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Research Article

Cesaŕo Summable Relative Uniform Difference Sequence of Positive Linear Functions

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In this article, we introduce the class of sequence of functions $C_1(\Delta, ru)$ of Cesaŕo summable relative uniform difference sequence of functions. We have studied the topological properties of $C_1(\Delta, ru)$. We also obtain the necessary and sufficient condition to characterize the matrix classes $(C_1(\Delta, ru), \ell_\infty(ru)), (C_1(\Delta, ru), c(ru), P)$.

1. Introduction

Throughout the study, $\omega(ru)$, $C_1(\Delta, ru)$, and $\ell_\infty(ru)$ denote the classes of all relative uniform sequence space, Cesaŕo summable relative uniform difference sequence space, and bounded relative uniform sequence space, respectively.

Moore in 1910 introduced the notion of uniform convergence of a sequence of functions relative to a scale function. Chittenden [1–3] gave the detailed definition of the notion as follows.

Definition 1 (see [1]). A sequence $(f_i(x))$ of single-valued, real-valued functions $f_i(x)$ of a variable x ranging over a compact subset D of real numbers is said to be relatively uniformly convergent on D w. r. t. a scale function $\sigma(x)$ in case there exist a limiting function $f(x)$ and scale function $\sigma(x)$ defined on D and for every ε, an integer $n_0 = n_0(\varepsilon)$ such that for every $n \geq n_0$ and for all $x \in D$,

$$|f_i(x) - f(x)| < \varepsilon|\sigma(x)|. \tag{1}$$

The notion was further discussed from various aspects by Demirci et al. [4], Demirci and Orhan [5], Devi and Tripathy [6], and many others.

Example 1. Let $0 < a < 1$ be a real number. Consider the sequence of functions $(f_i(x)), f_i : [a, 1] \longrightarrow R$, for all $i \in N$ defined by

$$f_i(x) = \frac{1}{ix}, \text{for all } x \in [a, 1], i \in N. \tag{2}$$

This sequence of functions does not converge uniformly to 0 on $[a, 1]$. However, $(f_i(x))$ converges to 0 uniformly with respect to the scale function $\sigma(x)$ defined by

$$\sigma(x) = \frac{1}{x}, \text{for all } x \in [a, 1]. \tag{3}$$

Kizmaz [7] defined the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ as follows:

$$Z(\Delta) = \{x = (x_i): (\Delta x_i) \in Z\}, \quad (4)$$

for $Z = \ell_\infty, c, c_0$ where $\Delta x_i = x_i - x_{i+1}, i \in N$.

These sequence spaces are Banach space under the norm

$$\|x_i\|_\Delta = |x_1| + \sup_{i\in N}|\Delta x_i|. \quad (5)$$

The notion was further studied from different aspects by many others [8–13].

The Cesaŕo sequence space $\mathrm{Ces}_\infty, \mathrm{Ces}_p (1 < p<\infty)$ were introduced by Shiue [14], and it has been shown that $\ell_\infty \subset \mathrm{Ces}_p$; the inclusion is strict for $1 < p < \infty$. Further, the Cesaŕo sequence spaces X_p and X_∞ of nonabsolute type were defined by Ng and Lee [15, 16]. For a detail account of Cesaŕo difference sequence space, one may refer to [17–20].

Let $A = (a_{ni})$ be an infinite matrix of real or complex numbers. Then, A transforms from the sequence space λ into the sequence space F, if $Ax \in F$ for each sequence $(x_i) \in \lambda$ that is $Ax = (A_n x) \in F$, where $A_n x = \sum_{i=1}^{\infty} a_{ni} x_i$, provided that the infinite series converges for each $n \in N$.

Matrix transformation between sequence space was studied from different aspects by many others [21–25].

2. Definitions and Preliminaries

Definition 2. A sequence space λ is said to be solid or normal if $(x_i) \in \lambda$ implies $(\alpha_i x_i) \in \lambda$, for all (α_i) with $|\alpha_i| \le 1$, for all $i \in N$.

Definition 3. A sequence space λ is said to be *monotone* if it contains the canonical preimages of all its step spaces.

Remark 4. A sequence space λ is solid then, λ is monotone.

Definition 5. A sequence space λ is said to be *symmetric* if $(x_i) \in \lambda \Rightarrow (x_{\pi(i)}) \in \lambda$, for all $i \in N$, where π is a permutation of N, the set of natural numbers.

Definition 6. A sequence space λ is said to be *convergence free* if $(x_i) \in \lambda$ and $x_i = 0 \Rightarrow y_i = 0$ together with $(y_i) \in \lambda$, for all $i \in N$.

Definition 7. A sequence space λ is said to be a *sequence algebra* if $(x_i . y_i) \in \lambda$ whenever (x_i) and (y_i) belongs to λ, for all $i \in N$.

In this article we introduce the sequence space $C_1(\Delta, ru)$ of Cesaŕo summable relative uniform difference sequence of functions and it is defined as follows:

$$C_1(\Delta, ru) = \{f = (f_i(x)) \in \omega(ru): (\Delta f_i(x)) \in C_1 \ w.r.t. \text{the scale function } \sigma(x)\}, \quad (6)$$

where $\Delta f_i(x) \in C_1$ (ru) and $\Delta f_i(x) = f_i(x) - f_{i+1}(x)$.

3. Main Results

We state the following result without proof.

Theorem 8. *The sequence space $C_1(\Delta, ru)$ is a normed linear space.*

Theorem 9. *The sequence space $C_1(\Delta, ru)$ is a Banach space normed by*

$$\|f\|_{(\Delta,\sigma)} = \sup_{\|x\|\le 1} \frac{\|f_1(x)\|\|\sigma(x)\|}{\|x\|} + \sup_{p\ge 1} \sup_{\|x\|\le 1} \frac{1}{p} \frac{\sum_{i=1}^{p}\|\Delta f_i(x)\|\|\sigma(x)\|}{\|x\|}. \quad (7)$$

Proof. Let $(f^n(x))$ be a Cauchy sequence in $C_1(\Delta, ru)$ where

$$(f^n(x)) = (f_i^n(x)) = (f_1^n(x), f_2^n(x), \cdots) \in C_1(\Delta, ru), \text{for each } i \in N. \quad (8)$$

□

Then,

$$\|f^n(x) - f^m(x)\|_{(\Delta,\sigma)} = \sup_{\|x\|\le 1} \frac{\|f_1^n(x) - f_1^m(x)\|\|\sigma(x)\|}{\|x\|} + \sup_{p\ge 1} \sup_{\|x\|\le 1} \frac{1}{p} \cdot \frac{\sum_{i=1}^{p}\|\Delta f_i^n(x) - \Delta f_i^m(x)\|\|\sigma(x)\|}{\|x\|} \longrightarrow 0. \quad (9)$$

For all $n, m \ge n_0$,

$$\|f^n(x) - f^m(x)\|_{(\Delta,\sigma)} = \sup_{\|x\|\le 1} \frac{\|f_1^n(x) - f_1^m(x)\|\|\sigma(x)\|}{\|x\|} + \sup_{p\ge 1} \sup_{\|x\|\le 1} \frac{1}{p} \cdot \frac{\sum_{i=1}^{p}\|\Delta f_i^n(x) - \Delta f_i^m(x)\|\|\sigma(x)\|}{\|x\|} < \frac{\varepsilon}{2}. \quad (10)$$

$(f_1^n(x))$ is a Cauchy sequence in D w.r.t. $\sigma(x)$ for all $x \in D$.

$\Rightarrow (f_1^n(x))$ is convergent in D w.r.t. $\sigma(x)$ for all $x \in D$.

Let $\lim_{n\longrightarrow\infty} f_1^n(x) = f_1(x), x \in D$.

Similarly, $\lim_{n\longrightarrow\infty} (1/p)\sum_{i=1}^{p} \Delta f_i^n(x) = (1/p)\sum_{i=1}^{p} \Delta f_i(x), x \in D$.

From the above equations we get,

$$\lim_{n\longrightarrow\infty} f_i^n(x) = f_i(x), \quad (11)$$

for all $x \in D$, for all $i \in N$.

From (10) we have

$$\lim_{m\longrightarrow\infty}\frac{\|(f_1^n(x)-f_1^m(x))\sigma(x)\|}{\|x\|}=\frac{\|(f_1^n(x)-f_1(x))\sigma(x)\|}{\|x\|}<\frac{\varepsilon}{2}, \quad (12)$$

for all $x \in D$.

Similarly,

$$\lim_{m\longrightarrow\infty}\frac{1}{p}\frac{\sum_{i=1}^{p}\|(\Delta f_i^n(x)-\Delta f_i^m(x))\sigma(x)\|}{\|x\|}=\frac{1}{p}\frac{\sum_{i=1}^{p}\|(\Delta f_i^n(x)-\Delta f_i(x))\sigma(x)\|}{\|x\|}<\frac{\varepsilon}{2}, \quad (13)$$

for all $x \in D$ and $i \in N$.

Since $\varepsilon/2$ is not dependent on i, we have

$$\sup_{\|x\|\le 1}\frac{\|(f_1^n(x)-f_1(x))\sigma(x)\|}{\|x\|}<\frac{\varepsilon}{2},$$
$$\sup_{p\ge 1}\sup_{\|x\|\le 1}\frac{1}{p}\frac{\sum_{i=1}^{p}\|(\Delta f_i^n(x)-\Delta f_i(x))\sigma(x)\|}{\|x\|}<\frac{\varepsilon}{2}. \quad (14)$$

Evidently,

$$\|f^n(x)-f(x)\|_{(\Delta,\sigma)}=\sup_{\|x\|\le 1}\frac{\|f_1^n(x)-f_1(x)\|\|\sigma(x)\|}{\|x\|}+\sup_{p\ge 1}\sup_{\|x\|\le 1}\frac{1}{p}\cdot\frac{\sum_{i=1}^{p}\|\Delta f_i^n(x)-\Delta f_i(x)\|\|\sigma(x)\|}{\|x\|}\le\varepsilon. \quad (15)$$

$\Rightarrow\|f^n(x)-f(x)\|_{(\Delta,\sigma)}\le\varepsilon$, for all $n\ge n_0$, for all $x\in D$.

Therefore, $(f^n(x)-f(x))\in C_1(\Delta,ru)$, for all $n\ge n_0$, for all $x\in D$.

Then, $f(x)=f^n(x)-(f^n(x)-f(x))\in C_1(\Delta,ru)$ since $C_1(\Delta,ru)$ is a linear space.

Theorem 10. *The inclusion $C_1(ru)\subset C_1(\Delta,ru)$ strictly holds.*

Proof. The proof of the theorem is obvious and the strictness of the inclusion is shown in the following example. □

Example 1. Let $0<a<1$ be a real number and $D=[a,1]$. Consider the sequence of real valued functions $(f_i(x)), f_i:[a,1]\longrightarrow R$, for all $i\in N$, defined by

$$f_i(x)=ix, \text{ for all } x\in[a,1],$$
$$\Delta f_i(x)=f_i(x)-f_{i+1}(x)=x, \text{ for all } x\in[a,1]. \quad (16)$$

$(f_i(x))\in C_1(\Delta,ru)$ w.r.t. the scale function $\sigma(x)=1$, for all xin D, but $(f_i(x))\notin\ell_\infty(ru)$.

Hence, the inclusion is strict.

Theorem 11. *The inclusion $c(\Delta,ru)\subset C_1(\Delta,ru)$ strictly holds.*

Proof. The proof is obvious and the strictness of the inclusion is shown in the following example. □

Example 2. Let $0<a<1$ be a real number and $D=[a,1]$. Consider the sequence of real valued functions $(f_i(x)), f_i:[a,1]\longrightarrow R$, for all $i\in N$, defined by

$$f_i(x)=\begin{cases}x, & \text{for } i \text{ is odd},\\ 0, & \text{otherwise},\end{cases}$$
$$\Delta f_i(x)=\begin{cases}x, & \text{for } i \text{ is odd},\\ -x, & \text{otherwise}.\end{cases} \quad (17)$$

We have $(f_i(x))\in C_1(\Delta,ru)$ w.r.t. the scale function $\sigma(x)$ defined by

$$\sigma(x)=\left\{\frac{1}{x}, \text{for all } x\in[a,1],\right. \quad (18)$$

but $(f_i(x))\notin c(\Delta,ru)$. Hence, the inclusion is strict.

Theorem 12. *The sequence space $C_1(\Delta,ru)$ is not monotone.*

Proof. The proof is shown in the following example. □

Example 3. Let $0<a<1$ be a real number and $D=[a,1]$. Consider the sequence of real valued functions $(f_i(x)), f_i:[a,1]\longrightarrow R$, for all $i\in N$, defined by

$$f_i(x)=ix, \text{ for all } x\in[a,1],$$
$$\Delta f_i(x)=f_i(x)-f_{i+1}(x)=x, \text{ for all } x\in[a,1]. \quad (19)$$

$(f_i(x))\in C_1(\Delta,ru)$ w.r.t. the scale function defined on D by $\sigma(x)=1$.

Let $(g_i(x))$ be the preimage of $(f_i(x))$ defined by

$$g_i(x)=\begin{cases}i^2x, & \text{for } i=k^2, k\in N,\\ 0, & \text{otherwise}.\end{cases} \quad (20)$$

One cannot get a scale function that makes $(g_i(x))\in C_1(\Delta,ru)$.

Hence $C_1(\Delta,ru)$ is not monotone.

Remark 13. The sequence space $C_1(\Delta,ru)$ is not solid since $C_1(\Delta,ru)$ is not monotone.

Theorem 14. *The sequence space $C_1(\Delta,ru)$ is not symmetric.*

Proof. The proof of the theorem is shown with the help of the following example. □

Example 4. Let us consider the sequence of functions $(f_i(x))$ considered in Example 3. Let $(g_i(x))$ be the rearrangement sequence of functions of $(f_i(x))$ defined by

$$g_i(x) = \begin{cases} (i+1)x, & \text{for } i = 2j-1, j \in N, \\ (i-1)x, & \text{for } i = 2j, j \in N. \end{cases} \tag{21}$$

One cannot get a scale function that makes $(g_i(x)) \in C_1(\Delta, ru)$.

Hence, $C_1(\Delta, ru)$ is not symmetric.

Theorem 15. *The sequence space $C_1(\Delta, ru)$ is not sequence algebra.*

Proof. The proof of the theorem is shown in the following example. □

Example 5. Let $0 < a < 1$ be a real number and $D = [a, 1]$. Consider the sequences of real valued functions $(f_i(x)), f_i : [a, 1] \longrightarrow R$, and $(g_i(x)), g_i : [a, 1] \longrightarrow R$, for all $i \in N$, defined by

$$\begin{gathered} f_i(x) = g_i(x) = ix, \text{for all } i \in N. \\ f_i(x).g_i(x) = i^2 x^2. \end{gathered} \tag{22}$$

We get that $(f_i(x)), (g_i(x)) \in C_1(\Delta, ru)$ but one cannot get a scale function that makes $(f_i(x).g_i(x)) \in C_1(\Delta, ru)$.

Hence, $C_1(\Delta, ru)$ is not sequence algebra.

Theorem 16. *The sequence space $C_1(\Delta, ru)$ is not convergence free.*

Proof. The proof of the theorem follows from example below. □

Example 6. Let $0 < a < 1$ be a real number and $D = [a, 1]$. Consider the sequence of real valued functions $(f_i(x)), f_i : [a, 1] \longrightarrow R$, for all $i \in N$, defined by

$$\begin{gathered} (f_i(x)) = x, \text{for all } x \in [a, 1], \\ \Delta f_i(x) = 0. \end{gathered} \tag{23}$$

Therefore, $(f_i(x)) \in C_1(\Delta, ru)$ w.r.t. the constant scale function defined on D by $\sigma(x) = 1$.

Let us consider another sequence of functions $(g_i), g_i$: $[a, 1] \longrightarrow R$ defined by

$$g_i(x) = \begin{cases} g_1(x) = x\,; \\ g_{i+1}(x) = g_{m-1}(x) + (n+1)x, & \text{for } m \geq 2, m, n, i \in N, \end{cases}$$
$$C_1(\Delta, ru) = -nx, n \geq 2, n \in N. \tag{24}$$

One cannot find a scale function that makes $(g_i(x)) \in C_1(\Delta, ru)$.

Hence, the sequence space $C_1(\Delta, ru)$ is not convergence free.

3.1. Matrix Transformation between Sequence of Functions. In this section, we give certain matrix classes between the sequence of functions.

Theorem 17. *$A \in (C_1(\Delta, ru), \ell_\infty(ru))$ if and only if $\sup_{n\geq 1} \sum_{i=2}^{\infty} (i-1)|a_{ni}| < \infty$.*

Proof. Let $(f_i(x)) \in C_1(\Delta, ru)$ and $\sup_{n\geq 1} \sum_{i=2}^{\infty} (i-1)|a_{ni}| < \infty$.

$$\begin{aligned} \sum_{i=1}^{\infty} a_{ni} f_i(x)\sigma(x) &= -\sum_{i=2}^{\infty} (i-1) a_{ni} \left(\frac{1}{i-1} \sum_{t=1}^{i-1} \Delta f_t(x)\sigma(x) \right) \\ &\quad + f_1(x)\sigma(x) \sum_{i=1}^{\infty} a_{ni} = \sum_{i=2}^{\infty} a_{ni} \sum_{t=1}^{i-1} \Delta f_t(x)\sigma(x). \end{aligned} \tag{25}$$

□

From the relation between dual and matrix map, we know that $c^\alpha = \ell_1$. Hence,

$$\sum_{i=2}^{\infty} a_{ni} \sum_{t=1}^{i-1} \Delta f_t(x)\sigma(x) \text{ converges absolutely w.r.t.} \sigma(x). \tag{26}$$

For $p \in N$, we have, $\sum_{i-1}^{p} a_{ni} f_i(x)\sigma(x) = -\sum_{i=1}^{p} a_{ni} (\sum_{t=1}^{i-1} \Delta f_t(x)\sigma(x)) + f_1(x)\sigma(x) \sum_{i=1}^{p} a_{ni}$.

By the argument (10), we know that $\sum_{i=1}^{p} a_{ni} f_i(x)\sigma(x)$ is absolutely convergent w.r.t. scale function $\sigma(x)$.

Then,

$$\begin{aligned} \sum_{i=1}^{\infty} |a_{ni} f_i(x)\sigma(x)| &\leq \left(\sup_n \sum_{i=2}^{\infty} (i-1)|a_{ni}| \right) \\ &\quad \cdot \left(\sup_{x\leq 1} \sup_{i\geq 2} \frac{1}{i-1} \sum_{t=1}^{i-1} |\Delta f_t(x)\sigma(x)| \right) \\ &\quad + |f_1(x)\sigma(x)| \sup_n \sum_{i=2}^{\infty} (i-1)|a_{ni}|. \end{aligned} \tag{27}$$

We have, $\sum_{i=1}^{\infty} a_{ni} f_i(x)\sigma(x) < \infty$ since $\sup_{n\geq 1} \sum_{i=2}^{\infty} (i-1)|a_{ni}| < \infty$.

Conversely, we know that A is a bounded linear function from $C_1(\Delta, ru)$ to $\ell_\infty(ru)$, so we can write,

$$\begin{aligned} \sum_{i=1}^{\infty} |a_{ni} f_i(x)\sigma(x)| &= |(A_n f)\sigma(x)| \leq \sup_n |(A_n f)\sigma(x)| \\ &= \|(Af)\sigma(x)\|_\infty \leq \|A\| \|f\|_{(\Delta,\sigma).} \end{aligned} \tag{28}$$

We choose a sequence of functions $(f_i), f_i$: $[0, 1] \longrightarrow R$ defined by

$$f_i(x) = \begin{cases} (i-1)x sgn a_{ni}, & \text{for } 1 < i \leq r, \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

We get $(f_i(x)) \in C_1(\Delta, ru)$ w.r.t. scale function $\sigma(x) = 1/x$ with the norm $\|f\|_{(\Delta,\sigma)} = 1$.

Putting the value of $(f_i(x))$ in equation (26) and letting the limit $r \longrightarrow \infty$, we get

$$\sum_{i=2}^{\infty}(i-1)|a_{ni}| \le \|A\|. \quad (30)$$

Theorem 18. *$A \in (C_1(\Delta, ru), c(ru), P)$ if and only if*

(i) $\sup_n \sum_{i=2}^{\infty}(i-1)|a_{ni}| < \infty$

(ii) $\lim_n \sum_{i=2}^{\infty}(i-1)a_{ni} = -1$

(iii) $\lim_n a_{ni} = 0$ *for each* i

(iv) $\lim_n \sum_i a_{ni} = 0$.

Proof. Let us assume that the conditions (i)-(iv) hold true and let $(f_i(x)) \in C_1(\Delta, ru)$, i.e., $\lim_i (1/i)\sum_{t=1}^{i}\Delta f_t(x)\sigma(x) = f(x)$ (say).

We know from (i) that for each $x \in D$ and $n \in N$, $\sum_i (i-1)|a_{ni}|$ converges. It follows that $\sum_{i=2}^{\infty}(i-1)a_{ni}((1/(i-1)i-1)\sum_{i=2}^{\infty}\Delta f_t(x)\sigma(x))$ converges.

$$\sum_i a_{ni} f_i(x)\sigma(x) = -\sum_{i=2}^{\infty}(i-1)a_{ni}\left(\frac{1}{i-1}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)\right) - f(x)\sum_i (i-1)a_{ni} + f_1(x)\sigma(x)\sum_i a_{ni}. \quad (31)$$

For any $n_0 \in N$, we have

$$\left|\sum_{i=2}^{\infty}(i-1)a_{ni}\left(\frac{1}{i-1}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)\right)\right| \le \sup_{x\le 1}\sup_i\left(\frac{1}{i-1}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)\right)\sum_{i=2}^{n_0}(i-1)|a_{ni}| + \sup_n \sum_{i=2}^{\infty}(i-1)a_{ni}\sup_{x\le 1}\sup_{i>n_0} \cdot \left(\frac{1}{i-1}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)\right)$$

$$\lim_n \sup\left|\sum_{i=2}^{\infty}(i-1)a_{ni}\left(\frac{1}{i-1}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)\right)\right| \le \sup_n \sum_{i=2}^{\infty}(i-1)a_{ni}\sup_{x\le 1}\sup_{i>n_0}\left(\frac{1}{i-1}\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)\right). \quad (32)$$

Let $n_0 = 0$, we get $\sum_{i=2}^{\infty}(i-1)a_{ni}((1/(i-1)i-1)\sum_{t=1}^{i-1}\Delta f_t(x)\sigma(x) - f(x)) \longrightarrow 0$.

Substituting this value in equation (31) and using conditions (ii) and (iv), we get $\sum_i a_{ni} f_i(x)\sigma(x)$ converges to $f(x)$ w.r.t. the scale function $\sigma(x)$. Conversely, let $A \in (C_1(\Delta, ru), c(ru), P)$. Then, $(\sum_i a_{ni} f_i(x)\sigma(x)) \in c(ru)$, for all

$$(f_i(x)) \in C_1(\Delta, ru). \quad (33)$$

Condition (i) can be proceed as same as shown in Theorem 17.

(ii) Let $(f_i), f_i\colon [0,1] \longrightarrow R$ defined by

$$f_i(x) = ix, \text{for all } i \in N. \quad (34)$$

Then, $f_i(x) \in C_1(\Delta, ru)\ w.r.t.\sigma(x) = 1/x$.
Since $\Delta f_i(x) \longrightarrow -1$, we have $\lim_n \sum_i (i-1)a_{ni} = -1$.
(iii) Let $(f_i), f_i\colon [0,1] \longrightarrow R$ defined by

$$f_i(x) = e_i x, \text{for all } i \in N. \quad (35)$$

Then, $f_i(x) \in C_1(\Delta, ru)\ w.r.t.\sigma(x) = 1/x$.
Since $\Delta f_i(x) \longrightarrow 0$, we have $\lim_n a_{ni} = 0$.
(iv) Let $(f_i), f_i\colon [0,1] \longrightarrow R$ defined by

$$f_i(x) = x, \text{for all } i \in N. \quad (36)$$

Then, $f_i(x) \in C_1(\Delta, ru)\ w.r.t.\sigma(x) = 1/x$.
Since $\Delta f_i(x) \longrightarrow 0$, we have $\lim_n \sum_i (i-1)a_{ni} = 0$.
The following theorem is stated without proof and can be proceeded the same as in Theorem 18. □

Theorem 19. *$A \in (C_1(\Delta, ru), c_0(ru))$ if and only if*

(i) $\sup_n \sum_{i=2}^{\infty}(i-1)|a_{ni}| < \infty$

(ii) $\lim_n \sum_{i=2}^{\infty}(i-1)a_{ni} = 0$

(iii) $\lim_n a_{ni} = 0$ *for each* i

(iv) $\lim_n \sum_i a_{ni} = 0$.

4. Conclusions

In this article, we studied the concept of Cesaŕo summability from the aspects of relative uniform convergence of difference sequence of positive linear functions w.r.t. a scale function $\sigma(x)$ on a compact domain D. The class of difference sequence of functions $C_1(\Delta, ru)$ is introduced, and its properties like solid, monotone, symmetric, sequence algebra, and convergence free are discussed. We have also further introduced characterization of matrix classes of $(C_1(\Delta, ru), \ell_\infty(ru)), (C_1(\Delta, ru), c(ru), P)$ and $(C_1(\Delta, ru), c_0(ru))$.

Data Availability

No data were used to support to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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