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Research Article

Certain Analytic Formulas Linked to Locally Fractional Differential and Integral Operators

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The present investigation is aimed at defining different classes of analytic functions and conformable differential operators in view of the concept of locally fractional differential and integral operators. We present a novel generalized class of analytic functions, which we call it locally fractional analytic functions in the open unit disk. For the suggested class, we look at conditions to get the starlikeness and convexity properties.

1. Introduction

The idea of local fractional calculus (LFC) is an innovative differentiation and integration model for functions affecting on special fractal sets. Experts and scientists have been attracted by this theory. The LFC of a complex variable was established by Yang [1] (see [2] for additional material). The issue of explanation of fractal operators over analytic functions is glowered to be solved since the effort of Viswanathan and Navascues [3]. They presented a technique to define α-fractal operator on $\mathbb{C}^k(I)$, the space of all $k-$real-valued continuous functions defined on a compact interval I. In our work, we extend this idea into a complex domain, which is already compact in the z-plane. Therefore, in our opinion, the current study contributes to the theory of fractal functions and makes it easier for them to find new applications in a variety of domains, such as numerical analysis, functional analysis, and harmonic analysis; for example, in relation to PDEs. We anticipate that the current investigation will open the door to further research on shape-preserving fractal approximation in the different function spaces that are being taken into consideration. The reader is encouraged to see [4] for a fractal approximation that preserves shape in the space of differential functions.

Let $\vartheta \in (0,1]$ be a fractional number and $\xi = \chi + i\eta$ be a complex number. Then, the fractal complex number ξ^ϑ can be defined by [2]

$$\xi^\vartheta := \chi^\vartheta + i^\nu \eta^\vartheta, \quad \chi, \eta \in \mathbb{R}. \tag{1}$$

The local fractional derivative (LFD) at a random point ξ_0 for a complex function $\upsilon(\xi)$ can be formulated as follows:

$$\Delta^\vartheta \upsilon(\xi) = \Gamma(1+\vartheta) \lim_{\xi \longrightarrow \xi_0} \left(\frac{\upsilon(\xi) - \upsilon(\xi_0)}{(\xi - \xi_0)^\vartheta} \right), \quad (\vartheta \in (0,1], \xi, \xi_0 \in \mathbb{C}, \upsilon : \mathbb{C} \longrightarrow \mathbb{C}). \tag{2}$$

Let $\upsilon(\xi) = (\xi)^{n\vartheta}$, for example; then, the LFD can be calculated as follows:

$$\Delta^\vartheta (\xi)^{n\vartheta} = \left(\frac{\Gamma(1+n\vartheta)}{\Gamma(1+(n-1)\vartheta)} \right) \xi^{(n-1)\vartheta}. \tag{3}$$

Obviously, when $\vartheta = 1$, the LFD reduces to the normal derivative $\xi^n = n\xi^{n-1}$. Moreover, for $n = 1$, the LFD becomes

$$\Delta^{\vartheta}(\xi)^{\vartheta} = \Gamma(1+\vartheta). \tag{4}$$

On Cantor sets, the local fractional differential operator (LFDO) can be used to construct a variety of different transformations and summations (see, e.g., Yang et al. [5–7]).

Let $\mathbb{A}_{\vartheta}$, $\vartheta \in (0, 1]$ be a class of locally fractional normalized functions in $\Omega \coloneqq \{\xi \in \mathbb{C} : |\xi| < 1\}$ such that

$$\upsilon\left(\xi^{\vartheta}\right) = \xi^{\vartheta} + \sum_{n=2}^{\infty} v_n \xi^{n\vartheta}, \quad \xi \in \Omega. \tag{5}$$

Clearly, when $\vartheta = 1$, we attain the usual normalized class $\mathbb{A}$:

$$\upsilon(\xi) = \xi + \sum_{n=2}^{\infty} v_n \xi^n, \quad \xi \in \Omega. \tag{6}$$

Two functions $\upsilon, \omega \in \mathbb{A}_{\vartheta}$ are convoluted if they accept

$$\upsilon\left(\xi^{\vartheta}\right) * \omega\left(\xi^{\vartheta}\right) = \xi^{\vartheta} + \sum_{n=2}^{\infty} v_n w_n \xi^{\vartheta n}, \quad \xi \in \Omega, \tag{7}$$

where

$$\omega\left(\xi^{\vartheta}\right) = \xi^{\vartheta} + \sum_{n=2}^{\infty} w_n \xi^{\vartheta n}. \tag{8}$$

Moreover, they are subordinated $\upsilon \prec \omega$ if they satisfy the equality $\upsilon(\xi) = (\omega(\lambda(\xi)))$, where λ is analytic with $|\lambda(\xi)| \le |\xi| < 1$ (see [8]). This definition can be extended to LDC by suggesting $\xi \longrightarrow \xi^{\vartheta}$.

1.1. Convoluted Operators. The local derivative suggests the subsequent functional operator: for $n \ge 1$.

$$\begin{aligned}\frac{\Delta^{\nu}\xi^{\vartheta n}}{\Gamma(1+\vartheta)\xi^{n(\vartheta-1)-\vartheta}} &= \frac{((\Gamma(1+n\vartheta))/(\Gamma(1+(n-1)\vartheta)))\xi^{(n-1)\vartheta}}{\Gamma(1+\nu)\xi^{n(\vartheta-1)-\vartheta}} \\ &= \left(\frac{\Gamma(1+n\vartheta)}{\Gamma(1+\vartheta)\Gamma(1+(n-1)\vartheta)}\right)\xi^n.\end{aligned} \tag{9}$$

As a result, we define the following function for all $n \ge 1$,

$$\widehat{E}^{\vartheta}(\xi) = \xi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(1+n\vartheta)}{\Gamma(1+\vartheta)\Gamma(1+(n-1)\vartheta)}\right)\xi^n. \tag{10}$$

Obviously, $\widehat{\mathbb{E}} \in \mathbb{A}$. By using the convolution operator, we define the the following LFDO: $\mathbb{D}^{\vartheta} : \Omega \longrightarrow \Omega$, such that

$$\mathbb{D}^{\vartheta}\upsilon(\xi) \coloneqq \left(\widehat{\mathrm{E}}^{\vartheta} * \upsilon\right)(\xi) = \xi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(1+n\vartheta)}{\Gamma(1+\vartheta)\Gamma(1+(n-1)\vartheta)}\right)v_n\xi^n. \tag{11}$$

Correspondingly, the fractional locally integral operator (FLIO) is defined as follows:

$$\mathbb{J}^{\vartheta}(\xi) \coloneqq \left(\mathrm{E}^{\vartheta} * \upsilon\right)(\xi) = \xi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(1+\vartheta)\Gamma(1+(n-1)\vartheta)}{\Gamma(1+n\vartheta)}\right)v_n\xi^n. \tag{12}$$

We proceed to define the $k-$LFDO and $k-$LFIO, as follows:

$$\begin{aligned}\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi) &\coloneqq \underbrace{\widehat{\mathrm{E}}^{\vartheta} * \cdots * \widehat{\mathrm{E}}^{\vartheta}}_{k-\text{times}} * \upsilon(\xi) = \xi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(1+n\vartheta)}{\Gamma(1+\vartheta)\Gamma(1+(n-1)\vartheta)}\right)^k v_n\xi^n \\ &\coloneqq \xi + \sum_{n=2}^{\infty}\left(\widehat{\mathrm{E}}_n\right)^k v_n\xi^n. \\ \left(\mathbb{J}_k^{\vartheta}\upsilon\right)(\xi) &\coloneqq \underbrace{\mathrm{E}^{\vartheta} * \cdots * \mathrm{E}^{\vartheta}}_{k-\text{times}} * \upsilon(\xi) = \xi + \sum_{n=2}^{\infty}\left(\frac{\Gamma(1+\vartheta)\Gamma(1+(n-1)\vartheta)}{\Gamma(1+n\vartheta)}\right)^k v_n\xi^n \\ &\coloneqq \xi + \sum_{n=2}^{\infty}\left(\mathrm{E}_n\right)^k v_n\xi^n.\end{aligned} \tag{13}$$

It is obvious that we reach the well-known Salagean differential and integral operators, respectively, when $\vartheta = 1$.

Definition 1. Define two formulas of analytic functions $\upsilon \in \mathbb{A}$ as follows:

$$\begin{aligned}\mathcal{S}_{\upsilon}(\xi) &\coloneqq \frac{\xi\upsilon'(\xi)}{\upsilon(\xi)}, \quad \xi \in \Omega, \\ \mathcal{K}_{\upsilon}(\xi) &\coloneqq 1 + \frac{\xi\upsilon''(\xi)}{\upsilon'(\xi)}, \quad \xi \in \Omega.\end{aligned} \tag{14}$$

Consequently, there are two classes for this investigation, the starlike and convex classes with

$$\begin{aligned}&\Re(\mathcal{S}_{\upsilon}(\xi)) > 0, \\ &\Re(\mathcal{K}_{\upsilon}(\xi)) > 0, \quad \xi \in \Omega,\end{aligned} \tag{15}$$

respectively.

1.2. Locally Fraction Conformable Operator

Definition 2. Let $\wp$ be a nonnegative number, such that $[[\wp]]$ be the integer part of $\wp$. By using the $k-$LFDO, we have the following locally fractional conformable differential operator:

$$\wedge^{\wp}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]=\wedge^{\wp-[[\wp]]}\left(\wedge^{[[\wp]]}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]\right)$$
$$=\frac{p_1(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)}\left(\wedge^{[[\wp]]}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]\right)$$
$$+\frac{p_0(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\zeta)}\left(\xi\left(\wedge^{[[\wp]]}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]\right),\right. \tag{16}$$

where for $q:=\wp-[[\wp]]\in[0,1)$,

$$\wedge^{0}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]=\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right],$$
$$\wedge^{q}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]=\frac{p_1(q,\xi)}{p_1(q,\xi)+p_0(q,\xi)}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]$$
$$+\frac{p_0(q,\xi)}{p_1(q,\xi)+p_0(q,\xi)}\left(\xi\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]'\right)$$
$$=\frac{p_1(q,\xi)}{p_1(q,\xi)+p_0(q,\xi)}\left[\xi+\sum_{n=2}^{\infty}\left(\widehat{\mathrm{E}}_n\right)^k\nu_n\xi^n\right]$$
$$+\frac{p_0(q,\xi)}{p_1(q,\xi)+p_0(\nu,\xi)}\left(\left[\xi+\sum_{n=2}^{\infty}\left(\widehat{\mathrm{E}}_n\right)^k n\nu_n\xi^n\right]\right)$$
$$=\xi+\sum_{n=2}^{\infty}\left(\frac{p_1(q,\xi)+n\,p_0(q,\xi)}{p_1(q,\xi)+p_0(q,\xi)}\right)\left(\widehat{\mathrm{E}}_n\right)^k\nu_n\xi^n$$
$$:=\xi+\sum_{n=2}^{\infty}\theta_n(q,\xi)\left(\widehat{\mathrm{E}}_n\right)^k\nu_n\xi^n\wedge^1\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]$$
$$=\xi\left(\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]\right)$$
$$\vdots$$
$$\wedge^{[[\wp]]}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]=\wedge^1\left(\wedge^{[[\wp]]-1}\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]\right), \tag{17}$$

where

$$\theta_n(q,\xi):=\left(\frac{p_1(q,\xi)+n\,p_0(q,\xi)}{p_1(\nu,\xi)+p_0(q,\xi)}\right), \tag{18}$$

and the functions $p_1,p_0:[0,1]\times\Omega\longrightarrow\Omega$ are analytic in Ω with

$$\begin{gathered}p_1(q,\xi)\neq-p_0(q,\xi),\\ \lim_{q\longrightarrow0}p_1(q,\xi)=1,\\ \lim_{q\longrightarrow1}p_1(q,\xi)=0,\quad p_1(q,\xi)\neq0,\forall\xi\in\Omega,q\in(0,1),\\ \lim_{q\longrightarrow0}p_0(q,\xi)=0,\\ \lim_{q\longrightarrow1}p_0(q,\xi)=1,\quad p_0(q,\xi)\neq0,\forall\xi\in\Omega,q\in(0,1).\end{gathered} \tag{19}$$

Remark 3.

(i) For constant coefficients, the operator $\wedge^q[(\mathbb{D}_k^{\vartheta}\upsilon)(\xi)]$ is normalized in Ω. Moreover, if $\wp-[[\wp]]=0,\vartheta=1$, then we realize the Sàlàgean differential operator [9]. If $k=0$, we attain the conformable differential operator in [10], which is based on the same assumptions. Similarly, we can replace the local fractional integral operator using the $k-\mathrm{LFJO}((\mathbb{J}_k^{\vartheta}\upsilon)(\xi))$

(ii) The authors in [11] presented a conformable fractional differential operator by using a combination of fractional integral and differential operators, as follows:

$$D^q\upsilon(\xi)=\xi+\sum_{n=2}^{\infty}\theta_n(q,\xi)\left(\frac{\Gamma(3-q)\Gamma(n+1)}{\Gamma(n+2-q)}\right)^k\nu_n\xi^n,\quad\xi\in\Omega, \tag{20}$$

where

$$\theta_n(q,\xi)=\left(\frac{p_1(q,\xi)+n\,p_0(q,\xi)}{p_1(\nu,\xi)+p_0(q,\xi)}\right) \tag{21}$$

We proceed to discuss the most important geometric properties of $(\mathbb{D}_k^{\vartheta}\upsilon)(\xi)$ and $\wedge^{\wp}[(\mathbb{D}_k^{\vartheta}\upsilon)(\xi)]$, in the next section.

1.3. Lemmas. We request the next preliminaries.

Lemma 4 (see [8]/p135). *Let υ be analytic and ω be univalent in Ω such that $\upsilon(0)=\omega(0)$. Furthermore, let ϕ be analytic in a domain involving $\omega(\Omega)$ and $\omega(\Omega)$. If $\xi\omega'(\xi)\phi(\omega(\xi))$ is starlike, then the subordination*

$$\xi\upsilon'(\xi)\phi(\upsilon(\xi))\prec\xi\omega'(\xi)\phi(\omega(\xi)) \tag{22}$$

yields

$$\upsilon(\xi)\prec\omega(\xi), \tag{23}$$

and ω is the best dominant.

Lemma 5 (see [12]). *For two analytic functions υ_1 and υ_2 in a complex domain such that $\upsilon_1(0)=\upsilon_2(0)$ and*

$$\upsilon_1(\xi)\prec\upsilon_2(\xi), \tag{24}$$

then

$$\int_0^{2\pi}|\upsilon_1(\xi)|^t d\mu\leq\int_0^{2\pi}|\upsilon_2(\xi)|^t d\mu, \tag{25}$$

where

$$\xi=\nu\exp(i\mu),\quad\nu\in(0,1),t\in\mathbb{R}^+. \tag{26}$$

The next section is about our results of the operator $\wedge^q[(\mathbb{D}_k^{\vartheta}\upsilon)(\xi)]$, and as a special consequence, we connect the operator $[(\mathbb{D}_k^{\vartheta}\upsilon)(\xi)]$.

2. Results

This section deals with the operator $\wedge^q[(\mathbb{D}_k^\vartheta \upsilon)(\xi)] := \wedge^q(\xi)$.

2.1. Starlikeness of $\wedge^q(\xi)$

Theorem 6. *If the following conditions occur:*

(i) Λ *is univalent in* Ω

(ii) $(\xi\Lambda'(\xi))/(\Lambda(\xi)(\Lambda(\xi)-1))$ *is starlike in* Ω

(iii) $(\mathscr{K}_{\wedge^q}(\xi)-1)/(\mathscr{S}_{\wedge^q}(\xi)-1) \prec 1+((\xi\Lambda'(\xi))/(\Lambda(\xi)(\Lambda(\xi)-1)))$

Then,

$$\mathscr{S}_{\wedge^q}(\xi) \prec \Lambda(\xi), \quad \xi \in \Omega, \tag{27}$$

and Λ is the best dominant. Moreover,

$$\int_0^{2\pi} |\mathscr{S}_{\wedge^q}(\xi)|^t d\mu \le \int_0^{2\pi} |\Lambda(\xi)|^t d\mu, \quad t \in \mathbb{R}^+. \tag{28}$$

Proof. Put the function H, as follows:

$$H(\xi) := \mathscr{S}_{\wedge^q}(\xi), \quad \xi \in \Omega. \tag{29}$$

A simple calculation yields

$$\mathscr{S}_H(\xi) = \mathscr{K}_{\wedge^q}(\xi) - H(\xi). \tag{30}$$

Consequently, we attain

$$\frac{\mathscr{K}_{\wedge^q}(\xi)-1}{\mathscr{S}_{\wedge^q}(\xi)-1} = \frac{\mathscr{S}_H(\xi)+H(\xi)-1}{H(\xi)-1} = 1 + \frac{\xi H'(\xi)}{H(\xi)(H(\xi)-1)}. \tag{31}$$

Thus, we have

$$\frac{\xi H'(\xi)}{H(\xi)(H(\xi)-1)} \prec \frac{\xi\Lambda'(\xi)}{\Lambda(\xi)(\Lambda(\xi)-1)}, \quad \xi \in \Omega. \tag{32}$$

Based on Lemma 4, we attain the requested result. The second part is a direct application of Lemma 5. □

Theorem 7. *If the following conditions hold:*

(i) Λ *is univalent in* Ω

(ii) $(\xi\Lambda'(\xi))/(\Lambda(\xi)-1)$ *is starlike in* Ω

(iii) $\mathscr{S}_{\wedge^q}(\xi)(((\mathscr{K}_{\wedge^q}(\xi)-1)/(\mathscr{S}_{\wedge^q}(\xi)-1)\mathscr{S}_{\wedge^q}(\xi)-1)-1) \prec ((\xi\Lambda'(\xi))/(\Lambda(\xi)-1))$

Then,

$$\mathscr{S}_{\wedge^q}(\xi) \prec \Lambda(\xi), \quad \xi \in \Omega, \tag{33}$$

and Λ is the best dominant. Furthermore,

$$\int_0^{2\pi} |\mathscr{S}_{\wedge^q}(\xi)|^t d\mu \le \int_0^{2\pi} |\Lambda(\xi)|^t d\mu, \quad t \in \mathbb{R}^+. \tag{34}$$

Proof. Formulate the function H as follows:

$$H(\xi) := \mathscr{S}_{\wedge^q}(\xi), \quad \xi \in \Omega. \tag{35}$$

Then, we get the equality

$$\mathscr{S}_H(\xi) + H(\xi) = \mathscr{K}_{\wedge^q}(\xi). \tag{36}$$

Replacing produces the following results:

$$\mathscr{S}_{\wedge^q}(\xi)\left(\frac{\mathscr{K}_{\wedge^q}(\xi)-1}{\mathscr{S}_{\wedge^q}(\xi)-1} - 1\right) = \frac{\xi H'(\xi)}{H(\xi)-1}. \tag{37}$$

Hence,

$$\frac{\xi H'(\xi)}{H(\xi)-1} \prec \frac{\xi\Lambda'(\xi)}{\Lambda(\xi)-1}, \quad \xi \in \Omega. \tag{38}$$

In view of Lemma 4, we have the outcome. Lemma 5 implies the integral inequality. □

Theorem 8. *If the following conditions are fulfilled:*

(i) Λ *is univalent in* Ω

(ii) $\mathscr{S}_\Lambda$ *is starlike in* Ω

(iii) $\mathscr{K}_{\wedge^q}(\xi) - \mathscr{S}_{\wedge^q}(\xi) \prec \mathscr{S}_\Lambda(\xi)$

Then,

$$\mathscr{S}_{\Lambda_q}(\xi) \prec \Lambda(\xi), \quad \xi \in \Omega, \tag{39}$$

and Λ is the best dominant. In addition,

$$\int_0^{2\pi} |\mathscr{S}_{\wedge^q}(\xi)|^t d\mu \le \int_0^{2\pi} |\Lambda(\xi)|^t d\mu, \quad t \in \mathbb{R}^+. \tag{40}$$

Proof. Present the function H as follows:

$$H(\xi) := \mathscr{S}_{\wedge^q}(\xi), \quad \xi \in \Omega. \tag{41}$$

Therefore, we get

$$\mathscr{S}_H(\xi) + H(\xi) = \mathscr{K}_{\wedge^q}(\xi). \tag{42}$$

Replacing brings that

$$\mathscr{K}_{\wedge^q}(\xi) - \mathscr{S}_{\wedge^q}(\xi) = \mathscr{S}_H(\xi). \tag{43}$$

Hence,

$$\mathscr{S}_H(\xi) \prec \mathscr{S}_\Lambda(\xi), \quad \xi \in \Omega. \tag{44}$$

Hence, Lemma 4 implies $H(\xi) \prec \Lambda(\xi)$. And Lemma 5 yields the last integral inequality

$$\int_0^{2\pi} |H(\xi)|^t d\mu \le \int_0^{2\pi} |\Lambda(\xi)|^t d\mu, \quad t \in \mathbb{R}^+. \tag{45}$$

□

Theorem 9. *If the following conditions are applied:*

(i) Λ is univalent in Ω

(ii) $\xi\Lambda'(\xi)$ is starlike in Ω

(iii) $\mathcal{S}_{\wedge^q}(\xi)(\mathcal{K}_{\wedge^q}(\xi) - \mathcal{S}_{\wedge^q}(\xi)) \prec \xi\Lambda'(\xi)$

Then,

$$\mathcal{S}_{\wedge^q}(\xi) \prec \Lambda(\xi), \quad \xi \in \Omega, \tag{46}$$

and Λ is the best dominant. Also,

$$\int_0^{2\pi} |\mathcal{S}_{\wedge^q}(\xi)|^t d\mu \le \int_0^{2\pi} |\Lambda(\xi)|^t d\mu, \quad t \in \mathbb{R}^+. \tag{47}$$

Proof. Define the function H as follows:

$$H(\xi) \coloneqq \mathcal{S}_{\wedge^q}(\xi), \quad \xi \in \Omega. \tag{48}$$

Consequently, we have

$$\mathcal{S}_H(\xi) + H(\xi) = \mathcal{K}_{\wedge^q}(\xi). \tag{49}$$

Substituting implies

$$\mathcal{S}_{\wedge^q}(\xi)(\mathcal{K}_{\wedge^q}(\xi) - \mathcal{S}_{\wedge^q}(\xi)) = \xi H'(\xi). \tag{50}$$

Hence,

$$\xi H'(\xi) \prec \xi\Lambda'(\xi), \quad \xi \in \Omega. \tag{51}$$

According to Lemma 4, we have result $H(\xi) \prec \Lambda(\xi)$, while Lemma 5 gives

$$\int_0^{2\pi} |H(\xi)|^t d\mu \le \int_0^{2\pi} |\Lambda(\xi)|^t d\mu, \quad t \in \mathbb{R}^+. \tag{52}$$

□

Remark 10. Note that, when $q = 0$ in Theorems 6–9, we have the starlikeness of the operator $[(\mathbb{D}_k^\vartheta \upsilon)(\xi)]$, as follows:

$$\mathcal{S}_{[(\mathbb{D}_k^\vartheta \upsilon)]}(\xi) \prec \Lambda(\xi), \quad \xi \in \Omega. \tag{53}$$

Moreover, when $k = 0, q = 0$, we obtain the Ma-Minda class of starlikeness [13], as follows:

$$\mathcal{S}_\upsilon(\xi) \prec \Lambda(\xi), \quad \xi \in \Omega. \tag{54}$$

The last class is studied widely by many researchers for different types of functions [8] $\Lambda(\xi)$. For example, the inequality $\Lambda(\xi) \coloneqq (1 + i\xi)/(1 - j\xi), -1 \le j < i \le 1$, which indicates that the image of Ω under the description $\mathcal{S}_\upsilon(\xi)$ is centered on the x-axis owning diameter of end points $(1 - i)/(1 - j)$ and $(1 + i)/(1 + j)$ (see [14]). Another recent example is that $\Lambda(\xi) = \cos(\xi)$ [15].

We proceed to discover more properties of the locally fractional conformable operator.

Theorem 11. *If the following conditions hold:*

(i) Λ is convex univalent in Ω

(ii) $\mathcal{S}_{\wedge^q}(\xi) \prec \Lambda(\xi), \Lambda(0) = 1$

Then,

$$\wedge^q(\xi) \prec \xi \exp\left(\int_0^\xi \frac{\Lambda(\Xi(z))}{z} dz\right), \tag{55}$$

where Ξ fulfills $\Xi(0) = 0$ and $|\Xi(\xi)| < 1$. In addition, the inequality $|\xi| \coloneqq \tau < 1$ yields

$$\exp\left(\int_0^1 \frac{\Lambda(-\tau)}{\tau} d\tau\right) \le \left|\frac{\wedge^q(\xi)}{\xi}\right| \le \exp\left(\int_0^1 \frac{\Lambda(\tau)}{\tau} d\tau\right). \tag{56}$$

Proof. The conditions (i)-(ii) yield

$$\frac{(\wedge^q(\xi))}{\wedge^q(\xi)} - \frac{1}{\xi} = \frac{\Lambda(\Xi(\xi)) - 1}{\xi}. \tag{57}$$

Integration implies that

$$\wedge^q(\xi) \prec \xi \exp\left(\int_0^\xi \frac{\Lambda(\Xi(z))}{z} dz\right), \tag{58}$$

which is equivalent to

$$\frac{\wedge^q(\xi)}{\xi} \prec \exp\left(\int_0^\xi \frac{\Lambda(\Xi(z))}{z} dz\right). \tag{59}$$

But, we have

$$\Lambda(-\tau|\xi|) \le \mathfrak{R}(\Lambda(\Xi(\xi\tau))) \le \Lambda(\tau|\xi|). \tag{60}$$

Then, we obtain

$$\int_0^1 \frac{\Lambda(-\tau|\xi|)}{\rho} d\tau \le \int_0^1 \frac{\mathfrak{R}(\Lambda(\Xi(\xi\tau)))}{\tau} d\tau \le \int_0^1 \frac{\Lambda(\tau|\xi|)}{\tau} d\tau. \tag{61}$$

The above conclusion leads to

$$\int_0^1 \frac{\Lambda(-\tau|\xi|)}{\rho} d\tau \le \log\left|\frac{\wedge^q(\xi)}{\xi}\right| \le \int_0^1 \frac{\Lambda(\tau|\xi|)}{\tau} d\tau. \tag{62}$$

This leads to

$$\exp\left(\int_0^1 \frac{\Lambda(-\tau)}{\tau} d\tau\right) \le \left|\frac{\wedge^q(\xi)}{\xi}\right| \le \exp\left(\int_0^1 \frac{\Lambda(\tau)}{\tau} d\tau\right). \quad (63)$$

□

Corollary 12. *Let the conditions of Theorem 11 hold for $q = 0$. Then,*

$$\left[\left(\mathbb{D}_k^{\vartheta} \upsilon\right)(\xi)\right] \prec \xi \exp\left(\int_0^{\xi} \frac{\Lambda(\Xi(z))}{z} dz\right), \quad (64)$$

where Ξ achieves $\Xi(0) = 0$ and $|\Xi(\xi)| < 1$. Moreover, the inequality $|\xi| \coloneqq \tau < 1$ yields

$$\exp\left(\int_0^1 \frac{\Lambda(-\tau)}{\tau} d\tau\right) \le \left|\frac{\left[\left(\mathbb{D}_k^{\vartheta}\upsilon\right)(\xi)\right]}{\xi}\right| \le \exp\left(\int_0^1 \frac{\Lambda(\tau)}{\tau} d\tau\right). \quad (65)$$

Corollary 13. *Let the conditions of Theorem 11 hold for $q = 0$ and $k = 0$. Then,*

$$[\upsilon(\xi)] \prec \xi \exp\left(\int_0^{\xi} \frac{\Lambda(\Xi(z))}{z} dz\right), \quad (66)$$

where Ξ satisfies $\Xi(0) = 0$ and $|\Xi(\xi)| < 1$. Moreover, the inequality $|\xi| \coloneqq \tau < 1$ yields

$$\exp\left(\int_0^1 \frac{\Lambda(-\tau)}{\tau} d\tau\right) \le \left|\frac{\upsilon(\xi)}{\xi}\right| \le \exp\left(\int_0^1 \frac{\Lambda(\tau)}{\tau} d\tau\right). \quad (67)$$

2.2. Positive Real Part of $\wedge^q(\xi)$. In this part, we aim to present the sufficient conditions for the operator $\wedge^q$ to be in the class of the real positive part ($\mathscr{P}$.).

Theorem 14. *Let $\ell \in (0, 1)$ and $\wp \in [0, \infty)$. If*

$$p_0(\wp - [[\wp]], \xi) = \left(\frac{\ell}{1-\ell}\right) p_1(\wp - [[\wp]], \xi)\ ; q = \wp - [[\wp]], \quad (68)$$

then

$$\frac{\wedge^{\wp+2}(\xi)}{\wedge^{\wp+1}(\xi)} \in \mathscr{P} \Longrightarrow \frac{\wedge^{\wp+1}(\xi)}{\wedge^{\wp}(\xi)} \in \mathscr{P}. \quad (69)$$

Proof. By the condition $p_0(\wp - [[\wp]], \xi) = (\ell/(1-\ell)) p_1(\wp - [[\wp]], \xi)$ and by Definition 2, we have

$$\begin{aligned} \wedge^{\wp}(\xi) &= (1-\ell)\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi\left(\wedge^{[[\wp]]}(\xi)]\right), \\ \wedge^{\wp+1}(\xi)] &= \xi\left(\wedge^{[[\wp]]}(\xi)]\right) + \ell\xi^2\left(\wedge^{[[\wp]]}(\xi)]\right), \\ \wedge^{\wp+2}(\xi)] &= \xi\left(\wedge^{[[\wp]]}(\xi)]\right) + (1+2\ell)\wp^2\left(\wedge^{[[\wp]]}(\xi)\big]\right) + \ell\xi^3\left(\wedge^{[[\wp]]}(\xi)]\right). \end{aligned} \quad (70)$$

We realize that

$$\Re\left(\frac{\wedge^{\wp+2}(\xi)}{\wedge^{\wp+1}(\xi)}\right) > 0, \quad (71)$$

whenever the following real is positive:

$$\Re\left\{1 + \frac{(1+\ell)\xi\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi^2\left(\wedge^{[[\wp]]}(\xi)\right)}{\left(\wedge^{[[\wp]]}(\xi)\big]\right) + \ell\xi\left(\wedge^{[[\wp]]}(\xi)\right)}\right\} > 0. \quad (72)$$

Hence, from the inequality (74) with the idea of convex functional

$$(1-\ell)\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi\left[\wedge^{[[\wp]]}(\xi)\right], \quad (73)$$

we have

$$\Re\left\{1 + \frac{\xi\left[(1-\ell)\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi\left(\wedge^{[[\wp]]}(\xi)\right)\right]}{\left[(1-\ell)\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi\left(\wedge^{[[\wp]]}(\xi)\right)\right]}\right\} > 0. \quad (74)$$

Since convexity implies starlikeness, then we realize

$$\Re\left\{\frac{\xi\left[(1-\ell)\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi\left(\wedge^{[[\wp]]}(\xi)\right)\right]}{(1-\ell)\left(\wedge^{[[\wp]]}(\xi)\right) + \ell\xi\left(\wedge^{[[\wp]]}(\xi)\right)}\right\} > 0. \quad (75)$$

The inequality (75) holds whenever

$$\Re\left(\frac{\wedge^{\wp+1}(\xi)]}{\wedge^{\wp}(\xi)]}\right) > 0. \quad (76)$$

□

Note that when $k = 0$ in Theorem 14, we have [10]—Theorem 3.1.

Next, we deal with the functional

$$\frac{\wedge^{\wp+1}(\xi)}{\xi\left(\wedge^{[[\wp]]}(\xi)\right)'} \quad (77)$$

to be in the class of positive real part $\mathscr{P}$.

Theorem 15. *Let $\ell \in (0, 1)$, $\wp \in [0, \infty)$, and*

$$p_1(\wp - [[\wp]], \xi) = \left(\frac{\ell}{1-\ell}\right) p_0(\wp - [[\wp]], \xi). \quad (78)$$

If $\wedge^{[[\wp]]}(\xi) \in \mathscr{C}$ (the class of convex univalent functions in Ω), then

$$\frac{\wedge^{\wp+1}(\xi)]}{\xi\left(\wedge^{[[\wp]]}(\xi)\big]\right)} \in \mathscr{P}(\ell). \quad (79)$$

Proof. Applying the differential operator rule to

$$\wedge^{\wp+1}(\xi)] = \wedge^1\left(\wedge^{\wp}(\xi)\right) \quad (80)$$

yields

$$\begin{aligned}
\wedge^{\wp+1}(\xi)] &= \wedge^{\wp-[[\wp]]}\left(\wedge^{[[\wp]]+1}(\xi)\right) = \wedge^{\wp-[[\wp]]}\left\{\wedge\left[\wedge^{[[\wp]]}(\xi)\right]\right\} \\
&= \wedge^{\wp-[[\wp]]}\left\{\xi\left[\wedge^{[[\wp]]}(\xi)\right]\right\} \\
&= \frac{p_1(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)}\left\{\xi\left[\wedge^{[[\wp]]}(\xi)\right]\right\} \\
&\quad + \frac{p_0(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)} \\
&\quad \times \left\{\xi\left[\left(\wedge^{[[\wp]]}(\xi)\right)+\xi\left(\wedge^{[[\wp]]}(\xi)\right)\right]\right\} \\
&= \frac{p_1(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)}\left\{\xi\left[\wedge^{[[\wp]]}(\xi)\right]\right\} \\
&\quad + \frac{p_0(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)}\left\{\xi\left[\wedge^{[[\wp]]}(\xi)\right]\right\} \\
&\quad + \frac{p_0(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)}\left\{\xi^2\left[\wedge^{[[\wp]]}(\xi)\right]\right\} \\
&= \xi\left[\wedge^{[[\wp]]}(\xi)\right] + \frac{p_0(\wp-[[\wp]],\xi)}{p_1(\wp-[[\wp]],\xi)+p_0(\wp-[[\wp]],\xi)}\left\{\xi^2\left[\wedge^{[[\wp]]}(\xi)\right]\right\}.
\end{aligned} \tag{81}$$

Dividing equation (80) by the formula $\xi(\wedge^{[[\wp]]}(\xi))$ and employing the condition

$$p_1(\wp-[[\wp]],\xi) = \left(\frac{\ell}{1-\ell}\right)p_0(\wp-[[\wp]],\xi), \tag{82}$$

we attain

$$\frac{\wedge^{\wp+1}(\xi)}{\xi\left(\wedge^{[[\wp]]}(\xi)\right)} = 1 + (1-\ell)\frac{\xi\left(\wedge^{[[\wp]]}(\xi)\right)}{\left(\wedge^{[[\wp]]}(\xi)\right)}. \tag{83}$$

The convexity of $\wedge^{[[\wp]]}(\xi), \xi \in \Omega$, brings

$$\Re\left\{1 + \frac{\xi\left(\wedge^{[[\wp]]}(\xi)\right)}{\left(\wedge^{[[\wp]]}(\xi)\right)}\right\} > 0. \tag{84}$$

Hence, it yields that

$$\Re\left\{\frac{\wedge^{\wp+1}(\xi)}{\xi\left(\wedge^{[[\wp]]}(\xi)\right)}\right\} > \ell. \tag{85}$$

□

Note that when $k = 0$ in Theorem 15, we have [10]—Theorem 3.2.

3. Conclusion

By using the concept of locally fractional differential and integral of a complex variable, we illustrated a set of differential and integral operators acting on a class of normalized analytic functions. Moreover, we presented a conformable differential operator linking with the suggested locally fractional differential operator (similarly if we take the locally fractional integral). Different investigations are introduced, including the stralikeness and convexity properties.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally and significantly to writing this article. All authors read and agreed to the published version of the manuscript.

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