Hindawi
Journal of Function Spaces
Volume 2022, Article ID 3811326, 22 pages
https://doi.org/10.1155/2022/3811326

Research Article

Fixed Point Property of Variable Exponent Cesàro Complex Function Space of Formal Power Series under Premodular

Awad A. Bakery [1,2] and Elsayed A. E. Mohamed [1,3]

[1]University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Egypt
[3]Department of Mathematics, Faculty of Education, Alzaeim Alazhari University, Khartoum, Sudan

Correspondence should be addressed to Elsayed A. E. Mohamed; elsayed.bayen2020@yahoo.com

Received 5 February 2022; Accepted 7 March 2022; Published 26 March 2022

Academic Editor: Hüseyin Işık

We have defined the variable exponent of the Cesàro complex function space of formal power series. We have constructed the prequasi-ideal generated by s -numbers and this new space of complex functions. We present some topological and geometric structures of this class of ideal. The existence of Caristi's fixed point is examined. Some geometric properties related to the fixed point theory are presented. Finally, real-world examples and applications show solutions to some nonlinear difference equations.

1. Introduction

Since the publishing of the book [1] on the Banach fixed point theorem, several mathematicians have studied possible extensions to the Banach fixed point theorem. The nonlinear analysis relies heavily on the Banach contraction principle, a powerful nonlinear analysis tool. The variable exponent Lebesgue spaces $L_{(r)}$ contain Nakano sequence spaces. Variable exponent spaces were thought to offer adequate frameworks for the mathematical components of several issues. Standard Lebesgue spaces were inadequate throughout the second half of the twentieth century. Since these spaces and their effects have become a well-known and efficient instrument for solving a range of problems, they have become a flourishing topic of research, with ramifications that extend into a wide variety [2] of mathematical disciplines. The study of variable exponent Lebesgue spaces $L_{(r)}$ received additional impetus from the mathematical description of non-Newtonian fluid hydrodynamics [3, 4]. Non-Newtonian fluids, also known as electrorheological fluids, have various applications ranging from military science to civil engineering and orthopedics. Guo and Zhu [5] investigated a class of stochastic Volterra-Levin equations with Poisson jumps. Mao et al. [6] were concerned with neutral stochastic functional differential equations driven by pure jumps (NSFDEwPJs). They proved the existence and uniqueness of the solution to NSFDEwPJs whose coefficients satisfy the local Lipschitz condition and established the pth exponential estimations and almost surely asymptotic estimations of the solution for NSFDEwJs. Yang and Zhu [7] concerned with a class of stochastic neutral functional differential equations of Sobolev type with Poisson jumps. The mapping ideal theory is well regarded in functional analysis. Using s-numbers is an essential technique. Pietsch [8–11] developed and studied the theory of s-numbers of linear bound mappings between Banach spaces. He offered and explained some topological and geometric structures of the quasi ideals of ℓ_p -type mappings. Then, Constantin [12] generalized the class of ℓ_p -type mappings to the class of ces_p -type mappings. Makarov and Faried [13] showed some inclusion relations of ℓ_p -type mappings. As a generalization of ℓ_p -type mappings, Stolz mappings and mappings' ideal were examined by Tita [14, 15]. In [16], Maji and Srivastava studied the class $A_p^{(s)}$ of s-type ces_p mappings using s-number sequence and Cesàro sequence spaces and they introduced a new class $A_{p,q}^{(s)}$ of s-type $\text{ces}(p, q)$ mappings by weighted ces_p with $1 < p < \infty$. In [17], the class of s-type $Z(u, v; \ell_p)$

mappings was defined and some of their properties were explained. Yaying et al. [18] defined and studied χ_r^η, whose its r-Cesàro matrix in ℓ_n, with $r \in (0, 1]$ and $1 < \eta < \infty$. They explained the quasi-Banach ideal of type χ_r^η, with $r \in (0, 1]$ and $1 < \eta < \infty$. Kannan [19] gave an example of a class of mappings with the same fixed point actions as contractions, though that fails to be continuous. The only attempt to describe Kannan operators in modular vector spaces was once made in Reference [20]. Bakery and Mohamed [21] investigated the concept of a prequasinorm on Nakano sequence space with a variable exponent in the range $(0; 1]$. They discussed the adequate circumstances for it to generate prequasi-Banach and closed space when endowed with a definite prequasinorm and the Fatou property of various prequasinorms on it. Additionally, they established a fixed point for Kannan prequasinorm contraction mappings on it and the prequasi-Banach mappings' ideal generated from s-numbers belonging to this sequence space. Also, in [22], they found some fixed points results of Kannan nonexpansive mappings on generalized Cesàro backward difference sequence space of the nonabsolute type. The set of nonnegative integers, real, and complex numbers will be denoted by $\mathcal{N}$, $\mathfrak{R}$, and $\mathbb{C}$, respectively. By $\mathfrak{R}^{\mathcal{N}}$ and $\mathfrak{R}+^{\mathcal{N}}$, we denote the space of real and positive real sequences. By ℓ_∞ and ℓ_r, we denote the spaces of bounded and r-absolutely summable sequences of $\mathfrak{R}$.

Lemma 1 (see [23]). *Suppose $\tau_q > 0$ and $y_q \in \mathfrak{R}$ for all $q \in \mathcal{N}$, then*

$$\left|y_q + z_q\right|^{\tau_q} \le 2^{K-1}\left(\left|y_q\right|^{\tau_q} + \left|z_q\right|^{\tau_q}\right), \tag{1}$$

where $K = \max\{1, \sup_q \tau_q\}$.

If $\tau = (\tau_a) \in \mathfrak{R}^{+\mathcal{N}}$ and $\tau_a \ge 1$, for all $a \in \mathcal{N}$, the variable exponent Cesàro complex function space is denoted by

$$\mathfrak{C}_{\tau(\cdot)} = \left\{ f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{v=0}^{\infty} \widehat{f}_v y^v \text{ and } h(\mu f) < \infty, \text{ for some } \mu > 0 \right\}, \text{ when } h(f) = \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^{\infty} \left|\widehat{f}_k\right|}{a+1} \right)^{\tau_a}. \tag{2}$$

For more information on formal power series spaces and their behaviors, see [24–27]. Many fixed point theorems in a particular space work by either expanding the self-mapping acting on it or expanding the space itself. In this paper, we have introduced the concept of premodular special spaces of formal power series, which are important extensions of the concept of modular spaces. We have built large spaces of solutions to many nonlinear summable and difference equations. It is the first attempt to examine the fixed point theory and Caristi's fixed point in certain premodular special spaces of formal power series. The purpose of this study is arranged, as follows: In Section 2, we present and study the space $(\mathfrak{C}_{\tau(\cdot)})_h$ equipped with a definite function h. In Section 3, we suggest a generalization of Caristi's fixed point theorem. In Section 4, the mapping ideals formed by s-numbers and this function space are constructed, and their geometric and topological properties are presented. Speccifically, we explore, in Section 5, some geometric properties connected with fixed point theory in $(\mathfrak{C}_{\tau(\cdot)})_h$. Finally, in Section 6, we discuss several applications of solutions to summable equations and illustrate our findings with some instances.

2. Some Properties of $\mathfrak{C}_{\tau(\cdot)}$

In this section, we investigate sufficient setups of $\mathfrak{C}_{\tau(\cdot)}$ equipped with definite function h to be prequasiclosed and Banach (ssfps). We also present the Fatou property of various h on $\mathfrak{C}_{\tau(\cdot)}$.

Theorem 2. *If $(\tau_q) \in \ell_\infty$ and $\tau_a > 1$, for all $a \in \mathcal{N}$, then*

$$\mathfrak{C}_{\tau(\cdot)} = \left\{ f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{v=0}^{\infty} \widehat{f}_v y^v \textit{ and } h(\mu f) < \infty, \textit{ for any } \mu > 0 \right\}. \tag{3}$$

Proof.

$$\begin{aligned} \mathfrak{C}_{\tau(\cdot)} &= \left\{ f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{v=0}^{\infty} \widehat{f}_v y^v \text{ and } h(\mu f) < \infty, \text{ for some } \mu > 0 \right\} \\ &= \left\{ f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{v=0}^{\infty} \widehat{f}_v y^v, \inf |\mu|^{\tau_a} \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^{a} \left|\widehat{f}_k\right|}{a+1} \right)^{\tau_a} \right. \\ &\le \left. \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^{a} \left|\mu \widehat{f}_k\right|}{a+1} \right)^{\tau_a} < \infty, \text{ for some } \mu > 0_a \right\} \\ &= \left\{ f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{v=0}^{\infty} \widehat{f}_v y^v, \sum_{a=0}^{\infty} \left(\frac{\sum_{k=0}^{a} \left|\widehat{f}_k\right|}{a+1} \right)^{\tau_a} < \infty \right\} \\ &= \left\{ f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{v=0}^{\infty} \widehat{f}_v y^v \text{ and } h(\mu f) < \infty, \text{ for any } \mu > 0 \right\}. \end{aligned} \tag{4}$$

Let us indicate ϑ, the zero function of $\mathscr{H}$ and the space of finite formal power series by $\mathfrak{F}$, i.e, when $f \in \mathfrak{F}$, then there is $k \in \mathcal{N}$ so that $f(y) = \sum_{a=0}^{k} \widehat{f}_a y^a$. Nakano [28] introduced the concept of modular vector spaces. □

Definition 3. Suppose $\mathscr{H}$ is a vector space. A function $h : \mathscr{H}$ $[0,\infty)$ is said to be modular, if the next conditions hold

(a) If $g \in \mathscr{H}$, then $h(g) \geq 0$ and $g = \vartheta \Longleftrightarrow h(g) = 0$

(b) $h(\eta g) = h(g)$ holds, for all $g \in \mathscr{H}$ and $|\eta| = 1$

(c) The inequality $h(\alpha g + (1-\alpha) f) \leq h(g) + h(f)$ satisfies, for all $g, f \in \mathscr{H}$ and $\alpha \in [0,1]$

Definition 4 (see [29]). The space $\mathscr{H} = \{f \in \mathbb{C}^{\mathbb{C}} : f(y) = \sum_{a=0}^{\infty} \widehat{f}_a y^a\}$ is said to be a special space of formal power series (or in short ssfps), if it verifies the following settings:

(1) $e^{(p)} \in \mathscr{H}$, for every $p \in \mathcal{N}$, where $e^{(p)}(y) = \sum_{a=0}^{\infty} e_a^{(p)} y^a = y^p$

(2) For all $g \in \mathscr{H}$ and $|\widehat{f}_a| \leq |\widehat{g}_a|$, for every $a \in \mathcal{N}$, then $f \in \mathscr{H}$

(3) If $g \in \mathscr{H}$ then $g_{[\cdot]} \in \mathscr{H}$, where $g_{[\cdot]}(y) = \sum_{p=0}^{\infty} \widehat{g[p/2]} y^p$ and $[p/2]$ indicates the integral part of $p/2$

Definition 5 (see [29]). A subspace $\mathscr{H}_h$ of the ssfps is said to be a premodular ssfps, if there is a function $h : \mathscr{H} \longrightarrow [0,\infty)$ verifies the following conditions:

(i) If $g \in \mathscr{H}$, then $h(g) \geq 0$ and $g = \vartheta \Leftrightarrow h(g) = 0$

(ii) When $f \in \mathscr{H}$ and $\lambda \in \mathbb{C}$, then there are $Q \geq 1$ such that $h(\lambda f) \leq |\lambda| Q h(f)$

(iii) Suppose $f, g \in \mathscr{H}$, then there are $P \geq 1$ such that $h(f+g) \leq P(h(f) + h(g))$

(iv) Suppose $|\widehat{f_b}| \leq |\widehat{g_b}|$, for all $b \in \mathcal{N}$, then $h(f) + h(g)$

(v) There are $P_0 \geq 1$ such that $h(f) \leq h(f_{[\cdot]} \leq P_0 h(f))$

(vi) The closure of $\mathfrak{F} = \mathscr{H}_h$

(vii) There are $\xi > 0$ so that $h(\lambda e^{(0)}) \geq \xi |\lambda| h(e^{(0)})$, where $\lambda \in \mathbb{C}$

Clearly, the concept of premodular vector spaces is more general than modular vector spaces, an example of premodular vector space but not modular vector space.

Example 1. The function $h(f) = \sum_{q=0}^{\infty} (\sum_{p=0}^{q} |\widehat{f}_p| / (q+1))^{(2q+3)/(q+4)}$ is a premodular (not a modular) on the vector space $\mathfrak{C}(((2q+3)/(q+4))_{q=0}^{\infty})$. As for every $f, g \in \mathfrak{C}(((2q+3)/(q+4))_{q=0}^{\infty})$, one has

$$h\left(\frac{f+g}{2}\right) = \sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^{q} \left|\left(\widehat{f}_p + \widehat{g}_p\right)/2\right|}{q+1}\right)^{2q+3/q+4} \leq \frac{2}{\sqrt[4]{8}} (h(f) + h(g)), \tag{5}$$

an example of premodular vector space and modular vector space.

Example 2. The function $h(f) = \inf\{\alpha > - : \sum_{q=0}^{\infty} (\sum_{p=0}^{q} |\widehat{f}_p / \alpha| / (q+1))^{(2q+3)/(q+4)} \leq 1\}$ is a premodular (modular) on the vector space $\mathfrak{C}(((2q+3)/(q+2))_{q=0}^{\infty})$.

Definition 6 (see [29]). A subspace $\mathscr{H}_h$ of the ssfps is said to be a prequasinormed ssfps, if there is a function $h : \mathscr{H} \longrightarrow [0,\infty)$ verifies the following conditions:

(i) If $g \in \mathscr{H}$, then the $h(g) \geq 0$ and $g = \vartheta \Longleftrightarrow h(g) = 0$

(ii) When $f \in \mathscr{H}$ and $\lambda \in \mathbb{C}$, then there are $Q \geq 1$ such that $h(\lambda f) \leq |\lambda| Q h(f)$

(iii) Suppose $f, g \in \mathscr{H}$ then there are $P \geq 1$ such that $h(f+g) \leq P(h(f) + h(g))$

Recall that $\mathscr{H}_h$ is said to be a prequasi-Banach ssfps, when $\mathscr{H}_h$ is complete.

Theorem 7 (see [30]). *All premodular ssfps $\mathscr{H}_h$ is a prequasinormed ssfps.*

Theorem 8 (see [30]). *All quasinormed (ssfps) is a prequasinormed (ssfps).*

Definition 9.

(a) The function h on $\mathfrak{C}_{\tau(\cdot)}$ is said to be h-convex, if

$$h(\alpha f + (1-\alpha) g) \leq \alpha h(f) + (1-\alpha) h(g), \tag{6}$$

for every $\alpha \in [0,1]$ and $f, g \mathfrak{C}_{\tau(\cdot)}$

(b) $\{g_q\}_{q \in \mathcal{N}} \subseteq (\mathfrak{C}_{\tau(\cdot)})_h$ is h-convergent to $g \in (\mathfrak{C}_{\tau(\cdot)})_h$, if and only if, $\lim_{q \longrightarrow \infty} h(g_q - g) = 0$. When the h-limit exists, then it is unique

(c) $\{g_q\}_{q \in \mathcal{N}} \subseteq (\mathfrak{C}_{\tau(\cdot)})_h$ is h-Cauchy, if $\lim_{q,r \longrightarrow \infty} h(g_q - g_r) = 0$

(d) $\Gamma \subset (\mathfrak{C}_{\tau(\cdot)})_h$ is h-closed, when for all h-converges, $\{g_1\}_{a \in \mathcal{N}} \subset \Gamma$ to g, then $g \in \Gamma$

(e) $\Gamma \subset (\mathfrak{C}_{\tau(\cdot)})_h$ is h-bounded, if $\delta_h(\Gamma) = \sup\{h(f-g) : f, g \in \Gamma\} < \infty$

(f) The h-ball of radius $\varepsilon \geq 0$ and center f, for every $f \in (\mathfrak{C}_{\tau(\cdot)})_h$, is described as

$$B_h(f,\varepsilon)=\left\{g\in\left(\mathfrak{C}_{\tau(\cdot)}\right)_h : h(f-g)\leq\varepsilon\right\} \tag{7}$$

(g) A prequasinorm h on $\mathfrak{C}_{\tau(\cdot)}$ holds the Fatou property, if for every sequence $\{g^q\}\subseteq(\mathfrak{C}_{\tau(\cdot)})_h$ under $\lim_{q\longrightarrow\infty} h(g^q-g)=0$ and all $f\in(\mathfrak{C}_{\tau(\cdot)})_h$, one has $h(f-g) \leq \sup_r \inf_{q\geq r} h(f-g^q)$

Recall that the Fatou property explains the h-closedness of the h-balls. We will mark the space of all increasing sequences of real numbers by $\mathbf{I}$.

Theorem 10. *$(\mathfrak{C}_{\tau(\cdot)})_h$, where $h(f)[\sum_{q=0}^{\infty}(\sum_{p=0}^{q}|\widehat{f_p}|/(q+1))^{\tau_q}]^{1/K}$. for all $f\in\mathfrak{C}_{\tau(\cdot)}$, is a premodular (ssfps), when $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ with $\tau_0>1$.*

Proof. Evidently, $h(f)\geq 0$ and $h(f)=0 \Longleftrightarrow =\vartheta$.

Let $f,g\in\mathfrak{C}_{\tau(\cdot)}$. One has $(f+g)(y)=\sum_{v=0}^{\infty}(\widehat{f_v}+\widehat{g_v})y^v\in\mathbb{C}$ with

$$\begin{aligned} h(f+g)&\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p+g_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &+\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{g_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &=h(f)+h(g)<\infty. \end{aligned} \tag{8}$$

As $\alpha f\in\mathfrak{C}_{\tau(\cdot)}$, hence from conditions (1-i) and (1-ii), one has $\mathfrak{C}_{\tau(\cdot)}$ is linear. Also $e^{(r)}\in\mathfrak{C}_{\tau(\cdot)}$, for all $r\in\mathcal{N}$, since

$$h\left(e^{(r)}\right)=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{e_p^{(e)}}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}=\left[\sum_{r=0}^{\infty}\left(\frac{1}{r+1}\right)^{\tau_0}\right]^{1/K}<\infty. \tag{9}$$

There is $Q=\max\{1,\sup_q|\alpha|^{(\tau_q/K)-1}\}\geq 1$ with $h(\alpha f)\leq Q|\alpha|h(f)$ for all $f\in\mathfrak{C}_{\tau(\cdot)}$ and $\alpha\in\mathbb{C}$

Assume $|f_q|\leq|g_q|$, for all $q\in\mathcal{N}$ and $g\in\mathfrak{C}_{\tau(\cdot)}$. One finds

$$h(f)=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{g_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}=h(g)<\infty, \tag{10}$$

then $f\in\mathfrak{C}_{\tau(\cdot)}$.

Obviously, from (58).

Let $(f_q)\in\mathfrak{C}_{\tau(\cdot)}$, we get

$$\begin{aligned} h\left(\left(f_{[p/2]}\right)\right)&=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_{[p/2]}}\right|}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{2q}\left|\widehat{f_{[p/2]}}\right|}{2q+1}\right)^{\tau_{2q}}+\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{2q+1}\left|\widehat{f_{[p/2]}}\right|}{2q+1}\right)^{\tau_{2q+1}}\right]^{1/K} \\ &\leq\left[\sum_{q=0}^{\infty}\left(\frac{\left|\widehat{f_q}\right|+2\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}+\sum_{q=0}^{\infty}\left(\frac{2\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq\left[\sum_{q=0}^{\infty}\left(\frac{3\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}+\sum_{q=0}^{\infty}\left(\frac{2\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq\left(3^K+2^K\right)^{1/K}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &=\left(3^K+2^K\right)^{1/K}h\left(\left(Y_q\right)\right), \end{aligned} \tag{11}$$

then $(f_{[p/2]})\in\mathfrak{C}_{\tau(\cdot)}$.

From (59), we obtain $P_0=(3^K+2^K)^{1/K}\geq 1$.

Evidently the closure of $\mathfrak{F}=\mathfrak{C}_{\tau(\cdot)}$.

There is $0<\sigma\leq\sup_q|\alpha|^{(\tau_q/K)-1}$, for $\alpha\neq 0$ or $\sigma>0$, for $\alpha=0$ with $h(\alpha e^{(0)})\geq\sigma|\alpha|h(e^{(0)})$. □

Theorem 11. *If $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ with $\tau_0>1$, then $(\mathfrak{C}_{\tau(\cdot)})_h$ is a prequasi-Banach (ssfps), where*

$$h(f)=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}, \tag{12}$$

for every $f\in\mathfrak{C}_{\tau(\cdot)}$.

Proof. According to Theorems 10 and 7, the space $(\mathfrak{C}_{\tau(\cdot)})_h$ is a prequasinormed (ssfps). Assume $f^l=(f_q^l)_{q=0}^{\infty}$ is a Cauchy sequence in $(\mathfrak{C}_{\tau(\cdot)})_h$, hence for every $\varepsilon\in(0,1)$, one has $l_0\in\mathcal{N}$ such that for all $l,m\geq l_0$, one gets

$$h\left(f^l - f^m\right) = \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p^l - f_p^m}\right|}{q+1}\right)^{\tau_q}\right]^{1/K} < \varepsilon. \quad (13)$$

□

This implies $|\widehat{f_q^l} - \widehat{f_q^m}| < \varepsilon$. Hence, $(\widehat{f_q^m})$ is a Cauchy sequence in $\mathbb{C}$, for constant $q \in \mathcal{N}$, which implies $\lim_{m\longrightarrow\infty} |\widehat{f_q^m} - \widehat{f_q^0}| = 0$, for constant $q \in \mathcal{N}$. Hence, $h(f^l - f^0) < \varepsilon$, for every $l \geq l_0$. Since $h(f^0) = h(f^0 - f^l + f^l) \leq h(f^l - f^0) + h(f^l) < \infty$. So, $f^0 \in \mathfrak{C}_{\tau(\cdot)}$.

Theorem 12. *Suppose $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then $(\mathfrak{C}_{\tau(\cdot)})_h$ is a pre-quasi closed (ssfps), where*

$$h(f) = \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}, \quad (14)$$

for every $f \in \mathfrak{C}_{\tau(\cdot)}$.

Proof. According to Theorems 10 and 7, the space $(\mathfrak{C}_{\tau(\cdot)})_h$ is a prequasinormed (ssfps). Assume $f^l = (f_q^l)_{q=0}^{\infty} \in (\mathfrak{C}_{\tau(\cdot)})_h$ and $\lim_{l\longrightarrow\infty} h(f^l - f^0) = 0$, then for all, $\varepsilon \in (0,1)$, there is $l_0 \in \mathcal{N}$ such that for all $l \geq l_0$, we obtain

$$\varepsilon > h\left(f^l - f^0\right) = \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p^l} - \widehat{f_p^0}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}, \quad (15)$$

which implies $|\widehat{f_q^l} - \widehat{f_q^0}| < \varepsilon$, as $\mathbb{C}$ is a complete space. Therefore, $(\widehat{f_q^l})$ is a convergent sequence in $\mathbb{C}$, for fixed $q \in \mathcal{N}$. So $\lim_{l\longrightarrow\infty}|\widehat{f_q^l} = \widehat{f_q^0}|$, for fixed $q \in \mathcal{N}$. Since, $h(f^0) \leq h(f^l - f^0) + h(f^l) < \infty$. So, $f^0 \in \mathfrak{C}_{\tau(\cdot)}$. □

Theorem 13. *The function*

$$h(f) = \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}, \quad (16)$$

holds the Fatou property, when $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, for all $f \in \mathfrak{C}_{\tau(\cdot)}$.

Proof. Let $\{g^r\} \subseteq (\mathfrak{C}_{\tau(.)})_h$ such that $\lim_{r\longrightarrow\infty} h(g^r - g) = 0$. Since $(\mathfrak{C}_{\tau(.)})_h$ is a pre-quasi closed space, one has $g \in (\mathfrak{C}_{\tau(.)})_h$. For all $f \in (\mathfrak{C}_{\tau(.)})_h$, one gets

$$\begin{aligned} h(f-g) &= \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p} - \widehat{g_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p} - \widehat{g_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\quad + \left[\sum_{q=0}^{\infty}\left(\frac{\left(\sum_{p=0}^{q}\left|\widehat{g_p^r} - \widehat{g_p}\right|\right)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq \sup_m \inf_{r\geq m} h(f - g^r). \end{aligned} \quad (17)$$

□

Theorem 14. *The function*

$$h(f) = \sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}, \quad (18)$$

does not hold the Fatou property, for all $f \in \mathfrak{C}_{\tau(\cdot)}$, when $(\tau_q) \in \ell_\infty$ and $\tau_q > 1$, for all $q \in \mathcal{N}$.

Proof. Let $\{g^r\} \subseteq (\mathfrak{C}_{\tau(\cdot)})_h$ so that $\lim_{r\longrightarrow\infty} h(g^r - g) = 0$. Since $(\mathfrak{C}_{\tau(\cdot)})_h$ is a pre-quasi closed space, one gets $g \in (\mathfrak{C}_{\tau(\cdot)})_h$. For every $f \in \mathfrak{C}_{\tau(\cdot)}$, we obtain

$$\begin{aligned} h(f-g) &= \sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p} - \widehat{g_p}\right|}{q+1}\right)^{\tau_q} \\ &\leq 2^{K-1}\left(\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p} - \widehat{g_p}\right|}{q+1}\right)^{\tau_q} + \sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{g_p^r} - \widehat{g_p}\right|}{q+1}\right)^{\tau_q}\right) \\ &\leq 2^{K-1}\sup_m \inf_{r\geq m} h(f - g^r). \end{aligned} \quad (19)$$

□

Example 3. For $(\tau_q) \in [1,\infty)^{\mathcal{N}}$, the function

$$h(f) = \inf\left\{\alpha > 0 : \sum_{q\in\mathcal{N}}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}/\alpha\right|}{q+1}\right)^{\tau_q} \leq 1\right\}, \quad (20)$$

is a norm on $\mathfrak{C}_{\tau(\cdot)}$.

Example 4. The function

$$h(f) = \sqrt[3]{\sum_{q\in\mathcal{N}}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{3q+2/q+1}}, \quad (21)$$

is a prequasinorm (not a quasinorm) on $\mathfrak{C}(((3q+2)/(q+1))_{q=0}^{\infty})$.

Example 5. The function

$$h(f) = \sum_{q\in\mathcal{N}} \left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{3q+2/q+1}, \tag{22}$$

is a prequasinorm (not a quasinorm) on $\mathfrak{C}(((3q+2)/(q+1))_{q=0}^{\infty})$.

Example 6. The function

$$h(f) = \sqrt[d]{\sum_{q\in\mathcal{N}} \left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{d}}, \tag{23}$$

is a prequasinorm, quasinorm, and not a norm on $\mathfrak{C}_d$, for $0<d<1$.

3. Caristi's Fixed Point Theorem in $(\mathfrak{C}_{\tau(\cdot)})_h$

In this section, the existence of Caristi's fixed point in $(\mathfrak{C}_{\tau(\cdot)})_h$ is presented according to Farkas [31], where

$$h(f) = \left[\sum_{q\in\mathcal{N}} \left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}, \tag{24}$$

for all $f\in\mathfrak{C}_{\tau(\cdot)}$.

Definition 15. The function $\Psi_1 : (\mathfrak{C}_{\tau(\cdot)})_h \longrightarrow (-\infty,\infty]$ is said to be lower semicontinuous at $G^{(0)}\in(\mathfrak{C}_{\tau(\cdot)})_h$ if $\lim_{G\longrightarrow G^{(0)}}\inf \Psi_1(G)=\Psi_1(G^{(0)})$, where $\lim_{G\longrightarrow G^{(0)}}\inf\Psi_1(G) = \sup_{v\in v(G^{(0)})}\inf_{G\in V}\Psi_1(G)$, where $V(G^{(0)})$ is a neighborhood system of $G^{(0)}$.

Definition 16. The function $\Psi_1 : (\mathfrak{C}_{\tau(\cdot)})_h \longrightarrow (-\infty,\infty]$ is said to be proper, when

$$\mathcal{D}(\Psi_1) = \left\{G\in\left(\mathfrak{C}_{\tau(\cdot)}\right)_h : \Psi_1(G)<\infty\right\}\neq 0. \tag{25}$$

Theorem 17. *Suppose $\Xi\neq 0$ and Ξ is a h-closed subset of $(\mathfrak{C}_{\tau(\cdot)})_h$, and $\Psi_1 : \Xi\longrightarrow(-\infty,\infty]$ is a proper, h-lower semicontinuous function with $\inf_{G\in\Xi}\Psi_1(G)>-\infty$. If $\gamma>0$, $\{\varpi_q\}\subset(0,\infty)$, and $G^{(0)}\in\Xi$ so that $\Psi_1(G^{(0)})\leq\inf_{G\in\Xi}\Psi_1(G)+\gamma$. One gets $\{G^{(q)}\}\in\Xi$ which h-converges to some $G^{(\gamma)}$, and*

(i) $h(G^{(\gamma)}-G^{(q)})\leq\gamma/2^q\varpi_0$, with $q\in\mathcal{N}$

$$\Psi_1\left(G^{(\gamma)}\right)+\sum_{q=0}^{\infty}\varpi_q h\left(G^{(\gamma)}-G^{(q)}\right)\leq\Psi_1 G^{(0)} \tag{26}$$

(ii) when $G\neq G^{(\gamma)}$, then

$$\Psi_1\left(G^{(\gamma)}\right)+\sum_{q=0}^{\infty}\varpi_q h\left(G^{(\gamma)}-G^{(q)}\right)<\Psi_1(G)+\sum_{q=0}^{\infty}\varpi_q h\left(G-G^{(q)}\right) \tag{27}$$

Proof. If $S(G^{(0)})=\{G\in\Xi:\Psi_1(G)+\varpi_0 h(G-G^{(0)})\leq\Psi_1(G^{(0)})\}$. Since $G^{(0)}\in S(G^{(0)})$, then $S(G^{(0)})\neq 0$. As Ψ_1 is h-lower semicontinuous, h holds the Fatou property and Ξ is h-closed, then $S(G^{(0)})$ is h-closed. Take $G^{(1)}\in S(G^{(0)})$ and

$$\Psi_1\left(G^{(1)}\right)+\varpi_0 h\left(G^{(1)}-G^{(0)}\right)\leq\inf_{G\in S\left(G^{(0)}\right)}\left\{\Psi_1(G)+\varpi_0 h\left(G-G^{(0)}\right)\right\}+\frac{\gamma\varpi_1}{2\varpi_0}. \tag{28}$$

Choose

$$S\left(G^{(1)}\right)=\left\{G\in S\left(G^{(0)}\right):\Psi_1(G)+\sum_{j=0}^{1}\varpi_j h\left(G-G^{(j)}\right)\right.\\ \left.\leq\Psi_1\left(G^{(1)}\right)+\varpi_0 h\left(G^{(1)}-G^{(j)}\right)\right\}. \tag{29}$$

As $S(G^{(0)})$, we get $S(G^{(1)})\neq 0$ and h-closed. Suppose that one has built $\{G^{(0)},G^{(1)},G^{(2)},\cdots,G^{(q)}\}$ and $\{S(G^{(0)}),S(G^{(1)}),S(G^{(2)}),\cdots,S(G^{(q)})\}$. Next, take $G^{(q+1)}\in S(G^{(q)})$ and

$$\Psi_1\left(G^{(q+1)}\right)+\sum_{j=0}^{q}\varpi_j h\left(G^{(q+1)}-G^{(j)}\right)\\ \leq\inf_{G\in S\left(G^{(q)}\right)}\left\{\Psi_1(G)+\sum_{j=0}^{q}\varpi_j h\left(G-G^{(j)}\right)\right\}+\frac{\gamma\varpi_q}{2^q\varpi_0}. \tag{30}$$

Let

$$S\left(G^{(q+1)}\right)\coloneqq\left\{G\in S\left(G^{(q)}\right):\Psi_1(G)+\sum_{j=0}^{q+1}\varpi_j h\left(G-G^{(j)}\right)\right.\\ \left.\leq\Psi_1\left(G^{(q+1)}\right)+\sum_{j=0}^{q}\varpi_j h\left(G^{(q+1)}-G^{(j)}\right)\right\}, \tag{31}$$

hence we form by induction, the sequences $\{G^{(q)}\}$ and $\{S(G^{(q)})\}$. Fix $q\in\mathcal{N}$. Suppose $W\in S(G^{(q)})$. One obtains

$$\Psi_1(W) + \sum_{j=0}^{q} \varpi_j h\left(W - G^{(j)}\right) \le \Psi_1\left(G^{(q)}\right) + \sum_{j=0}^{q-1} \varpi_j h\left(G^{(q)} - G^{(j)}\right), \tag{32}$$

then

$$\begin{aligned} \varpi_q h\left(W - G^{(q)}\right) &\le \Psi_1\left(G^{(q)}\right) + \sum_{j=0}^{q-1} \varpi_j h\left(G^{(q)} - G^{(j)}\right) \\ &\quad - \left[\Psi_1(W) + \sum_{j=0}^{q-1} \varpi_j h\left(W - G^{(j)}\right)\right] \\ &\le \Psi_1\left(G^{(q)}\right) + \sum_{j=0}^{q-1} \varpi_j h\left(G^{(q)} - G^{(j)}\right) - \inf_{G \in S(G(q-1))} \\ &\quad \cdot \left[\Psi_1(G) + \sum_{j=0}^{q-1} \varpi_j h\left(G - G^{(j)}\right)\right] \le \frac{\gamma \varpi_q}{2^q \varpi_0}. \end{aligned} \tag{33}$$

As $\{S(G^{(q)})\}$ is decreasing with $G^{(q)} \in S(G^{(q)})$, for all $q \in \mathcal{N}$, one gets

$$h\left(G^{(q+p)} - G^{(q)}\right) \le \frac{\gamma}{2^q \varpi_0}, \tag{34}$$

with $q, p \in \mathcal{N}$. This implies $\{G^{(q)}\}$ is h -Cauchy. Since $(\mathfrak{C}_{\tau(\cdot)})_h$ is h -Banach space; hence, $\{G^{(q)}\}$ has h–limits $G^{(\gamma)}$ and $\cap_{q \in \mathcal{N}} S(G^{(q)}) = \{G^{(\gamma)}\}$. Since $G^{(q+1)} \in S(G^{(q)})$, we can see

$$\Psi_1\left(G^{(q+1)}\right) + \sum_{j=0}^{q} \varpi_j h\left(G^{(q+1)} - G^{(j)}\right) \le \Psi_1\left(G^{(q)}\right) + \sum_{j=0}^{q-1} \varpi_j h\left(G^{(q)} - G^{(j)}\right), \tag{35}$$

hence, $\{\Psi_1(G^{(q)}) + \sum_{j=0}^{q-1} \varpi_j h(G^{(q+1)} - G^{(j)})\}$ is decreasing. After, let $G \ne G^{(\gamma)}$. One gets $m \in \mathcal{N}$ with $G \in S(G^{(q)})$, with $q \ge m$, i.e.,

$$\Psi_1\left(G^{(q)}\right) + \sum_{j=0}^{q-1} \varpi_j h\left(G^{(q)} - G^{(j)}\right) < \Psi_1(G) + \sum_{j=0}^{q} \varpi_j h\left(G - G^{(j)}\right). \tag{36}$$

Since $G^{(\gamma)} + \in S(G^{(q)})$, with $q \ge m$, we get

$$\begin{aligned} \Psi_1\left(G^{(\gamma)}\right) &+ \sum_{j=0}^{q} \varpi_j h\left(G^{(\gamma)} - G^{(j)}\right) \\ &\le \Psi_1\left(G^{(q)}\right) + \sum_{j=0}^{q-1} \varpi_j h\left(G^{(q)} - G^{(j)}\right) \\ &\le \Psi_1\left(G^{(m)}\right) + \sum_{j=0}^{m-1} \varpi_j h\left(G^{(m)} - G^{(j)}\right). \end{aligned} \tag{37}$$

Put $q \longrightarrow \infty$ in the previous inequality, then

$$\begin{aligned} \Psi_1\left(G^{(\gamma)}\right) &+ \sum_{j=0}^{\infty} \varpi_j h\left(G^{(\gamma)} - G^{(j)}\right) \\ &\le \Psi_1(x_m) + \sum_{j=0}^{m-1} \varpi_j h\left(G^m - G^{(j)}\right) \\ &< \Psi_1(G) + \sum_{j=0}^{m} \varpi_j h\left(G - G^{(j)}\right) \\ &\le \Psi_1(G) + \sum_{j=0}^{\infty} \varpi_j h\left(G - G^{(j)}\right). \end{aligned} \tag{38}$$

This gives

$$\Psi_1\left(G^{(\gamma)}\right) + \sum_{q=0}^{q} \varpi_q h\left(G^{(\gamma)} - G^{(q)}\right) < \Psi_1(G) + \sum_{q=0}^{\infty} \varpi_q h\left(G - G^{(q)}\right). \tag{39}$$

□

Theorem 18. *Suppose $\Xi \ne 0$ and Ξ is a h-closed subset of $(\mathfrak{C}_{\tau(\cdot)})_h$. By taking $\gamma > 0$ and $\{\varpi_n\}$ and $0 < \omega = \sum_{n=0}^{\infty} \varpi_n < \infty$. If $H : \Xi \longrightarrow \Xi$ is a mapping and there is a function $\Psi_1 : \Xi \longrightarrow (-\infty, \infty]$ holds a proper and h-lower semicontinuous with $\inf_{G \in \Xi} \Psi_1(G) > -\infty$ and*

(1) $h(H(G) - Y) - h(G - Y) \le h(H(G) - Y)$, for any G, $Y \in \Xi$

(2) $h(H(G) - G) \le \Psi_1(G) - \Psi_1(H(G))$, with $G \in \Xi$

Then, H has a fixed point in Ξ.

Proof. As $0 < \omega = \sum_{n=0}^{\infty} \varpi_n < \infty$, one has $\Psi_2 \coloneqq \omega \Psi_1$ is also proper, h-lower semicontinuous and bounded from below. If $G \in \Xi$, one gets

$$\omega h(H(G) - G) \le \Psi_2(G) - \Psi_2(H(G)). \tag{40}$$

As $\inf_{G \in \Xi} \Psi_2(G) > -\infty$, one obtains $G^{(0)} \in \Xi$ with $\Psi_2(G^{(0)}) < \inf_{G \in \Xi} \Psi_2(G) + \gamma$. From Theorem 17, there is $\{G^{(q)}\}$ which h-converges to some $G^{(\gamma)} \in \Xi$, and

$$\Psi_2\left(G^{(\gamma)}\right) + \sum_{q=0}^{\infty} \varpi_q h\left(G^{(\gamma)} - G^{(q)}\right) < \Psi_2(G) + \sum_{q=0}^{\infty} \varpi_q h\left(G - G^{(q)}\right), \tag{41}$$

for every $G \ne G^{(\gamma)}$. Assume that $H(G^{(\gamma)}) \ne G^{(\gamma)}$, we have

$$\Psi_2\left(G^{(\gamma)}\right) + \sum_{q=0}^{\infty} \varpi_q h\left(G^{(\gamma)} - G^{(q)}\right) < \Psi_2\left(H\left(G^{(\gamma)}\right)\right) + \sum_{q=0}^{\infty} \varpi_q h\left(H\left(G^{(\gamma)}\right) - G^{(q)}\right), \quad (42)$$

then

$$\begin{aligned}\Psi_2\left(G^{(\gamma)}\right) - \Psi_2\left(H\left(G^{(\gamma)}\right)\right) &< \sum_{q=0}^{\infty} \varpi_q h\left(H\left(G^{(\gamma)}\right) - G^{(q)}\right) - \sum_{q=0}^{\infty} \varpi_q h\left(G^{(\gamma)} - G^{(q)}\right) \\ &= \sum_{q=0}^{\infty} \varpi_q \left(h\left(H\left(G^{(\gamma)}\right) - G^{(q)}\right) - h\left(G^{(\gamma)} - G^{(q)}\right)\right).\end{aligned} \quad (43)$$

From condition (40), then

$$\Psi_2\left(G^{(\gamma)}\right) - \Psi_2\left(H\left(G^{(\gamma)}\right)\right) < \sum_{q=0}^{\infty} \varpi_q h\left(H\left(G^{(\gamma)}\right) - G^{(\gamma)}\right) = \omega h\left(H\left(G^{(\gamma)}\right) - G^{(\gamma)}\right). \quad (44)$$

The inequality (40) implies that

$$\omega h\left(H\left(G^{(\gamma)}\right) - G^{(\gamma)}\right) \le \Psi_2\left(G^{(\gamma)}\right) - \Psi_2\left(H\left(G^{(\gamma)}\right)\right) < \omega h\left(H\left(G^{(\gamma)}\right) - G^{(\gamma)}\right). \quad (45)$$

This is a contradiction, hence $H(G^{(\gamma)}) = G^{(\gamma)}$. □

4. Structure of Mappings' Ideal

The structure of the mappings' ideal by $(\mathfrak{C}_{\tau(\cdot)})_h$, where

$$h(f) = \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}, \quad (46)$$

for all $f \in \mathfrak{C}_{\tau(\cdot)}$, and s -numbers have been explained. We study enough setups on $(\mathfrak{C}_{\tau(\cdot)})_h$ such that the class $^{Ŧ}(\mathfrak{C}_{\tau(\cdot)})_h$ is complete and closed. We investigate enough setups (not necessary) on $(\mathfrak{C}_{\tau(\cdot)})_h$ such that the closure of $F = {}^{Ŧ\alpha}(\mathfrak{C}_{\tau(\cdot)})_h$. This gives a negative answer of Rhoades' [32] open problem about the linearity of s -type $(\mathfrak{C}_{\tau(\cdot)})_h$ spaces. We explain enough setups on $(\mathfrak{C}_{\tau(\cdot)})_h$ such that $^{Ŧ}(\mathfrak{C}_{\tau(\cdot)})_h$ is strictly contained for different powers, $^{Ŧ\alpha}(\mathfrak{C}_{\tau(\cdot)})_h$ is the minimum, the class $^{Ŧ}(\mathfrak{C}_{\tau(\cdot)})_h$ is simple, and $(^{Ŧ}(\mathfrak{C}_{\tau(\cdot)})_h)^{\lambda} = {}^{Ŧ}(\mathfrak{C}_{\tau(\cdot)})_h$.

We denote the space of all bounded, finite rank linear mappings from an infinite-dimensional Banach space Δ into an infinite-dimensional Banach space Λ by $\mathscr{L}(\Delta, \Lambda)$, and $\mathbf{F}(\Delta, \Lambda)$ and when $\Delta = \Lambda$, we inscribe $\mathscr{L}(\Delta)$ and $\mathbf{F}(\Delta)$. The space of approximable and compact-bounded linear mappings from a Banach space Δ into a Banach space Λ will be indicated by $\Upsilon(\Delta, \Lambda)$ and $\mathscr{L}_c(\Delta, \Lambda)$, and if $\Delta = \Lambda$, we mark $\Upsilon(\Delta)$ and $\mathscr{L}_c(\Delta)$, respectively.

Definition 19 (see [33]). An s-number function is a mapping $s : \mathscr{L}(\Delta, \Lambda) \longrightarrow \mathfrak{R}^{+\mathscr{N}}$ that sorts every $V \in \mathscr{L}(\Delta, \Lambda)$ unique sequence $(s_d(V))_{d=0}^{\infty}$ validates the following settings:

(a) $\|V\| = s_0(V) \ge s_1(V) \ge s_2(V) \ge \cdots \ge 0$, for all $V \in \mathscr{L}(\Delta, \Lambda)$

(b) $s_{l+d-1}(V_1 + V_2) \le s_l(V_1) + s_d(V_2)$, for all $V_1, V_2 \in \mathscr{L}(\Delta, \Lambda)$ and $l, d \in \mathscr{N}$

(c) $s_d(VYW) \le \|V\| s_d(Y) \|W\|$, for all $W \in \mathscr{L}(\Delta_0, \Delta)$, $Y \in \mathscr{L}(\Delta, \Lambda)$, and $V \in \mathscr{L}(\Lambda, \Lambda_0)$, where Δ_0 and Λ_0 are arbitrary Banach spaces

(d) when $V \in \mathscr{L}(\Delta, \Lambda)$ and $\gamma \in \mathfrak{R}$, then $s_d(\gamma V) = |\gamma| s_d(V)$

(e) suppose rank $(V) \le d$, then $s_d(V) = 0$, for each $V \in \mathscr{L}(\Delta, \Lambda)$

(f) $s_{l \ge q}(I_q) = 0$ or $s_{l<q}(I_q) = 1$, where I_q denotes the unit map on the q-dimensional Hilbert space ℓ_2^q

Some examples of s-numbers are as follows:

(1) The qth Kolmogorov number, described by $d_q(X)$, is marked by

$$d_q(X) = \inf_{\dim J \le q} \sup_{\|f\| \le 1} \inf_{g \in J} \|Xf - g\|. \quad (47)$$

(2) The qth approximation number, described by $\alpha_q(X)$, is marked by

$$\alpha_q(X) = \inf \{\|X - Y\| : Y \in \mathscr{L}(\Delta, \Lambda) \text{and rank } (Y) \le q\}. \quad (48)$$

Definition 20 (see [10]). Assume $\mathscr{L}$ is the class of all bounded linear mappings within any two arbitrary Banach spaces. A subclass $\mathscr{U}$ of $\mathscr{L}$ is said to be a mappings' ideal, when all $\mathscr{U}(\Delta, \Lambda) = \mathscr{U} \cap \mathscr{L}(\Delta, \Lambda)$ verifies the following conditions:

(i) $I_\Gamma \in \mathscr{U}$, where Γ marks Banach space of one dimension

(ii) The space $\mathscr{U}(\Delta, \Lambda)$ is linear over $\mathfrak{R}$

(iii) If $W \in \mathscr{L}(\Delta_0, \Delta)$, $X \in \mathscr{U}(\Delta, \Lambda)$, and $Y \in \mathscr{L}(\Lambda, \Lambda_0)$ then, $YXW \in \mathscr{U}(\Delta_0, \Lambda_0)$

Notations 21 (see [30]).

$$\mathfrak{F}_{\mathscr{H}} \coloneqq \{\mathfrak{F}_{\mathscr{H}}$$

$(\Delta, \Lambda)\}$, where $\mathfrak{F}_{\mathscr{H}}(\Delta, \Lambda) \coloneqq \{V \in \mathscr{L}\ (\Delta, \Lambda)\colon f_s \in \mathscr{H}$, where $f_s(y) = \sum_{n=0}^{\infty} s_n(V) y^n\}, \mathfrak{F}^{\alpha}_{\mathscr{H}} \coloneqq \{\mathfrak{F}^{\alpha}_{\mathscr{H}}(\Delta, \Lambda)\}$, where $\mathfrak{F}^{\alpha}_{\mathscr{H}}(\Delta, \Lambda) \coloneqq \{V \in \mathscr{L}(\Delta, \Lambda)\colon \ \colon f_\alpha \in \mathscr{H}$, where $f_\alpha(y) = \sum_{n=0}^{\infty} \alpha_n(V) y^n\}, \mathfrak{F}^{d}_{\mathscr{H}} \coloneqq \{\mathfrak{F}^{d}_{\mathscr{H}}(\Delta, \Lambda)\}$, where $\mathfrak{F}^{d}_{\mathscr{H}}(\Delta, \Lambda) \coloneqq \{V \in \mathscr{L}(\Delta, \Lambda)\colon \ \colon f_d \in \mathscr{H}$, where $f_d(\ y) = \sum_{n=0}^{\infty} d_n(V) y^n\}$.**Theorem 22.** (see [29]). *Suppose $\mathscr{H}$ is a (ssfps), then $\mathfrak{F}_{\mathscr{H}}$ is mappings' ideal.*

According to Theorems 10 and 22, one concludes the next theorem.

Theorem 23. *Suppose $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then $\mathfrak{F}(\mathfrak{C}_{\tau(\cdot)})_h$ is a mappings' ideal.*

Definition 24 (see [34]). A function $H \in [0,\infty)^{\mathscr{U}}$ is said to be a pre-quasi norm on the ideal $\mathscr{U}$, if it verifies the following setups:

(1) Let $V \in \mathscr{U}(\Delta, \Lambda)$, $H(V) \geq 0$, and $H(V) = 0$, if and only if, $V = 0$

(2) we have $Q \geq 1$ so as to $H(\alpha V) \leq D|\alpha| H(V)$, for every $V \in \mathscr{U}(\Delta, \Lambda)$ and $\alpha \in \mathfrak{R}$

(3) we have $P \geq 1$ so that $H(V_1 + V_2) \leq P[H(V_1) + H(V_2)]$, for each $V_1, V_2 \in \mathscr{U}(\Delta, \Lambda)$

(4) we have $\sigma \geq 1$ when $V \in \mathscr{L}(\Delta_0, \Delta)$, $X \in \mathscr{U}(\Delta, \Lambda)$, and $Y \in \mathscr{L}(\Lambda, \Lambda_0)$ then $H(YXV) \leq \sigma \|Y\| H(X) \|V\|$

Theorem 25 (see [35]). *Every quasinorm on the ideal $\mathscr{U}$ is a prequasinorm on the same ideal.*

Theorem 26. *If $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then H is a prequasinorm on $\mathfrak{F}(\mathfrak{C}_{\tau(\cdot)})_h$, so that $H(Z) = h(f_s)$, where f_s $(\mathfrak{C}_{\tau(\cdot)})_h$ and $f_s(y) = \sum_{n=0}^{\infty} s_n(Z) y^n$.*

Proof.

(1) When $X \in \mathfrak{F}(\mathfrak{C}_{\tau(\cdot)})_h(\Delta, \Lambda)$, $H(X) = h(f_s) \geq 0$, and $H(X) = h(f_s) = 0$, if and only if, $s_n(X) = 0$, for all $n \in \mathscr{N}$; if and only if, $X = 0$

(2) There is $Q \geq 1$ with $H(\varepsilon X) \leq h(\varepsilon f_s) \leq Q \|\varepsilon\| H(X)$ for every $X \in \mathfrak{F}(\mathfrak{C}_{\tau(\cdot)})_h(\Delta, \Lambda)$ and $\varepsilon \in \mathbb{C}$

(3) One has $PP_0 \geq 1$ so that for $X_1, X_2 \in \mathfrak{F}(\mathfrak{C}_{\tau(\cdot)})_h(\Delta, \Lambda)$; hence, there are $f1_s, f2_s (\mathfrak{C}_{\tau(\cdot)})_h$ with $f1_s(y) = \sum_{n=0}^{\infty} s_n(X_1) y^n$ and $f2_s(y) = \sum_{n=0}^{\infty} s_n(X_2) y^n$. Therefore, for $g_s(y) =: \sum_{n=0}^{\infty} s_n(X_1 + X_2) y^n$, we have $KK_0 \geq 1$ so that

$$\begin{aligned} H(X_1 + X_2) = h(g_s) &\leq h\left((f1_s)_{[\cdot]} + (f2_s)_{[\cdot]}\right) \\ &\leq P\left(h(f1_s)_{[\cdot]} + h(f2_s)_{[\cdot]}\right) \\ &\leq PP_0(H(X_1) + H(X_2)) \end{aligned} \tag{50}$$

(4) We have $\varrho \geq 1$ if $X \in \mathscr{L}(\Delta_0, \Delta)$, $Y \in \mathfrak{F}(\mathfrak{C}_{\tau(\cdot)})_h(\Delta, \Lambda)$, and $Z \in L(\Lambda, \Lambda_0)$; hence, there is $f_s \in (\mathfrak{C}_{\tau(\cdot)})_h$ with $f_s(y) = \sum_{n=0}^{\infty} s_n(Y) y^n$. Then, for $g_s(y) =: \sum_{n=0}^{\infty} s_n(ZYX) y^n$, one has

$$H(ZYX) = h(g_s) \leq h(\|X\| \|Z\| f_s) \leq \varrho \|X\| H(Y) \|Z\|. \tag{51}$$

□

Theorem 27. *Suppose $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$ one has $(\mathfrak{F}(\mathfrak{C}_{\tau(\cdot)})_h, H)$ is a prequasi-Banach mappings' ideal.*

Proof. Suppose $(V_a)_{a\in\mathscr{N}}$ is a Cauchy sequence in $\mathfrak{F}(\mathfrak{C}_{\tau(\cdot)})_h(\Delta, \Lambda)$. As $\mathscr{L}(\Delta, \Lambda) \supseteq S_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda)$, hence, there is $f^a_s \in (\mathfrak{C}_{\tau(\cdot)})_h$ with $f^a_s \in (y) = \sum_{n=0}^{\infty} s_n(V_a) y^n$ for very $a \in \mathscr{N}$, then

$$\begin{aligned} H(V_r - V_a) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^{q} s_p(V_r - V_a)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\geq \inf_q \| V_r - V_a \|^{\tau_q/K} \left[\sum_{q=0}^{\infty} \left(\frac{1}{q+1}\right)^{\tau_q}\right]^{1/K}, \end{aligned} \tag{52}$$

hence, $(V_a)_{a\in\mathscr{N}}$ is a Cauchy sequence in $\mathscr{L}(\Delta, \Lambda)$ is a Banach space, so there exists $V \in \mathscr{L}(\Delta, \Lambda)$ so that $\lim_{a\longrightarrow\infty} \| V_a - V \| = 0$ and since $f^a_s (\mathfrak{C}_{\tau(\cdot)})_h$, for all $a \in \mathscr{N}$ and $(\mathfrak{C}_{\tau(\cdot)})_h$ is a premodular (ssfps), hence, one can see

$$\begin{aligned} H(V) &= \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^{q} s_p(V)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^{q} s_{[p/2]}(V - V_a)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\quad + \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^{q} s_{[p/2]}(V_a)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\leq \left[\sum_{q=0}^{\infty} \| V_a - V \|^{\tau_q}\right]^{1/K} \left(3^K + 2^K\right)^{1/K} \left[\sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^{q} s_p(V_a)}{q+1}\right)^{\tau_q}\right]^{1/K} < \varepsilon. \end{aligned} \tag{53}$$

We obtain $f^a_s (\mathfrak{C}_{\tau(\cdot)})_h$, hence $V \in \mathfrak{F}(\mathfrak{C}_{\tau(\cdot)})_h(\Delta, \Lambda)$. □

Theorem 28. *If $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, one has $(\mathfrak{F}(\mathfrak{C}_{\tau(\cdot)})_h, H)$ is a prequasiclosed mappings' ideal.*

Proof. Suppose $V_a \in \text{Ŧ}(\mathfrak{C}_{\tau(\cdot)})_h(\Delta, \Lambda)$, for all $a \in \mathcal{N}$ and $\lim_{a\longrightarrow\infty} H\|V_a - V\| = 0$, hence, there is $f_s^a \in (\mathfrak{C}_{\tau(\cdot)})_h$ with $f_s^a(y) = \sum_{n=0}^{\infty} s_n(V_a)y^n$, for all $a \in \mathcal{N}$, there is $\varsigma > 0$ and as $\mathcal{L}(\Delta, \Lambda) \supseteq S_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda)$, one has

$$H(V_a - V) = \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q} s_p(V_a - V)}{q+1}\right)^{\tau_q}\right]^{1/K} \le \inf_q \|V_a - V\|^{\tau_q/K}\left[\sum_{q=0}^{\infty}\left(\frac{1}{q+1}\right)^{\tau_q}\right]^{1/K}. \quad (54)$$

□

So $(V_a)_{a\in\mathcal{N}}$ is convergent in $\mathcal{L}(\Delta, \Lambda)$, i.e., $\lim_{a\longrightarrow\infty}\|V_a - V\| = 0$ and since $f_s^a(\mathfrak{C}_{\tau(\cdot)})_h$, for all $a \in \mathcal{N}$ and $(\mathfrak{C}_{\tau(\cdot)})_h$ is a premodular (ssfps), hence, one can see

$$\begin{aligned} H(V) &= \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q} s_p(V)}{q+1}\right)^{\tau_q}\right]^{1/K} \le \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q} s_{[p/2]}(V - V_a)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\quad + \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q} s_{[p/2]}(V_a)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\le \left[\sum_{q=0}^{\infty}(\|V_a - V\|)^{\tau_q}\right]^{1/K} \\ &\quad + \left(3^K + 2^K\right)^{1/K}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q} s_p(V_a)}{q+1}\right)^{\tau_q}\right]^{1/K} < \varepsilon. \end{aligned} \quad (55)$$

We obtain $f_s \in (\mathfrak{C}_{\tau(\cdot)})_h$, hence $V \in \text{Ŧ}_{(\mathfrak{C}_{\tau(\cdot)})_h}(\Delta, \Lambda)$.

Definition 29. A prequasinorm H on the ideal $\text{Ŧ}_{\mathcal{H}_h}$ verifies the Fatou property if for every $\{T_q\}_{q\in\mathcal{N}} \subseteq \text{Ŧ}_{\mathcal{H}_h}(\Delta, \Lambda)$ so that $\lim_{q\longrightarrow\infty} H(T_q - T) = 0$ and $M \in \text{Ŧ}_{\mathcal{H}_h}(\Delta, \Lambda)$, one gets

$$H(M - T) \le \sup_q \inf_{j\ge q} H\left(M - T_j\right). \quad (56)$$

Theorem 30. *Suppose $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then $(\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}, H)$ does not verify the Fatou property.*

Proof. Assume $\{T_q\}_{q\in\mathcal{N}} \subseteq \text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta, \Lambda)$ with $\lim_{q\longrightarrow\infty} H(T_q - T) = 0$. Since $\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}$ is a prequasiclosed ideal, then $T \in \text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta, \Lambda)$. So for every $M \in \text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta, \Lambda)$, one has

$$\begin{aligned} H(M - T) &= \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q} s_p(M - T)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\le \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q} s_{[p/2]}\left(M - T_j\right)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\quad + \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q} s_{[p/2]}\left(T_j - T\right)}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &\le \left(3^K + 2^K\right)^{1/K}\sup_r \inf_{j\ge r}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q} s_p\left(M - T_j\right)}{q+1}\right)^{\tau_q}\right]^{1/K}. \end{aligned} \quad (57)$$

□

Theorem 31. $\text{Ŧ}_{(\mathfrak{C}_{\tau(.)}})_h^{\alpha}(\Delta, \Lambda) =$ *the closure of* $\mathbf{F}(\Delta, \Lambda)$, *if* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 1$. *But the converse is not necessarily true.*

Proof. As $e^{(q)} \in (\mathfrak{C}_{\tau(.)})_h$, for every $q \in \mathcal{N}$ and $(\mathfrak{C}_{\tau(.)})_h$ is a linear space. Suppose $Z \in \mathbf{F}(\Delta, \Lambda)$ with rank $(Z) = m$, where $m \in \mathcal{N}$, hence $f_\alpha \in \mathfrak{F}$ with $f_\alpha(y) = \sum_{n=0}^{m-1}\alpha_n(Z)y^n$, one has $f_\alpha \in (\mathfrak{C}_{\tau(.)})_h$. Therefore, the closure of $\mathbf{F}(\Delta, \Lambda) \subseteq \text{Ŧ}^{\alpha}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta, \Lambda)$. Assume $Z \in \text{Ŧ}^{\alpha}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta, \Lambda)$, we have $g_\alpha \in (\mathfrak{C}_{\tau(.)})_h$. As $h(g_\alpha) < \infty$, assume $\rho \in (0, 1)$, then there is $q_0 \in \mathcal{N} - \{0\}$ with $h(g_\alpha - \sum_{n=0}^{q_0-1} e^{(n)}) < \rho/2^{K+3}\eta d$, for some $d \ge 1$, where $\eta = \max\{1, \sum_{q=q_0}^{\infty}(1/q+1)^{\tau_q}\}$. Since $(\alpha_q(Z))$ is decreasing, we have

$$\begin{aligned} \sum_{q=q_0+1}^{2q_0}\left(\frac{\sum_{p=0}^{q}\alpha_{2q_0}(Z)}{q+1}\right)^{\tau_q} &\le \sum_{q=q_0+1}^{2q_0}\left(\frac{\sum_{p=0}^{q}\alpha_p(Z)}{q+1}\right)^{\tau_q} \\ &\le \sum_{q=q_0}^{\infty}\left(\frac{\sum_{p=0}^{q}\alpha_p(Z)}{q+1}\right)^{\tau_q} < \frac{\rho}{2^{K+3}\eta d}. \end{aligned} \quad (58)$$

Hence, there is $Y \in \mathbf{F}_{2q_0}(\Delta, \Lambda)$ so that $\text{rank}(Y) \le 2q_0$ and

$$\sum_{q=2q_0+1}^{3q_0}\left(\frac{\sum_{p=0}^{q}\|Z - Y\|}{q+1}\right)^{\tau_q} \le \sum_{q=q_0+1}^{2q_0}\left(\frac{\sum_{p=0}^{q}\|Z - Y\|}{q+1}\right)^{\tau_q} < \frac{\rho}{2^{K+3}\eta d}, \quad (59)$$

since $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, we have

$$\sup_{q=q_0}^{\infty}\left(2q_0\|Z - Y\|\right)^{\tau_q} < \frac{\rho}{2^{2K+2}\eta}. \quad (60)$$

Therefore, one has

$$\sum_{q=0}^{q_0}(\|Z - Y\|)^{\tau_q} < \frac{\rho}{2^{K+3}\eta d}. \quad (61)$$

As $Z - Y \in \text{Ŧ}^{\alpha}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta, \Lambda)$, hence $g_\alpha \in (\mathfrak{C}_{\tau(.)})_h$, where g_α

$(y) \coloneqq \sum_{n=0}^{\infty} \alpha_n (Z - Y) y^n$. In view of inequalities (58)–(61), one has

$$\begin{aligned}
d(Z, Y) = h(g_\alpha) &= \sum_{q=0}^{3q_0-1} \left(\frac{\sum_{p=0}^{q} \alpha_p (Z - Y)}{q+1} \right)^{\tau_q} + \sum_{q=3q_0}^{\infty} \left(\frac{\sum_{p=0}^{q} \alpha_p (Z - Y)}{q+1} \right)^{\tau_q} \\
&\leq \sum_{q=0}^{3q_0} \left(\frac{\sum_{p=0}^{q} \|Z - Y\|}{q+1} \right)^{\tau_q} + \sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^{q+2q_0} \alpha_p (Z - Y)}{q+2q_0+1} \right)^{\tau_{q+2q_0}} \\
&\leq \sum_{q=0}^{3q_0} (\|Z - Y\|)^{\tau_q} + \sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^{q+2q_0} \alpha_p (Z - Y)}{q+1} \right)^{\tau_q} \\
&\leq 3 \sum_{q=0}^{q_0} (\|Z - Y\|)^{\tau_q} + \sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^{2q_0-1} \alpha_p (Z - Y) + \sum_{p=2q_0}^{q+2q_0} \bar{\rho}\left(\overline{\alpha_p (Z - Y), \bar{0}}\right)}{q+1} \right)^{\tau_q} \\
&\leq 3 \sum_{q=0}^{q_0} (\|Z - Y\|)^{\tau_q} + 2^{K-1} \\
&\quad \cdot \left[\sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^{2q_0-1} \alpha_p (Z - Y)}{q+1} \right)^{\tau_q} + \sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=2q_0}^{q+2q_0} \alpha_p (Z - Y)}{q+1} \right)^{\tau_q} \right] \\
&\leq 3 \sum_{q=0}^{q_0} (\|Z - Y\|)^{\tau_q} + 2^{K-1} \\
&\quad \cdot \left[\sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^{2q_0} \alpha_p (Z - Y)}{q+1} \right)^{\tau_q} + \sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^{q} \alpha_{p+2q_0} (Z - Y)}{q+1} \right)^{\tau_q} \right] \\
&\leq 3 \sum_{q=0}^{q_0} (\|Z - Y\|)^{\tau_q} + 2^{K-1} \left[\eta \sup_{q=q_0}^{\infty} (2q_0 \|Z - Y\|)^{\tau_q} + \sum_{q=q_0}^{\infty} \left(\frac{\sum_{p=0}^{q} \alpha_p (Z)}{q+1} \right)^{\tau_q} \right]
\end{aligned} \tag{62}$$

□

Therefore, $\text{Ŧ}^{\alpha}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta, \Lambda) \subseteq$ the closure of $\mathbf{F}(\Delta, \Lambda)$. Contrarily, one has a counter example as $I_2 \in \text{Ŧ}^{\alpha}_{(\mathfrak{C}((0,0,2,2,2,\cdots)))_h}(\Delta, \Lambda)$, but $\tau_0 > 1$ is not verified.

Theorem 32. *Suppose* $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $1 < \tau_x^{(1)} < \tau_x^{(2)}$, *for all* $x \in \mathscr{N}$, *hence*

$$\text{Ŧ}_{\left(\mathfrak{C}\left(\left(\tau_x^{(1)}\right)\right)\right)_h}(\Delta, \Lambda) \text{Ŧ}_{\left(\mathfrak{C}\left(\left(\tau_x^{(2)}\right)\right)\right)_h}(\Delta, \Lambda) \subsetneqq \mathscr{L}(\Delta, \Lambda). \tag{63}$$

Proof. Let $Z \in \text{Ŧ}_{(\mathfrak{C}((\tau_x^{(1)})))_h}(\Delta, \Lambda)$, hence $(g_s) \in (\mathfrak{C}((\tau_x^{(1)})))_h$, where $g_s(y) \coloneqq \sum_{n=0}^{\infty} s_n(Z) y^n$. One gets

$$\sum_{x=0}^{\infty} \left(\frac{\sum_{p=0}^{x} s_p(Z)}{x+1} \right)^{\tau_x^{(2)}} < \sum_{x=0}^{\infty} \left(\frac{\sum_{p=0}^{x} s_p(Z)}{x+1} \right)^{\tau_x^{(1)}} < \infty, \tag{64}$$

then $(g_s) \in (\mathfrak{C}((\tau_x^{(2)})))_h$ this implies $Z \in \text{Ŧ}_{(\mathfrak{C}((\tau_x^{(2)})))_h}(\Delta, \Lambda)$. After, if we choose $(s_x(Z))_{x=0}^{\infty}$ with $\sum_{p=0}^{x} s_p(Z) = (x+1)^{1-(1/\tau_x^{(1)})}$, we have $Z \in \mathscr{L}(\Delta, \Lambda)$ such that

$$\begin{aligned}
\sum_{x=0}^{\infty} \left(\frac{\sum_{p=0}^{x} s_p(Z)}{x+1} \right)^{\tau_x^{(1)}} &= \sum_{x=0}^{\infty} \frac{1}{x+1} = \infty, \\
\sum_{x=0}^{\infty} \left(\frac{\sum_{p=0}^{x} s_p(Z)}{x+1} \right)^{\tau_x^{(2)}} &\leq \sum_{x=0}^{\infty} \left(\frac{1}{x+1} \right)^{\tau_x^{(2)}/\tau_x^{(1)}} < \infty.
\end{aligned} \tag{65}$$

□

Then, $Z \notin \text{Ŧ}_{(\mathfrak{C}((\tau_x^{(1)})))_h}(\Delta, \Lambda)$ and $Z \in \text{Ŧ}_{(\mathfrak{C}((\tau_x^{(2)})))_h}(\Delta, \Lambda)$.

Clearly, $\text{Ŧ}_{(\mathfrak{C}((\tau_x^{(2)})))_h}(\Delta, \Lambda) \subset \mathscr{L}(\Delta, \Lambda)$. Next, if we put $(s_x(Z))_{x=0}^{\infty}$ with $\sum_{p=0}^{x} s_p(Z) = (x+1)^{1-(1/\tau_x^{(2)})}$. We have $Z \in \mathscr{L}(\Delta, \Lambda)$ such that $Z \notin \text{Ŧ}_{(\mathfrak{C}((\tau_x^{(2)})))_h}(\Delta, \Lambda)$.

Theorem 33. *Assume* $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 1$, *hence* $\text{Ŧ}^{\alpha}_{(\mathfrak{C}_{\tau(.)})_h}$ *is minimum.*

Proof. Let $\text{Ŧ}^{\alpha}_{\mathfrak{C}_{\tau(.)}}(\Delta, \Lambda) = \mathscr{L}(\Delta, \Lambda)$, one has $\eta > 0$ so that $H(Z) \leq \eta \|Z\|$, where

$$H(Z) = \sum_{q=0}^{\infty} \left(\frac{\sum_{p=0}^{q} \alpha_p (Z)}{q+1} \right)^{\tau_q}, \tag{66}$$

for all $Z \in \mathscr{L}(\Delta, \Lambda)$. According to Dvoretzky's theorem [36], with $r \in \mathscr{N}$, we get quotient spaces Δ/Y_r and subspaces M_r of Λ which can be transformed onto ℓ_2^r by isomorphisms V_r and X_r with $\|V_r\|\|V_r^{-1}\| \leq 2$ and $\|X_r\|\|X_r^{-1}\| \leq 2$. If I_r is the identity map on ℓ_2^r, T_r is the quotient map from Δ onto Δ/Y_r and J_r is the natural embedding map from M_r into Λ. □

Assume m_q is the Bernstein numbers [9], then

$$\begin{aligned}
1 = m_q(I_r) &= m_q\left(X_r X_r^{-1} I_r V_r V_r^{-1}\right) \leq \|X_r\| m_q\left(X_r^{-1} I_r V_r\right) \left\|V_r^{-1}\right\| \\
&= \|X_r\| m_q\left(J_r X_r^{-1} I_r V_r\right) \left\|V_r^{-1}\right\| \leq \|X_r\| d_q\left(J_r X_r^{-1} I_r V_r\right) \left\|V_r^{-1}\right\| \\
&= \|X_r\| d_q\left(J_r X_r^{-1} I_r V_r T_r\right) \left\|V_r^{-1}\right\| \leq \|X_r\| \alpha_q\left(J_r X_r^{-1} I_r V_r T_r\right) \left\|V_r^{-1}\right\|,
\end{aligned} \tag{67}$$

for $0 \leq q \leq r$. Then, we have

$$1 \leq \left(\|X_r\| \left\|V_r^{-1}\right\| \right)^{\tau_q} \left(\frac{\sum_{p=0}^{q} \alpha_p \left(J_r X_r^{-1} I_r V_r T_r\right)}{q+1} \right)^{\tau_q}. \tag{68}$$

So, there are $\varrho \geq 1$, we obtain

$$\begin{aligned}\sum_{q=0}^{r} 1 &\le \varrho\|X_r\|\|V_r^{-1}\| \sum_{q=0}^{r}\left(\frac{\sum_{p=0}^{q}\alpha_p\left(J_rX_r^{-1}I_rV_rT_r\right)}{q+1}\right)^{\tau_q} \Longrightarrow \sum_{q=0}^{r} 1\\ &\le \varrho\|X_r\|\|V_r^{-1}\|H\left(J_rX_r^{-1}I_rV_rT_r\right) \Longrightarrow \sum_{q=0}^{r} 1\\ &\le \varrho\eta\|X_r\|\|V_r^{-1}\|\|J_rX_r^{-1}I_rV_rT_r\| \Longrightarrow \sum_{q=0}^{r} 1\\ &\le \varrho\eta\|X_r\|\|V_r^{-1}\|\|J_rX_r^{-1}\|\|I_r\|\|V_rT_r\|\\ &= \varrho\eta\|X_r\|\|V_r^{-1}\|\|X_r^{-1}\|\|I_r\|\|V_r\| \le 4\varrho\eta.\end{aligned} \tag{69}$$

So there is a contradiction, if $r \longrightarrow \infty$. Therefore, Δ and Λ both cannot be infinite dimensional if $\text{Ŧ}_{\mathfrak{C}_{\tau(.)}}^{\alpha}(\Delta,\Lambda) = \mathscr{L}(\Delta,\Lambda)$.

As with the previous theorem, we can easily prove the following theorem.

Theorem 34. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, hence $\text{Ŧ}_{\mathfrak{C}_{\tau(.)}}^{d}$ is minimum.*

Lemma 35 (see [10]). *If $B \in \mathscr{L}(\Delta,\Lambda)$ and $B \notin \Upsilon(\Delta,\Lambda)$, then $D \in \mathscr{L}(\Delta)$ and $M \in \mathscr{L}(\Lambda)$ with $MBDI_b = I_b$, with $b \in \mathcal{N}$.*

Theorem 36 (see [10]). *In general, we have*

$$\mathbf{F}(\Delta) \subsetneq \Upsilon(\Delta) \subsetneq \mathscr{L}_c(\Delta) \subsetneq \mathscr{L}(\Delta). \tag{70}$$

Theorem 37. *Let $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $1 < \tau_x^{(1)} < \tau_x^{(2)}$, for all $x \in \mathcal{N}$, hence*

$$\begin{aligned}&\mathscr{L}\left(\text{Ŧ}_{\left(\mathfrak{C}\left(\left(\tau_x^{(2)}\right)\right)\right)_h}(\Delta,\Lambda), \text{Ŧ}_{\left(\mathfrak{C}\left(\left(\tau_x^{(1)}\right)\right)\right)_h}(\Delta,\Lambda)\right)\\ &\quad= \Upsilon\left(\text{Ŧ}_{\left(\mathfrak{C}\left(\left(\tau_x^{(2)}\right)\right)\right)_h}(\Delta,\Lambda), \text{Ŧ}_{\left(\mathfrak{C}\left(\left(\tau_x^{(1)}\right)\right)\right)_h}(\Delta,\Lambda)\right).\end{aligned} \tag{71}$$

Proof. Assume $X \in \mathscr{L}(\text{Ŧ}_{(\mathfrak{C}((\tau_x^{(2)})))_h}(\Delta,\Lambda), \text{Ŧ}_{(\mathfrak{C}((\tau_x^{(1)})))_h}(\Delta,\Lambda))$ and $X \notin \Upsilon(\text{Ŧ}_{(\mathfrak{C}((\tau_x^{(2)})))_h}(\Delta,\Lambda), \text{Ŧ}_{(\mathfrak{C}((\tau_x^{(1)})))_h}(\Delta,\Lambda))$. By using Lemma 35, we have $Y \in \mathscr{L}(\text{Ŧ}_{(\mathfrak{C}((\tau_x^{(2)})))_h}(\Delta,\Lambda))$ and $Z \in \mathscr{L}(\text{Ŧ}_{(\mathfrak{C}((\tau_x^{(1)})))_h}(\Delta,\Lambda))$ so that $ZXYI_b = I_b$, hence with $b \in \mathcal{N}$, one has

$$\begin{aligned}\|I_b\|_{\text{Ŧ}_{\left(\mathfrak{C}\left(\left(\tau_x^{(1)}\right)\right)\right)_h}(\Delta,\Lambda)} &= \sum_{x=0}^{\infty}\left(\frac{\sum_{p=0}^{x}s_p(I_b)}{x+1}\right)^{\tau_x^{(1)}}\\ &\le \|ZXY\|\|I_b\|_{\text{Ŧ}_{\left(\mathfrak{C}\left(\left(\tau_x^{(2)}\right)\right)\right)_h}(\Delta,\Lambda)}\\ &\le \sum_{x=0}^{\infty}\left(\frac{\sum_{p=0}^{x}s_p(I_b)}{x+1}\right)^{\tau_x^{(2)}}.\end{aligned} \tag{72}$$

□

This fails Theorem 32. So, $X \in \Upsilon(\text{Ŧ}_{(\mathfrak{C}((\tau_x^{(2)})))_h}(\Delta,\Lambda), \text{Ŧ}_{(\mathfrak{C}((\tau_x^{(1)})))_h}(\Delta,\Lambda))$.

Corollary 38. *Assume $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $1 < \tau_x^{(1)} < \tau_x^{(2)}$, for all $x \in \mathcal{N}$, hence,*

$$\begin{aligned}&\mathscr{L}\left(\text{Ŧ}_{\left(\mathfrak{C}\left(\left(\tau_x^{(2)}\right)\right)\right)_h}(\Delta,\Lambda), \text{Ŧ}_{\left(\mathfrak{C}\left(\left(\tau_x^{(1)}\right)\right)\right)_h}(\Delta,\Lambda)\right)\\ &\quad= \mathscr{L}_c\left(\text{Ŧ}_{\left(\mathfrak{C}\left(\left(\tau_x^{(2)}\right)\right)\right)_h}(\Delta,\Lambda), \text{Ŧ}_{\left(\mathfrak{C}\left(\left(\tau_x^{(1)}\right)\right)\right)_h}(\Delta,\Lambda)\right).\end{aligned} \tag{73}$$

Proof. Evidently, as $\Upsilon \subset \mathscr{L}_c$. □

Definition 39 (see [10]). A Banach space Δ is said to be simple, if there is an unique nontrivial closed ideal in $\mathscr{L}(\Delta)$.

Theorem 40. *Let $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, hence $\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}$ is simple.*

Proof. Let $X \in \mathscr{L}_c(\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta,\Lambda))$ and $X \notin \Upsilon(\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta,\Lambda))$. From Lemma 35, there exist $Y, Z \in \mathscr{L}(\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta,\Lambda))$ with $ZXYI_b = I_b$, which gives that $I_{\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta,\Lambda)} \in \mathscr{L}_c(\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta,\Lambda))$. Then, $\mathscr{L}(\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta,\Lambda)) = \mathscr{L}_c(\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta,\Lambda))$; hence, $\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}$ is simple Banach space. □

Notations 41.

$$(\text{Ŧ}_{\mathscr{H}})^{\lambda} \coloneqq \left\{(\text{Ŧ}_{\mathscr{H}})^{\lambda}(\Delta,\Lambda)\,;\,\Delta \text{ and } \Lambda \text{ are Banach Spaces}\right\}, \text{where} \tag{74}$$

$(\text{Ŧ}_{\mathscr{H}})^{\lambda}(\Delta,\Lambda) \coloneqq \{X \in \mathscr{L}(\Delta,\Lambda)\colon\ f_\lambda \in \mathscr{H}_h, \text{where } f_\lambda(y) = \sum_{n=0}^{\infty}\lambda_n(T)y^n \text{ and } \|X - \lambda_x(X)I\| = 0, \text{for every } x \in \mathcal{N}\}$.

Theorem 42. *Assume $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, hence,*

$$\left(\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}\right)^{\lambda}(\Delta,\Lambda) = \text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta,\Lambda). \tag{75}$$

Proof. Let $X \in (\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h})^{\lambda}(\Delta,\Lambda)$, hence $f_\lambda \in (\mathfrak{C}_{\tau(.)})_h$, where $f_\lambda(y) = \sum_{n=0}^{\infty}\lambda_n(T)y^n$ and $\|X - \lambda_x(X)I\| = 0$, with $x \in \mathcal{N}$. We have $X = \lambda_x(X)I$, for all $x \in \mathcal{N}$, so

$$s_x(X) = s_x(\lambda_x(X)I) = |\lambda_x(X)|, \tag{76}$$

with $x \in \mathcal{N}$. One gets $f_s \in (\mathfrak{C}_{\tau(.)})_h$; hence, $X \in \text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta,\Lambda)$. Next, suppose $X \in \text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta,\Lambda)$. Hence, $f_s \in (\mathfrak{C}_{\tau(.)})_h$. One gets

$$\sum_{x=0}^{\infty}(s_x(X))^{\tau_x} \le \sum_{x=0}^{\infty}\left(\frac{\sum_{p=0}^{x}s_p(X)}{x+1}\right)^{\tau_x} < \infty. \tag{77}$$

□

Then, $\lim_{x\longrightarrow\infty} s_x(X)=0$. If $\|X-s_x(X)I\|^{-1}$ exists, with $x\in\mathcal{N}$. Then, $\|X-s_x(X)I\|^{-1}$ exists and bounded, for all $x\in\mathcal{N}$. So, $\lim_{x\longrightarrow\infty}\|X-s_x(X)I\|^{-1}=\|X\|^{-1}$ exists and bounded. Since $(\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}, H)$ is a pre-quasi mappings' ideal, one has

$$I=XX^{-1}\in \text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h}(\Delta,\Lambda) \Rightarrow g_s\in\mathfrak{C}_{\tau(.)} \Longrightarrow \lim_{x\longrightarrow\infty} s_x(I)=0, \tag{78}$$

where $g_s(y)=\sum_{n=0}^{\infty} s_n(I)y^n$. This gives a contradiction, as $\lim_{x\longrightarrow\infty} s_x(I)=1$. Therefore, $\|X-s_x(X)I\|=0$, with $x\in\mathcal{N}$, which explains $X\in(\text{Ŧ}_{(\mathfrak{C}_{\tau(.)})_h})^{\lambda}(\Delta,\Lambda)$.

5. Nonexpansive Mappings on $(\mathfrak{C}_{\tau(.)})_h$

In this section, we have presented some geometric properties connected with the fixed point theory in $(\mathfrak{C}_{\tau(.)})_h$.

In the next part of this section, we will use the function h as

$$h(f)=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}, \tag{79}$$

for all $f\in\mathfrak{C}_{\tau(.)}$.

Definition 43 (see [37]). A sequence $\{g_p\}\subseteq\mathscr{H}_h$, is said to be ε-separated sequence for some $\varepsilon>0$, if

$$\operatorname{sep}\left(g_p\right)=\inf\left\{h\left(g_p-g_q\right): p\neq q\right\}>\varepsilon. \tag{80}$$

Definition 44. [37]. If $k\geq 2$ is an integer, a Banach space $\mathscr{H}_h$ is said to be k-nearly uniformly convex (k-NUC) when for all $\varepsilon>0$ one has $\delta\in(0,1)$ so that for every sequence $\{g_p\}\subseteq B(\mathscr{H}_h)$, with $\operatorname{sep}(g_p)\geq\varepsilon$, we have $p_1,p_2,p_3,\cdots,p_k\in\mathcal{N}$.

Such that

$$h\left(\frac{g_{p_1}+g_{p_2}+g_{p_3}+\cdots+g_{p_k}}{k}\right)<1-\delta. \tag{81}$$

Definition 45 [38]. A function h is said to be hold the δ_2-condition ($h\in\delta_2$), if for any $\varepsilon>0$, there exists a constant $k\geq 2$ and $a>0$ such that,

$$h(2g)\leq kh(g)+\varepsilon \text{ for each } g\in\mathscr{H}_h, \text{ with } h(g)\leq a. \tag{82}$$

If h satisfies the δ_2-condition for any $a>0$ with $k\geq 2$ depending on a, we say that h satisfies the strong δ_2-condition ($\rho\in\delta_2^s$).

Theorem 46 ((see [38]), Lemma 2.1). *Suppose $h\in\delta_2^s$, then for any $L>0$ and $\varepsilon>0$ one has $\delta>0$ with $|h(f+g)-h(f)|<\varepsilon$,$f,g\in\mathscr{H}_h$, with $h(f)\leq L$ and $h(g)\leq\delta$.*

Theorem 47. *Pick an $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ with $\tau_0>1$, then for any $L>0$ and $\varepsilon>0$ one has $\delta>0$ with $|h(f+g)-h(f)|<\varepsilon$, for every $f,g\in(\mathfrak{C}_{\tau(.)})_h$, so that $h(f)\leq L$ and $h(g)\leq\delta$.*

Proof. Since $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ with $\tau_0>1$, then $h\in\delta_2^s$. According to Theorem 46, the proof follows.

We denote $S(\mathscr{H}_h)$ and $B(\mathscr{H}_h)$ for the unit sphere and the unit ball of $\mathscr{H}_h$, respectively. □

Theorem 48. *Suppose $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ with $\tau_0>1$, then is k-NUC, for any integer $k\geq 2$.*

Proof. Assume $\varepsilon\in(0,1)$ and $\{f_n\}\subseteq B((\mathfrak{C}_{\tau(.)})_h)$, where $f_n(y)=\sum_{i=0}^{\infty}\widehat{f_n(i)}y^i$ so that $sep(f_n)\geq\varepsilon$. For all $m\in\mathcal{N}$, suppose $f_n^m(y)=\sum_{i=0}^{\infty}\widehat{f_n^m(i)}y^i$, where $(\widehat{f_n^m(i)})_{i=0}^{\infty}=(0,0,0,\cdots,\widehat{f_n(m)},\widehat{f_n(m+1)},\cdots)$. As for all $i\in\mathcal{N}$, $(\widehat{f_n(i)})_{n=0}^{\infty}\in\ell_\infty$, from the diagonal method, one has a subsequence (f_{n_j}) of (f_n) with $(\widehat{f_{n_j}(i)})$ converges for all $i\in\mathcal{N}$, $0\leq i\leq m$. One obtains an increasing sequence of positive integers (t_m) so that *sep* $((f_{n_j}^m)_{j>t_m})\geq\varepsilon$. Therefore, one has a sequence of positive integers $(r_m)_{m=0}^{\infty}$ with $r_0<r_1<r_2<\cdots$, so that

$$h^K\left(f_{r_m}^m\right)\geq\frac{\varepsilon}{2}, \tag{83}$$

for all $m\in\mathcal{N}$. For constant integer $k\geq 2$, assume $\varepsilon_1=((k^{p_0-1}-1)/((k-1)k^{p_0}))(\varepsilon/4)$ from Theorem 47, one gets $\delta>0$ with

$$\left|h^K(f+g)-h^K(f)\right|<\varepsilon_1. \tag{84}$$

□

If $h^K(g)\leq\delta$. As $h^K(f_n)\leq 1$, for every $n\in\mathcal{N}$, one has positive integers $m_i (i=0,1,2,\cdots,k-2)$ with $m_0<m_1<m_2<\cdots<m_{k-2}$ with $h^K(f_i^{m_i})\leq\delta$. Define $m_{k-1}=m_{k-2}+1$. From inequality (83), one can see $h(f_{r_{m_k}}^{m_k})\geq\varepsilon/2$. Suppose $p_i=i$ for $0\leq i\leq k-2$ and $p_{k-1}=r_{m_{k-1}}$. According to inequalities (83), (84), and convexity of $J_n(u)=|u|^{\tau_n}$ for every $n\in\mathcal{N}$, one has

$$\begin{aligned}
&h^K\left(\frac{f_{p_0}+f_{p_1}+f_{p_2}+.\cdots+f_{p_{k-1}}}{k}\right)\\
&=\sum_{n=0}^{\infty}\left(\frac{\sum_{i=0}^{n}\left|\left(\widehat{f_{p_2}(i)}+f_{p_3}(i)+\cdots+\widehat{f_{p_{k-1}}(i)}\right)/k\right|}{n+1}\right)^{\tau_n}\\
&=\sum_{n=0}^{m_1-1}\left(\frac{\sum_{i=0}^{n}\left|\left(\widehat{f_{p_2}(i)}+f_{p_3}(i)+\cdots+\widehat{f_{p_{k-1}}(i)}\right)/k\right|}{n+1}\right)^{\tau_n}\\
&+\sum_{n=m_1}^{\infty}\left(\frac{\sum_{i=0}^{n}\left|\left(\widehat{f_{p_2}(i)}+f_{p_3}(i)+\cdots+\widehat{f_{p_{k-1}}(i)}\right)/k\right|}{n+1}\right)^{\tau_n}
\end{aligned}$$

$$\leq \sum_{n=0}^{m_1-1}\left(\frac{\sum_{i=0}^{n}\left|\left(\widehat{f_{p_2}(i)}+f_{p_3}(i)+\cdots+\widehat{f_{p_{k-1}}(i)}\right)/k\right|}{n+1}\right)^{\tau_n}$$

$$+\sum_{n=m_1}^{\infty}\left(\frac{\sum_{i=0}^{n}\left|\left(\widehat{f_{p_2}(i)}+f_{p_3}(i)+\cdots+\widehat{f_{p_{k-1}}(i)}\right)/k\right|}{n+1}\right)^{\tau_n}+\varepsilon_1$$

$$\leq \sum_{n=0}^{m_1-1}\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_j}(i)}\right|}{n+1}\right)^{\tau_n}$$

$$+\sum_{n=m_1}^{m_2-1}\left(\frac{\sum_{i=0}^{n}\left|\left(\widehat{f_{p_2}(i)}+f_{p_3}(i)+\cdots+\widehat{f_{p_{k-1}}(i)}\right)/k\right|}{n+1}\right)^{\tau_n}$$

$$+\sum_{n=m_2}^{\infty}\left(\frac{\sum_{i=0}^{n}\left|\left(\widehat{f_{p_2}(i)}+f_{p_3}(i)+\cdots+\widehat{f_{p_{k-1}}(i)}\right)/k\right|}{n+1}\right)^{\tau_n}+\varepsilon_1$$

$$\leq \sum_{n=0}^{m_1-1}\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_j}(i)}\right|}{n+1}\right)^{\tau_n}$$

$$+\sum_{n=m_1}^{m_2-1}\left(\frac{\sum_{i=0}^{n}\left|\left(\widehat{f_{p_2}(i)}+f_{p_3}(i)+\cdots+\widehat{f_{p_{k-1}}(i)}\right)/k\right|}{n+1}\right)^{\tau_n}$$

$$+\sum_{n=m_2}^{\infty}\left(\frac{\sum_{i=0}^{n}\left|\left(\widehat{f_{p_2}(i)}+f_{p_3}(i)+\cdots+\widehat{f_{p_{k-1}}(i)}\right)/k\right|}{n+1}\right)^{\tau_n}+2\varepsilon_1$$

$$\leq \sum_{n=0}^{m_1-1}\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_j}(i)}\right|}{n+1}\right)^{\tau_n}+\sum_{n=m_1}^{m_2-1}\frac{1}{k}\sum_{j=1}^{k-1}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_j}(i)}\right|}{n+1}\right)^{\tau_n}$$

$$+\sum_{n=m_2}^{m_3-1}\frac{1}{k}\sum_{j=2}^{k-1}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_k}(i)}\right|}{n+1}\right)^{\tau_n}+\cdots+\sum_{n=m_{k-1}}^{m_k-1}\frac{1}{k}\sum_{j=k-2}^{k-1}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_k}(i)}\right|}{n+1}\right)^{\tau_n}$$

$$+\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_k}(i)}\right|/k}{n+1}\right)^{\tau_n}+(k-1)\varepsilon_1$$

$$\leq \frac{h^K\left(f_{p_0}+f_{p_1}+f_{p_2}+.\cdots+f_{p_{k-2}}\right)}{k}+\frac{1}{k}\sum_{n=0}^{m_k-1}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_k}(i)}\right|}{n+1}\right)^{\tau_n}$$

$$+\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_k}(i)}/k\right|}{n+1}\right)^{\tau_n}+(k-1)\varepsilon_1$$

$$\leq \frac{k-1}{k}+\frac{1}{k}\sum_{n=0}^{m_k-1}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_k}(i)}\right|}{n+1}\right)^{\tau_n}$$

$$+\frac{1}{k^{\tau_0}}\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_k}(i)}\right|}{n+1}\right)^{\tau_n}+(k-1)\varepsilon_1$$

$$\leq 1-\frac{1}{k}+\frac{1}{k}\left(1-\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_k}(i)}\right|}{n+1}\right)^{\tau_n}\right)$$

$$+\frac{1}{k^{\tau_0}}\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_k}(i)}\right|}{n+1}\right)^{\tau_n}+(k-1)\varepsilon_1$$

$$=1+(k-1)\varepsilon_1-\left(\frac{k^{\tau_0-1}-1}{k^{\tau_0}}\right)\sum_{n=m_k}^{\infty}\left(\frac{\sum_{i=0}^{n}\left|\widehat{f_{p_k}(i)}\right|}{n+1}\right)^{\tau_n}$$

$$\leq 1+(k-1)\varepsilon_1-\left(\frac{k^{\tau_0-1}-1}{k^{\tau_0}}\right)\frac{\varepsilon}{2}=1-\left(\frac{k^{\tau_0-1}-1}{k^{\tau_0}}\right)\frac{\varepsilon}{4}. \tag{85}$$

So, $\left(\mathfrak{C}_{\tau(.)}\right)_h$ is k-NUC.

Recall that k-NUC implies reflexivity.

Definition 49 (see [39]). A Banach space $\mathscr{H}_h$ holds the uniform Opial property, if for all $\varepsilon>0$ one has $\gamma>0$ so that for every weakly null sequence $\{f_n\}\subset S(\mathscr{H}_h)$ and $f\in\mathscr{H}_h$ so that $h(f)\geq\varepsilon$, then

$$1+\gamma\leq\liminf_{n\longrightarrow\infty}h(f_n+f). \tag{86}$$

Definition 50 (see [40]). For a bounded subset $E\subset\mathscr{H}_h$, the set-measure of noncompactness defined by

$$\alpha(E)=\inf\{\xi>0: E \text{ can be covered by finitely many sets of diameter} \leq\xi\}. \tag{87}$$

Definition 51 (see [41, 42]). The ball-measure of noncompactness is defined by

$$\beta(E)=\inf\{\xi>0: E \text{ can be covered by finitely many balls of diameter} \leq\xi\}. \tag{88}$$

Definition 52 (see [43]). For a subset $E\subset\mathscr{H}_h$ is said to be α-minimal if $\alpha(C)=\alpha(E)$, for any infinite subset C of E.

Definition 53 (see [43]). The packing rate of a Banach space $\mathscr{H}_h$ is denoted by $\gamma(\mathscr{H}_h)$, and the formula defines it

$$\gamma(\mathscr{H}_h)=\frac{\delta(\mathscr{H}_h)}{\sigma(\mathscr{H}_h)}, \tag{89}$$

where $\delta(\mathscr{H}_h)$ and $\sigma(\mathscr{H}_h)$ are defined as the supremum and the infimum, respectively, of the set

$$\left\{\frac{\beta(E)}{\alpha(E)}: E\subset\mathscr{H}_h, E \text{ is } \alpha-\text{minimal}, \alpha(E)>0\right\}. \tag{90}$$

Definition 54 (see [41]). The function $\varDelta$ is said to be the modulus of noncompact convexity, if for every $\xi>0$ define

$$\varDelta(\xi)=\inf\left\{1-\inf_{f\in E}h(f): E \text{is a closed convex subset of } B(\mathscr{H}_h) \text{ with } \beta(E)\geq\xi\right\}. \tag{91}$$

Definition 55 (see [39]). A Banach space $\mathscr{H}_h$ is said to be hold property (L), when $\lim_{\varepsilon\longrightarrow 1^-}\varDelta(\varepsilon)=1$..

Definition 56. An operator $V : \mathscr{H}_h \longrightarrow \mathscr{H}_h$ is said to be a h-contraction, if one gets $\alpha \in [0, 1)$ with $h(Vg - Vf) \leq \alpha h(g - f)$, for all $g, f \in \mathscr{H}_h$. The operator V is said to be h-nonexpansive, when $\alpha = 1$. An element $g \in \mathscr{H}_h$ is said to be a fixed point of V, when $V(g) = g$..

Theorem 57 (see [39]).

(1) *Suppose a Banach space $\mathscr{H}_h$ holds property (L), then it has the fixed point property, i.e., for every nonexpansive self-mapping of a nonempty, closed, bounded, convex subset has a fixed point*

(2) *A Banach space $\mathscr{H}_h$ holds property (L), if and only if, it is reflexive and has the uniform Opial property*

Theorem 58. *Suppose $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then $(\mathfrak{C}_{\tau(.)})_h$ has the uniform Opial property.*

Proof. Let $\varepsilon > 0$ one finds a positive number $\varepsilon_0 \in (0, \varepsilon)$ with

$$1 + \frac{\varepsilon^K}{2} > (1 + \varepsilon_0)^K. \quad (92)$$

If $f \in (\mathfrak{C}_{\tau(.)})_h$ and $h(f) \geq \varepsilon$., one has $n_1 \in \mathscr{N}$ with

$$\sum_{n=n_1+1}^{\infty}\left(\frac{1}{n+1}\sum_{i=0}^{n}\left|\widehat{f(i)}\right|\right)^{\tau_n} < \left(\frac{\varepsilon_0}{4}\right)^K. \quad (93)$$

Therefore, one gets

$$h\left(\sum_{i=n_1+1}^{\infty}\widehat{f(i)}e^{(i)}\right) < \frac{\varepsilon_0}{4} < \frac{\varepsilon}{4}. \quad (94)$$

Also, one has

$$\begin{aligned}\varepsilon^K &\leq \sum_{n=0}^{n_1}\left(\frac{1}{n+1}\sum_{i=0}^{n}\left|\widehat{f(i)}\right|\right)^{\tau_n} + \sum_{n=n_1+1}^{\infty}\left(\frac{1}{n+1}\sum_{i=0}^{n}\left|\widehat{f(i)}\right|\right)^{\tau_n}\\ &< \sum_{n=0}^{n_1}\left(\frac{1}{n+1}\sum_{i=0}^{n}\left|\widehat{f(i)}\right|\right)^{\tau_n} + \left(\frac{\varepsilon_0}{4}\right)^K\\ &< \sum_{n=0}^{n_1}\left(\frac{1}{n+1}\sum_{i=0}^{n}\left|\widehat{f(i)}\right|\right)^{\tau_n} + \frac{\varepsilon^K}{4},\end{aligned} \quad (95)$$

if

$$\frac{3\varepsilon^K}{4} \leq \sum_{n=0}^{n_1}\left(\frac{1}{n+1}\sum_{i=0}^{n}\left|\widehat{f(i)}\right|\right)^{\tau_n}. \quad (96)$$

For any weakly null sequence $\{f_m\} \subset S((\mathfrak{C}_{\tau(.)})_h)$, in virtue of $\widehat{f_m(i)} \longrightarrow 0$ for $i = 0, 1, 2, \cdots$, one has $m_0 \in \mathscr{N}$ with

$$h\left(\sum_{i=0}^{n_1}\widehat{f_m(i)}e^{(i)}\right) < \frac{\varepsilon_0}{4}, \quad (97)$$

for $m > m_0$. One can see

$$\begin{aligned}h(f_m + f) &= h\left(\sum_{i=0}^{n_1}\left(\widehat{f_m(i)} + \widehat{f(i)}\right)e^{(i)} + \sum_{i=n_1+1}^{\infty}\left(\widehat{f_m(i)} + \widehat{f(i)}\right)e^{(i)}\right)\\ &\geq h\left(\sum_{i=0}^{n_1}\widehat{f(i)}e^{(i)} + \sum_{i=n_1+1}^{\infty}\widehat{f_m(i)}e^{(i)}\right)\\ &\quad - h\left(\sum_{i=0}^{n_1}\widehat{f_m(i)}e^{(i)}\right) - h\left(\sum_{i=n_1+1}^{\infty}\widehat{f(i)}e^{(i)}\right)\\ &\geq h\left(\sum_{i=0}^{n_1}\widehat{f(i)}e^{(i)} + \sum_{i=n_1+1}^{\infty}\widehat{f_m(i)}e^{(i)}\right) - \frac{\varepsilon_0}{2},\end{aligned} \quad (98)$$

if $m > m_0$. For $a := \sum_{i=0}^{n_1}|\widehat{f(i)}|$, one obtains

$$\begin{aligned}h^K\left(\sum_{i=0}^{n_1}\widehat{f(i)}e^{(i)} + \sum_{i=n_1+1}^{\infty}\widehat{f_m(i)}e^{(i)}\right) &= \sum_{n=0}^{n_1}\left(\frac{1}{n+1}\sum_{i=0}^{n}\left|\widehat{f(i)}\right|\right)^{\tau_n}\\ &\quad + \sum_{n=n_1+1}^{\infty}\left(\frac{1}{n+1}\sum_{i=0}^{n}\left(a + \left|\widehat{f_m(i)}\right|\right)\right)^{\tau_n}\\ &\geq \sum_{n=0}^{n_1}\left(\frac{1}{n+1}\sum_{i=0}^{n}\left|\widehat{f(i)}\right|\right)^{\tau_n} + \sum_{n=n_1+1}^{\infty}\left(\frac{1}{n+1}\sum_{i=0}^{n}\left|\widehat{f_m(i)}\right|\right)^{\tau_n}\\ &\geq \frac{3\varepsilon^K}{4} + \left(1 - \frac{\varepsilon^K}{4}\right) = 1 + \frac{\varepsilon^K}{2} > (1 + \varepsilon_0)^K.\end{aligned} \quad (99)$$

Combining this with the previous inequality, one has

$$\begin{aligned}h(f_m + f) &\geq h\left(\sum_{i=0}^{n_1}\widehat{f(i)}e^{(i)} + \sum_{i=n_1+1}^{\infty}\widehat{f_m(i)}e^{(i)}\right) - \frac{\varepsilon_0}{2}\\ &\geq 1 + \varepsilon_0 - \frac{\varepsilon_0}{2} = 1 + \frac{\varepsilon_0}{2}.\end{aligned} \quad (100)$$

□

Therefore, the space $(\mathfrak{C}_{\tau(.)})_h$ has the uniform Opial property.

From Theorem 58 and the reflexivity of the space $(\mathfrak{C}_{\tau(.)})_h$, by applying Theorem 47, we get the following.

Corollary 59. *If $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then $(\mathfrak{C}_{\tau(.)})_h$ has the property (L) and the fixed point property.*

Definition 60. $\mathscr{H}_h$ holds the h-normal structure property, if and only if, for every nonempty h-bounded, h-convex, and h-closed subset Γ of $\mathscr{H}_h$ not decreased to one point, one has $f \in \Gamma$ with

$$\sup_{g\in\Gamma} h(f-g) < \delta_h(\Gamma) \coloneqq \sup \{h(f-g): f, g \in \Gamma\} < \infty. \tag{101}$$

Definition 61 (see [44]). The weakly convergent sequence coefficient of a Banach space $\mathscr{H}_h$, denoted by WCS($\mathscr{H}_h$), is defined as follows:

$$\begin{aligned} WCS(\mathscr{H}_h) = \inf \{A(\{f_n\}): \{f_n\}_{n=1}^{\infty} \subset S(\mathscr{H}_h), A(\{f_n\}) \\ = A_1(\{f_n\}), f_n \xrightarrow{w} 0\}, \end{aligned} \tag{102}$$

where

$$\begin{aligned} A(\{f_n\}) &= \limsup_{n\longrightarrow\infty} \left\{h\left(f_i - f_j\right): i, j \geq n, i \neq j\right\}, \\ A_1(\{f_n\}) &= \liminf_{n\longrightarrow\infty} \left\{h\left(f_i - f_j\right): i, j \geq n, i \neq j\right\}. \end{aligned} \tag{103}$$

Theorem 62 (see [45]). *A reflexive Banach space $\mathscr{H}_h$ such that WCS($\mathscr{H}_h$) > 1 has the normal structure property.*

Theorem 63. *If $(\tau_q)_{q\in\mathscr{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then $(\mathfrak{C}_{\tau(.)})_h$ holds the h-normal structure property.*

Proof. Take any $\varepsilon > 0$ and an asymptotic equidistant sequence $\{f_n\} \subset S((\mathfrak{C}_{\tau(.)})_h)$ with $f_n \longrightarrow^w 0$ and let $v_1 = f_1$. One has $i_1 \in \mathscr{N}$ with

$$h\left(\sum_{i=i_1+1}^{\infty} \widehat{v_1(i)} e^{(i)}\right) < \varepsilon. \tag{104}$$

As $f_n \longrightarrow 0$ coordinate-wise, one gets $n_2 \in \mathscr{N}$ with

$$h\left(\sum_{i=1}^{i_1} \widehat{f_n(i)} e^{(i)}\right) < \varepsilon. \tag{105}$$

For $n \geq n_2$, put $v_2 = f_{n_2}$, one gets $i_2 > i_1$ with

$$h\left(\sum_{i=i_2+1}^{\infty} \widehat{v_1(i)} e^{(i)}\right) < \varepsilon. \tag{106}$$

As $f_n(i) \longrightarrow 0$ coordinate-wise, one obtains $n_3 \in \mathscr{N}$ with

$$h\left(\sum_{i=1}^{i_2} \widehat{f_n(i)} e^{(i)}\right) < \varepsilon. \tag{107}$$

For $n \geq n_3$. By induction, one has a subsequence $\{v_n\}$ of $\{f_n\}$ with

$$h\left(\sum_{i=i_n+1}^{\infty} \widehat{v_n(i)} e^{(i)}\right) < \varepsilon, h\left(\sum_{i=1}^{i_n} \widehat{v_{n+1}(i)} e^{(i)}\right) < \varepsilon. \tag{108}$$

Take

$$z_n = \sum_{i=i_{n-1}+1}^{i_n} \widehat{v_n(i)} e^{(i)}, \tag{109}$$

for $n = 2, 3, \cdots$. So,

$$\begin{aligned} 1 \geq h(z_n) &= h\left(\sum_{i=1}^{\infty} \widehat{v_n(i)} e^{(i)} - \sum_{i=1}^{i_n-1} \widehat{v_n(i)} e^{(i)} - \sum_{i=i_n+1}^{\infty} \widehat{v_n(i)} e^{(i)}\right) \\ &\geq h\left(\sum_{i=1}^{\infty} \widehat{v_n(i)} e^{(i)}\right) - h\left(\sum_{i=1}^{i_n-1} \widehat{v_n(i)} e^{(i)}\right) - h\left(\sum_{i=i_n+1}^{\infty} \widehat{v_n(i)} e^{(i)}\right) > 1 - 2\varepsilon. \end{aligned} \tag{110}$$

For every $n, m \in \mathscr{N}$ so that $n \neq m$, one can see

$$\begin{aligned} h(v_n - v_m) &= h\left(\sum_{i=1}^{\infty} \widehat{v_n(i)} e^{(i)} - \sum_{i=1}^{\infty} \widehat{v_m(i)} e^{(i)}\right) \\ &\geq h\left(\sum_{i=i_{n-1}+1}^{i_n} \widehat{v_n(i)} e^{(i)} - \sum_{i=i_{m-1}+1}^{i_m} \widehat{v_m(i)} e^{(i)}\right) \\ &\quad - h\left(\sum_{i=1}^{i_{n-1}} \widehat{v_n(i)} e^{(i)}\right) - h\left(\sum_{i=i_n+1}^{\infty} \widehat{v_n(i)} e^{(i)}\right) \\ &\quad - h\left(\sum_{i=1}^{i_{m-1}} \widehat{v_m(i)} e^{(i)}\right) - h\left(\sum_{i=i_m+1}^{\infty} \widehat{v_m(i)} e^{(i)}\right) \\ &\geq h(z_n - z_m) - 4\varepsilon, \end{aligned} \tag{111}$$

which gives $A(\{f_n\}) = A(\{v_n\}) \geq A(\{z_n\}) - 4\varepsilon$. Take $u_n = z_n / \|z_n\|$, for $n = 2, 3, \cdots$. Then,

$$u_n \in S\left(\left(\mathfrak{C}_{\tau(.)}\right)_h\right); \tag{112}$$

$$A(\{f_n\}) \geq 1 - \varepsilon A(\{u_n\}) - 4\varepsilon. \tag{113}$$

On the other hand,

$$h(v_n - v_m) \leq h(z_n - z_m) + 4\varepsilon \leq h(u_n - u_m) + 4\varepsilon, \tag{114}$$

for any $n, m \in \mathscr{N}$ with $n \neq m$. Therefore,

$$A(\{u_n\}) \geq A(\{f_n\}) - 4\varepsilon. \tag{115}$$

By the arbitrariness of $\varepsilon > 0$, we have from the relations (112), (113), and (115) that

$$WCS\left(\left(\mathfrak{C}_{\tau(.)}\right)_h\right) = \inf \{A(\{u_n\})\}, \tag{116}$$

such that

$$u_n = \sum_{i=i_{n-1}+1}^{i_n} \widehat{u_n(i)} e^{(i)} \in S\left(\left(\mathfrak{C}_{\tau(.)}\right)_h\right), 0 = i_0 < i_1 < \cdots, u_n \xrightarrow{w} 0 \text{ and} \{u_n\} \text{is asymptotic equidistant.} \tag{117}$$

Take $m \in \mathcal{N}$ large enough such that

$$\sum_{k=i_{m-1}+1}^{\infty} \left(\frac{b}{k}\right)^{\tau_k} < \varepsilon, \tag{118}$$

where $b \coloneqq \sum_{i=i_{n-1}+1}^{i_n} |u_n(i)|$. One gets for

$$\begin{aligned} h^K(u_n - u_m) &= \sum_{k=i_{n-1}+1}^{i_{m-1}} \left(\frac{1}{k}\sum_{i=1}^{k}\left|\widehat{u_n(i)}\right|\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\left(b + \sum_{i=1}^{k}\left|\widehat{u_m(i)}\right|\right)\right)^{\tau_k} \\ &\geq \sum_{k=i_{n-1}+1}^{i_{m-1}} \left(\frac{1}{k}\sum_{i=1}^{k}\left|\widehat{u_n(i)}\right|\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}\left|\widehat{u_m(i)}\right|\right)^{\tau_k} \\ &= \sum_{k=i_{n-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}\left|\widehat{u_n(i)}\right|\right)^{\tau_k} - \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{b}{k}\right)^{\tau_k} \\ &\quad + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}\left|\widehat{u_m(i)}\right|\right)^{\tau_k} > 1 - \varepsilon + 1 = 2 - \varepsilon, \end{aligned} \tag{119}$$

that is $A_n(\{u_n\}) \geq (2-\varepsilon)^{1/K}$. Note that

$$\begin{aligned} &\left[\sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\left(b + \sum_{i=1}^{k}\left|\widehat{u_m(i)}\right|\right)\right)^{\tau_k}\right]^{1/K} \\ &\leq \left[\sum_{k=i_{m-1}+1}^{\infty} \left(\frac{b}{k}\right)^{\tau_k}\right]^{1/K} + \left[\sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}\left|\widehat{u_m(i)}\right|\right)^{\tau_k}\right]^{1/K} < \varepsilon^{1/K} + 1. \end{aligned} \tag{120}$$

Therefore,

$$\begin{aligned} h^K(u_n - u_m) &= \sum_{k=i_{n-1}+1}^{i_{m-1}} \left(\frac{1}{k}\sum_{i=1}^{k}\left|\widehat{u_m(i)}\right|\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\left(b + \sum_{i=1}^{k}\left|\widehat{u_m(i)}\right|\right)\right)^{\tau_k} \\ &\leq \sum_{k=i_{n-1}+1}^{\infty} \left(\frac{1}{k}\sum_{i=1}^{k}\left|\widehat{u_m(i)}\right|\right)^{\tau_k} + \sum_{k=i_{m-1}+1}^{\infty} \left(\frac{1}{k}\left(b + \sum_{i=1}^{k}\left|\widehat{u_m(i)}\right|\right)\right)^{\tau_k} \\ &\leq 1 + \left(1+\varepsilon^{1/K}\right)^K, \end{aligned} \tag{121}$$

with $n, m \in \mathcal{N}$ and $n \neq m$. Therefore, $A_n(\{u_n\}) \leq (1 + (1+\varepsilon^{1/K})^K)^{1/K}$ and, by the arbitrariness of $\varepsilon > 0$, one has $\mathrm{WCS}((\mathfrak{C}_{\tau(.)})_h) = 2^{1/K}$. From Theorems 48 and 62, then, the function space $(\mathfrak{C}_{\tau(.)})_h$ has the h-normal structure property. □

Theorem 64 (see [46]). *If $\mathscr{H}_h$ is reflexive Banach space with the uniform Opial property, one has $\gamma(\mathscr{H}_h) = 2/WCS(\mathscr{H}_h)$..*

Theorem 65. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then $\gamma((\mathfrak{C}_{\tau(.)})_h) = 2^{1-(1/K)}$..*

Proof. Since $(\mathfrak{C}_{\tau(.)})_h$ is reflexive Banach space with the uniform Opial property, one obtains

$$\gamma\left(\left(\mathfrak{C}_{\tau(.)}\right)_h\right) = \frac{2}{\mathrm{WCS}\left(\left(\mathfrak{C}_{\tau(.)}\right)_h\right)} = 2^{1-1/K}. \tag{122}$$

□

Theorem 66. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$ and $\mathfrak{W} : (\mathfrak{C}_{\tau(.)})_h \longrightarrow (\mathfrak{C}_{\tau(.)})_h$ is h-contraction mapping, where $h(f) = [\sum_{q=0}^{\infty}(\sum_{p=0}^{q}|\widehat{f_p}|/(q+1))^{\tau_q}]^{1/K}$, for every $f \in \mathfrak{C}_{\tau(.)}$, then $\mathfrak{W}$ has a unique fixed point.*

Proof. Let the setups be satisfied. For every $f \in (\mathfrak{C}_{\tau(.)})_h$, then $\mathfrak{W}^p f \in (\mathfrak{C}_{\tau(.)})_h$. As $\mathfrak{W}$ is a h-contraction mapping, one gets

$$\begin{aligned} h\left(\mathfrak{W}^{p+1}f - \mathfrak{W}^p f\right) &\leq \alpha h\left(\mathfrak{W}^p f - \mathfrak{W}^{p-1} f\right) \\ &\leq \alpha^2 h\left(\mathfrak{W}^{p-1} f - \mathfrak{W}^{p-2} f\right) \leq \cdots \leq \alpha^p h(\mathfrak{W}f - f). \end{aligned} \tag{123}$$

So, for all $p, q \in \mathcal{N}$ so that $q > p$, one has

$$h(\mathfrak{W}^q f - \mathfrak{W}^p f) \leq \alpha^p h(\mathfrak{W}^{q-p} f - f). \tag{124}$$

Therefore, $\{\mathfrak{W}^p f\}$ is a Cauchy sequence in $(\mathfrak{C}_{\tau(.)})_h$. Since the space $(\mathfrak{C}_{\tau(.)})_h$ is prequasi-Banach (ssfps). One gets $g \in (\mathfrak{C}_{\tau(.)})_h$ with $\lim_{p\longrightarrow\infty} \mathfrak{W}^p f = g$, to prove that $\mathfrak{W}g = g$. According to Theorem 13, h verifies the Fatou property; one can see

$$h(\mathfrak{W}g - g) \leq \sup_i \inf_{p\geq i} h\left(\mathfrak{W}^{p+1} f - \mathfrak{W}^p f\right) \leq \sup_i \inf_{p\geq i} \alpha^p h(\mathfrak{W}f - f) = 0, \tag{125}$$

so $\mathfrak{W}g = g$. Then, g is a fixed point of $\mathfrak{W}$. To prove that the fixed point is unique, let us have two different fixed points $f, g \in (\mathfrak{C}_{\tau(.)})_h$ of $\mathfrak{W}$. One obtains

$$h(f - g) \leq h(\mathfrak{W}f - \mathfrak{W}g) \leq \alpha h(f - g). \tag{126}$$

So, $f = g$. □

Example 7. Assume

$$V:\left(\mathfrak{C}\left(\left(\frac{2q+3}{q+2}\right)_{q=0}^{\infty}\right)\right)_h\longrightarrow\left(\mathfrak{C}\left(\left(\frac{2q+3}{q+2}\right)_{q=0}^{\infty}\right)\right)_h, \tag{127}$$

where

$$h(g)=\sqrt{\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{g_p}\right|}{q+1}\right)^{2q+3/q+2}}, \tag{128}$$

for every $g\in\mathfrak{C}(((2q+3)/(q+2))_{q=0}^{\infty})$ and $V(g)=g/4$.

Since for all $f_1,f_2\in(\mathfrak{C}(((2q+3)/(q+2))_{q=0}^{\infty}))_h$, one gets

$$h(Vf_1-Vf_2)=h\left(\frac{f_1}{4}-\frac{f_2}{4}\right)\le\frac{1}{\sqrt[4]{64}}(h(f_1-f_2)). \tag{129}$$

So V is h-contraction. Assume $V:\Gamma\longrightarrow\Gamma$ with $V(g)=g/4$, where

$$\Gamma=\left\{f\in\left(\mathfrak{C}\left(\left(\frac{2q+3}{q+2}\right)_{q=0}^{\infty}\right)\right)_h:\widehat{f_0}=\widehat{f_1}=0\right\}. \tag{130}$$

Since V is h-contraction. So, it is h-nonexpansive. By Corollary 59, V holds a fixed point ϑ in Γ.

6. Applications to Nonlinear Summable Equations

Numerous authors, for example in [47], have examined nonlinear summable equations such as (132). This section is dedicated to locating a solution to (132) in $(\mathfrak{C}_{\tau(.)})_h$, where the conditions $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ with $\tau_0>1$ are satisfied and

$$h(f)=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{p=0}^{q}\left|\widehat{f_p}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}, \tag{131}$$

for every $f\in\mathfrak{C}_{\tau(.)}$. Take a look at the equations that are summable:

$$\widehat{g_a}=\widehat{r_a}+\sum_{m=0}^{\infty}A(a,m)f\left(m,\widehat{g_m}\right), \tag{132}$$

and assume $W:(\mathfrak{C}_{\tau(.)})_h\longrightarrow(\mathfrak{C}_{\tau(.)})_h$ defined by

$$(W(g))(z)=\sum_{a=0}^{\infty}\left(\widehat{r_a}+\sum_{m=0}^{\infty}A(a,m)f\left(m,\widehat{g_m}\right)\right)z^a. \tag{133}$$

Theorem 67. *The summable equations (132) have only one solution in $(\mathfrak{C}_{\tau(.)})_h$ if $A:\mathcal{N}^2\longrightarrow\mathbb{C}$, $f:\mathcal{N}\times\mathbb{C}\longrightarrow\mathbb{C}$, $\widehat{r}:\mathcal{N}\longrightarrow\mathbb{C}$, $\widehat{t}:\mathcal{N}\longrightarrow\mathbb{C}$, assume there is $\kappa\in\mathbb{C}$ so that $\sup_q|\kappa|^{\tau_q/K}\in[0,1)$ and for every $a\in\mathcal{N}$, we have*

$$\left|\sum_{m=0}^{\infty}A(a,m)\left(f\left(m,\widehat{g_m}\right)-f\left(m,\widehat{t_m}\right)\right)\right|\le|\kappa|\left|\widehat{g_a}-\widehat{t_a}\right|. \tag{134}$$

Proof. Let the setups be verified. Consider the mapping $W:(\mathfrak{C}_{\tau(.)})_h\longrightarrow(\mathfrak{C}_{\tau(.)})_h$ defined by (133). We have

$$\begin{aligned}h(Wg-Wt)&=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\widehat{(Wg)}_a-\widehat{(Wt)}_a\right|}{q+1}\right)^{\tau_q}\right]^{1/K}\\&=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\sum_{m=0}^{\infty}A(a,m)\left[f\left(m,\widehat{g_m}\right)-f\left(m,\widehat{t_m}\right)\right]\right|}{q+1}\right)^{\tau_q}\right]^{1/K}\\&\le\sup_q|\kappa|^{\tau_q/K}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\widehat{g_a}-\widehat{t_a}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}.\end{aligned} \tag{135}$$

□

According to Theorem 66, one obtains a unique solution of equation (132) in $(\mathfrak{C}_{\tau(.)})_h$.

Example 1. Assume the function space $(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$, where

$$h(f)=\sqrt[3]{\sum_{a=0}^{\infty}\left(\frac{\sum_{b=0}^{a}\left|\widehat{f_b}\right|}{a+1}\right)^{3a+2/a+1}}, \tag{136}$$

for all $f\in\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty})$.

$$\widehat{g_a}=5^{-(2a+3i)}+\sum_{m=0}^{\infty}(-1)^{ai+3m}\left(\frac{\cos\left|\widehat{g_a}\right|}{\sinh\left|\widehat{g_a}\right|+\sin ma+1}\right)^q, \tag{137}$$

where $q>0$, $i^2=-1$ and let $W:(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h\longrightarrow(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$ defined by

$$(W(g))(z)=\sum_{a=0}^{\infty}\left(5^{-(2a+3i)}+\sum_{m=0}^{\infty}(-1)^{ai+3m}\left(\frac{\cos\left|\widehat{g_a}\right|}{\sinh\left|\widehat{g_a}\right|+\sin ma+1}\right)^q\right)z^a. \tag{138}$$

It is easy to see that

$$\left|\sum_{m=0}^{\infty}(-1)^{ai}\left(\frac{\cos\left|\widehat{g_a}\right|}{\sinh\left|\widehat{g_a}\right|+\sin ma+1}\right)^q\left((-1)^{3m}-(-1)^{3m}\right)\right|\le\frac{1}{3}\left|\widehat{g_a}-\widehat{t_a}\right|. \tag{139}$$

By Theorem 67, the summable equations (137) have one solution in $(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$.

Example 2. Given the function space $(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$, where

$$h(f)=\sqrt{\sum_{a=0}^{\infty}\left(\frac{\sum_{b=0}^{a}\left|\widehat{f_b}\right|}{a+1}\right)^{2a+3/a+2}}, \tag{140}$$

for all $f\in\mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty})$. Consider the summable equations (137) with $a\geq 2$ and let $W:\Xi\longrightarrow\Xi$, where $\Xi=\{f\in(\mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty}))_h:\widehat{f_0}=\widehat{f_1}=0\}$, defined by

$$(W(f))(z)=\sum_{a=2}^{\infty}\left(5^{-(2a+3i)}+\sum_{m=0}^{\infty}(-1)^{ai+3m}\left(\frac{\cos\left|\widehat{f_a}\right|}{\sinh\left|\widehat{f_a}\right|+\sin ma+1}\right)^{q}\right)z^{a}. \tag{141}$$

Clearly, Ξ is a nonempty, h-convex, h-closed, and h-bounded subset of $(\mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty}))_h$. It is easy to see that

$$\left|\sum_{m=0}^{\infty}(-1)^{ai}\left(\frac{\cos\left|\widehat{g_a}\right|}{\sinh\left|\widehat{g_a}\right|+\sin ma+1}\right)^{q}\left((-1)^{3m}-(-1)^{3m}\right)\right|\leq\frac{1}{9}\left|\widehat{g_a}-\widehat{t_a}\right|. \tag{142}$$

By Theorem 67 and Corollary 59, the summable equations (137) with $a\geq 2$ have a solution in Ξ.

Example 3. Assume the function space $(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$, where

$$h(g)=\sqrt[3]{\sum_{a=0}^{\infty}\left(\frac{\sum_{b=0}^{a}\left|\widehat{g_b}\right|}{a+1}\right)^{3a+2/a+1}}, \tag{143}$$

for all $g\in\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty})$.

Consider the non-linear difference equations,

$$\widehat{g_a}=e^{-(2a+3i)}+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3mi-a)\cos^{p}\left|\widehat{g_{a-2}}\right|}{\sinh^{q}\left|\widehat{g_{a-1}}\right|+\sin ma+1}, \tag{144}$$

where $\widehat{g_{-2}},\widehat{g_{-1}},p,q>0$, $i^2=-1$ and let $W:(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h\longrightarrow(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$ defined by

$$(W(g))(z)=\sum_{a=0}^{\infty}\left(e^{-(2a+3i)}+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3mi-a)\cos^{p}\left|\widehat{g_{a-2}}\right|}{\sinh^{q}\left|\widehat{g_{a-1}}\right|+\sin ma+1}\right)z^{a}. \tag{145}$$

It is easy to see that

$$\left|\sum_{m=0}^{\infty}\frac{\cosh(3mi-a)\cos^{p}\left|\widehat{g_{a-2}}\right|}{\sinh^{q}\left|\widehat{g_{a-1}}\right|+\sin ma+1}(\tan(2m+1)-\tan(2m+1))\right|\leq\frac{1}{5}\left|\widehat{g_a}-\widehat{t_a}\right|. \tag{146}$$

By Theorem 67, the nonlinear difference equations (144) have one solution in $(\mathfrak{C}(((3a+2)/(a+1))_{a=0}^{\infty}))_h$.

Example 4. Given the function space $(\mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty}))_h$, where $h(g)=\sqrt{\sum_{a=0}^{\infty}(\sum_{b=0}^{a}|\widehat{g_b}|/(a+1))^{(2a+3)/(a+2)}}$, for all $g\in\mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty})$. Consider the non-linear difference equations (144) with $a\geq 1$ and let $W:\Xi\longrightarrow\Xi$, where $\Xi=\{g\in(\mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty}))_h:\widehat{g_0}=0\}$, defined by

$$(W(g))(z)=\sum_{a=1}^{\infty}\left(e^{-(2a+3i)}+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3mi-a)\cos^{p}\left|\widehat{g_{a-2}}\right|}{\sinh^{q}\left|\widehat{g_{a-1}}\right|+\sin ma+1}\right)z^{a}. \tag{147}$$

Clearly, Ξ is a nonempty, h-convex, h-closed, and h-bounded subset of $(\mathfrak{C}(((2a+3)/(a+2))_{a=0}^{\infty}))_h$. It is easy to see that

$$\left|\sum_{m=0}^{\infty}\frac{\cosh(3mi-a)\cos^{p}\left|\widehat{g_{a-2}}\right|}{\sinh^{q}\left|\widehat{g_{a-1}}\right|+\sin ma+1}(\tan(2m+1)-\tan(2m+1))\right|\leq\frac{1}{5}\left|\widehat{g_a}-\widehat{t_a}\right|. \tag{148}$$

By Theorem 67 and Corollary 59, the nonlinear difference equations (144) with $a\geq 1$ have a solution in Ξ.

Example 5. The summable equations (132) have a solution in $(\mathfrak{C}_{\tau(.)})_h$ if

$$K\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\widehat{r_a}-\widehat{g_a}+\sum_{m=0}^{\infty}A(a,m)f\left(m,\widehat{g_m}\right)\right|}{q+1}\right)^{\tau_q}\right]^{1/K}\leq\ln\frac{\sum_{q=0}^{\infty}\left(\left(\sum_{a=0}^{q}\left|\widehat{r_a}+\sum_{m=0}^{\infty}A(a,m)f\left(m,\widehat{g_m}\right)\right|\right)/(q+1)\right)^{\tau_q}}{\sum_{q=0}^{\infty}\left(\sum_{a=0}^{q}\left|\widehat{g_a}\right|/(q+1)\right)^{\tau_q}}. \tag{149}$$

Evidently, we have

$$h(Wg-g)=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\widehat{r_a}-\widehat{g_a}+\sum_{m=0}^{\infty}A(a,m)f\left(m,\widehat{g_m}\right)\right|}{q+1}\right)^{\tau_q}\right]^{1/K}$$
$$\leq\frac{1}{K}\ln\frac{\sum_{q=0}^{\infty}\left(\left(\sum_{a=0}^{q}\left|\widehat{r_a}+\sum_{m=0}^{\infty}A(a,m)f\left(m,\widehat{g_m}\right)\right|\right)/(q+1)\right)^{\tau_q}}{\sum_{q=0}^{\infty}\left(\sum_{a=0}^{q}\left|\widehat{g_a}\right|/(q+1)\right)^{\tau_q}}$$
$$=\ln\left(h(Wg)\right)-\ln\left(h(g)\right). \tag{150}$$

By Theorem 18, one gets a solution of equation (132) in $(\mathfrak{C}_{\tau(.)})_h$.

Example 6. The summable equations (132) have a solution in $(\mathfrak{C}_{\tau(.)})_h$, if

$$\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\widehat{r_a}-\widehat{g_a}+\sum_{m=0}^{\infty}A(a,m)f\left(m,\widehat{g_m}\right)\right|}{q+1}\right)^{\tau_q}\right]^{1/K}$$
$$\leq\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\widehat{g_a}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}-\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\widehat{r_a}+\sum_{m=0}^{\infty}A(a,m)f\left(m,\widehat{g_m}\right)\right|}{q+1}\right)^{\tau_q}\right]^{1/K}. \tag{151}$$

Clearly, we have

$$h(Wg-g)=\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\widehat{r_a}-\widehat{g_a}+\sum_{m=0}^{\infty}A(a,m)f\left(m,\widehat{g_m}\right)\right|}{q+1}\right)^{\tau_q}\right]^{1/K}$$
$$\leq\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\widehat{g_a}\right|}{q+1}\right)^{\tau_q}\right]^{1/K}-\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\widehat{r_a}+\sum_{m=0}^{\infty}A(a,m)f\left(m,\widehat{g_m}\right)\right|}{q+1}\right)^{\tau_q}\right]^{1/K}$$
$$=h(g)-h(Wg). \tag{152}$$

By Theorem 18, one gets a solution of equation(132) in $(\mathfrak{C}_{\tau(.)})_h$.

Assume Ω is the set of all closed and bounded intervals on the real line $\mathfrak{R}$. For $t=[t_1,t_2]$ and $g=[g_1,g_2]$ in Ω, suppose

$$t\leq g \text{ if and only if } t_1\leq g_1 \text{ and } t_2\leq g_2. \tag{153}$$

Define a metric ρ on Ω by

$$\rho(t,g)=\max\left\{|t_1-g_1|,|t_2-g_2|\right\}. \tag{154}$$

Matloka [48] showed that ρ is a metric on Ω, and (Ω,ρ) is a complete metric space.

Definition 68. A fuzzy number g is a fuzzy subset of $\mathfrak{R}$, i.e., a mapping $g:\mathfrak{R}\longrightarrow[0,1]$ which verifies the following four settings:

(a) g is fuzzy convex, i.e., for $x,y\in\mathfrak{R}$ and $\alpha\in[0,1]$, $g(\alpha x+(1-\alpha)y)\geq\min\{g(x),g(y)\}$

(b) g is normal, i.e., there is $y_0\in\mathfrak{R}$ such that $g(y_0)=1$

(c) g is an upper semicontinuous, i.e., for all $\alpha>0$, $g^{-1}([0,x+\alpha))$ for all $x\in[0,1]$ is open in the usual topology of $\mathfrak{R}$

(d) The closure of $g^0\coloneqq\{y\in\mathfrak{R}:g(y)>0\}$ is compact

Recall that the β-level set of a fuzzy real number g, $0<\beta<1$ indicated by g^β is defined as

$$g^\beta=\{y\in\mathfrak{R}:g(y)\geq\beta\}. \tag{155}$$

The set of every upper semicontinuous, normal, convex fuzzy number, and is compact and is denoted by $\mathfrak{R}([0,1])$. The set $\mathfrak{R}$ can be embedded in $\mathfrak{R}([0,1])$, if we define $r\in\mathfrak{R}([0,1])$ by

$$\bar{r}(t)=\begin{cases}1, & t=r,\\ 0, & t\neq r.\end{cases} \tag{156}$$

Consider the summable equations of fuzzy reals (132) and assume $W:(\mathfrak{C}_{\tau(.)})_h\longrightarrow(\mathfrak{C}_{\tau(.)})_h$ defined by

$$(W(g))(z)=\sum_{a=0}^{\infty}\bar{\rho}\left(\widehat{r_a}+\sum_{m=0}^{\infty}A(a,m)f\left(m,\widehat{g_m}\right),\bar{0}\right)z^a, \tag{157}$$

where $\bar{\rho}:\mathfrak{R}[0,1]\times\mathfrak{R}[0,1]\longrightarrow\mathfrak{R}^+\cup\{0\}$ is defined by $\bar{\rho}(t,g)=\sup_{0\leq\beta\leq1}\rho(t^\beta,g^\beta)$. For more details about the fuzzy numbers and their properties, see Zadeh [49].

Theorem 69. *The summable equations (132) have an unique solution in $(\mathfrak{C}_{\tau(.)})_h$ if $A:\mathcal{N}^2\longrightarrow\mathfrak{R}^+$, $f:\mathcal{N}\times\mathfrak{R}^+[0,1]\longrightarrow\mathfrak{R}^+[0,1]$, $\widehat{r_a}:\mathcal{N}\longrightarrow\mathfrak{R}^+[0,1]$, $\widehat{t_a}:\mathcal{N}\longrightarrow\mathfrak{R}^+[0,1]$, assume there is $\kappa\in\mathbb{C}$ so that $\sup_q|\kappa|^{\tau_q/K}\in[0,1)$ and for every $a\in\mathcal{N}$, we have*

$$\left|\sum_{m=0}^{\infty}A(a,m)\left(f\left(m,\widehat{g_m}\right)-f\left(m,\widehat{t_m}\right)\right)\right|\leq|\kappa|\left|\widehat{g_a}-\widehat{t_a}\right|. \tag{158}$$

Proof. Let the setups be verified. Consider the mapping $W : (\mathfrak{C}_{\tau(.)})_h \longrightarrow (\mathfrak{C}_{\tau(.)})_h$ defined by (157). We have

$$\begin{aligned} h(Wg - Wt) &= \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|(\widehat{Wg})_a - (\widehat{Wt})_a\right|}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &= \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\bar{\rho}(\widehat{r_a} + \sum_{m=0}^{\infty}A(a,m)f(m,\widehat{g_m}),\bar{0}) - \bar{\rho}\left(\widehat{r_a} + \sum_{m=0}^{\infty}A(a,m)f\left(m,\widehat{t_m}\right),\bar{0}\right)\right|}{q+1}\right)^{\tau_q}\right]^{1/K} \\ &= \left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\sum_{m=0}^{\infty}A(a,m)\left[f(m,\widehat{g_m}) - f\left(m,\widehat{t_m}\right)\right]\right|}{q+1}\right)^{\tau_q}\right]^{1/K} \leq \sup_q|\kappa|^{\tau_q/K}\left[\sum_{q=0}^{\infty}\left(\frac{\sum_{a=0}^{q}\left|\bar{\rho}(\widehat{g_a},\bar{0}) - \bar{\rho}\left(\widehat{t_a},\bar{0}\right)\right|}{q+1}\right)^{\tau_q}\right]^{1/K} = \sup_q|\kappa|^{\tau_q/K}h(g-t). \end{aligned} \tag{159}$$

□

According to Theorem 66, one obtains a unique solution of equation (132) in $(\mathfrak{C}_{\tau(.)})_h$.

7. Conclusion

We discuss in this paper some topological and geometric structure of $(\mathfrak{C}_{\tau(.)})_h$, the existence of Caristi's fixed point in it, of the class $\mathfrak{F}_{(\mathfrak{C}_{\tau(.)})_h}$, and of the class $(\mathfrak{F}_{(\mathfrak{C}_{\tau(.)})_h})^{\lambda}$. Moreover, some geometric properties related to the fixed point theory in $(\mathfrak{C}_{\tau(.)})_h$ are introduced. Finally, we investigate several solutions applications to summable equations and illustrate our findings with some instances. This article has several advantages for researchers, such as studying the fixed points of any contraction mapping on this prequasispace, which is a generalization of the quasinormed spaces, a new general space of solutions for many difference equations, examining the eigenvalue problem in these new settings, and noting that the closed mappings' ideals are certain to play an important function in the principle of Banach lattices, hence since many fixed point theorems in a particular space work by either expanding the self-mapping acting on it or expanding the space itself, as future work, we can enlarge the space $(\mathfrak{C}_{\tau(.)})_h$ by q-analogue or generalize the self-mapping acting on it.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work was funded by the University of Jeddah, Saudi Arabia, under grant No. UJ-21-DR-76. The authors, therefore, acknowledge with thanks the university technical and financial support.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Berlin, 2011.

[3] K. Rajagopal and M. Ruzicka, "On the modeling of electrorheological materials," *Mechanics Research Communications*, vol. 23, no. 4, pp. 401–407, 1996.

[4] M. Ruzicka, "Electrorheological fluids. modeling and mathematical theory," in *Lecture Notes in Mathematics*, p. 1748, Springer, Berlin, Germany, 2000.

[5] L. Guo and Q. Zhu, "Stability analysis for stochastic Volterra Levin equations with Poisson jumps: fixed point approach," *Journal of Mathematical Physics*, vol. 52, no. 4, article 042702, 2011.

[6] W. Mao, Q. Zhu, and X. Mao, "Existence, uniqueness and almost surely asymptotic estimations of the solutions to neutral stochastic functional differential equations driven by pure jumps," *Applied Mathematics and Computation*, vol. 254, pp. 252–265, 2015.

[7] X. Yang and Q. Zhu, "Existence, uniqueness, and stability of stochastic neutral functional differential equations of Sobolev-type," *Journal of Mathematical Physics*, vol. 56, no. 12, article 122701, 2015.

[8] A. Pietsch, "Einigie neu Klassen von Kompakten linearen Abbildungen," *REVUE ROUMAINE DE MATHÉMATIQUES PURES ET APPLIQUÉES*, vol. 8, pp. 427–447, 1963.

[9] A. Pietsch, "S-numbers of operators in Banach spaces," *Studia Mathematica*, vol. 51, no. 3, pp. 201–223, 1974.

[10] A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1980.

[11] A. Pietsch, "Small ideals of operators," *Studia Mathematica*, vol. 51, no. 3, pp. 265–267, 1974.

[12] G. Constantin, "Operators of ces–p type," *Rend. Acc. Naz. Lincei.*, vol. 52, no. 8, pp. 875–878, 1972.

[13] B. M. Makarov and N. Faried, "Some properties of operator ideals constructed by s numbers," in *Theory of Operators in Functional Spaces*, pp. 206–211, Academy of Science. Siberian section, Novosibirsk, Russia, 1977.

[14] N. Tita, "On Stolz mappings," *MATHEMATICA JAPONICA*, vol. 26, no. 4, pp. 495-496, 1981.

[15] N. Tita, *Ideale de operatori generate de s numere*, Univ. Tranilvania. Brasov, 1998.

[16] A. Maji and P. D. Srivastava, "Some results of operator ideals on s-type $|A, p|$ operators," *Tamkang Journal of Mathematics*, vol. 45, no. 2, pp. 119–136, 2014.

[17] E. E. Kara and M. İlkhan, "On a new class of s-type operators," *Konuralp Journal of Mathematics (KJM)*, vol. 3, no. 1, pp. 1–11, 2015.

[18] T. Yaying, B. Hazarika, and M. Mursaleen, "On sequence space derived by the domain of q-Cesàro matrix in ℓ_p space and the associated operator ideal," *Journal of Mathematical Analysis and Applications*, vol. 493, article 124453, 2021.

[19] R. Kannan, "Some results on fixed points—II," *The American Mathematical Monthly*, vol. 76, no. 4, pp. 405–408, 1969.

[20] S. J. H. Ghoncheh, "Some fixed point theorems for Kannan mapping in the modular spaces," *Ciência e Natura*, vol. 37, pp. 462–466, 2015.

[21] A. A. Bakery and O. S. K. Mohamed, "Kannan prequasi contraction maps on Nakano sequence spaces," *Journal of Function Spaces*, vol. 2020, Article ID 8871563, 10 pages, 2020.

[22] A. A. Bakery and O. S. K. Mohamed, "Kannan nonexpansive maps on generalized Cesàro backward difference sequence space of non-absolute type with applications to summable equations," *Journal of Inequalities and Applications*, vol. 2021, no. 1, 2021.

[23] B. Altay and F. Başar, "Generalization of the sequence space $\ell_{(p)}$ derived by weighted means," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 147–185, 2007.

[24] A. L. Shields, "Weighted shift operators and analytic function theory," in *Topics of Operator Theory*, vol. 13 of Math. Surveys Monographs, Amer. Math., Providence, RI, 1974.

[25] K. Hedayatian, "On cyclicity in the space $H^p(\beta)$," *Taiwanese Journal of Mathematics*, vol. 8, no. 3, pp. 429–442, 2004.

[26] H. Emamirad and G. S. Heshmati, "Chaotic weighted shifts in Bargmann space," *Journal of Mathematical Analysis and Applications*, vol. 308, no. 1, pp. 36–46, 2005.

[27] N. Faried, A. Morsy, and Z. A. Hassanain, "S-numbers of shift operators of formal entire functions," *Journal of Approximation Theory*, vol. 176, pp. 15–22, 2013.

[28] H. Nakano, "Modulared sequence spaces," *Proceedings of the Japan Academy*, vol. 27, pp. 508–512, 1951.

[29] A. A. Bakery and M. H. El Dewaik, "A generalization of Caristi's fixed point theorem in the variable exponent weighted formal power series space," *Journal of Function Spaces*, vol. 2021, Article ID 9919420, 18 pages, 2021.

[30] A. A. Bakery, E. A. E. Mohamed, and E. A. E. Mohamed, "Some applications of new complex function space constructed by different weights and exponents," *Journal of Mathematics*, vol. 2021, Article ID 7570145, 18 pages, 2021.

[31] C. Farkas, "A generalized form of Ekeland's variational principle," *Analele Universitatii" Ovidius" Constanta-Seria Matematica*, vol. 20, no. 1, pp. 101–112, 2012.

[32] B. E. Rhoades, "Operators of A-*p* type," *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni.*, vol. 59, no. 3-4, pp. 238–241, 1975.

[33] A. Pietsch, *Eigenvalues and S-Numbers*, Cambridge University Press, New York. NY, USA, 1986.

[34] N. Faried and A. A. Bakery, "Small operator ideals formed by *s* numbers on generalized Cesáro and Orlicz sequence spaces," *Journal of Inequalities and Applications*, vol. 2018, no. 1, 2018.

[35] A. A. Bakery, E. A. E. Mohamed, and O. S. K. Mohamed, "On the domain of Cesàro matrix defined by weighted means in $\ell_{t(.)}$. , and its pre-quasi ideal with some applications," *Journal of Mathematics and Computer Science*, vol. 26, no. 1, pp. 41–66, 2021.

[36] A. Pietsch, *Operator Ideals*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[37] D. Kutzarova, "_k_ - _β_ and _k_ -nearly uniformly convex Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 162, no. 2, pp. 322–338, 1991.

[38] Y. Cui and H. Hudzik, "On the uniform Opial property in some modular sequence spaces," *Uniwersytet im*, vol. 26, pp. 93–102, 1998.

[39] S. Prus, "Banach spaces with the uniform opial property," *Nonlinear Analysis*, vol. 18, no. 8, pp. 697–704, 1992.

[40] K. Kuratowski, "Sur les espaces complets," *Fundamenta Mathematicae*, vol. 15, pp. 301–309, 1930.

[41] R. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, 1990.

[42] W. A. Kirk, "A fixed point theorem for mappings which do not increase distances," *The American Mathematical Monthly*, vol. 72, no. 9, pp. 1004–1006, 1965.

[43] J. M. Ayerbe Toledano, T. Dominguez Benavides, and G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, vol. 99 of Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 1997.

[44] G. Zhang, "Weakly convergent sequence coefficient of product space," *Proceedings of the American Mathematical Society*, vol. 117, no. 3, pp. 637–643, 1993.

[45] W. Bynum, "Normal structure coefficients for Banach spaces," *Pacific Journal of Mathematics*, vol. 86, no. 2, pp. 427–436, 1980.

[46] J. M. Ayerbe Toledano and T. Dominguez Benavides, "Connections between some Banach space coefficients concerning normal structure," *Journal of Mathematical Analysis and Applications*, vol. 172, no. 1, pp. 53–61, 1993.

[47] P. Salimi, A. Latif, and N. Hussain, "Modified α-ψ-contractive mappings with applications," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[48] M. Matloka, "Sequences of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 28, pp. 28–37, 1986.

[49] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.