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Research Article

(p, h)-Convex Functions Associated with Hadamard and Fejér-Hadamard Inequalities via k-Fractional Integral Operators

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In this article, generalized versions of the k-fractional Hadamard and Fejér-Hadamard inequalities are constructed. To obtain the generalized versions of these inequalities, k-fractional integral operators including the well-known Mittag-Leffler function are utilized. The class of (p, h)-convex functions for Hadamard-type inequalities give the generalizations of results which have been proved in literature for p-convex, h-convex, and several functions deducible from these two classes.

1. Introduction

The subject of fractional calculus which includes the study of fractional-order integrals and derivatives has become a very prominent research topic in recent years. It has got the attention of researchers working in almost all fields of science and engineering. Integral operators are a very important part of fractional calculus; they play an important role in the mathematical treatment of different kinds of real-world problems. In the theory of mathematical inequalities, fractional integral operators appeared as a tool for the generalizations of classical inequalities. In the last two decades, a lot of such generalizations have been published by several authors.

Iscan and Wu [1] have generalized Hermite-Hadamard inequalities for harmonically convex functions by applying fractional integrals. Akkurt et al. [2] have obtained inequalities for $(k-h)$-Riemann-Liouville fractional integrals with the aid of synchronous and monotonic functions. Mubeen and Habibullah [3] have introduced k-analogue of Riemann-Liouville fractional integrals which are used for establishing k-fractional versions of well-known inequalities. A study of k-fractional integrals has been presented by Tunç et al. in [4], and fractional inequalities have been proved. Farid et al. in [5] demonstrated some novel Fejér-Hadamard and Hadamard inequalities for harmonically convex functions by using fractional integrals containing the Mittag-Leffler function.

Kunt and Iscan [6] presented Hermite-Hadamard-Fejér-type inequalities for p-convex functions using fractional integrals. Rashid et al. [7] demonstrated the Hadamard and the Fejér-Hadamard-type inequalities for the generalized fractional integral operator involving Mittag-Leffler functions for preinvexity and m-preinvexity. Klnç et al. [8] proved Hadamard and Fejér-Hadamard inequalities for (h, m)-strongly convex functions via generalized fractional integrals with Mittag-Leffler functions. Jia et al. [9] demonstrated new types of Hadamard and Fejér-Hadamard fractional integral inequalities for Riemann-Liouville fractional integrals. Yussouf et al. [10] presented generalized types of Hadamard and Fejér-Hadamard-type fractional integral inequalities by utilizing generalized fractional integrals including Mittag-Leffler functions. Faisal et al. [11] proved new Hermite-Hadamard-Jensen-Mercer-type inequalities for convex functions by using Riemann-Liouville fractional integrals. The other recent studies in this area were presented by Zhao et al. [12–14] and Chu et al. [15].

The purpose of this article is to investigate the Hadamard and the Fejér-Hadamard-type inequalities for (p, h)-convex functions via generalized k-fractional integrals including Mittag-Leffler functions. Inequalities for several kinds of convex functions are special cases of results of this paper. Also, several new inequalities can be deduced from presented results in specific settings. First, we give definitions of integral operators and some results which will set the foundation for this article.

Definition 1 (see [16]). Let $w, \alpha, l, \gamma, c, \in \mathbb{C}$, and $\mathfrak{R}(l), \mathfrak{R}(\alpha) > 0, \mathfrak{R}(c) > \mathfrak{R}(\gamma) > 0$ with $\tilde{p} \geq 0, \delta, \mu > 0$, and $0 < \nu \leq \delta + \mu$. Also, let $\sigma \in L_1[\varepsilon, \nu]$ and $\zeta \in [\varepsilon, \nu]$. In that case, the generalized fractional operators $F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,\varepsilon} + \sigma$ and $F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,\nu} - \sigma$ are defined by

$$\left(F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,\varepsilon+}\sigma\right)(\zeta;\tilde{p}) = \int_{\varepsilon}^{\zeta} (\zeta - t)^{\alpha-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\zeta - t)^{\mu};\tilde{p}\right)\sigma(t)dt,$$
$$\left(F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,\upsilon-}\sigma\right)(\zeta;\tilde{p}) = \int_{\zeta}^{\upsilon} (t - \zeta)^{\alpha-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(t - \zeta)^{\mu};\tilde{p}\right)\sigma(t)dt, \tag{1}$$

where

$$E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(t;\tilde{p}) = \sum_{n=0}^{\infty} \frac{\beta_{\tilde{p}}(\gamma + n\nu, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{n\nu}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}} \tag{2}$$

is the enlarged generalized Mittag-Leffler function and $\beta_{\tilde{p}}$ is the expansion of beta function described as noted below:

$$\beta_{\tilde{p}}(\zeta, \eta) = \int_0^1 t^{\zeta-1}(1 - t)^{\eta-1} e^{-(\tilde{p}/t(1-t))} dt, \tag{3}$$

where $\mathfrak{R}(\zeta), \mathfrak{R}(\eta), \mathfrak{R}(c\tilde{p}) > 0$.

Definition 2 (see [16]). Let $\sigma, \varsigma : [\varepsilon, \upsilon] \longrightarrow \mathbb{R}$ with $0 < \varepsilon < \upsilon$ be the functions such that σ is positive and $\sigma \in L_1[\varepsilon, \upsilon]$ and ς be differentiable and absolutely increasing. Also, let ϕ/ζ be an increasing function on $[\varepsilon, \infty)$, and $\gamma, c, w, \alpha, l, \in \mathbb{C}$, $\mathfrak{R}(l)$, $\mathfrak{R}(\alpha) > 0, \mathfrak{R}(c) > \mathfrak{R}(\gamma) > 0$ with $\tilde{p} \geq 0, \delta, \mu > 0$, and $0 < \nu \leq \mu + \delta$. In that case, for $\zeta \in [\varepsilon, \upsilon]$, the fractional operators are defined by

$$\left({}_{\varsigma}F^{\phi,\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,\varepsilon+}\sigma\right)(\zeta;\tilde{p}) = \int_{\varepsilon}^{\zeta} \frac{\phi(\varsigma(\zeta) - \varsigma(t))}{\varsigma(\zeta) - \varsigma(t)} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\varsigma(\zeta) - \varsigma(t))^{\mu};\tilde{p}\right)\varsigma'(t)\sigma(t)dt,$$
$$\left({}_{\varsigma}F^{\phi,\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,\upsilon-}\sigma\right)(\zeta;\tilde{p}) = \int_{\zeta}^{\upsilon} \frac{\phi(\varsigma(t) - \varsigma(\zeta))}{\varsigma(t) - \varsigma(\zeta)} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\varsigma(t) - \varsigma(\zeta))^{\mu};\tilde{p}\right)\varsigma'(t)\sigma(t)dt. \tag{4}$$

Definition 3 (see [17]). Let $\sigma, \varsigma : [\varepsilon, \upsilon] \longrightarrow \mathbb{R}$ with $0 < \varepsilon < \upsilon$ be the functions such that σ is positive and $\sigma \in L_1[\varepsilon, \upsilon]$ and ς be differentiable and absolutely increasing. Also, let w, $\gamma, c, \alpha, l, \in \mathbb{C}$ *and* $\mathfrak{R}(l), \mathfrak{R}(\alpha) > 0, \mathfrak{R}(c) > \mathfrak{R}(\gamma) > 0$ *with* $\tilde{p} \geq 0, \delta, \mu > 0$ *and* $0 < \nu \leq \mu + \delta$. In that case, for $\zeta \in [\varepsilon, \upsilon]$, the unified integral operators are defined by

$$\left({}_{\varsigma}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,\varepsilon+}\sigma\right)(\zeta;\tilde{p}) = \int_{\varepsilon}^{\zeta} (\varsigma(\zeta) - \varsigma(t))^{\alpha-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\varsigma(\zeta) - \varsigma(t))^{\mu};\tilde{p}\right)\varsigma'(t)\sigma(t)dt,$$
$$\left({}_{\varsigma}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,\upsilon-}\sigma\right)(\zeta;\tilde{p}) = \int_{\zeta}^{\upsilon} (\varsigma(t) - \varsigma(\zeta))^{\alpha-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\varsigma(t) - \varsigma(\zeta))^{\mu};\tilde{p}\right)\varsigma'(t)\sigma(t)dt. \tag{5}$$

Recently, Yue et al. [18] described generalized k-fractional operators including the further extension of the Mittag-Leffler function as noted below.

Definition 4. Let $\sigma, \varsigma : [\varepsilon, \upsilon] \longrightarrow \mathbb{R}$ with $0 < \varepsilon < \upsilon$ be the functions such that σ be positive and $\sigma \in L_1[\varepsilon, \upsilon]$ and ς be differentiable and absolutely increasing. Let $\alpha > k$ and γ, c, $w, \alpha, l, \in \mathbb{R}, \alpha, l > 0$, and $c > \gamma > 0$, with $\tilde{p} \geq 0, \delta, \mu > 0$ *and* $0 < \nu \leq \mu + \delta$. In that case, for $\zeta \in [\varepsilon, \upsilon]$, the left-right generalized k-fractional operators $\left({}^{k}_{\varsigma}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,\varepsilon+}\sigma\right)$ and $\left({}^{k}_{\varsigma}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,\upsilon-}\sigma\right)$ are defined by

$$\left({}^{k}_{\varsigma}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,\varepsilon+}\sigma\right)(\zeta;\tilde{p}) = \int_{\varepsilon}^{\zeta} (\varsigma(\zeta) - \varsigma(t))^{\alpha/k-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\varsigma(\varepsilon) - \varsigma(t))^{\mu};\tilde{p}\right)\varsigma'(t)\sigma(t)dt, \tag{6}$$

$$\left({}^{k}_{\varsigma}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,\upsilon-}\sigma\right)(\zeta;\tilde{p}) = \int_{\zeta}^{\upsilon} (\varsigma(t) - \varsigma(\zeta))^{\alpha/k-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\varsigma(t) - \varsigma(\zeta))^{\mu};\tilde{p}\right)\varsigma'(t)\sigma(t)dt. \tag{7}$$

Next, we define convex and related functions.

Definition 5 (see [19]). A function $\sigma : [\varepsilon, \upsilon] \longrightarrow \mathbb{R}$ is said to be convex, if

$$\sigma(t\zeta + (1 - t)\eta) \leq t\sigma(\zeta) + (1 - t)\sigma(\eta), \tag{8}$$

for all $\zeta, \eta \in [\varepsilon, \upsilon]$ and $t \in [0, 1]$.

Definition 6 (see [20]). Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $\sigma : I \longrightarrow \mathbb{R}$ is said to be a p-convex function, if

$$\sigma\left(\left[t\zeta^p + (1 - t)\eta^p\right]^{1/p}\right) \leq t\sigma(\zeta) + (1 - t)\sigma(\eta), \tag{9}$$

for all $\zeta, \eta \in I$ and $t \in [0, 1]$.

Definition 7 (see [6]). Let $p \in \mathbb{R}\backslash\{0\}$. In that case, a function $\sigma : [\varepsilon, \upsilon] \subset (0,\infty) \longrightarrow \mathbb{R}$ is called p-symmetric in accordance with $[\varepsilon^p + \upsilon^p/2]^{1/p}$, if

$$\sigma\left(t^{1/p}\right) = \sigma\left(\left[\varepsilon^p - \upsilon^p - t\right]^{1/p}\right), \tag{10}$$

for $t \in [0, 1]$.

Definition 8 (see [21]). Let $h : J \longrightarrow \mathbb{R}$ be a nonnegative and nonzero function and $p \in \mathbb{R}\backslash\{0\}$. We say that $\sigma : I \longrightarrow \mathbb{R}$ is a (p, h)-convex function, if σ is nonnegative and

$$\sigma\left(\left[t\zeta^p + (1-t)\eta^p\right]^{1/p}\right) \le h(t)\sigma(\zeta) + h(1-t)\sigma(\eta), \quad (11)$$

for all $\zeta, \eta \in I$ and $t \in (0, 1)$. If the above inequality is reversed, then σ is said to be a (p, h)-concave function.

Remark 9.

(i) By taking $h(t) = t$ in Definition 8, we obtain the definition of p-convex function

(ii) By taking $h(t) = t$ and $p = 1$ in Definition 8, we obtain the definition of convex function

(iii) By taking $h(t) = t^s$ and $p = 1$ in Definition 8, we obtain the definition of s-convex function

(iv) By taking $h(t) = t^{-1}$ and $p = 1$ in Definition 8, we obtain the definition of Godunova-Levin-type convex function

(v) By taking $p = 1$ in Definition 8, we obtain the definition of h-convex function

(vi) By taking $h(t) = 1$ and $p = 1$ in Definition 8, we obtain the definition of p-function

The following inequality is the well-known Hadamard inequality.

Theorem 10 (see [22]). *Let $\sigma : [\varepsilon, \upsilon] \longrightarrow \mathbb{R}$ be a convex function for $\varepsilon < \upsilon$. Then, the following inequality holds:*

$$\sigma\left(\frac{\varepsilon+\upsilon}{2}\right) \le \frac{1}{\upsilon-\varepsilon}\int_\varepsilon^\upsilon \sigma(\zeta)d\zeta \le \frac{\sigma(\varepsilon)+\sigma(\upsilon)}{2}. \quad (12)$$

The Fejér-Hadamard inequality is a weighted type of the Hadamard inequality presented by Fejér in [23].

Theorem 11. *Let $\sigma : [\varepsilon, \upsilon] \longrightarrow \mathbb{R}$ be a convex function and $\varsigma : [\varepsilon, \upsilon] \longrightarrow \mathbb{R}$ be nonnegative, integrable, and symmetric with respect to $((\varepsilon + \upsilon))/2$. In that case, the below inequality takes:*

$$\sigma\left(\frac{\varepsilon+\upsilon}{2}\right)\int_\varepsilon^\upsilon \varsigma(\zeta)d\zeta \le \int_\varepsilon^\upsilon \sigma(\zeta)\varsigma(\zeta)d\zeta \le \frac{\sigma(\varepsilon)+\sigma(\upsilon)}{2}\int_\varepsilon^\upsilon \varsigma(\zeta)d\zeta. \quad (13)$$

For other classes of functions defined after motivating from convex function, the above inequalities have been studied extensively (see [1, 5, 6, 24–26]).

In the upcoming section, first, we construct the Hadamard inequality for (p, h)-convex function via generalized k-fractional integrals. Then, an identity is used to construct the Fejér-Hadamard inequality for (p, h)-convex function via generalized k-fractional integrals. After, another version of the Hadamard inequality and the Fejér-Hadamard inequality is presented. Moreover, the presented results generalize many already published results.

2. k-Fractional Integral Inequalities of Hadamard and Fejér-Hadamard Type

The generalized k-fractional Hadamard inequality is stated and proved in the following theorem.

Theorem 12. *Let $h : J \longrightarrow \mathbb{R}$ be nonnegative, nonzero, and integrable function. Also, let $\sigma, \varsigma : [\varepsilon, \upsilon] \longrightarrow \mathbb{R}$ with $0 < \varepsilon < \upsilon$ be the functions such that σ is positive and $\sigma \in L_1[\varepsilon, \upsilon]$, and ς is differentiable and absolutely increasing. If σ is (p, h)-convex, and $p \in R\backslash\{0\}$, in that case, the below inequalities for k-fractional operators (6) and (7) occur:*

(i) If $p > 0$, in that case,

$$\begin{aligned}
&\sigma\left(\left[\frac{\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)}{2}\right]^{1/p}\right)\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))+}1\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon))\,;\tilde{p}\right)\\
&\le h\left(\frac{1}{2}\right)\left[\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))+}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon))\,;\tilde{p}\right)\right.\\
&\quad\left.+\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\upsilon))-}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\varepsilon))\,;\tilde{p}\right)\right]\\
&\le h\left(\frac{1}{2}\right)[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]\left[\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},1-}h\right)(0\,;\tilde{p})\right.\\
&\quad\left.+\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},0+}h\right)(1\,;\tilde{p})\right],
\end{aligned} \quad (14)$$

where $\bar{w} = w/(\varsigma^p(\upsilon) - \varsigma^p(\varepsilon))^\mu$ and $\theta(t) = \varsigma^{1/p}(t)$ for $t \in [\varepsilon^p, \upsilon^p]$

(ii) If $p < 0$, in that case,

$$\begin{aligned}
&\sigma\left(\left[\frac{\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)}{2}\right]^{1/p}\right)\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))-}1\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon))\,;\tilde{p}\right)\\
&\le h\left(\frac{1}{2}\right)\left[\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\upsilon))+}\sigma\circ\varsigma\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon))\,;\tilde{p}\right)\right.\\
&\quad\left.+\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))-}\sigma\circ\varsigma\right)\left(\varsigma^{-1}(\varsigma^p(\varepsilon))\,;\tilde{p}\right)\right]\\
&\le h\left(\frac{1}{2}\right)[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]\left[\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},0-}h\right)(0\,;\tilde{p})\right.\\
&\quad\left.+\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},1+}h\right)(1\,;\tilde{p})\right],
\end{aligned} \quad (15)$$

where $\bar{w} = w/(\varsigma^p(\varepsilon) - \varsigma^p(\upsilon))^\mu$ and $\theta(t) = \varsigma^{1/p}(t)$ for $t \in [\upsilon^p, \varepsilon^p]$

Proof. We demonstrate claim (i) as noted below:

(i) Since σ is (p, h)-convex function on $[\varepsilon, \upsilon]$, for $\zeta, \eta \in I$, we get

$$\sigma\left(\left[\frac{\varsigma^p(\zeta)+\varsigma^p(\eta)}{2}\right]^{1/p}\right)\le h\left(\frac{1}{2}\right)[\sigma(\varsigma(\zeta))+\sigma(\varsigma(\eta))]. \quad (16)$$

Taking $\varsigma(\zeta)=(t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon))^{1/p}$ and $\varsigma(\eta)=(t\varsigma^p(\upsilon)+(1-t)\varsigma^p(\varepsilon))^{1/p}$ in the above inequality, we get

$$\sigma\left(\left[\frac{\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)}{2}\right]^{1/p}\right)\le h\left(\frac{1}{2}\right)\left[\sigma\left(\left[t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)\right]^{1/p}\right)+\sigma\left(\left[t\varsigma^p(\upsilon)+(1-t)\varsigma^p(\varepsilon)\right]^{1/p}\right)\right]. \quad (17)$$

Multiplying both sides of (17) by $t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})$ and integrating over $[0,1]$, we get

$$\sigma\left(\left[\frac{\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)}{2}\right]^{1/p}\right)\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})dt \le h\left(\frac{1}{2}\right)\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})\sigma\left(\left[t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)\right]^{1/p}\right)dt + h\left(\frac{1}{2}\right)\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})\sigma\left(\left[t\varsigma^p(\upsilon)+(1-t)\varsigma^p(\varepsilon)\right]^{1/p}\right)dt. \quad (18)$$

By choosing $\varsigma(\zeta)=t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)$ and $\varsigma(\eta)=t\varsigma^p(\upsilon)+(1-t)\varsigma^p(\varepsilon)$ in (18), we get

$$\sigma\left(\left[\frac{\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)}{2}\right]^{1/p}\right)\int_{\varsigma^{-1}(\varsigma^p(\varepsilon))}^{\varsigma^{-1}(\varsigma^p(\upsilon))}(\varsigma^p(\upsilon)-\varsigma(\zeta))^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(\bar{w}(\varsigma^p(\upsilon)-\varsigma(\zeta))^\mu;\tilde{p})\varsigma'(\zeta)d\zeta \le h\left(\frac{1}{2}\right)\int_{\varsigma^{-1}(\varsigma^p(\varepsilon))}^{\varsigma^{-1}(\varsigma^p(\upsilon))}(\varsigma^p(\upsilon)-\varsigma(\zeta))^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}(\varsigma^p(\upsilon)-\varsigma(\zeta))^\mu;\tilde{p}\right)\sigma\left(\varsigma^{1/p}(\zeta)\right)\varsigma'(\zeta)d\zeta + h\left(\frac{1}{2}\right)\int_{\varsigma^{-1}(\varsigma^p(\varepsilon))}^{\varsigma^{-1}(\varsigma^p(\upsilon))}(\varsigma(\eta)-\varsigma^p(\varepsilon))^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(\bar{w}(\varsigma(\eta)-\varsigma^p(\varepsilon))^\mu;\tilde{p})\sigma\left(\varsigma^{1/p}(\eta)\right)\varsigma'(\eta)d\eta. \quad (19)$$

By utilizing k-fractional operators (6) and (7), the first side of (14) is acquired.

Now, to demonstrate the second side of (14), once again, (p,h)-convexity of f over $[\varepsilon,\upsilon]$, and for $t\in[0,1]$, we get

$$\sigma\left(\left[t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)\right]^{1/p}\right)+\sigma\left(\left[t\varsigma^p(\upsilon)+(1-t)\varsigma^p(\varepsilon)\right]^{1/p}\right)\le[h(t)+h(1-t)][\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]. \quad (20)$$

Multiplying both sides of (20) by $h(1/2)t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})$ and integrating over $[0,1]$, we get

$$h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})\sigma\left(\left[t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)\right]^{1/p}\right)dt+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})\sigma\left(\left[t\varsigma^p(\upsilon)+(1-t)\varsigma^p(\varepsilon)\right]^{1/p}\right)dt\right] \le h\left(\frac{1}{2}\right)[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]\times\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})h(t)dt+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})h(1-t)dt\right]. \quad (21)$$

Taking $\varsigma(\zeta)=t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)$ and $\varsigma(\eta)=t\varsigma^p(\upsilon)+(1-t)\varsigma^p(\varepsilon)$ in (21), in that case, by utilizing k-fractional operators (6) and (7), the second side of (14) is acquired.

(ii) Proof is the same as the proof of (i)

□

Corollary 13. *By utilizing (14) and (15), some more k-fractional inequalities are offered as noted below:*

(i) By choosing $\varsigma=I$ and $\tilde{p}=w=0$, we acquire

$$\sigma\left(\left[\frac{\varepsilon^p+\upsilon^p}{2}\right]^{1/p}\right)\int_{\varepsilon^p}^{\upsilon^p}(\upsilon^p-\zeta)^{\alpha/k-1}d\zeta\le h\left(\frac{1}{2}\right)\times\left[\int_{\varepsilon^p}^{\upsilon^p}(\upsilon^p-\zeta)^{\alpha/k-1}\sigma\left(\zeta^{1/p}\right)d\zeta+\int_{\varepsilon^p}^{\upsilon^p}(\eta-\varepsilon^p)^{\alpha/k-1}\sigma\left(\eta^{1/p}\right)d\eta\right]\le h\left(\frac{1}{2}\right)[\sigma(\varepsilon)+\sigma(\upsilon)]\left[\int_0^1 t^{\alpha/k-1}h(t)dt+\int_0^1 t^{\alpha/k-1}h(1-t)dt\right] \quad (22)$$

(ii) By choosing $\varsigma=I$, $\tilde{p}=0$, $p=-1$, we acquire

$$\sigma\left(\frac{2\varepsilon\upsilon}{\varepsilon+\upsilon}\right)\int_{1/\varepsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}\left(\frac{1}{\upsilon}-\zeta\right)^\mu\right)d\zeta\le h\left(\frac{1}{2}\right)\left[\int_{1/\varepsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}\left(\frac{1}{\upsilon}-\zeta\right)^\mu\right)\sigma\left(\frac{1}{\zeta}\right)d\zeta+\int_{1/\varepsilon}^{1/\upsilon}\left(\eta-\frac{1}{\varepsilon}\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}\left(\eta-\frac{1}{\varepsilon}\right)^\mu\right)\sigma\left(\frac{1}{\eta}\right)d\eta\right]\le h\left(\frac{1}{2}\right)[\sigma(\varepsilon)+\sigma(\upsilon)]\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu)h(t)dt+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu)h(1-t)dt\right] \quad (23)$$

(iii) *By choosing $p=-1$ and $\varsigma=I$, we acquire*

$$\begin{aligned}
&\sigma\left(\frac{2\varepsilon\upsilon}{\varepsilon+\upsilon}\right)\int_{1/\varepsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{w}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu};\tilde{p}\right)d\zeta\\
&\le h\left(\frac{1}{2}\right)\left[\int_{1/\varepsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{w}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu};\tilde{p}\right)\sigma\left(\frac{1}{\zeta}\right)d\zeta\right.\\
&\left.+\int_{1/\varepsilon}^{1/\upsilon}\left(\eta-\frac{1}{\varepsilon}\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{w}\left(\eta-\frac{1}{\varepsilon}\right)^{\mu};\tilde{p}\right)\sigma\left(\frac{1}{\eta}\right)d\eta\right]\\
&\le h\left(\frac{1}{2}\right)[\sigma(\varepsilon)+\sigma(\upsilon)]\left[\int_0^1 t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(wt^{\mu};\tilde{p})h(t)dt\right.\\
&\left.+\int_0^1 t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(wt^{\mu};\tilde{p})h(1-t)dt\right]
\end{aligned}\tag{24}$$

(iv) *By choosing $p=-1$, $w=\tilde{p}=0$, and $\varsigma=I$, we acquire*

$$\begin{aligned}
&\sigma\left(\frac{2\varepsilon\upsilon}{\varepsilon+\upsilon}\right)\int_{1/\varepsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}d\zeta\le h\left(\frac{1}{2}\right)\left[\int_{1/\varepsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}\sigma\left(\frac{1}{\zeta}\right)d\zeta\right.\\
&\left.+\int_{1/\varepsilon}^{1/\upsilon}\left(\eta-\frac{1}{\varepsilon}\right)^{\alpha/k-1}\sigma\left(\frac{1}{\eta}\right)d\eta\right]\le h\left(\frac{1}{2}\right)[\sigma(\varepsilon)+\sigma(\upsilon)]\\
&\times\left[\int_0^1 t^{\alpha/k-1}h(t)dt+\int_0^1 t^{\alpha/k-1}h(1-t)dt\right]
\end{aligned}\tag{25}$$

(v) *By choosing $p=-1$, we acquire*

$$\begin{aligned}
&\sigma\left(\left[\frac{\varsigma^{-1}(\varepsilon)+\varsigma^{-1}(\upsilon)}{2}\right]^{-1}\right)\times\int_{\varsigma^{-1}(\varsigma^{-1}(\varepsilon))}^{\varsigma^{-1}(\varsigma^{-1}(\upsilon))}\left(\varsigma^{-1}(\upsilon)\right.\\
&\left.-\varsigma(\zeta)\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{w}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\mu};\tilde{p}\right)\varsigma'(\zeta)d\zeta\\
&\le h\left(\frac{1}{2}\right)\left[\int_{\varsigma^{-1}(\varsigma^{-1}(\varepsilon))}^{\varsigma^{-1}(\varsigma^{-1}(\upsilon))}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\right.\\
&\times\left(\bar{w}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\mu};\tilde{p}\right)\sigma\left(\varsigma^{-1}(\zeta)\right)\varsigma'(\zeta)d\zeta\\
&+\int_{\varsigma^{-1}(\varsigma^{-1}(\varepsilon))}^{\varsigma^{-1}(\varsigma^{-1}(\upsilon))}\left(\varsigma(\eta)-\varsigma^{-1}(\varepsilon)\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\bar{w}(\varsigma(\eta)\\
&\left.-\varsigma^{-1}(\varepsilon))^{\mu};\tilde{p})\sigma\left(\varsigma^{-1}(\eta)\right)\varsigma'(\eta)d\eta\right]\le h\left(\frac{1}{2}\right)[\sigma(\varepsilon)+\sigma(\upsilon)]\\
&\times\left[\int_0^1 t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(wt^{\mu};\tilde{p})h(t)dt\right.\\
&\left.+\int_0^1 t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(wt^{\mu};\tilde{p})h(1-t)dt\right]
\end{aligned}\tag{26}$$

Remark 14. The well-known k-fractional inequalities are further noted as below:

(i) By choosing $h(t)=t$ and $k=1$ in Corollary 13 (i), Theorem 9 of [6] is acquired

(ii) By choosing $h(t)=t$ and $k=1$ in Corollary 13 (ii), Theorem 2.1 of [24] is acquired

(iii) By choosing $h(t)=t$ and $k=1$ in Corollary 13 (iii), Theorem 2.1 of [5] is acquired

(iv) By choosing $h(t)=t$ and $k=1$ in Corollary 13 (iv), Theorem 4 of [1] is acquired

(v) By choosing $h(t)=t$ and $k=1$ in Corollary 13 (v), Theorem 2.1 of [27] is acquired

The below lemma is beneficial to offer the Fejér-Hadamard inequality for generalized k-fractional integrals.

Lemma 15 (see [18]). *Let $\sigma,\varsigma:[\varepsilon,\upsilon]\longrightarrow\mathbb{R}$ with $0<\varepsilon<\upsilon$ be the functions such that σ is positive and $\sigma\in L_1[\varepsilon,\upsilon]$, and ς is differentiable and absolutely increasing. If $p\in R\backslash\{0\}$ and $\sigma(\varsigma^{1/p}(\zeta))=\sigma([\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)-\varsigma(\zeta)]^{1/p})$, in the case for generalized k-fractional operators (6) and (7), we have*

(i) *If $p>0$, in that case,*

$$\begin{aligned}
&\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))+}^{\gamma,\delta,\nu,c}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)\\
&=\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\upsilon))-}^{\gamma,\delta,\nu,c}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\varepsilon));\tilde{p}\right)\\
&=\frac{1}{2}\left[\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))+}^{\gamma,\delta,\nu,c}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)\right.\\
&\left.+\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\upsilon))-}^{\gamma,\delta,\nu,c}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\varepsilon));\tilde{p}\right)\right],
\end{aligned}\tag{27}$$

with $\theta(t)=\varsigma^{1/p}(t)$ for all $t\in[\varepsilon^p,\upsilon^p]$

(ii) *If $p<0$, in that case,*

$$\begin{aligned}
&\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\upsilon))+}^{\gamma,\delta,\nu,c}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\varepsilon));\tilde{p}\right)\\
&=\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))-}^{\gamma,\delta,\nu,c}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)\\
&=\frac{1}{2}\left[\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\upsilon))+}^{\gamma,\delta,\nu,c}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\varepsilon));\tilde{p}\right)\right.\\
&\left.+\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))-}^{\gamma,\delta,\nu,c}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)\right],
\end{aligned}\tag{28}$$

with $\theta(t)=\varsigma^{1/p}(t)$ for all $t\in[\upsilon^p,\varepsilon^p]$

The Fejér-Hadamard inequality for generalized k-fractional integrals is stated and proved in the following theorem.

Theorem 16. *Let $h:J\longrightarrow\mathbb{R}$ be a nonnegative and nonzero function. Also, let $\sigma,\varsigma,r:[\varepsilon,\upsilon]\longrightarrow\mathbb{R}$, $0<\varepsilon<\upsilon$, be the functions such that σ is positive and $\sigma\in L_1[\varepsilon,\upsilon]$, ς is differentiable* *and absolutely increasing, and r is a nonnegative and integrable function. If σ is (p, h)-convex with $p \in R\backslash\{0\}$, and $\sigma(\varsigma^{1/p}(\zeta)) = \sigma([\varsigma^p(\varepsilon) + \varsigma^p(\upsilon) - \varsigma(\zeta)]^{1/p})$, then the following inequalities for generalized k-fractional integral operators (6) and (7) hold:*

(i) *If $p > 0$, in that case,*

$$\sigma\left(\left[\frac{\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)}{2}\right]^{1/p}\right)\left[\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))+}r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)+\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\upsilon))-}r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\varepsilon));\tilde{p}\right)\right]$$
$$\le h\left(\frac{1}{2}\right)\left[\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))+}\sigma\circ r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)+\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\upsilon))-}\sigma\circ r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\varepsilon));\tilde{p}\right)\right]$$
$$\le h\left(\frac{1}{2}\right)[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]\times\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\times\left([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p}\right)h(t)dt+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\left([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p}\right)h(1-t)dt\right], \tag{29}$$

where $\bar{w} = w/(\varsigma^p(\upsilon) - \varsigma^p(\varepsilon))^\mu$, *and* $\theta(t) = \varsigma^{1/p}(t)$ *for* $t \in [\varepsilon^p, \upsilon^p]$

(ii) *If $p < 0$, in that case,*

$$\sigma\left(\left[\frac{\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)}{2}\right]^{1/p}\right)\left[\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\upsilon))+}r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\varepsilon));\tilde{p}\right)+\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))-}r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)\right]$$
$$\le h\left(\frac{1}{2}\right)\left[\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\upsilon))+}\sigma\circ r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\varepsilon));\tilde{p}\right)+\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))-}\sigma\circ r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)\right]$$
$$\le h\left(\frac{1}{2}\right)[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\times\left([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p}\right)h(t)dt+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\times\left([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p}\right)h(1-t)dt\right], \tag{30}$$

where $\bar{w} = w/(\varsigma^p(\varepsilon) - \varsigma^p(\upsilon))^\mu$, *and* $\theta(t) = \varsigma^{1/p}(t)$ *for* $t \in [\upsilon^p, \varepsilon^p]$

Proof. We demonstrate claim (i) as noted below:

(i) Multiplying both sides of (17) by $t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p})$ and then integrating over $[0,1]$, we have

$$\sigma\left(\left[\frac{\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)}{2}\right]^{1/p}\right)\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\left([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p}\right)dt$$
$$\le h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})\sigma\left([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p}\right)r\times\left([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p}\right)dt+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})\sigma\times\left([t\varsigma^p(\upsilon)+(1-t)\varsigma^p(\varepsilon)]^{1/p}\right)r\left([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p}\right)dt\right]. \tag{31}$$

By choosing $\varsigma(x) = t\varsigma^p(\varepsilon) + (1-t)\varsigma^p(\upsilon)$, that is, $\varsigma^p(\varepsilon) + \varsigma^p(\upsilon) - \varsigma(\zeta) = t\varsigma^p(\upsilon) + (1-t)\varsigma^p(\varepsilon)$, in (31) and using $\sigma(\varsigma^{1/p}(\zeta)) = \sigma([\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)-\varsigma(\zeta)]^{1/p})$, we have

$$\sigma\left(\left[\frac{\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)}{2}\right]^{1/p}\right)\times\int_{\varsigma^{-1}(\varsigma^p(\varepsilon))}^{\varsigma^{-1}(\varsigma^p(\upsilon))}(\varsigma^p(\upsilon)-\varsigma(\zeta))^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(\bar{w}(\varsigma^p(\upsilon)-\varsigma(\zeta))^\mu;\tilde{p})(r\circ\theta)(\zeta)\varsigma'(\zeta)d\zeta\le h\left(\frac{1}{2}\right)\left[\int_{\varsigma^{-1}(\varsigma^p(\varepsilon))}^{\varsigma^{-1}(\varsigma^p(\upsilon))}(\varsigma^p(\upsilon)-\varsigma(\zeta))^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}(\varsigma^p(\upsilon)-\varsigma(\zeta))^\mu;\tilde{p}\right)(\sigma\circ\theta)(\zeta)(r\circ\theta)(\zeta)\varsigma'(\zeta)d\zeta+\int_{\varsigma^{-1}(\varsigma^p(\varepsilon))}^{\varsigma^{-1}(\varsigma^p(\upsilon))}(\varsigma(\zeta)-\varsigma^p(\varepsilon))^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(\bar{w}(\varsigma(\zeta)-\varsigma^p(\varepsilon))^\mu;\tilde{p})(\sigma\circ\theta)(\zeta)(r\circ\theta)(\zeta)\varsigma'(\zeta)d\zeta\right]. \tag{32}$$

This implies

$$\sigma\left(\left[\frac{\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)}{2}\right]^{1/p}\right)\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))+}r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)\le h\left(\frac{1}{2}\right)\left[\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\varepsilon))+}\sigma\circ r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)+\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^p(\upsilon))-}\sigma\circ r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\varepsilon));\tilde{p}\right)\right]. \tag{33}$$

By using Lemma 15 (i) in the above inequality, we get the first inequality of (29).

Now, to demonstrate the second side of (29), multiplying both sides of (20) with $t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p})$, and next integrating over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\left([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p}\right)\sigma\times\left([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p}\right)dt+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\times\left([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p}\right)\sigma\left([t\varsigma^p(\upsilon)+(1-t)\varsigma^p(\varepsilon)]^{1/p}\right)dt$$
$$\le h\left(\frac{1}{2}\right)[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r([t\varsigma^p(\varepsilon)+(1-t)\varsigma^p(\upsilon)]^{1/p})[h(t)+h(1-t)]dt. \tag{34}$$

Setting $\varsigma(\zeta)=t\varsigma^{p}(\varepsilon)+(1-t)\varsigma^{p}(\upsilon)$ and using $\sigma(\varsigma^{1/p}(\zeta))=\sigma([\varsigma^{p}(\varepsilon)+\varsigma^{p}(\upsilon)-\varsigma(\zeta)]^{1/p})$ in (34), we have

$$\begin{aligned}&\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^{p}(\varepsilon))+}\sigma\circ r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^{p}(\upsilon));\tilde{p}\right)\\&\quad+\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},\varsigma^{-1}(\varsigma^{p}(\upsilon))-}\sigma\circ r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^{p}(\varepsilon));\tilde{p}\right)\\&\quad\le h\left(\frac{1}{2}\right)[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]\times\int_{0}^{1}t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})r\\&\quad\times\left(\left[t\varsigma^{p}(\varepsilon)+(1-t)\varsigma^{p}(\upsilon)\right]^{1/p}\right)[h(t)+h(1-t)]dt.\end{aligned}\tag{35}$$

By utilizing Lemma 15 (i) in the above inequality, we get the second side of (29).

(ii) The proof is similar to the proof of (i)

□

Corollary 17. *By utilizing (29) and (30), some more k-fractional inequalities are offered as noted below:*

(i) By choosing $\varsigma=I$ and $\tilde{p}=0$, we acquire

$$\begin{aligned}&\sigma\left(\left[\frac{\varepsilon^{p}+\upsilon^{p}}{2}\right]^{1/p}\right)\int_{\varepsilon^{p}}^{\upsilon^{p}}(\upsilon^{p}-\zeta)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(\bar{w}(\upsilon^{p}-\zeta)^{\mu})(r\circ\theta)(\zeta)d\zeta\\&\le h\left(\frac{1}{2}\right)\left[\int_{\varepsilon^{p}}^{\upsilon^{p}}(\upsilon^{p}-\zeta)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(\bar{w}(\upsilon^{p}-\zeta)^{\mu})(\sigma\circ\theta)(\zeta)(r\circ\theta)(\zeta)d\zeta\right.\\&\quad\left.+\int_{\varepsilon^{p}}^{\upsilon^{p}}(\zeta-\varepsilon^{p})^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(\bar{w}(\zeta-\varepsilon^{p})^{\mu})(\sigma\circ\theta)(\zeta)(r\circ\theta)(\zeta)d\zeta\right]\\&\le h\left(\frac{1}{2}\right)[\sigma(\varepsilon)+\sigma(\upsilon)]\left[\int_{0}^{1}t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu})r\left(\left[t\varepsilon^{p}+(1-t)\upsilon^{p}\right]^{1/p}\right)h(t)dt\right.\\&\quad\left.+\int_{0}^{1}t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu})r\left(\left[t\varepsilon^{p}+(1-t)\upsilon^{p}\right]^{1/p}\right)h(1-t)dt\right]\end{aligned}\tag{36}$$

(ii) By choosing $\varsigma=I$ and $w=\tilde{p}=0$, we acquire

$$\begin{aligned}&\sigma\left(\left[\frac{\varepsilon^{p}+\upsilon^{p}}{2}\right]^{1/p}\right)\int_{\varepsilon^{p}}^{\upsilon^{p}}(\upsilon^{p}-\zeta)^{\alpha/k-1}(r\circ\theta)(\zeta)d\zeta\\&\le h\left(\frac{1}{2}\right)\left[\int_{\varepsilon^{p}}^{\upsilon^{p}}(\upsilon^{p}-\zeta)^{\alpha/k-1}(\sigma\circ\theta)(\zeta)(r\circ\theta)(\zeta)d\zeta\right.\\&\quad\left.+\int_{\varepsilon^{p}}^{\upsilon^{p}}(\zeta-\varepsilon^{p})^{\alpha/k-1}(\sigma\circ\theta)(\zeta)(r\circ\theta)(\zeta)d\zeta\right]\\&\le h\left(\frac{1}{2}\right)[\sigma(\varepsilon)+\sigma(\upsilon)]\left[\int_{0}^{1}t^{\alpha/k-1}r\left(\left[t\varepsilon^{p}+(1-t)\upsilon^{p}\right]^{1/p}\right)h(t)dt\right.\\&\quad\left.+\int_{0}^{1}t^{\alpha/k-1}r\left(\left[t\varepsilon^{p}+(1-t)\upsilon^{p}\right]^{1/p}\right)h(1-t)dt\right]\end{aligned}\tag{37}$$

(iii) By choosing $\tilde{p}=0$, $r(\zeta)=1$, $\varsigma=I$, and $p=-1$, we acquire

$$\begin{aligned}&\sigma\left(\frac{2\varepsilon\upsilon}{\varepsilon+\upsilon}\right)\int_{1/\varepsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu}\right)\theta(\zeta)d\zeta\\&\le h\left(\frac{1}{2}\right)\left[\int_{1/\varepsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu}\right)(\sigma\circ\theta)(\zeta)\theta(\zeta)d\zeta\right.\\&\quad\left.+\int_{1/\varepsilon}^{1/\upsilon}\left(\zeta-\frac{1}{\varepsilon}\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}\left(\zeta-\frac{1}{\varepsilon}\right)^{\mu}\right)(\sigma\circ\theta)(\zeta)\theta(\zeta)d\zeta\right]\\&\le h\left(\frac{1}{2}\right)[\sigma(\varepsilon)+\sigma(\upsilon)]\left[\int_{0}^{1}t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu})h(t)dt\right.\\&\quad\left.+\int_{0}^{1}t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu})h(1-t)dt\right]\end{aligned}\tag{38}$$

(iv) By choosing $\varsigma=I$, $p=-1$, and $r(\zeta)=1$, we acquire

$$\begin{aligned}&\sigma\left(\frac{2\varepsilon\upsilon}{\varepsilon+\upsilon}\right)\int_{1/\varepsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu};\tilde{p}\right)\theta(\zeta)d\zeta\\&\le h\left(\frac{1}{2}\right)\left[\int_{1/\varepsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu};\tilde{p}\right)(\sigma\circ\theta)(\zeta)\theta(\zeta)d\zeta\right.\\&\quad\left.+\int_{1/\varepsilon}^{1/\upsilon}\left(\zeta-\frac{1}{\varepsilon}\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}\left(\zeta-\frac{1}{\varepsilon}\right)^{\mu};\tilde{p}\right)(\sigma\circ\theta)(\zeta)\theta(\zeta)d\zeta\right]\\&\le h\left(\frac{1}{2}\right)[\sigma(\varepsilon)+\sigma(\upsilon)]\left[\int_{0}^{1}t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})h(t)dt\right.\\&\quad\left.+\int_{0}^{1}t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})h(1-t)dt\right]\end{aligned}\tag{39}$$

(v) By choosing $w=\tilde{p}=0$, $r(\zeta)=1$, $p=-1$, and $\varsigma=I$, we acquire

$$\begin{aligned}&\sigma\left(\frac{2\varepsilon\upsilon}{\varepsilon+\upsilon}\right)\int_{1/\varepsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}\theta(\zeta)d\zeta\\&\le h\left(\frac{1}{2}\right)\left[\int_{1/\varepsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}(\sigma\circ\theta)(\zeta)\theta(\zeta)d\zeta\right.\\&\quad\left.+\int_{1/\varepsilon}^{1/\upsilon}\left(\zeta-\frac{1}{\varepsilon}\right)^{\alpha/k-1}(\sigma\circ\theta)(\zeta)\theta(\zeta)d\zeta\right]\\&\le h\left(\frac{1}{2}\right)[\sigma(\varepsilon)+\sigma(\upsilon)]\left[\int_{0}^{1}t^{\alpha/k-1}h(t)dt+\int_{0}^{1}t^{\alpha/k-1}h(1-t)dt\right]\end{aligned}\tag{40}$$

(vi) By choosing $p=-1$, we acquire

$$\sigma\left(\left[\frac{\varsigma^{-1}(\varepsilon)+\varsigma^{-1}(\upsilon)}{2}\right]^{-1}\right)\times\int_{\varsigma^{-1}(\varsigma^{-1}(\varepsilon))}^{\varsigma^{-1}(\varsigma^{-1}(\upsilon))}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{w}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\mu};\tilde{p}\right)(r\circ\theta)(\zeta)\varsigma'(\zeta)d\zeta$$
$$\le h\left(\frac{1}{2}\right)\left[\int_{\varsigma^{-1}(\varsigma^{-1}(\varepsilon))}^{\varsigma^{-1}(\varsigma^{-1}(\upsilon))}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\times\left(\bar{w}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\mu};\tilde{p}\right)(\sigma\circ\theta)(\zeta)(r\circ\theta)(\zeta)\varsigma'(\zeta)d\zeta\right.$$
$$\left.+\int_{\varsigma^{-1}(\varsigma^{-1}(\varepsilon))}^{\varsigma^{-1}(\varsigma^{-1}(\upsilon))}\left(\varsigma(\zeta)-\varsigma^{-1}(\varepsilon)\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\times\left(\bar{w}\left(\varsigma(\zeta)-\varsigma^{-1}(\varepsilon)\right)^{\mu};\tilde{p}\right)(\sigma\circ\theta)(\zeta)(r\circ\theta)(\zeta)\varsigma'(\zeta)d\zeta\right]$$
$$\le h\left(\frac{1}{2}\right)[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]\left[\int_0^1 t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(wt^{\mu};\tilde{p})r\times\left(\frac{\varsigma(\varepsilon)\varsigma(\upsilon)}{t\varsigma(\upsilon)+(1-t)\varsigma(\varepsilon)}\right)h(t)dt+\int_0^1 t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(wt^{\mu};\tilde{p})r\times\left(\frac{\varsigma(\varepsilon)\varsigma(\upsilon)}{t\varsigma(\upsilon)+(1-t)\varsigma(\varepsilon)}\right)h(1-t)dt\right] \quad (41)$$

Remark 18. The mentioned k-fractional inequalities are further connected with foreknown conclusions as noted below:

(i) By setting $h(t)=t$ and $k=1$ in Corollary 17 (ii), Theorem 9 of [6] is obtained

(ii) By setting $h(t)=t$ and $k=1$ in Corollary 17 (iii), Theorem 2.1 of [24] is obtained

(iii) By setting $h(t)=t$ and $k=1$ in Corollary 17 (iv), Theorem 2.1 of [5] is obtained

(iv) By setting $h(t)=t$ and $k=1$ in Corollary 17 (v), Theorem 4 of [1] is obtained

(v) By setting $h(t)=t$ and $k=1$ in Corollary 17 (vi), Theorem 2.5 of [27] is obtained

In the next theorem, we offer another type of the Hadamard inequality.

Theorem 19. *Let $h : J \longrightarrow \mathbb{R}$ is a nonnegative and nonzero function. Also, let $\sigma, \varsigma : [\varepsilon, \upsilon] \longrightarrow \mathbb{R}$, $0<\varepsilon<\upsilon$, be the functions such that σ is positive and $\sigma \in L_1[\varepsilon, \upsilon]$ and ς is differentiable and absolutely increasing. If σ is (p, h)-convex, $p \in R\backslash\{0\}$, then for generalized k-fractional integral operators (6) and (7), we have the following:*

(i) If $p>0$, in that case,

$$\sigma\left(\left[\frac{\varsigma^{p}(\varepsilon)+\varsigma^{p}(\upsilon)}{2}\right]^{1/p}\right)\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{\bar{w}},\varsigma^{-1}((\varsigma^{p}(\varepsilon)+\varsigma^{p}(\upsilon))/2)+}^{\gamma,\delta,\nu,c}1\right)\left(\varsigma^{-1}(\varsigma^{p}(\upsilon));\tilde{p}\right)$$
$$\le h\left(\frac{1}{2}\right)\left[\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{\bar{w}},\varsigma^{-1}((\varsigma^{p}(\varepsilon)+\varsigma^{p}(\upsilon))/2)+}^{\gamma,\delta,\nu,c}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^{p}(\upsilon));\tilde{p}\right)+\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{\bar{w}},\varsigma^{-1}((\varsigma^{p}(\varepsilon)+\varsigma^{p}(\upsilon))/2)-}^{\gamma,\delta,\nu,c}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^{p}(\varepsilon));\tilde{p}\right)\right]$$
$$\le h\left(\frac{1}{2}\right)[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]\times\left[\int_0^1 t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(wt^{\mu};\tilde{p})h\left(\frac{t}{2}\right)dt+\int_0^1 t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(wt^{\mu};\tilde{p})h\left(\frac{2-t}{2}\right)dt\right], \quad (42)$$

where $\bar{\bar{w}}=2^{\mu}w/(\varsigma^{p}(\upsilon)-\varsigma^{p}(\varepsilon))^{\mu}$, and $\theta(t)=\varsigma^{1/p}(t)$ for $t\in[\varepsilon^{p}, \upsilon^{p}]$

(ii) If $p<0$, in that case,

$$\sigma\left(\left[\frac{\varsigma^{p}(\varepsilon)+\varsigma^{p}(\upsilon)}{2}\right]^{1/p}\right)\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{\bar{w}},\varsigma^{-1}((\varsigma^{p}(\varepsilon)+\varsigma^{p}(\upsilon))/2)-}^{\gamma,\delta,\nu,c}1\right)\left(\varsigma^{-1}(\varsigma^{p}(\upsilon));\tilde{p}\right)$$
$$\le h\left(\frac{1}{2}\right)\left[\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{\bar{w}},\varsigma^{-1}((\varsigma^{p}(\varepsilon)+\varsigma^{p}(\upsilon))/2)+}^{\gamma,\delta,\nu,c}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^{p}(\varepsilon));\tilde{p}\right)+\left({}_{\varsigma}^{k}F_{\mu,\alpha,l,\bar{\bar{w}},\varsigma^{-1}((\varsigma^{p}(\varepsilon)+\varsigma^{p}(\upsilon))/2)-}^{\gamma,\delta,\nu,c}\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^{p}(\upsilon));\tilde{p}\right)\right]$$
$$\le h\left(\frac{1}{2}\right)[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]\times\left[\int_0^1 t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(wt^{\mu};\tilde{p})h\left(\frac{t}{2}\right)dt+\int_0^1 t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(wt^{\mu};\tilde{p})h\left(\frac{2-t}{2}\right)dt\right], \quad (43)$$

where $\bar{\bar{w}}=2^{\mu}w/(\varsigma^{p}(\varepsilon)-\varsigma^{p}(\upsilon))^{\mu}$, and $\theta(t)=\varsigma^{1/p}(t)$ for $t\in[\upsilon^{p}, \varepsilon^{p}]$

Proof. We demonstrate claim (i) as noted below:

(i) Choosing $\varsigma(\zeta)=[(t/2)\varsigma^{p}(\varepsilon)+(2-t/2)\varsigma^{p}(\upsilon)]^{1/p}$ and $\varsigma(\eta)=[(t/2)\varsigma^{p}(\upsilon)+(2-t/2)\varsigma^{p}(\varepsilon)]^{1/p}$ in (16), we have

$$\sigma\left(\left[\frac{\varsigma^{p}(\varepsilon)+\varsigma^{p}(\upsilon)}{2}\right]^{1/p}\right)\le h\left(\frac{1}{2}\right)\left[\sigma\left(\left[\left(\frac{t}{2}\right)\varsigma^{p}(\varepsilon)+\left(\frac{2-t}{2}\right)\varsigma^{p}(\upsilon)\right]^{1/p}\right)+\sigma\left(\left[\left(\frac{t}{2}\right)\varsigma^{p}(\upsilon)+\left(\frac{2-t}{2}\right)\varsigma^{p}(\varepsilon)\right]^{1/p}\right)\right]. \quad (44)$$

Multiplying both sides of (44) by $t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(wt^{\mu};\tilde{p})$ and then integrating over $[0,1]$, we have

$$\sigma\left(\left[\frac{\varsigma^{p}(\varepsilon)+\varsigma^{p}(\upsilon)}{2}\right]^{1/p}\right)\int_{0}^{1}t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu};\tilde{p})dt \le h\left(\frac{1}{2}\right)\left[\int_{0}^{1}t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu};\tilde{p})\sigma\left(\left[\frac{t}{2}\varsigma^{p}(\varepsilon)+\left(\frac{2-t}{2}\right)\varsigma^{p}(\upsilon)\right]^{1/p}\right)dt + \int_{0}^{1}t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu};\tilde{p})\sigma\left(\left[\frac{t}{2}\varsigma^{p}(\upsilon)+\left(\frac{2-t}{2}\right)\varsigma^{p}(\varepsilon)\right]^{1/p}\right)dt\right]. \quad (45)$$

By choosing $\varsigma(\zeta)=[(t/2)\varsigma^{p}(\varepsilon)+(2-t/2)\varsigma^{p}(\upsilon)]^{1/p}$ and $\varsigma(\eta)=[(t/2)\varsigma^{p}(\upsilon)+(2-t/2)\varsigma^{p}(\varepsilon)]^{1/p}$ in (45), then by using k-fractional integral operators (6) and (7), the first inequality of (42) is obtained.

Now, to demonstrate the second side of (42), once again (p,h)-convexity of σ over $[\varepsilon,\upsilon]$ and for $t\in[0,1]$, we obtain

$$\sigma\left(\left[\frac{t}{2}\varsigma^{p}(\varepsilon)+\left(\frac{2-t}{2}\right)\varsigma^{p}(\upsilon)\right]^{1/p}\right)+\sigma\left(\left[\frac{t}{2}\varsigma^{p}(\upsilon)+\left(\frac{2-t}{2}\right)\varsigma^{p}(\varepsilon)\right]^{1/p}\right) \le [\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]\left[h\left(\frac{t}{2}\right)+h\left(\frac{2-t}{2}\right)\right]. \quad (46)$$

Multiplying both sides of (46) by $h(1/2)t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu};\tilde{p})$ and then integrating over $[0,1]$, we have

$$h\left(\frac{1}{2}\right)\left[\int_{0}^{1}t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu};\tilde{p})\sigma\left(\left[\frac{t}{2}\varsigma^{p}(\varepsilon)+\left(\frac{2-t}{2}\right)\varsigma^{p}(\upsilon)\right]^{1/p}\right)dt + \int_{0}^{1}t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu};\tilde{p})\sigma\left(\left[\frac{t}{2}\varsigma^{p}(\upsilon)+\left(\frac{2-t}{2}\right)\varsigma^{p}(\varepsilon)\right]^{1/p}\right)dt\right] \le h\left(\frac{1}{2}\right)[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]\times\left[\int_{0}^{1}t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu};\tilde{p})h\left(\frac{t}{2}\right)dt + \int_{0}^{1}t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu};\tilde{p})h\left(\frac{2-t}{2}\right)dt\right]. \quad (47)$$

Choosing $\varsigma(\zeta)=(t/2)\varsigma^{p}(\varepsilon)+(2-t/2)\varsigma^{p}(\upsilon)$ and $\varsigma(\eta)=(t/2)\varsigma^{p}(\upsilon)+(2-t/2)\varsigma^{p}(\varepsilon)$ in (47) and by utilizing k-fractional operators (6) and (7), the second side of (42) is acquired.

(ii) Proof is the same as the proof of (i)

□

Corollary 20. *By utilizing (42) and (43), some more k-fractional inequalities are offered as noted below:*

(i) By choosing $\varsigma=I$ and $\tilde{p}=0$, $p=-1$, we acquire

$$\sigma\left(\frac{2\varepsilon\upsilon}{\varepsilon+\upsilon}\right)\int_{2\varepsilon\upsilon/\varepsilon+\upsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{\omega}}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu}\right)d\zeta \le h\left(\frac{1}{2}\right)\left[\int_{2\varepsilon\upsilon/\varepsilon+\upsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{\omega}}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu}\right)\sigma\left(\zeta^{-1}\right)d\zeta + \int_{1/\varepsilon}^{2\varepsilon\upsilon/\varepsilon+\upsilon}\left(\eta-\frac{1}{\varepsilon}\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{\omega}}\left(\eta-\frac{1}{\varepsilon}\right)^{\mu}\right)\sigma\left(\eta^{-1}\right)d\eta\right] \le h\left(\frac{1}{2}\right)[\sigma(\varepsilon)+\sigma(\upsilon)]\times\left[\int_{0}^{1}t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu})h\left(\frac{t}{2}\right)dt + \int_{0}^{1}t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu})h\left(\frac{2-t}{2}\right)dt\right] \quad (48)$$

(ii) By choosing $p=-1$ and $\varsigma=I$, we acquire

$$\sigma\left(\frac{2\varepsilon\upsilon}{\varepsilon+\upsilon}\right)\int_{2\varepsilon\upsilon/\varepsilon+\upsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{\omega}}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu};\tilde{p}\right)d\zeta \le h\left(\frac{1}{2}\right)\left[\int_{2\varepsilon\upsilon/\varepsilon+\upsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{\omega}}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu};\tilde{p}\right)\sigma\left(\zeta^{-1}\right)d\zeta + \int_{1/\varepsilon}^{2\varepsilon\upsilon/\varepsilon+\upsilon}\left(\eta-\frac{1}{\varepsilon}\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{\omega}}\left(\eta-\frac{1}{\varepsilon}\right)^{\mu};\tilde{p}\right)\sigma\left(\eta^{-1}\right)d\eta\right] \le h\left(\frac{1}{2}\right)[\sigma(\varepsilon)+\sigma(\upsilon)]\times\left[\int_{0}^{1}t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu};\tilde{p})h\left(\frac{t}{2}\right)dt + \int_{0}^{1}t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu};\tilde{p})h\left(\frac{2-t}{2}\right)dt\right] \quad (49)$$

(iii) By choosing $p=-1$, we acquire

$$\sigma\left(\left[\frac{\varsigma^{-1}(\varepsilon)+\varsigma^{-1}(\upsilon)}{2}\right]^{-1}\right)\times\int_{\varsigma^{-1}(\varsigma^{-1}(\varepsilon)+\varsigma^{-1}(\upsilon)/2)}^{\varsigma^{-1}(\varsigma^{-1}(\upsilon))}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\times\left(\bar{\bar{\omega}}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\mu};\tilde{p}\right)\varsigma'(\zeta)d\zeta \le h\left(\frac{1}{2}\right)\times\left[\int_{\varsigma^{-1}(\varsigma^{-1}(\varepsilon)+\varsigma^{-1}(\upsilon)/2)}^{\varsigma^{-1}(\varsigma^{-1}(\upsilon))}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\times\left(\bar{\bar{\omega}}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\mu};\tilde{p}\right)\sigma\left(\varsigma^{-1}(\zeta)\right)\varsigma'(\zeta)d\zeta + \int_{\varsigma^{-1}(\varsigma^{-1}(\varepsilon))}^{\varsigma^{-1}(\varsigma^{-1}(\varepsilon)+\varsigma^{-1}(\upsilon)/2)}\left(\varsigma(\eta)-\varsigma^{-1}(\varepsilon)\right)^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\times\left(\bar{\bar{\omega}}\left(\varsigma(\eta)-\varsigma^{-1}(\varepsilon)\right)^{\mu};\tilde{p}\right)\sigma\left(\varsigma^{-1}(\eta)\right)\varsigma'(\eta)d\eta\right] \le h\left(\frac{1}{2}\right)[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))]\times\left[\int_{0}^{1}t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu};\tilde{p})h\left(\frac{t}{2}\right)dt + \int_{0}^{1}t^{\alpha/k-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}(\omega t^{\mu};\tilde{p})h\left(\frac{2-t}{2}\right)dt\right] \quad (50)$$

Remark 21. The mentioned k-fractional inequalities are farther connected with foreknown conclusions as noted below:

(i) By choosing $h(t) = t$ and $k = 1$ in Corollary 20 (i), Theorem 2.3 of [24] is acquired

(ii) By choosing $h(t) = t$ and $k = 1$ in Corollary 20 (ii), Theorem 2.3 of [5] is acquired

(iii) By choosing $h(t) = t$ and $k = 1$ in Corollary 20 (iii), Theorem 2.7 of [27] is acquired

The second type of the Fejér-Hadamard inequality for generalized k-fractional integrals is given as noted below.

Theorem 22. *Let $h : J \longrightarrow \mathbb{R}$ be a nonnegative and nonzero function. Also, let $\sigma, \varsigma, r : [\varepsilon, \upsilon] \longrightarrow \mathbb{R}$ with $0 < \varepsilon < \upsilon$ be the functions such that σ is positive and $\sigma \in L_1[\varepsilon, \upsilon]$, ς is differentiable and absolutely increasing, and r is a nonnegative and integrable function. If σ is (p, h)-convex, $p \in R\backslash\{0\}$ and $\sigma(\varsigma^{1/p}(\zeta)) = \sigma([\varsigma^p(\varepsilon) + \varsigma^p(\upsilon) - \varsigma(\zeta)]^{1/p})$, then the following inequalities for generalized k-fractional integral operators (6) and (7) hold:*

(i) If $p > 0$, in that case,

$$\begin{aligned}
&\sigma\left(\left[\frac{\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)}{2}\right]^{1/p}\right)\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{\bar{w}},\varsigma^{-1}(\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)/2)+} r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)\\
&\quad\le h\left(\frac{1}{2}\right)\left[\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{\bar{w}},\varsigma^{-1}(\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)/2)+} r\circ\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)\right.\\
&\qquad\left.+\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{\bar{w}},\varsigma^{-1}(\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)/2)-} r\circ\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\varepsilon));\tilde{p}\right)\right]\\
&\quad\le h\left(\frac{1}{2}\right)\left[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))\right]\times\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\right.\\
&\qquad\times\left(\left[\frac{t}{2}\varsigma^p(\varepsilon)+\frac{2-t}{2}\varsigma^p(\upsilon)\right]^{1/p}\right)h\left(\frac{t}{2}\right)dt\\
&\qquad+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\left(\left[\frac{t}{2}\varsigma^p(\varepsilon)+\frac{2-t}{2}\varsigma^p(\upsilon)\right]^{1/p}\right)h\\
&\qquad\left.\times\left(\frac{2-t}{2}\right)dt\right],
\end{aligned}\tag{51}$$

where $\bar{\bar{w}} = 2^\mu w/(\varsigma^p(\upsilon) - \varsigma^p(\varepsilon))^\mu$, and $\theta(t) = \varsigma^{1/p}(t)$ for $t \in [\varepsilon^p, \upsilon^p]$

(ii) If $p < 0$, in that case,

$$\begin{aligned}
&\sigma\left(\left[\frac{\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)}{2}\right]^{1/p}\right)\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{\bar{w}},\varsigma^{-1}(\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)/2)-} r\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)\\
&\quad\le h\left(\frac{1}{2}\right)\left[\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{\bar{w}},\varsigma^{-1}(\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)/2)+} r\circ f\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\varepsilon));\tilde{p}\right)\right.\\
&\qquad\left.+\left({}_{\varsigma}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{\bar{w}},\varsigma^{-1}(\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)/2)-} r\circ\sigma\circ\theta\right)\left(\varsigma^{-1}(\varsigma^p(\upsilon));\tilde{p}\right)\right]\\
&\quad\le h\left(\frac{1}{2}\right)\left[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))\right]\times\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\right.\\
&\qquad\times\left(\left[\frac{t}{2}\varsigma^p(\varepsilon)+\frac{2-t}{2}\varsigma^p(\upsilon)\right]^{1/p}\right)h\left(\frac{t}{2}\right)dt\\
&\qquad+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\left(\left[\frac{t}{2}\varsigma^p(\varepsilon)+\frac{2-t}{2}\varsigma^p(\upsilon)\right]^{1/p}\right)h\\
&\qquad\left.\times\left(\frac{2-t}{2}\right)dt\right],
\end{aligned}\tag{52}$$

where $\bar{\bar{w}} = 2^\mu w/(\varsigma^p(\varepsilon) - \varsigma^p(\upsilon))^\mu$, and $\theta(t) = \varsigma^{1/p}(t)$ for $t \in [\upsilon^p, \varepsilon^p]$

Proof. We demonstrate the first claim as noted below:

(i) Multiplying (44) by $t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r([(t/2)\,\varsigma^p(\varepsilon) + (2-t/2)\varsigma^p(\upsilon)]^{1/p})$ and then integrating over $[0, 1]$, we have

$$\begin{aligned}
&\sigma\left(\left[\frac{\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)}{2}\right]^{1/p}\right)\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\\
&\quad\cdot\left(\left[\frac{t}{2}\varsigma^p(\varepsilon)+\left(\frac{2-t}{2}\right)\varsigma^p(\upsilon)\right]^{1/p}\right)dt\le h\left(\frac{1}{2}\right)\\
&\quad\cdot\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\left(\left[\frac{t}{2}\varsigma^p(\varepsilon)+\left(\frac{2-t}{2}\right)\varsigma^p(\upsilon)\right]^{1/p}\right)\sigma\right.\\
&\quad\cdot\left(\left[\frac{t}{2}\varsigma^p(\varepsilon)+\left(\frac{2-t}{2}\right)\varsigma^p(\upsilon)\right]^{1/p}\right)dt+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\\
&\quad\cdot\left(\left[\frac{t}{2}\varsigma^p(\varepsilon)+\left(\frac{2-t}{2}\right)\varsigma^p(\upsilon)\right]^{1/p}\right)\sigma\\
&\quad\left.\cdot\left(\left[\frac{t}{2}\varsigma^p(\upsilon)+\left(\frac{2-t}{2}\right)\varsigma^p(\varepsilon)\right]^{1/p}\right)dt\right].
\end{aligned}\tag{53}$$

By choosing $\varsigma(\zeta) = (t/2)\varsigma^p(\varepsilon) + (2-t/2)\varsigma^p(\upsilon)$ and $\varsigma(\eta) = (t/2)\varsigma^p(\upsilon) + (2-t/2)\varsigma^p(\varepsilon)$, that is, $\varsigma^p(\varepsilon) + \varsigma^p(\upsilon) - \varsigma(\zeta) = (t/2)\varsigma^p(\upsilon) + (2-t/2)\varsigma^p(\varepsilon)$, in (53), in that case, by utilizing $\sigma(\varsigma^{1/p}(\zeta)) = \sigma([\varsigma^p(\varepsilon) + \varsigma^p(\upsilon) - \varsigma(\zeta)]^{1/p})$ and k-fractional integral operators (6) and (7), the first side of (51) is obtained.

Now, to demonstrate the second side of (51), multiplying both sides of (46) with $h(1/2)t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r([(t/2)\varsigma^p(\varepsilon) + (2-t/2)\varsigma^p(\upsilon)]^{1/p})$ and integrating over $[0, 1]$, we obtain

$$\begin{aligned}
&h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\left(\left[\frac{t}{2}\varsigma^p(\varepsilon)+\left(\frac{2-t}{2}\right)\varsigma^p(\upsilon)\right]^{1/p}\right)\sigma\right.\\
&\quad\times\left(\left[\frac{t}{2}\varsigma^p(\varepsilon)+\left(\frac{2-t}{2}\right)\varsigma^p(\upsilon)\right]^{1/p}\right)dt+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\\
&\quad\left.\times\left(\left[\frac{t}{2}\varsigma^p(\varepsilon)+\left(\frac{2-t}{2}\right)\varsigma^p(\upsilon)\right]^{1/p}\right)\sigma\left(\left[\frac{t}{2}\varsigma^p(\upsilon)+\left(\frac{2-t}{2}\right)\varsigma^p(\varepsilon)\right]^{1/p}\right)dt\right]\\
&\le h\left(\frac{1}{2}\right)\left[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))\right]\times\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})r\right.\\
&\quad\times\left(\left[\frac{t}{2}\varsigma^p(\varepsilon)+\left(\frac{2-t}{2}\right)\varsigma^p(\upsilon)\right]^{1/p}\right)h\left(\frac{t}{2}\right)dt
\end{aligned}$$

$$+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})r\left(\left[\frac{t}{2}\varsigma^p(\varepsilon)+\left(\frac{2-t}{2}\right)\varsigma^p(\upsilon)\right]^{1/p}\right)h\left(\frac{2-t}{2}\right)dt\Bigg]. \tag{54}$$

By choosing $\varsigma(\zeta)=(t/2)\varsigma^p(\varepsilon)+(2-t/2)\varsigma^p(\upsilon)$ and $\varsigma(\eta)=(t/2)\varsigma^p(\upsilon)+(2-t/2)\varsigma^p(\varepsilon)$, that is, $\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)-\varsigma(\zeta)=(t/2)\varsigma^p(\upsilon)+(2-t/2)\varsigma^p(\varepsilon)$, in (54), then by using $\sigma(\varsigma^{1/p}(\zeta))=\sigma([\varsigma^p(\varepsilon)+\varsigma^p(\upsilon)-\varsigma(\zeta)]^{1/p})$ and k-fractional integral operators (6) and (7), the second inequality of (51) is acquired.

(ii) Proof is the same as the proof of (i)

□

Corollary 23. *By utilizing (51) and (52), some more k-fractional inequalities are offered as noted below:*

(i) By choosing $\varsigma=I$ and $\tilde{p}=0,p=-1$, we acquire

$$\begin{aligned}&\sigma\left(\frac{2\varepsilon\upsilon}{\varepsilon+\upsilon}\right)\int_{2\varepsilon\upsilon/\varepsilon+\upsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{\bar{w}}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu}\right)r\left(\zeta^{-1}\right)d\zeta\\&\le h\left(\frac{1}{2}\right)\left[\int_{2\varepsilon\upsilon/\varepsilon+\upsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{\bar{w}}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu}\right)\sigma\right.\\&\times\left(\zeta^{-1}\right)r\left(\zeta^{-1}\right)d\zeta+\int_{1/\varepsilon}^{2\varepsilon\upsilon/\varepsilon+\upsilon}\left(\zeta-\frac{1}{\varepsilon}\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\\&\left.\times\left(\bar{\bar{w}}\left(\zeta-\frac{1}{\varepsilon}\right)^{\mu}\right)\sigma\left(\zeta^{-1}\right)r\left(\zeta^{-1}\right)d\zeta\right]\le h\left(\frac{1}{2}\right)\left[\sigma(\varepsilon)+\sigma(\upsilon)\right]\\&\times\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu})r\left(\frac{2\varepsilon\upsilon}{t\upsilon+(2-t)\varepsilon}\right)h\left(\frac{t}{2}\right)dt\right.\\&\left.+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu})\left(\frac{2\varepsilon\upsilon}{t\upsilon+(2-t)\varepsilon}\right)h\left(\frac{2-t}{2}\right)dt\right]\end{aligned} \tag{55}$$

(ii) By choosing $p=-1$ and $\varsigma=I$, we acquire

$$\begin{aligned}&\sigma\left(\frac{2\varepsilon\upsilon}{\varepsilon+\upsilon}\right)\int_{2\varepsilon\upsilon/\varepsilon+\upsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{\bar{w}}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu};\tilde{p}\right)r\left(\zeta^{-1}\right)d\zeta\\&\le h\left(\frac{1}{2}\right)\left[\int_{2\varepsilon\upsilon/\varepsilon+\upsilon}^{1/\upsilon}\left(\frac{1}{\upsilon}-\zeta\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{\bar{w}}\left(\frac{1}{\upsilon}-\zeta\right)^{\mu};\tilde{p}\right)\sigma\right.\\&\times\left(\zeta^{-1}\right)r\left(\zeta^{-1}\right)d\zeta+\int_{1/\varepsilon}^{2\varepsilon\upsilon/\varepsilon+\upsilon}\left(\zeta-\frac{1}{\varepsilon}\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\\&\left.\times\left(\bar{\bar{w}}\left(\zeta-\frac{1}{\varepsilon}\right)^{\mu};\tilde{p}\right)\sigma\left(\zeta^{-1}\right)r\left(\zeta^{-1}\right)d\zeta\right]\le h\left(\frac{1}{2}\right)\left[\sigma(\varepsilon)+\sigma(\upsilon)\right]\\&\times\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})r\left(\frac{2\varepsilon\upsilon}{t\upsilon+(2-t)\varepsilon}\right)h\left(\frac{t}{2}\right)dt\right.\\&\left.+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})r\left(\frac{2\varepsilon\upsilon}{t\upsilon+(2-t)\varepsilon}\right)h\left(\frac{2-t}{2}\right)dt\right]\end{aligned} \tag{56}$$

(iii) By choosing $p=-1$, we acquire

$$\begin{aligned}&\sigma\left(\left[\frac{\varsigma^{-1}(\varepsilon)+\varsigma^{-1}(\upsilon)}{2}\right]^{-1}\right)\times\int_{\varsigma^{-1}(\varsigma^{-1}(\varepsilon)+\varsigma^{-1}(\upsilon)/2)}^{\varsigma^{-1}(\varsigma^{-1}(\upsilon))}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\\&\times\left(\bar{\bar{w}}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\mu};\tilde{p}\right)r\left(\varsigma^{-1}(\zeta)\right)\varsigma'(\zeta)d\zeta\le h\left(\frac{1}{2}\right)\\&\times\left[\int_{\varsigma^{-1}(\varsigma^{-1}(\varepsilon)+\varsigma^{-1}(\upsilon)/2)}^{\varsigma^{-1}(\varsigma^{-1}(\upsilon))}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\right.\\&\times\left(\bar{\bar{w}}\left(\varsigma^{-1}(\upsilon)-\varsigma(\zeta)\right)^{\mu};\tilde{p}\right)\sigma\left(\varsigma^{-1}(\zeta)\right)r\left(\varsigma^{-1}(\zeta)\right)\varsigma'(\zeta)d\zeta\\&+\int_{\varsigma^{-1}(\varsigma^{-1}(\varepsilon))}^{\varsigma^{-1}(\varsigma^{-1}(\varepsilon)+\varsigma^{-1}(\upsilon)/2)}\left(\varsigma(\zeta)-\varsigma^{-1}(\varepsilon)\right)^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\\&\left.\times\left(\bar{\bar{w}}\left(\varsigma(\zeta)-\varsigma^{-1}(\varepsilon)\right)^{\mu};\tilde{p}\right)\sigma\left(\varsigma^{-1}(\zeta)\right)r\left(\varsigma^{-1}(\zeta)\right)\varsigma'(\zeta)d\zeta\right]\\&\le h\left(\frac{1}{2}\right)\left[\sigma(\varsigma(\varepsilon))+\sigma(\varsigma(\upsilon))\right]\left[\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})r\right.\\&\times\left(\frac{2\varsigma(\varepsilon)\varsigma(\upsilon)}{t\varsigma(\upsilon)+(2-t)\varsigma(\varepsilon)}\right)h\left(\frac{t}{2}\right)dt+\int_0^1 t^{\alpha/k-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})r\\&\left.\times\left(\frac{2\varsigma(\varepsilon)\varsigma(\upsilon)}{t\varsigma(\upsilon)+(2-t)\varsigma(\varepsilon)}\right)h\left(\frac{2-t}{2}\right)dt\right]\end{aligned} \tag{57}$$

Remark 24. The well-known k-fractional inequalities are further noted as below:

(i) By choosing $h(t)=t$ and $k=1$ in Corollary 23 (i), Theorem 2.6 of [24] is acquired

(ii) By choosing $h(t)=t$ and $k=1$ in Corollary 23 (ii), Theorem 2.6 of [5] is acquired

(iii) By choosing $h(t)=t$ and $k=1$ in Corollary 23 (iii), Theorem 2.10 of [27] is acquired

3. Conclusion

In this article, generalized versions of k-fractional Hadamard and Fejér-Hadamard inequalities are presented. To obtain these inequalities, k-fractional integral operators including the Mittag-Leffler function have been used for (p,h)-convex functions. Many published results in the literature are directly connected with the findings of this paper. Some corollaries have been formulated as new k-fractional Hadamard and Fejér-Hadamard inequalities.

Data Availability

There is no data required for this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] I. Iscan and S. Wu, "Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals," *Applied Mathematics and Computation*, vol. 238, pp. 237–244, 2014.

[2] A. Akkurt, M. E. Yildirim, and H. Yildirim, "On some integral inequalities for (k, h)-Riemann-Liouville fractional integral," *New Trends in Mathematical Sciences*, vol. 4, no. 2, pp. 138–146, 2016.

[3] S. Mubeen and G. M. Habibullah, "k-fractional integrals and applications," *International Journal of Contemporary Mathematical Sciences*, vol. 7, no. 2, pp. 89–94, 2012.

[4] T. Tunç, H. Budak, F. Usta, and M. Z. Sarikaya, "On new generalized fractional integral operators and related fractional inequalities," *Konuralp Journal of Mathematics (KJM)*, vol. 8, no. 2, pp. 268–278, 2020.

[5] G. Farid, A. U. Rehman, and S. Mehmood, "Hadamard and Fejer-Hadamard type integral inequalities for harmonically convex functions via an extended generalized Mittag-Leffler function," *Journal of Mathematical and Computational Science*, vol. 8, pp. 630–643, 2018.

[6] M. Kunt and I. Iscan, "Hermite-Hadamard-Fejér type inequalities for p-convex functions via fractional integrals," *Iranian Journal of Science and Technology, Transactions A: Science*, vol. 42, no. 4, pp. 2079–2089, 2018.

[7] S. Rashid, A. O. Akdemir, M. A. Noor, and K. I. Noor, "Generalization of inequalities analogous to preinvex functions via extended generalized Mittag-Leffler functions," in *2019 International Conference on Applied and Engineering Mathematics (ICAEM)*, pp. 256–263, 2019.

[8] S. Klnç, A. Akkurt, and H. Yldrm, "Generalization Mittag-Leffler function associated with of the Hadamard and Fejér-Hadamard inequalities for $(h - m)$-strongly convex functions via fractional integrals," *Journal of Universal Mathematics*, vol. 2, no. 1, pp. 8–15, 2019.

[9] W. Jia, M. Yussouf, G. Farid, and K. A. Khan, "Hadamard and Fejer–Hadamard inequalities for $(\alpha, h - m) - p$-convex functions via Riemann–Liouville fractional integrals," *Mathematical Problems in Engineering*, vol. 2021, Article ID 9945114, 12 pages, 2021.

[10] M. Yussouf, G. Farid, K. A. Khan, and C. Y. Jung, "Hadamard and Fejer–Hadamard inequalities for further generalized fractional integrals involving Mittag-Leffler functions," *Journal of Mathematics*, vol. 2021, Article ID 5589405, 13 pages, 2021.

[11] S. Faisal, M. A. Khan, T. U. Khan, T. Saeed, A. M. Alshehri, and E. R. Nwaeze, "New "Conticrete" Hermite-Hadamard-Jensen-Mercer fractional inequalities," *Symmetry*, vol. 14, no. 2, p. 294, 2022.

[12] T. H. Zhao, Z. Y. He, and Y. M. Chu, "On some refinements for inequalities involving zero-balanced hypergeometric function," *AIMS Mathematics*, vol. 5, no. 6, pp. 6479–6495, 2020.

[13] T. H. Zhao, M. K. Wang, and Y. M. Chu, "Monotonicity and convexity involving generalized elliptic integral of the first kind," *RACSAM*, vol. 115, no. 2, pp. 1–13, 2021.

[14] T. H. Zhao, M. K. Wang, and Y. M. Chu, "Concavity and bounds involving generalized elliptic integral of the first kind," *Journal of Mathematical Inequalities*, vol. 15, no. 2, pp. 701–724, 2007.

[15] H. H. Chu, T. H. Zhao, and Y. M. Chu, "Sharp bounds for the Toader mean of order 3 in terms of arithmetic, quadratic and contraharmonic means," *Mathematica Slovaca*, vol. 70, no. 5, pp. 1097–1112, 2020.

[16] G. Farid, "A unified integral operator and further its consequences," *Open Journal of Mathematical Analysis*, vol. 4, no. 1, pp. 1–7, 2020.

[17] M. Yussouf, G. Farid, K. A. Khan, and C. Y. Jung, "Hadamard and Fejér-Hadamard inequalities for further generalized fractional integrals involving Mittag-Leffler functions," *Journal of Mathematics*, vol. 2021, Article ID 5589405, 13 pages, 2021.

[18] Y. Yue, G. Farid, A. K. Demirel, W. Nazeer, and Y. Zhao, "Hadamard and Fejér-Hadamard inequalities for generalized k-fractional integrals involving further extension of Mittag-Leffler functions," *AIMS Mathematics*, vol. 7, no. 1, pp. 681–703, 2021.

[19] C. P. Niculescu and L. E. Persson, *Convex Functions and Their Applications A Contemporary Approach*, Springer, New York, 2006.

[20] I. Iscan, "Ostrowski type inequalities for p-convex functions," *New Trends in Mathematical Sciences*, vol. 4, no. 3, pp. 140–150, 2016.

[21] Z. B. Fang and R. Shi, "On the (p,h)-convex function and some integral inequalities," *Journal of Inequalities and Applications*, vol. 2014, no. 1, 16 pages, 2014.

[22] B. G. Pachpatte, *Mathematical Inequalities*, vol. 67, Elsevier, 2005.

[23] L. Fejér, "Uberdie Fourierreihen II," *Math Naturwiss Anz Ungar. Akad. Wiss*, vol. 24, pp. 369–390, 1906.

[24] G. Abbas and G. Farid, "*Hadamard and Fejér-Hadamard type inequalities for harmonically convex functions via generalized fractional integrals*," *Journal of Analysis*, vol. 25, no. 1, pp. 107–119, 2017.

[25] H. Ullah, M. A. Khan, and T. Saeed, "Determination of bounds for the Jensen gap and its applications," *Mathematics*, vol. 9, no. 23, p. 3132, 2021.

[26] M. A. Khan, S. Anwar, S. Khalid, and Z. M. M. M. Sayed, "Inequalities of the type Hermite-Hadamard-Jensen-Mercer for strong convexity," *Mathematical Problems in Engineering*, vol. 2021, Article ID 5386488, 16 pages, 2021.

[27] X. Qiang, G. Farid, M. Yussouf, K. A. Khan, and A. U. Rehman, "A note on the boundedness of sublinear operators on grand variable Herz spaces," *Journal of Inequalities and Applications*, vol. 2020, no. 1, 13 pages, 2020.