(0,\infty)$ and $f:\Omega\longrightarrow\mathbb{R}$ be continuous, and $[h_t]$ be a logarithmic family of Wulff shapes associated with (h_0,f). If $K\in\mathscr{K}_o^n$, then for $q\neq 0$,*

$$\lim_{t\longrightarrow 0}\frac{V_q(K,[h_t])-V_q(K,[h_0])}{t}=q\int_\Omega f(v)dC_q(K,[h_0],v),\tag{174}$$

for $q=0$,

$$\begin{aligned}
\lim_{t\longrightarrow 0}\frac{E(K,[h_t])-E(K,[h_0])}{t}&=\int_\Omega f(v)dC_0(K,[h_0],v),\\
\lim_{t\longrightarrow 0}\frac{\log\bar{V}_0(K,[h_t])-\log\bar{V}_0(K,[h_0])}{t}&=\frac{1}{V(Q)}\int_\Omega f(v)dC_0(K,[h_0],v).
\end{aligned}\tag{175}$$

Proof. The logarithmic family of Wulff shape $[h_t]$ is defined as the Wulff shape of h_t, where h_t is given by the following:

$$\log h_t=\log h_t+tf+o(t,\cdot).\tag{176}$$

Let $\rho_t=h_t^{-1}$. Then,

$$\log\rho_t=\log\rho_t-tf-o(t,\cdot).\tag{177}$$

Let $\langle\rho_t\rangle$ be the logarithmic family of convex hulls associated with $(\rho_0,-f)$. Then from Lemma 7, we obtain that

$$[h_t]=\langle\rho_t\rangle^*,\tag{178}$$

and the desired conclusions now follow from Theorem 27. □

We describe the special cases of Theorem 27 and Theorem 28 for logarithmic families of convex hull and Wulff shape generated by convex bodies.

Theorem 29. *If $K,Q\in\mathscr{K}_o^n$ and $g:S^{n-1}\longrightarrow\mathbb{R}$ is continuous, then for $q\neq 0$,*

$$\begin{aligned}
&\lim_{t\longrightarrow 0}\frac{V_q(K,\langle Q^*,g,t\rangle^*)-V_q(K,Q)}{t}\\
&=-q\int_{S^{n-1}}g(v)dC_q(K,Q,v),
\end{aligned}\tag{179}$$

for $q=0$,

$$\begin{aligned}
&\lim_{t\longrightarrow 0}\frac{E(K,\langle Q^*,g,t\rangle^*)-E(K,Q)}{t}=-\int_{S^{n-1}}g(v)dC_0(K,Q,v),\\
&\lim_{t\longrightarrow 0}\frac{\log\bar{V}_0(K,\langle Q^*,g,t\rangle^*)-\log\bar{V}_0(K,Q)}{t}\\
&=-\frac{1}{V(K)}\int_{S^{n-1}}g(v)dC_0(K,Q,v).
\end{aligned}\tag{180}$$

Proof. In Theorem 27, let $\rho_0=1/h_Q=\rho_{Q^*}$. Then, $\langle\rho_t\rangle^*=$

$\langle Q^*, g, t\rangle^*$. In particular, from (53) we have $\langle \rho_0\rangle^* = \langle \rho_{Q^*}\rangle^* = Q^{**} = Q$.□ □

Above variational formulas for convex hulls imply variational formulas for Wulff shapes.

Theorem 30. *If $K, Q \in \mathscr{K}_o^n$ and $f : S^{n-1} \longrightarrow \mathbb{R}$ is continuous, then for $q \neq 0$,*

$$\lim_{t\longrightarrow 0} \frac{V_q(K,[Q,f,t]) - V_q(K,Q)}{t} = q\int_{S^{n-1}} f(v)dC_q(K,Q,v), \tag{181}$$

for $q = 0$,

$$\begin{aligned} \lim_{t\longrightarrow 0} \frac{E(K,[Q,f,t]) - E(K,Q)}{t} &= \int_{S^{n-1}} f(v)dC_0(K,Q,v), \\ \lim_{t\longrightarrow 0} \frac{\log \bar{V}_0(K,[Q,f,t]) - \log \bar{V}_0(K,Q)}{t} &= \frac{1}{V(Q)}\int_{S^{n-1}} f(v)dC_0(K,Q,v). \end{aligned} \tag{182}$$

Proof. The logarithmic family of Wulff shapes $[Q, f, o, t]$ is defined by the Wulff shape $[h_t]$, where

$$\log h_t = \log h_Q + tf + o(t,\cdot). \tag{183}$$

This, and the fact that $1/h_Q = \rho_{Q^*}$, allows us to define

$$\log \rho_t^* = \log \rho_{Q^*} - tf - o(t,\cdot), \tag{184}$$

and ρ_t^* will generate a logarithmic family of convex hulls $\langle Q^*, -f, -o, t\rangle$. Since $\rho_t^* = 1/h_t$, Lemma 7 gives the following:

$$[Q, f, o, t] = \langle Q^*, -f, -o, t\rangle^*. \tag{185}$$

Therefore, Theorem 30 now follows directly from Theorem 29.□ □

The following theorem gives the variational formulas of L_p-mixed volumes and mixed entropies with respect to L_p Minkowski combinations.

Theorem 31. *If $p, q \in \mathbb{R}^n$ and $K, L, Q \in \mathscr{K}_o^n$, then for $p \neq 0$, $q \neq 0$,*

$$\begin{aligned} &\lim_{t\longrightarrow 0} \frac{V_q(K, Q +_p t\cdot L) - V_q(K,Q)}{t} \\ &= \frac{q}{p}\int_{S^{n-1}} h_L^p(v)dC_{p,q}(K,Q,v), \end{aligned} \tag{186}$$

for $p = 0$ and $q \neq 0$,

$$\begin{aligned} &\lim_{t\longrightarrow 0} \frac{V_q(K, Q +_0 t\cdot L) - V_q(K,Q)}{t} \\ &= q\int_{S^{n-1}} \log h_L(v)dC_q(K,Q,v), \end{aligned} \tag{187}$$

for $p \neq 0$ and $q = 0$,

$$\begin{aligned} &\lim_{t\longrightarrow 0} \frac{E(K, Q +_p t\cdot L) - E(K,Q)}{t} \\ &= \frac{1}{p}\int_{S^{n-1}} h_L^p(v)dC_{p,0}(K,Q,v), \end{aligned} \tag{188}$$

$$\begin{aligned} &\lim_{t\longrightarrow 0} \frac{\log \bar{V}_0(K, Q +_p t\cdot L) - \log \bar{V}_0(K,Q)}{t} \\ &= \frac{1}{pV(K)}\int_{S^{n-1}} h_L^p(v)dC_{p,0}(K,Q,u), \end{aligned} \tag{189}$$

and if $p = q = 0$,

$$\begin{aligned} &\lim_{t\longrightarrow 0} \frac{E(K, Q +_0 t\cdot L) - E(K,Q)}{t} \\ &= \int_{S^{n-1}} \log h_L(v)dC_0(K,Q,v), \\ &\lim_{t\longrightarrow 0} \frac{\log \bar{V}_0(K, Q +_0 t\cdot L) - \log \bar{V}_0(K,Q)}{t} \\ &= \frac{1}{V(K)}\int_{S^{n-1}} \log h_L(v)dC_0(K,Q,u). \end{aligned} \tag{190}$$

Proof. For small t, h_t is defined by the following:

$$h_t^p = h_Q^p + th_L^p \text{ for } p \neq 0, h_t = h_Q h_L^t \text{ for } p = 0. \tag{191}$$

From Equations (61) and (62), the Wulff shape $[h_t] = Q +_p t\cdot L$. For sufficiently small t, it follows from Equation (191) that

$$\begin{aligned} \log h_t &= \log h_Q + \frac{t}{p}\frac{h_L^p}{h_Q^p} + o(t,\cdot), \quad p \neq 0, \\ \log h_t &= \log h_Q + t\log h_L, \quad p = 0. \end{aligned} \tag{192}$$

Let $f = (1/p)(h_L^p/h_Q^p)$ when $p \neq 0$, and let $f = \log h_L$ when $p = 0$. The required formulas now follow Theorem 30 and Equation (105).□ □

We use the normalized power function, and we can write the formula in Theorem 31 as a single formula.

Theorem 32. *Suppose $p, q \in \mathbb{R}$. For $K, L, Q \in \mathscr{K}_o^n$,*

$$\frac{d}{dt} V_{\bar{q}}(K, Q +_p t\cdot L)\big|_{t=0} = \int_{S^{n-1}} h_L^{\bar{p}}(v)dC_{p,q}(K,Q,v). \tag{193}$$

For L_0 Minkowski linear combinations, it would help to have an affine version of Theorem 31. This is contained in

Theorem 33. *Suppose $q \neq 0$. If $K, L, Q \in \mathscr{K}_o^n$, then*

$$\lim_{t\longrightarrow 0} \frac{V_q(K, (1-t)\cdot Q +_0 t\cdot L) - V_q(K,Q)}{t} = q\int_{S^{n-1}} \log \frac{h_L(v)}{h_Q(v)} dC_q(K,Q,v), \tag{194}$$

$$\lim_{t\longrightarrow 0} \frac{E(K, (1-t)\cdot Q +_0 t\cdot L) - E(K,Q)}{t} = \int_{S^{n-1}} \log \frac{h_L(v)}{h_Q(v)} dC_0(K,Q,v), \tag{195}$$

$$\lim_{t\longrightarrow 0} \frac{\log \bar{V}_0(K, (1-t)\cdot Q +_0 t\cdot L) - \log \bar{V}_0(K,Q)}{t} = \frac{1}{V(K)}\int_{S^{n-1}} \log \frac{h_L(v)}{h_Q(v)} dC_0(K,Q,v). \tag{196}$$

Proof. Let

$$h_t = h_Q^{1-t} h_L^t. \tag{197}$$

From Equation (58) we know the Wulff space $[h_t] = (1-t)\cdot Q +_0 t\cdot L$. From the above definition of h_t, it follows immediately that for sufficiently small t,

$$\log h_t = \log h_Q + t \log \frac{h_L}{h_Q}. \tag{198}$$

Let $f = \log h_L/h_Q$. The desired formulas now follow directly from Theorem 30.□ □

Theorem 34. *If $p \neq 0$ and $q \neq 0$, then for all $K, L, Q \in \mathscr{K}_o^n$ and $\phi \in SL(n)$,*

$$C_{p,q}(\phi K, \phi Q, \cdot) = \phi_p^t \dashv C_{p,q}(K,Q,\cdot), \tag{199}$$

$$C_{p,0}(\phi K, \phi Q, \cdot) = \phi_p^t \dashv C_{p,0}(K,Q,\cdot), \tag{200}$$

$$C_q(\phi K, \phi Q, \cdot) = \phi_0^t \dashv C_q(K,Q,\cdot), \tag{201}$$

$$C_0(\phi K, \phi Q, \cdot) = \phi_0^t \dashv C_0(K,Q,\cdot). \tag{202}$$

Proof. Obviously, the case $p \neq 0$ and $q = 0$ is handled by Equation (200). The case $p = 0$ and $q \neq 0$ is handled by Equation (201), while the case $p = 0$ and $q = 0$ is handled by Equation (202).

We adopt the methods and techniques of paper [4]. Recall that Haberl and Parapatits refer to the [9] classified measure-valued operators on $\mathscr{K}_o^n$, which are $SL(n)$-inverse degree p and corresponding to the transformation behavior in Theorem 34. From Equations (63), (65), and (186), we see that for all $K, L, Q \in \mathscr{K}_o^n$ and all $\phi \in SL(n)$,

$$\int_{S^{n-1}} h_{\phi L}^p(v) dC_{p,q}(\phi K, \phi Q, v) = \int_{S^{n-1}} h_L^p(v) dC_{p,q}(K,Q,v), \tag{203}$$

or equivalently for all $K, L, Q \in \mathscr{K}_o^n$ and all $\phi \in SL(n)$,

$$\int_{S^{n-1}} h_L^p(v) dC_{p,q}(\phi K, \phi Q, v) = \int_{S^{n-1}} h_{\phi^{-1}L}^p(v) dC_{p,q}(K,Q,v). \tag{204}$$

By Definition 8, and note the important fact that support functions are positively homogeneous of degree 1, from Equations (45) and (204), we have the following:

$$\begin{aligned}\int_{S^{n-1}} h_L^p(v) d\phi_p^t \dashv C_{p,q}(K,Q,v) &= \int_{S^{n-1}} h_L^p\left(\phi^{-t}v\right) dC_{p,q}(K,Q,v) \\ &= \int_{S^{n-1}} h_{\phi^{-1}L}^p(v) dC_{p,q}(K,Q,v) \\ &= \int_{S^{n-1}} h_L^p(v) dC_{p,q}(\phi K, \phi Q, v).\end{aligned} \tag{205}$$

This shows that the measures $\phi_p^t \dashv C_{p,q}(K,Q,\cdot)$ and $C_{p,q}(\phi K, \phi Q, \cdot)$ when integrated against the p-th power of support functions of bodies in $\mathscr{K}_o^n$ are identical; thus, Lemma 12 now indicates that

$$C_{p,q}(\phi K, \phi Q, \cdot) = \phi_p^t \dashv C_{p,q}(K,Q,\cdot), \tag{206}$$

it can be concluded that Equation (199).

The proof for Equation (200) is the same as the proof for Equation (199): As long as $p \neq 0$, it will be the case that Equation (204) continues to hold even if $q = 0$ provided we appeal to Equations (188) and (71) when previously we had turned to Equations (188) and (65).

From Equations (63), (65), and (194), we know that for all $K, L, Q \in \mathscr{K}_o^n$ and all $\phi \in SL(n)$,

$$\int_{S^{n-1}} \log \frac{h_{\phi^{-1}L}(v)}{h_K(v)} dC_q(K,Q,v) = \int_{S^{n-1}} \log \frac{h_L(v)}{h_{\phi K}(v)} dC_q(\phi K, \phi Q, v). \tag{207}$$

In Equation (207), choose $L = B$. Then, by Equation (45), we see that $h_{\phi^{-1}L}(v) = h_L(\phi^{-t}v) = |\phi^{-t}v|$, and (6.15) becomes the following form:

$$\begin{aligned}\int_{S^{n-1}} \log h_K(v)dC_q(K,Q,v) &= \int_{S^{n-1}} \log\left|\phi^{-t}v\right| dC_q(K,Q,v)\\ &\quad + \int_{S^{n-1}} \log h_{\phi K}(v)dC_q(\phi K,\phi Q,v),\end{aligned} \tag{208}$$

for all $\phi \in SL(n)$ and all $K, Q \in \mathscr{K}_o^n$. Together with Equations (207) and (208), we have the following:

$$\begin{aligned}\int_{S^{n-1}} \log \frac{h_{\phi^{-1}L}(v)}{\left|\phi^{-t}v\right|} dC_q(K,Q,v) = \int_{S^{n-1}} \log h_L(v)dC_q(\phi K,\phi Q,v),\end{aligned} \tag{209}$$

this and Equation (45) give that for all $\phi \in SL(n)$ and all $K, L, Q \in \mathscr{K}_o^n$,

$$\begin{aligned}\int_{S^{n-1}} \log h_L(\langle\phi^{-t}v\rangle)dC_q(K,Q,v) = \int_{S^{n-1}} \log h_L(v)dC_q(\phi K,\phi Q,v).\end{aligned} \tag{210}$$

Equivalently,

$$\begin{aligned}\int_{S^{n-1}} \log h_L(v)d\phi_0^t \lrcorner C_q(K,Q,v) = \int_{S^{n-1}} \log h_L(v)dC_q(\phi K,\phi Q,v),\end{aligned} \tag{211}$$

for all $\phi \in SL(n)$ and all $K, Q \in \mathscr{K}_o^n$. Using Lemma 12, we see that Equation (211) yields

$$C_q(\phi K,\phi Q,\cdot) = \phi_0^t \lrcorner C_q(\phi K,\phi Q,v), \tag{212}$$

for all $\phi \in SL(n)$ and all $K, Q \in \mathscr{K}_o^n$. This establishes Equation (201).

The proof of Equation (202) is identical to the proof of Equation (201) except that instead of appealing to Equations (194) and (65) we appeal to Equations (195) and (71).□ □

5. The $\mathbf{L_{p,q}}$-Mixed Volumes

For $K, L \in \mathscr{K}_o^n$, the L_p-mixed volume $V_p(K,L)$ has the integral representation

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(v)^p dS_p(K,v). \tag{213}$$

From Equation (115), with $q=p$ and $g=h_L^p$, we have that

$$\begin{aligned}\int_{S^{n-1}} h_L^p(v)dC_{p,p}(K,Q,v) = \frac{1}{n}\int_{S^{n-1}} h_L^p(\alpha_K(u))h_K^{-p}(\alpha_K(u))\rho_K^n(u)du.\end{aligned} \tag{214}$$

By Equation (131), the L_p-mixed volume $V_p(K,L)$ has a dual integral formulation. If $K, L \in \mathscr{K}_o^n$, then

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^p (\alpha_K(u))\rho_K^n(u)du. \tag{215}$$

The dual integral formulation of L_p-mixed volume was first introduced by Lutwak et al. in [4]. This leads us to define following $L_{p,q}$-mixed volumes.

Definition 35. Let $p, q \in \mathbb{R}$ and $K, L, Q \in \mathscr{K}_o^n$. The $L_{p,q}$-mixed volume $V_{p,q}(K,L,Q)$ is defined by the following:

$$V_{p,q}(K,L,Q) = \int_{S^{n-1}} h_L^p(v)dC_{p,q}(K,Q,v). \tag{216}$$

Using Equation (115) with $g=h_L^p$, Equation (216) can be written as follows:

$$V_{p,q}(K,L,Q) = \frac{1}{n}\int_{S^{n-1}} \left(\frac{h_L}{h_Q}\right)^p (\alpha_K(u))\left(\frac{h_Q}{h_K}\right)^q (\alpha_K(u))\rho_K^n(u)du. \tag{217}$$

From Equations (216) and (124), the L_p-mixed volume $V_{p,q}(K,L,Q)$ can be written as follows:

$$\begin{aligned}V_{p,q}(K,L,Q) = \frac{1}{n}\int_{S^{n-1}} & h_L^p(v)h_K(v)^{1-q}h_Q(v)h_Q^{q-p}\\ &\cdot(v_K(\nabla h_K(v)))dS(K,v),\end{aligned} \tag{218}$$

where the function $v_K : \{\nabla h_K(v) : v \in S^{n-1}\} \subset \partial' K \longrightarrow S^{n-1}$.

From $L_{p,q}$-mixed volume (Equation (30)) (or Equation (217)), the L_p-mixed volume (Equation (9)) (or Equation (22)) will be shown to be the special cases.

Proposition 36. *Suppose $p, q \in \mathbb{R}$. If $K, L, Q \in \mathscr{K}_o^n$, then*

$$V_{p,q}(K,L,K) = V_p(K,L), \tag{219}$$

$$V_{p,p}(K,L,Q) = V_p(K,L), \tag{220}$$

$$V_{p,q}(K,L,L) = V_q(K,L), \tag{221}$$

$$V_{p,p}(K,L,B) = V_p(K,L), \tag{222}$$

$$V_{0,q}(K,L,Q) = V_q(K,Q). \tag{223}$$

Proof. Identity (Equations (219)–(221)) follow from

Equation (22) and Equation (34) (or Equation (217)). Similarly, we can prove Equations (222) and (223). □

Proposition 37. *The $L_{p,q}$ -mixed volume $V_{p,q}$ is $SL(n)$ -invariant. That is, for $p, q \in \mathbb{R}$, $K, L, Q \in \mathcal{K}_o^n$, and $\phi \in SL(n)$,*

$$V_{p,q}(\phi K, \phi L, \phi Q) = V_{p,q}(K, L, Q). \tag{224}$$

Proof. For $p = 0$, the conclusion follows from Equation (223) and the $SL(n)$-invariance of L_p-mixed volumes (Equation (65)). We assume $p \neq 0$. By Definition 35, Equation (199), and Equation (200), the fact that support functions are positively homogeneous of degree 1, Equation (45), and Definition 8, we have the following:

$$\begin{aligned} V_{p,q}(\phi K, \phi L, \phi Q) &= \int_{S^{n-1}} h_{\phi L}^p(v) dC_{p,q}(\phi K, \phi Q, v) \\ &= \int_{S^{n-1}} h_{\phi L}^p(v) d\phi_p^t \dashv C_{p,q}(K, Q, v) \\ &= \int_{S^{n-1}} h_{\phi L}^p\left(\phi^{-t} v\right) dC_{p,q}(K, Q, v) \\ &= \int_{S^{n-1}} h_L^p(v) dC_{p,q}(K, Q, v) \\ &= V_{p,q}(K, L, Q). \end{aligned} \tag{225}$$

From the dual Equation (217) of $L_{p,q}$-mixed volume and Equation (44), we have for real $\lambda > 0$,

$$V_{p,q}(\lambda K, \lambda L, \lambda Q) = \lambda^n V_{p,q}(K, L, Q). \tag{226}$$

□

Proposition 37, together with Equations (216) and (226), shows that for $\phi \in GL(n)$,

$$V_{p,q}(\phi K, \phi L, \phi Q) = |\phi| V_{p,q}(K, L, Q). \tag{227}$$

For $L_{p,q}$-mixed volume, the following inequality is a generalization of the L_p-Minkowski inequality for L_p-mixed volume.

Theorem 38. *Suppose p, q are such that $q \geq 1$ and $p < 0$. If $K, L, Q \in \mathcal{K}_o^n$, then*

$$V_{p,q}(K, L, Q)^n \geq V(K)^{n-q} V(L)^p V(Q)^{q-p}, \tag{228}$$

with equality if and only if K, L, Q are dilates when $q > 1$ and K, Q are homothetic when $q = 1$.

Proof. From Equations (21) and (217), we have the following:

$$\begin{aligned} V_{p,q}(K, L, Q) &= \frac{1}{n} \int_{S^{n-1}} h_L(\alpha_K(u))^p h_Q(\alpha_K(u))^{q-p} h_K(\alpha_K)^{-q} \rho_K^n(u) du \\ &= \frac{1}{n} \int_{S^{n-1}} h_L(\alpha_K(u))^p h_Q(\alpha_K(u))^{q-p} dS_q(K, u) \\ &= \frac{1}{n} \int_{S^{n-1}} \left[h_L(\alpha_K(u))^q\right]^{p/q} \left[h_Q(\alpha_K(u))^q\right]^{q-p/q} dS_q(K, u). \end{aligned} \tag{229}$$

From this, by the Hölder inequality (see [47]), the dual integral formulation (Equation (22)) of L_p-mixed volume and L_p-Minkowski inequality (Equation (11)), we have the following:

$$\begin{aligned} V_{p,q}(K, L, Q) &\geq \left(\frac{1}{n} \int_{S^{n-1}} h_L(\alpha_K(u))^q dS_q(K, u)\right)^{p/q} \\ &\quad \cdot \left(\frac{1}{n} \int_{S^{n-1}} h_Q(\alpha_K(u))^q dS_q(K, u)\right)^{(q-p)/q} \\ &= \left(\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^q (\alpha_K(u)) \rho_K(u)^n du\right)^{p/q} \\ &\quad \cdot \left(\frac{1}{n} \int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)^q (\alpha_K(u)) \rho_K(u)^n du\right)^{(q-p)/q} \\ &= V_q(K, L)^{p/q} V_q(K, Q)^{(q-p)/q} \\ &\geq V(K)^{(n-q)p/nq} V(L)^{p/n} V(K)^{((n-q)(q-p))/nq} V(Q)^{(q-p)/n} \\ &= V(Q)^{(q-p)/n} V(L)^{p/n} V(K)^{(n-q)/n}. \end{aligned} \tag{230}$$

□

The equality conditions follow from the equality conditions of Hölder inequality and the L_p-Minkowski inequality (Equation (11)) for L_p-mixed volumes. Namely, the equality for the above inequality holds if and only if K, L, Q are dilates when $q > 1$ and K, Q are homothetic when $q = 1$.

Over the past three decades, valuation theory has become an ever more important part of convex body geometry. See, e.g., [11–13, 18, 48–53]. The convex $L_{p,q}$-mixed volume is the valuation for each entry.

Proposition 39. *The $L_{p,q}$-mixed volume $V_{p,q}(K, L, Q)$ is a valuation over $\mathcal{K}_o^n$ with respect to all $K, L,$ and Q.*

Proof. The $L_{p,q}$-mixed volume $V_{p,q}(K, L, Q)$ is a valuation on $\mathcal{K}_o^n$ respect to the third argument can be seen easily by writing Equation (216) as follows:

$$V_{p,q}(K, L, Q) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) dC_{p,q}(K, Q, u), \tag{231}$$

and from Equation (139) (or Theorem 25), observing that for $K_1, K_2 \in \mathscr{K}_o^n$, we have the following:

$$\begin{aligned} &dC_{p,q}(K_1, Q, \cdot) + dC_{p,q}(K_2, Q, \cdot) \\ &\quad = dC_{p,q}(K_1 \cup K_2, Q, \cdot) + dC_{p,q}(K_1 \cap K_2, Q, \cdot). \end{aligned} \tag{232}$$

Together with Equations (216) and (232), we have the following:

$$\begin{aligned} &V_{p,q}(K_1 \cup K_2, L, Q) + V_{p,q}(K_1 \cap K_2, L, Q) \\ &\quad = V_{p,q}(K_1, L, Q) + V_{p,q}(K_2, L, Q). \end{aligned} \tag{233}$$

Namely, $V_{p,q}(K, L, Q)$ is a valuation in the third argument.

Observing that for $L_1, L_2 \in \mathscr{K}_o^n$ such that $L_1 \cup L_2 \in \mathscr{K}_o^n$. Then, we have the following:

$$h^p_{L_1\cup L_2} + h^p_{L_1\cap L_2} = h^p_{L_1} + h^p_{L_2}, \text{on } S^{n-1}. \tag{234}$$

Note that $h_{L_1\cup L_2} = \max\{h_{L_1}, h_{L_2}\}$ and $h_{L_1\cap L_2} = \min\{h_{L_1}, h_{L_2}\}$. Together with Equations (216) and (234), we see that $V_{p,q}(K, L, Q)$ is a valuation in the second argument, i.e,

$$V_{p,q}(K, L_1 \cup L_2, Q) + V_{p,q}(K, L_1 \cap L_2, Q) = V_{p,q}(K, L_1, Q) + V_{p,q}(K, L_2, Q). \tag{235}$$

Note that if $Q_1, Q_2 \in \mathscr{K}_o^n$ are such that $Q_1 \cup Q_2 \in \mathscr{K}_o^n$, then we have the following:

$$h^{q-p}_{Q_1\cup Q_2} + h^{q-p}_{Q_1\cap Q_2} = h^{q-p}_{Q_1} + h^{q-p}_{Q_2}, \text{on } S^{n-1}. \tag{236}$$

Together with Equations (218) and (236), we see that $V_{p,q}(K, L, Q)$ is a valuation in the first argument, i.e,

$$V_{p,q}(K, L, Q_1 \cup Q_2) + V_{p,q}(K, L, Q_1 \cap Q_2) = V_{p,q}(K, L, Q_1) + V_{p,q}(K, L, Q_2). \tag{237}$$

Let $K, Q \in \mathscr{K}_o^n$. The q-th mixed cone-volume measure $C_q(K, Q, \omega)$ of K and Q is a Borel measure on the unit sphere S^{n-1} is defined by for a Borel $\omega \subseteq S^{n-1}$ and $u \in \omega$,

$$dC_q(K, Q, \omega) = \frac{1}{n}\left(\frac{h_Q}{h_K}\right)^q (\alpha_K(u))\rho_K(u)^n du. \tag{238}$$

Since the q-th mixed volume, $V_q(K, Q)$ has a dual integral formulation:

$$V_q(K, Q) = \frac{1}{n}\int_{S^{n-1}}\left(\frac{h_Q}{h_K}\right)^q (\alpha_K(u))\rho_K^n(u)du. \tag{239}$$

We can turn the q-th mixed cone-volume measure into the probability measure on the unit sphere by normalizing it by q-th mixed volume of the bodies. The q-th mixed cone-volume probability measure $\bar{C}_q(K, Q; \cdot)$ of K and Q is defined by the following:

$$d\bar{C}_q(K, Q, \omega) = \frac{1}{V_q(K, Q)} dC_q(K, Q, \omega). \tag{240}$$

If $K, L, Q \in \mathscr{K}_o^n$, then for each real $p, q \in \mathbb{R}$, we define the normalized $L_{p,q}$-mixed volume by the following:

$$\begin{aligned} \bar{V}_{p,q}(K, L, Q) &= \left(\frac{V_{p,q}(K, L, Q)}{V_q(K, Q)}\right)^{1/p} \\ &= \left(\int_{S^{n-1}}\left(\frac{h_L(u)}{h_Q(u)}\right)^p (\alpha_K(u)) d\bar{C}_q(K, Q, u)\right)^{1/p}. \end{aligned} \tag{241}$$

Let $p \longrightarrow 0$. We give the following:

$$\bar{V}_{0,q}(K, L, Q) = \exp\left(\int_{S^{n-1}} \log\left(\frac{h_L}{h_Q}\right)(\alpha_K(u)) d\bar{C}_q(K, Q, u)\right). \tag{242}$$

The q-th mixed entropy $E_q(K, L, Q)$ of convex bodies $K, L, Q \in \mathscr{K}_o^n$ is defined by the following:

$$E_q(K, L, Q) = \int_{S^{n-1}} \log\left(\frac{h_L}{h_Q}\right)(\alpha_K(u)) dC_q(K, Q, u). \tag{243}$$

In particular,

$$\bar{V}_{0,0}(K, L, K) = \bar{V}_0(K, L), E_q(Q, L, Q) = E(Q, L). \tag{244}$$

□ □

6. The $\mathbf{L_{p,q}}$-Minkowski Problems

The existence and uniqueness of $L_{p,q}$-Minkowski problem is the central problem to be investigated here. Its existence problem can be expressed as follows:

Problem 40. Let $p, q \in \mathbb{R}$, and $Q \in \mathscr{K}_o^n$ is fixed. Given a Borel measure $\mu \in \mathscr{M}(S^{n-1})$, what are necessary and sufficient conditions on μ such that there exists a $K \in \mathscr{K}_o^n$ whose L_p-curvature measures $C_{p,q}(K, Q, \cdot)$ is the given measure μ?

L_p-Minkowski problem when $q = p$. When the given data measure μ has a density f, it follows from Equation (125) that $L_{p,q}$-Minkowski problem is equivalent to solving the following Monge-Ampère-type equation on S^{n-1}:

$$h^{1-q}\|\nu \circ (\nabla h)\|^{q-p}_{Q^*} \det\left(\bar{\nabla}^2 h + hI\right) = f, \tag{245}$$

where h is the unknown function on S^{n-1}, and ∇h is the gradient vector function in $\mathbb{R}^n$ of the extension from h to $\mathbb{R}^n$ as a vector function that is positively homogeneous of degree 1.

If we assume that the range of the gradient function ∇h is D, then $\nu : D \longrightarrow S^{n-1}$ is also an unknown function related to h.

Our uniqueness result for the $L_{p,q}$-Minkowski problem is presented in the following:

Problem 41. For fixed $p, q \in \mathbb{R}$ and $Q \in \mathcal{K}_o^n$, if $K, L \in \mathcal{K}_o^n$ such that

$$C_{p,q}(K, Q, \cdot) = C_{p,q}(L, Q, \cdot), \tag{246}$$

then how is K related to L?

Now, we establish uniqueness of the solution to the problem with $q \geq n$ for the case of polytopes.

Theorem 42. *Let $P, P' \in \mathcal{K}_o^n$ be polytopes and let $Q \in \mathcal{K}_o^n$. Suppose*

$$C_{p,q}(P, Q, \cdot) = C_{p,q}\left(P', Q, \cdot\right). \tag{247}$$

Then, $P = P'$ when $q > n$ and P' is a dilate of P when $q = n$.

Proof. According to Equations (121) and (122), we get that the curvature measures of polytopes are discrete, and that $C_{p,q}(P, Q, \cdot) = C_{p,q}(P', Q, \cdot)$ implies that P and P' must have the same outer unit normal vectors $v_1, v_2, \cdots, v_m$ and

$$C_{p,q}(P, Q \cdot) = C_{p,q}\left(P', Q, \cdot\right) = \sum_{i=1}^{m} d_i \delta_{v_i}, \tag{248}$$

where δ_{v_i} denotes the delta measure concentrated at v_i, and

$$\begin{aligned} d_i &= \frac{1}{n} h_Q^{q-p}(v_i) h_P^{-q}(v_i) \int_{S^{n-1}\cap\Delta_i} \rho_P^n(u)du \\ &= \frac{1}{n} h_Q^{q-p}(v_i) h_{P'}^{-q}(v_i) \int_{S^{n-1}\cap{\Delta'}_i} \rho_{P'}^n(u)du. \end{aligned} \tag{249}$$

Here Δ_i and ${\Delta'}_i$ are the cones formed by the origin and the facets of P and P' with vector v_i, respectively.

Assume that $P \neq P'$. Tt is easy to see that $P \subseteq P'$ is not possible. Set λ be the maximal number with $\lambda P \subseteq P'$. This has $\lambda < 1$. Since λP and P' have the same outer unit normal vectors, there is a facet of λP which is contained in a facet of P'. The outer unit normal vector of those facets is denoted by v_{i_1}. It follows that

$$\begin{aligned} h_{\lambda P}\left(v_{i_1}\right) &= h_{P'}\left(v_{i_1}\right), \\ \Delta_{i_1} &\subseteq {\Delta'}_{i_1}, \\ \rho_{\lambda P}(u) &= \rho_{P'}(u) \text{for all } u \in \Delta_{i_1}. \end{aligned} \tag{250}$$

Thus,

$$\begin{aligned} &\frac{1}{n} h_Q^{q-p}\left(v_{i_1}\right) h_{\lambda P}^{-q}\left(v_{i_1}\right) \int_{S^{n-1}\cap\Delta_{i_1}} \rho_{\lambda P}^n(u)du \\ &\quad\leq \frac{1}{n} h_Q^{q-p}\left(v_{i_1}\right) h_{P'}^{-q}\left(v_{i_1}\right) \int_{S^{n-1}\cap{\Delta'}_{i_1}} \rho_{P'}^n(u)du. \end{aligned} \tag{251}$$

with equality if and only if $\Delta_{i_1} = {\Delta'}_{i_1}$. By this and Equation (249), we can obtain that

$$\lambda^{n-q} \leq 1. \tag{252}$$

But $\lambda < 1$ implies that $\lambda^{n-q} > 1$ if $q > n$. Obviously, this is a contradiction.□ □

If $q = n$, then Equation (249) forces equality in Equation (251). So, $\Delta_{i_1} = {\Delta'}_{i_1}$, and the facets of λP and P' with outer unit normal vector v_{i_1} are the same. Let v_{i_2} is the outer unit normal vector to a facet, which is adjacent to the facet whose outer unit normal vector is v_{i_1}. Thus, the facet of λP with outer unit normal vector v_{i_2} is contained in the facet of P' with outer unit normal vector v_{i_2}. A similar argument holds that the two facets are the same. Continuing in this manner, it follows that $\lambda P = P'$.

7. Several Other Problems

Here, we present several issues that need to be discussed in the future. Some of the definitions and problems below are different from the paper [40, 43, 44, 54].

7.1. $L_{p,q}$-Mixed Affine Surface Areas. In [7], Lutwak defined the L_p-affine surface area $\Omega_p(K)$ for $p \geq 1$ by the following:

$$n^{-p/n}\Omega_p(K)^{(n+p)/n} = \inf\left\{nV_p(K, L^*)V(L)^{p/n} : L \in \mathcal{S}_o^n\right\}. \tag{253}$$

Hug in [55] observed that the L_p-affine surface area is well defined for $0 < p < 1$.

The following affine isoperimetric inequality was established in [7] for $p \geq 1$, and in [56] for $0 < p < 1$. If $K \in \mathcal{K}_c^n$, then

$$\Omega_p(K)^{n+p} \leq n^{n+p}\omega_n^{2p}V(K)^{n-p}, \quad p > 0, \tag{254}$$

with equality if and only if K is an ellipsoid. Here, ω_n is the volume of the n dimensional unit sphere.

Definition 43. Suppose $q \in \mathbb{R}$. For $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$, the q-th curvature measure $C_q(K, Q^*, \cdot)$ of K (related to star body Q) is defined by the following:

$$C_q(K, Q^*, \eta) = \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} \frac{\rho_K^n(u)}{(\rho_Q h_K)^q(\alpha_K(u))}du, \tag{255}$$

for each Borel $\eta \subseteq S^{n-1}$, and L_p-curvature measure $C_{p,q}(K, Q^*, \cdot)$ of K is defined by the following:

$$\begin{aligned} C_{p,q}(K, Q^*, \eta) &= \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} \rho_Q^p(\alpha_K(u)) \cdot \frac{\rho_K^n(u)}{(\rho_Q h_K)^q(\alpha_K(u))} du \\ &= \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} \rho_Q^p(\alpha_K(u)) dC_q(K, Q^*, u), \end{aligned} \tag{256}$$

for each Borel $\eta \subseteq S^{n-1}$.

It follows from Definition 43 that

$$dC_{p,q}(K, Q^*, \cdot) = \rho_Q^p dC_q(K, Q^*, \cdot). \tag{257}$$

Definition 44. Suppose $q \in \mathbb{R}$. If $K \in \mathscr{K}_o^n, L \in \mathscr{S}_o^n$, the q-th mixed volume $V_q(K, L^*)$ is defined by the following:

$$\begin{aligned} V_q(K, L^*) &= \frac{1}{n}\int_{S^{n-1}} \frac{\rho_K^n(u)}{(\rho_L h_K)^q(\alpha_K(u))} du \\ &= \frac{1}{n}\int_{S^{n-1}} \rho_L^{-q}(u) dS_q(K, u). \end{aligned} \tag{258}$$

Definition 45. Suppose $p, q \in \mathbb{R}$. If $K \in \mathscr{K}_o^n$ and $Q, L \in \mathscr{S}_o^n$, the $L_{p,q}$-mixed volume $V_{p,q}(K, L^*, Q^*)$ of K and L^* (with respect to Q) is defined by the following:

$$V_{p,q}(K, L^*, Q^*) = \int_{S^{n-1}} \rho_L(v)^{-p} dC_{p,q}(K, Q^*, v). \tag{259}$$

Inspired by [40, 54], from Equations (258) and (259) we define $L_{p,q}$-mixed affine surface area as follows:

Definition 46. For $p \in \mathbb{R}, q > 0$ and $K \in \mathscr{K}_o^n, Q \in \mathscr{S}_o^n$, the $L_{p,q}$-mixed affine surface area $\Omega_{p,q}(K, Q)$ of K (relate to Q) is defined by the following:

$$\begin{aligned} &n^{-q/n}\Omega_{p,q}(K, Q)^{(n+q)/n} \\ &\quad = \inf\left\{nV_{p,q}(K, L^*, Q^*)V_q(L, Q)^{q/n} : L \in \mathscr{S}_o^n\right\}. \end{aligned} \tag{260}$$

When $Q = L$, from Equation (219) we have the following:

$$V_{p,q}(K, L^*, L^*) = V_q(K, L^*). \tag{261}$$

$\Omega_{p,q}(K, L)$ is the L_q-affine surface area $\Omega_q(K)$.

Problem 47. For the $L_{p,q}$-mixed affine surface area, does it maintain affine invariance and continuity? How to establish its affine isoperimetric inequality?

7.2. ***$L_{p,q}$**-Mixed Geominimal Surface Area.* In [7], Lutwak defined the L_p-geominimal surface area $G_p(K)$ by the following:

$$\omega_n^{p/n} G_p(K) = \inf\left\{nV_p(K, L)V(L^*)^{p/n} : L \in \mathscr{K}_o^n\right\}, \tag{262}$$

and proved the following affine isoperimetric inequality: If $K \in \mathscr{K}_o^n$, then

$$G_p(K)^n \le n^n \omega_n^p V(K)^{n-p}, \tag{263}$$

with equality if and only if K is an ellipsoid.

Motivated by the L_p-mixed geominimal surface area (Equation (257)), we define $L_{p,q}$-mixed geominimal surface area, $G_{p,q}(K, Q)$, of K relative to Q as follows:

Definition 48. For $p \in \mathbb{R}, q \ge 1$, and $K, Q \in \mathscr{K}_o^n$, the $L_{p,q}$-mixed geominimal surface area $G_{p,q}(K, Q)$ of K relative to Q is defined by the following:

$$\omega_n^{q/n} G_{p,q}(K, Q) = \inf\left\{nV_{p,q}(K, L, Q)V_q(L^*, Q^*)^{q/n} : L \in \mathscr{K}_o^n\right\}. \tag{264}$$

When $Q = L$, from Equation (219) we have the following:

$$V_{p,q}(K, L, L) = V_q(K, L). \tag{265}$$

$G_{p,q}(K, L)$ is the L_q-geominimal surface area $G_q(K)$.

Problem 49. For the $L_{p,q}$-mixed geominimal surface area, does it maintain affine invariance and continuity? How to establish its affine isoperimetric inequality?

7.3. ***$L_{p,q}$**-John Ellipsoids.* Suppose $p \in (0,\infty)$ and K is a convex body in $\mathbb{R}^n$ with the origin in its interior. Among all origin-symmetric ellipsoids E, the unique ellipsoid that solves the constrained maximization problem:

$$\begin{gathered} \max_E \left(\frac{V(E)}{\omega_n}\right)^{1/n} \\ \text{subject to } \bar{V}_p(K, E) \le 1 \end{gathered} \tag{266}$$

is called the L_p-John ellipsoid of K which defined in [45] and denoted by E_pK. Clearly, $E_pB = B$. Here,

$$\bar{V}_p(K, E) = \left(\frac{1}{nV(K)}\int_{S^{n-1}} \left(\frac{h_E(u)}{h_K(u)}\right)^p h_K(u) dS(K, u)\right)^{1/n}, \quad 0 < p < \infty \tag{267}$$

is the normalized L_p-mixed volume of K and E. In the case $p = \infty$, we define the following:

$$\bar{V}_\infty(K, E) = \sup\left\{\frac{h_E(u)}{h_K(u)} : u \in \operatorname{supp} S(K, \cdot)\right\}. \tag{268}$$

In general, the L_p-John ellipsoid E_pK is not contained in

K (except when $p=\infty$). However, when $1 \le p \le \infty$, it has $V(E_pK) \le V(K)$. In reverse, for $0<p\le\infty$, the L_p version of ball's volume-ratio inequality [45] states that

$$\frac{V(K)}{V(E_pK)} \le \frac{2^n}{\omega_n}, \tag{269}$$

with equality if and only if K is a parallelotope.

We know that from Equation (241), for $0<p<\infty$, $q\in\mathbb{R}$, the normalized $L_{p,q}$-mixed volume is calculated by the following:

$$\bar{V}_{p,q}(K,L,Q) = \left(\int_{S^{n-1}} \left(\frac{h_L(\alpha_K(u))}{h_Q(\alpha_K(u))}\right)^p d\bar{C}_q(K,Q,u)\right)^{1/p}. \tag{270}$$

In the case $p=\infty$, define the following:

$$\bar{V}_{\infty,q}(K,L,Q) = \max\left\{\frac{h_L(\alpha_K(u))}{h_Q(\alpha_K(u))} : u \in \operatorname{supp} C_q(K,Q,\cdot)\right\}. \tag{271}$$

By Equation (271), we have the following:

$$\bar{V}_{\infty,q}(K,L,Q) \le 1 \text{ if and only if } L \subseteq Q. \tag{272}$$

Let $\mathscr{E}^n$ denote the class of origin-symmetric ellipsoids in $\mathbb{R}^n$. Inspired by the constrained maximization problem (Equation (266)), the reader may consider its $L_{p,q}$-version.

Problem 50. Let $0<p\le\infty$, $q\in\mathbb{R}$. For $K,Q\in\mathscr{K}_o^n$, find an ellipsoid, among all origin-symmetric ellipsoids, which solves the following constrained maximization problem:

$$\max_{E\in\mathscr{E}^n}\left(\frac{V(E)}{\omega_n}\right)^{1/n} \quad \text{subject to } \bar{V}_{p,q}(K,E,Q) \le 1. \tag{273}$$

An ellipsoid that solves the constrained maximization problem will be called $L_{p,q}$-John ellipsoid for K,Q and denoted by $E_{p,q}(K,Q)$.

In particular, when $Q=K$, from Equations (219) and (22), we have the following:

$$V_{p,q}(K,E,K) = V_p(K,E), V_q(K,K) = V(K). \tag{274}$$

Thus, $\bar{V}_{p,q}(K,E,K) = \bar{V}_p(K,E)$. So, Problem 50 degenerates into the problem.

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

The author declares that there are no competing interests.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (11561020). The author is particularly grateful to Professor Weidong Wang, Dr. Yibin Feng, and Dr. Denghui Wu for their comments on various drafts of this work.

References

[1] Y. Huang, E. Lutwak, D. Yang, and G. Zhang, "The L_p-Aleksandrov problem for L_p-integral curvature," *Journal of Differential Geometry*, vol. 110, no. 1, pp. 1–29, 2018.

[2] E. Lutwak, "The Brunn-Minkowski-Firey theory. I: Mixed volumes and the Minkowski problem," *Journal of Differential Geometry*, vol. 38, no. 1, pp. 131–150, 1993.

[3] Y. Huang, E. Lutwak, D. Yang, and G. Zhang, "Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems," *Acta Mathematica*, vol. 216, no. 2, pp. 325–388, 2016.

[4] E. Lutwak, D. Yang, and G. Zhang, "L_p dual curvature measures," *Advances in Mathematics*, vol. 329, pp. 85–132, 2018.

[5] A. Cianchi, E. Lutwak, D. Yang, and G. Zhang, "Affine Moser-Trudinger and Morrey-Sobolev inequalities," *Calculus of Variations and Partial Differential Equations*, vol. 36, no. 3, pp. 419–436, 2009.

[6] R. J. Gardner, *Geometric Tomography*, Cambridge University Press, Cambridge, 2013.

[7] E. Lutwak, "The Brunn-Minkowski-Firey theory II," *Advances in Mathematics*, vol. 118, no. 2, pp. 244–294, 1996.

[8] C. Haberl and L. Parapatits, "Valuations and surface area measures," *Journal fur die Reine und Angewandte Mathematik*, vol. 687, pp. 225–245, 2014.

[9] C. Haberl and L. Parapatits, "The centro-affine Hadwiger theorem," *Journal of the American Mathematical Society*, vol. 27, no. 3, pp. 685–705, 2014.

[10] C. Haberl and M. Ludwig, "A characterization of Lp intersection bodies," *International Mathematics Research Notices*, vol. 2006, 2006.

[11] J. Li, S. F. Yuan, and G. S. Leng, "L_p-Blaschke valuations," *Transactions of the American Mathematical Society*, vol. 367, no. 5, pp. 3161–3187, 2015.

[12] J. Li and G. S. Leng, "L_p Minkowski valuations on polytopes," *Advances in Mathematics*, vol. 299, pp. 139–173, 2016.

[13] M. Ludwig and M. Reitzner, "A classification of $SL(n)$ invariant valuations," *Annals of Mathematics*, vol. 172, no. 2, pp. 1219–1267, 2010.

[14] E. Lutwak, D. Yang, and G. Zhang, "Sharp affine L_p Sobolev inequalities," *Journal of Differential Geometry*, vol. 62, pp. 17–38, 2002.

[15] T. Y. Ma, "The minimal dual Orlicz surface area," *Taiwanese Journal of Mathematics*, vol. 20, no. 2, pp. 287–309, 2016.

[16] M. Meyer and E. Werner, "On the p-affine surface area," *Advances in Mathematics*, vol. 152, no. 2, pp. 288–313, 2000.

[17] C. Schütt and E. Werner, "Surface bodies and p-affine surface area," *Advances in Mathematics*, vol. 187, no. 1, pp. 98–145, 2004.

[18] F. Schuster and T. Wannerer, "Minkowski valuations and generalized valuations," *Journal of the European Mathematical Society*, vol. 20, no. 8, pp. 1851–1884, 2018.

[19] E. Werner, "Renyi divergence and Lp-affine surface area for convex bodies," *Advances in Mathematics*, vol. 230, no. 3, pp. 1040–1059, 2012.

[20] E. Werner and D. Ye, "Inequalities for mixed p-affine surface area," *Mathematische Annalen*, vol. 347, no. 3, pp. 703–737, 2010.

[21] D. H. Wu, "A generalization of L_p -Brunn-Minkowski inequalities and L_p-Minkowski problems for measures," *Advances in Applied Mathematics*, vol. 89, pp. 156–183, 2017.

[22] D. H. Wu, "The isomorphic Busemann-Petty problem for s -concave measures," *Geometriae Dedicata*, vol. 204, no. 1, pp. 131–148, 2020.

[23] D. M. Xi, H. L. Jin, and G. S. Leng, "The Orlicz Brunn-Minkowski inequality," *Advances in Mathematics*, vol. 260, pp. 350–374, 2014.

[24] D. Zou and G. Xiong, "The minimal Orlicz surface area," *Advances in Applied Mathematics*, vol. 61, pp. 25–45, 2014.

[25] K. S. Chou and X. J. Wang, "The Lp-Minkowski problem and the Minkowski problem in centroaffine geometry," *Advances in Mathematics*, vol. 205, no. 1, pp. 33–83, 2006.

[26] D. Hug, E. Lutwak, D. Yang, and G. Zhang, "On the L_p Minkowski problem for polytopes," *Discrete & Computational Geometry*, vol. 33, pp. 699–715, 2005.

[27] K. J. Böröczky and F. Fodor, "The L_p dual Minkowski problem for $p > 1$ and $q > 0$," *Differential Equations*, vol. 266, no. 12, pp. 7980–8033, 2019.

[28] A. Stancu, "The discrete planar L_0-Minkowski problem," *Advances in Mathematics*, vol. 167, no. 1, pp. 160–174, 2002.

[29] A. Stancu, "On the number of solutions to the discrete two-dimensional L_0-Minkowski problem," *Advances in Mathematics*, vol. 180, no. 1, pp. 290–323, 2003.

[30] G. X. Zhu, "The logarithmic Minkowski problem for polytopes," *Advances in Mathematics*, vol. 262, pp. 909–931, 2014.

[31] G. X. Zhu, "The centro-affine Minkowski problem for polytopes," *Journal of Differential Geometry*, vol. 101, no. 1, pp. 159–174, 2015.

[32] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, "The logarithmic Minkowski problem," *Journal of the American Mathematical Society*, vol. 26, no. 3, pp. 831–852, 2013.

[33] C. Haberl and F. Schuster, "Asymmetric affine Lp Sobolev inequalities," *Journal of Functional Analysis*, vol. 257, no. 3, pp. 641–658, 2009.

[34] E. Lutwak, D. Yang, and G. Zhang, "Optimal Sobolev norms and the Lp Minkowski problem," *International Mathematics Research Notices*, vol. 2006, 2006.

[35] G. Zhang, "The affine Sobolev inequality," *Journal of Differential Geometry*, vol. 53, no. 1, pp. 183–202, 1999.

[36] C. Q. Chen, Y. Huang, and Y. M. Zhao, "Smooth solutions to the L_p dual Minkowski problem," *Mathematische Annalen*, vol. 373, pp. 953–976, 2018.

[37] H. D. Chen and Q. R. Li, "The L_p dual Minkowski problem and related parabolic flows," *Journal of Functional Analysis*, vol. 281, pp. 109–139, 2021.

[38] H. D. Chen, S.-B. Chen, and Q. R. Li, "Variations of a class of Monge–Ampère-type functionals and their applications," *Analysis and PDE*, vol. 14, no. 3, pp. 689–716, 2021.

[39] Y. Huang and Y. Zhao, "On the L_p dual Minkowski problem," *Advances in Mathematics*, vol. 332, pp. 57–84, 2018.

[40] X. Li, H. J. Wang, and J. Z. Zhou, "(p, q)-mixed geominimal surface area and (p, q)-mixed affine surface area," *Journal of Mathematical Analysis and Applications*, vol. 475, no. 2, pp. 1472–1492, 2019.

[41] Q. R. Li, J. K. Liu, and J. Lu, "Nonuniqueness of solutions to the Lp dual Minkowski problem," *International Mathematics Research Notices*, vol. 2022, no. 12, pp. 9114–9150, 2022.

[42] H. J. Wang, N. F. Fang, and J. Z. Zhou, "Continuity of the solution to the dual Minkowski problem for negative indices," *Proceedings of the American Mathematical Society*, vol. 147, no. 3, pp. 1299–1312, 2019.

[43] T. Y. Ma, D. H. Wu, and Y. B. Feng, "(p, q)-John ellipsoids," *Journal of Geometric Analysis*, vol. 31, no. 10, pp. 9597–9632, 2021.

[44] X. Li, H. J. Wang, and J. Z. Zhou, "On (p, q)-John ellipsoid," *Science China Mathematics*, vol. 50, no. 8, pp. 1–23, 2020.

[45] E. Lutwak, D. Yang, and G. Zhang, "L_p John ellipsoids," *Proceedings of the London Mathematical Society*, vol. 90, no. 2, pp. 497–520, 2005.

[46] R. Schneider, "Convex bodies: the Brunn-Minkowski theory," in *Encyclopedia Math. Appl*, Cambridge University Press, Cambridge, second edition, 2014.

[47] G. Hardy, J. Littlewood, and G. Po'lya, *Inequalities*, Cambridge University Press, Cambridge, 1952.

[48] K. J. Böröczky and M. Ludwig, "Minkowski valuations on lattice polytopes," *Journal of the European Mathematical Society*, vol. 21, pp. 163–197, 2019.

[49] C. Haberl, "Minkowski valuations intertwining the special linear group," *Journal of the European Mathematical Society*, vol. 14, no. 5, pp. 1565–1597, 2012.

[50] M. Ludwig, "Ellipsoids and matrix-valued valuations," *Duke Mathematical Journal*, vol. 119, no. 1, pp. 159–188, 2003.

[51] M. Ludwig, "Intersection bodies and valuations," *American Journal of Mathematics*, vol. 128, no. 6, pp. 1409–1428, 2006.

[52] M. Ludwig, "Minkowski areas and valuations," *Journal of Differential Geometry*, vol. 86, no. 1, pp. 133–161, 2010.

[53] Y. M. Zhao, "On Lp-affine surface area and curvature measures," *International Mathematics Research Notices*, vol. 2016, no. 5, pp. 1387–1423, 2016.

[54] W. D. Wang and X. Zhao, "Some inequalities for the (p,q)-mixed affine surface areas," *Quaestiones Mathematicae*, vol. 44, no. 5, pp. 599–613, 2021.

[55] D. Hug, "Contributions to affine surface area," *Manuscripta Mathematica*, vol. 91, no. 1, pp. 283–301, 1996.

[56] E. Werner and D. Ye, "New Lp affine isoperimetric inequalities," *Advances in Mathematics*, vol. 218, no. 3, pp. 762–780, 2008.