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Research Article

New Fractional Derivative Expression of the Shifted Third-Kind Chebyshev Polynomials: Application to a Type of Nonlinear Fractional Pantograph Differential Equations

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The main goal of this paper is to develop a new formula of the fractional derivatives of the shifted Chebyshev polynomials of the third kind. This new formula expresses approximately the fractional derivatives of these polynomials in the Caputo sense in terms of their original ones. The linking coefficients are given in terms of a certain ${}_4F_3(1)$ terminating hypergeometric function. The integer derivatives of the shifted third-kind Chebyshev polynomials can be calculated using this formula after performing some reductions. To solve a nonlinear fractional pantograph differential equation with quadratic nonlinearity, the fractional derivative formula is used in conjunction with the tau technique. The role of the tau method is to convert the pantograph differential equation with its governing initial/boundary conditions into a nonlinear system of algebraic equations that can be treated with the aid of Newton's iterative scheme. To test the method's convergence, certain estimations are included. The proposed numerical method is demonstrated by numerical results to ensure its applicability and efficiency.

1. Introduction

In the last three decades, many searches highlighted descriptions of a variety of phenomena by using fractional differential equations (FDEs) (see, for example, [1–3]). Accordingly, a lot of research was directed to solving these equations. Unfortunately, the exact solutions of many models of FDEs are not always available. So, finding numerical solutions to these equations was the only way to obtain results that are enabling us to study these phenomena in a practical way. In this regard, several numerical approaches for dealing with FDEs have been presented. Among these methods, but not limited to, we find wavelets methods [4, 5], operational matrix methods [6–8], Adomian's decomposition method [9], tau method [10–12], pseudospectral method [13], finite difference method [14], and other methods [15–18].

Throughout the history of numerical analysis research, it has been clarified that orthogonal polynomials are credited with developing these numerical methods. One of the most important orthogonal polynomials that contributed to developing these methods is the Jacobi polynomial. The most famous special cases of Jacobi polynomials are Chebyshev polynomials of first, second, third, and fourth kinds.

Due to the importance of all kinds of Chebyshev polynomials in different branches of mathematics, a great number of authors investigated them from both theoretical and practical points of view. From a theoretical point of view, and for example, regarding the Chebyshev functional, it has an old history in the study (see, for example, [19]) and an extensive repertoire of applications in many fields (see [20]). Furthermore, in the sequence of papers [21–23], the authors have developed some type inequalities related to the Chebyshev functional. From a numerical point of view, the authors in [24] presented a Galerkin operational matrix method for the numerical treatment of linear and nonlinear hyperbolic telegraph type equations based on utilizing certain combinations of Chebyshev polynomials of the first kind. In [25], the fractional derivative formula of the first kind of Chebyshev polynomials was established. In addition, a type of fractional delay differential equations was treated using the spectral tau method. Chebyshev polynomials of the second kind were used in [26] to find spectral solutions for the fractional Riccati differential equations. Regrading the third- and fourth-kinds of Chebyshev polynomials, they were utilized in many applications. The authors in [27] introduced the operational matrices of derivatives of third- and fourth-kinds Chebyshev polynomials and employed them to numerically solve the Lane-Emden type equations. The authors in [28, 29] have employed such polynomials to treat other types of differential and integral equations. Recently, other two types of Chebyshev polynomials, namely, Chebyshev polynomials of the fifth- and sixth-kinds, were employed in a variety of applications. The author in [12] has established explicit formulas for the derivatives of the sixth-kind Chebyshev polynomials and utilized them to find spectral solution of the nonlinear one-dimensional Burgers' equation.

Spectral methods are among the most widely used numerical techniques that have been developed and adapted to solve various forms of DEs. The use of many properties of orthogonal polynomials has contributed to developing these methods, which enabled researchers to obtain explicit expressions for a general-order derivative of an infinitely differentiable function in terms of those of the function. These expressions enabled them to develop many algorithms to solve these equations (see, for example, [24]). Also, the orthogonality property and the properties of the roots of these polynomials had a clear effect in obtaining high-precision numerical solutions using the different versions of these methods, like the tau method [30, 31], collocation method [32, 33], and Galerkin method [24]. The tau approach offers the benefit of avoiding some problems that the Galerkin method faces. This is because of the freedom with which basis functions can be chosen and the underlying conditions are set as constraints (see [25]).

Delay differential equations (DDEs) and fractional delay differential equations (FDDEs) have vital roles as they arise in several disciplines such as biology, economic, and automatic control (see, [34]). The pantograph differential equations and the fractional pantograph differential equations are important types of DDEs and many authors have interests in them. For example, Sedaghat et al. [35] suggested an approximation to a pantograph equation with the aid of Chebyshev polynomials. The authors in [36] have applied a Taylor method for obtaining an approximate solution of the generalized pantograph equations. The direct operational tau method was employed to solve the pantograph-type equation in [37]. Fractional pantograph differential equations were handled using the generalized fractional-order Bernoulli wavelets in [38]. A wavelet matrix approach was followed in [39] based on using Müntz-Legendre polynomials to treat the fractional pantograph differential equations. For some other contributions regarding the different types of pantograph equations, on can be referred to [40–44].

In this study, an explicit expression for the fractional derivatives of the third-kind Chebyshev polynomials is established. As far as we know, this expression is new, and it generalizes the formula of the integer derivatives of Chebyshev polynomials of the third kind that is previously established. We demonstrate that this formula involves a certain ${}_4F_3(1)$ terminating hypergeometric term. Using the new formula, the tau method is applied to solve the fractional pantograph differential equations.

The following is a breakdown of the current paper's structure. Section 2 is devoted to displaying some definitions of the fractional calculus theory. Moreover, in this section, we present some useful formulas concerned with the third-kind Chebyshev polynomials and their shifted ones. The definitions of the generalized hypergeometric functions and the regularized hypergeometric functions are also presented in this section. Section 3 is interested in deriving in detail a new formula that expresses approximately the fractional derivatives of the shifted third-kind Chebyshev polynomials. Also, in this section, the well-known integer derivative formula of the shifted third-kind Chebyshev polynomials is a specific result of the fractional ones. In Section 4, we describe the proposed numerical algorithm for solving a type of fractional pantograph differential equation with quadratic nonlinearity using the spectral tau method. Some error estimates are given in Section 5 to examine the proposed polynomial series expansion's convergence and error analysis. Some numerical simulations are presented in Section 6 to validate the theoretical results. Finally, Section 7 summarises the findings.

2. Some Essentials and Useful formulas

Some definitions of the fractional calculus theory are presented in this section. In addition, various properties and important formulas for third-kind Chebyshev polynomials and their shifted counterparts are presented. Also, the definitions of the generalized hypergeometric functions and some their basic properties are given.

2.1. Some Definitions of the Fractional Operators

Definition 1. Let I^{μ}denote the Riemann-Liouville fractional integral operator of order μ on the usual Lebesgue space L_1 $[0, 1]$. Then, I^{μ} is defined as

$$I^{\mu} g(t) = \begin{cases} \dfrac{1}{\Gamma(\mu)} \displaystyle\int_0^t (t-\tau)^{\mu-1} g(\tau)\, d\tau, & \mu > 0, \\ g(t), & \mu = 0. \end{cases} \tag{1}$$

The properties below are valid.

(i) $I^{\mu} I^{\beta} = I^{\mu+\beta}$

(ii) $I^{\mu} I^{\beta} = I^{\beta} I^{\mu}$

(iii) $I^{\mu} t^{\gamma} = (\Gamma(\gamma+1)/\Gamma(\gamma+\mu+1)) t^{\gamma+\mu}$

where $\mu, \beta \geq 0$, and $\gamma > -1$.

Definition 2. The Caputo fractional-order derivatives of a function u defined on the interval $I = [0, 1]$ are defined as:

$$\left({}_0^C D_x^\gamma u\right)(x) = \frac{1}{\Gamma(\ell-\gamma)} \int_0^x (x-\tau)^{\ell-\gamma-1} u^{(\ell)}(\tau) d\tau, \gamma > 0, t > 0, \tag{2}$$

where $\ell - 1 \le \gamma < \ell, \ell \in \mathbb{N}$.

The following property is useful

$${}_0^C D_x^\alpha t^\beta = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \alpha \rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \ge \lceil \alpha \rceil, \end{cases} \tag{3}$$

where $\lceil \alpha \rceil$ denotes the lowest integer more than or equal to α and $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$.

2.2. An Account on Third-Kind Chebyshev Polynomials and Their Shifted Ones. The Chebyshev polynomials of the third-kind $V_n(x)$ are polynomials in x that have the following trigonometric definition (see [45])

$$V_n(x) = \frac{\cos(n+1/2)\theta}{\cos(\theta/2)}, \tag{4}$$

with $x = \cos\theta$.

The polynomials $V_n(x)$ are special ones of the Jacobi polynomials. More definitely, we have

$$V_n(x) = \frac{2^{2n}}{\binom{2n}{n}} P_n^{(-1/2,1/2)}(x). \tag{5}$$

With respect to the weight function $w(x) = \sqrt{(1+x)/(1-x)}$, these polynomials are orthogonal on $(-1, 1)$, in the sense that

$$\int_{-1}^1 \sqrt{\frac{1+x}{1-x}} V_m(x) V_n(x) dx = \begin{cases} 0, & m \ne n, \\ \pi, & m = n, \end{cases} \tag{6}$$

and they may be constructed by means of the following recursive formula:

$$V_n(x) = 2x\, V_{n-1}(x) - V_{n-2}(x), V_0(x) = 1, V_1(x) = 2x - 1, n = 2, 3, \cdots. \tag{7}$$

The shifted Chebyshev polynomials of the third kind on $[0, 1]$ are defined as

$$V_n^*(x) = V_n(2x - 1). \tag{8}$$

All properties of third-kind Chebyshev polynomials may be readily converted to yield the properties of the analog of their shifted polynomials.

The orthogonality relation of $V_k^*(x)$ on $[0, 1]$ with respect to the weight function $\sqrt{x/(1-x)}$ is given by

$$\int_0^1 \sqrt{\frac{x}{1-x}} V_k^*(x) V_j^*(x) dx = \begin{cases} \dfrac{\pi}{2}, & k = j, \\ 0, & k \ne j. \end{cases} \tag{9}$$

The power form representation of the third-kind Chebyshev polynomials and its inversion formula can be represented respectively as (see [46])

$$V_k^*(x) = \sum_{i=0}^k \frac{2^{2i}(-1)^{k+i}(k+i)!}{(2i+1)!(k-i)!} x^i, \tag{10}$$

and

$$x^r = \frac{(2r+1)!}{2^{2r}} \sum_{\ell=0}^r \frac{1}{(r-\ell)!(\ell+r+1)!} V_\ell^*(x). \tag{11}$$

2.3. An Account on Generalized Hypergeometric Function. We display in this section the definition of the generalized hypergeometric functions and the regularized hypergeometric function which will be essential in the upcoming section.

We recall here the definition of the generalized hypergeometric function given by (see, for example, [46])

$${}_pF_q\left(\begin{matrix} A_1, A_2, \cdots, A_p \\ B_1, B_2, \cdots, B_q \end{matrix} \;\middle|\; x\right) = \sum_{k=0}^\infty \frac{(A_1)_k (A_2)_k \cdots (A_p)_k}{(B_1)_k (B_2)_k \cdots (B_q)_k} \frac{x^k}{k!}, \tag{12}$$

and the regularized hypergeometric function are defined as

$${}_p\tilde{F}_q\left(\begin{matrix} A_1, A_2, \cdots, A_p \\ B_1, B_2, \cdots, B_q \end{matrix} \;\middle|\; x\right) = \sum_{k=0}^\infty \frac{(A_1)_k (A_2)_k \cdots (A_p)_k}{\Gamma(B_1+k)\Gamma(B_2+k) \cdots \Gamma(B_q+k)} \frac{x^k}{k!}, \tag{13}$$

where p and q are nonnegative integers. In addition, the constants B_j, $1 \le j \le q$ are all neither zeros nor negative integers.

Note 1. In (12), if one of A_i is a negative integer $(-n)$, the generalized hypergeometric function reduces to a polynomial of degree n.

3. Derivation of the Fractional Derivatives of Chebyshev Polynomials of Third-Kind

This section is confined to deriving in detail the formula that expresses the fractional derivatives of the third-kind Chebyshev polynomials. In addition, the well-known integer derivative formula will be deduced as a special case. For our derivation, the following two lemmas are needed.

Lemma 3. *For every nonnegative integers k, ℓ, r, the following reduction formula is valid*

$$
{}_3F_2\left(\begin{matrix} -\ell, 2k-\ell+1, k-\ell-r+\dfrac{3}{2} \\ k-\ell+\dfrac{3}{2}, 2k-2\ell-2r+2 \end{matrix}\,\middle|\,1\right) = \frac{2}{\sqrt{\pi}\,(2k+1)} \times \begin{cases} \dfrac{\Gamma((\ell+1)/2)(r)_{\ell/2}}{(k-\ell+3/2)_{\ell/2-1}(k-\ell-r+1)_{\ell/2}}, & \ell \text{ even}, \\ \dfrac{-\Gamma(\ell/2+1)(r)_{(\ell+1)/2}}{(k-\ell+3/2)_{(\ell-1)/2}(k-\ell-r+1)_{(\ell+1)/2}}, & \ell \text{ odd}. \end{cases} \tag{14}
$$

Proof. First, set

$$
Y_{\ell,k,r} = {}_3F_2\left(\begin{matrix} -\ell, 2k-\ell+1, k-\ell-r+\dfrac{3}{2} \\ k-\ell+\dfrac{3}{2}, 2k-2\ell-2r+2 \end{matrix}\,\middle|\,1\right). \tag{15}
$$

The following recurrence relation of order two can be generated using Zeilberger's approach (see [47]).

$$
\begin{aligned}
&(\ell+1)(2k-\ell)(\ell+2r)(-k+\ell+r+1)(-2k+\ell+2r-1)Y_{\ell,k,r} \\
&\quad -2(2k-2\ell+1)(k-\ell-r)\left(2k(\ell+r+1)-2\ell r-\ell(\ell+1)-2r^2\right)Y_{\ell+1,k,r} \\
&\quad -4(2k-2\ell-1)(2k-2\ell+1)(k-\ell-r-1)(k-\ell-r)^2 Y_{\ell+2,k,r} = 0,
\end{aligned} \tag{16}
$$

with the initial values

$$
Y_{0,k,r} = 1, \qquad Y_{1,k,r} = \frac{-r}{(2k+1)(k-r)}. \tag{17}
$$

The recurrence relation (16) can be exactly solved to give

$$
Y_{\ell,k,r} = \frac{2}{\sqrt{\pi}(2k+1)} \begin{cases} \dfrac{\Gamma((\ell+1)/2)(r)_{\ell/2}}{(k-\ell+3/2)_{\ell/2-1}(k-\ell-r+1)_{\ell/2}}, & \ell \text{ even}, \\ \dfrac{-\Gamma(\ell/2+1)(r)_{(\ell+1)/2}}{(k-\ell+3/2)_{(\ell-1)/2}(k-\ell-r+1)_{(\ell+1)/2}}, & \ell \text{ odd}. \end{cases} \tag{18}
$$

This completes the proof of Lemma 3. □

Lemma 4. *Let k, r be any two nonnegative integers with $k \geq r$. The following transformation formula is valid:*

$$
{}_4\tilde{F}_3\left(\begin{matrix} -(k-r), 1, k+r+1, \dfrac{3}{2} \\ r+\dfrac{3}{2}, 1-p, p+2 \end{matrix}\,\middle|\,1\right) = \frac{(3/2)_p(r-k)_p(k+r+1)_p}{(2p+1)!\Gamma(p+r+3/2)} \times {}_3F_2\left(\begin{matrix} -(k-p-r), p+\dfrac{3}{2}, k+p+r+1 \\ 2p+2, p+r+\dfrac{3}{2} \end{matrix}\,\middle|\,1\right). \tag{19}
$$

Proof. In the left-hand side of (19), the terminating hypergeometric series can be expressed as

$$
{}_4\tilde{F}_3\left(\begin{matrix} r-k, 1, k+r+1, \dfrac{3}{2} \\ r+\dfrac{3}{2}, 1-p, p+2 \end{matrix}\,\middle|\,1\right) = \sum_{s=p}^{k-r} \frac{(3/2)_s(r-k)_s(k+r+1)_s}{(s-p)!(s+p+1)!\Gamma(s+r+3/2)}, \tag{20}
$$

which can also be written as

$$
{}_4\tilde{F}_3\left(\begin{matrix} r-k, 1, k+r+1, \dfrac{3}{2} \\ r+\dfrac{3}{2}, 1-p, p+2 \end{matrix}\,\middle|\,1\right) = \sum_{\ell=0}^{k-p-r} \frac{(3/2)_{p+\ell}(r-k)_{p+\ell}(k+r+1)_{p+\ell}}{\ell!(2p+\ell+1)!\Gamma(p+r+\ell+3/2)}. \tag{21}
$$

In virtue of the identity:

$$
(A)_{p+\ell} = (A)_p(A+p)_\ell, \tag{22}
$$

relation (21) can be written alternatively as

$$
\begin{aligned}
&{}_4\tilde{F}_3\left(\begin{matrix} r-k, 1, k+r+1, \dfrac{3}{2} \\ r+\dfrac{3}{2}, 1-p, p+2 \end{matrix}\,\middle|\,1\right) \\
&= \frac{(3/2)_p\,(r-k)_p\,(k+r+1)_p}{(2p+1)!\Gamma(p+r+3/2)} \sum_{\ell=0}^{k-p-r} \frac{(p+3/2)_\ell\,(-k+p+r)_\ell\,(k+p+r+1)_\ell}{(2p+2)_\ell(p+r+3/2)_\ell\,\ell!},
\end{aligned} \tag{23}
$$

which implies the validity of transformation (19).

The key theorem in this section is now stated and proved. □

Theorem 5. *The following formula can be used to approximate the fractional derivatives of the polynomials $V_k^*(x)$ as:*

$$D^{\gamma}V_k^*(x) \simeq \frac{(-1)^{k+n}(2k+1)(k+n)!\Gamma(n-\gamma+3/2)}{\Gamma(n+3/2)(k-n)!} \times \sum_{p=0}^{N}\frac{1}{\Gamma(n-p-\gamma+1)\Gamma(n+p-\gamma+2)} \times {}_4F_3\left(\begin{matrix}1, n-k, k+n+1, n-\gamma+\frac{3}{2}\\ n+\frac{3}{2}, n-\gamma-p+1, n-\gamma+p+2\end{matrix}\,\Big|\,1\right)V_p^*(x), \quad (24)$$

where $n=\lceil\gamma\rceil$ is the well-known ceiling notation and N is a sufficiently large positive integer.

Proof. The power form representation of $V_k^*(x)$ in (10), along with relation (2.1) yields

$$D^{\gamma}V_k^*(x)=\sqrt{\pi}\left(k+\frac{1}{2}\right)\sum_{s=n}^{k}\frac{(-1)^{k+s}(k+s)!}{\Gamma(s+3/2)(k-s)!\Gamma(s-\gamma+1)}x^{s-\gamma}. \quad (25)$$

The inversion formula of $V_p^*(x)$ in (11) can be used to approximate $D^{\gamma}V_k^*(x)$ as

$$x^{s-\gamma}\simeq\frac{\Gamma(2s-2\gamma+2)}{2^{2s-2\gamma}}\sum_{\ell=0}^{N}\frac{1}{\Gamma(s-\gamma-\ell+1)\Gamma(s-\gamma+\ell+2)}V_{\ell}^*(x), \quad (26)$$

where N is a sufficiently large positive integer.

The following approximation for $D^{\gamma}V_k^*(x)$ is obtained by inserting (26) into (25)

$$D^{\gamma}V_k^*(x)\simeq(2k+1)\sum_{s=n}^{k}\frac{(-1)^{k+s}(k+s)!\Gamma(s-\gamma+3/2)}{\Gamma(s+3/2)(k-s)!} \times\sum_{\ell=0}^{N}\frac{1}{\Gamma(s-\gamma-\ell+1)\Gamma(s-\gamma+\ell+2)}V_{\ell}^*(x). \quad (27)$$

If the right-hand side of relation (27) is expanded and rearranged, then we get

$$D^{\gamma}V_k^*(x)\simeq(2k+1)\sum_{p=0}^{N}\sum_{\ell=0}^{k-n}\frac{(-1)^{k+n+\ell}(k+n+\ell)!}{\Gamma(\ell+n+3/2)(k-n-\ell)!} \times\frac{\Gamma(\ell+n-\gamma+3/2)}{\Gamma(\ell+n-p-\gamma+1)\Gamma(\ell+n+p-\gamma+2)}V_p^*(x). \quad (28)$$

In hypergeometric form, the last relation can be written as follows:

$$D^{\gamma}V_k^*(x) \simeq \frac{(-1)^{k+n}(2k+1)(k+n)!\Gamma(n-\gamma+3/2)}{\Gamma(n+3/2)(k-n)!} \times \sum_{p=0}^{N}\frac{1}{\Gamma(n-p-\gamma+1)\Gamma(n+p-\gamma+2)} \times {}_4F_3\left(\begin{matrix}1, n-k, k+n+1, n-\gamma+\frac{3}{2}\\ n+\frac{3}{2}, n-\gamma-p+1, n-\gamma+p+2\end{matrix}\,\Big|\,1\right)V_p^*(x). \quad (29)$$

This ends the proof of Theorem 5. □

Remark 6. The integer derivatives formula of the polynomials $V_k^*(x)$ may be extracted from Theorem 5 as a special case. The following corollary exhibits this formula.

Corollary 7. *Let r be a positive integer. Then, for all $k\geq r$, one has:*

$$D^{r}V_k^*(x)=\frac{2^{2r}}{(r-1)!}\sum_{\ell=0}^{\lfloor(k-r)/2\rfloor}\frac{(k-\ell)!(\ell+r-1)!}{\ell!(k-\ell-r)!}V_{k-2\ell-r}^*(x) + \frac{2^{2r}}{(r-1)!}\sum_{\ell=0}^{\lfloor(k-r-1)/2\rfloor}\frac{(k-\ell-1)!(\ell+r)!}{\ell!(k-\ell-r)!}V_{k-2\ell-r-1}^*(x). \quad (30)$$

Proof. Setting $\gamma=r$, r is a positive integer. In this case $\gamma=n=r$, and therefore, formula (24) can be converted into

$$D^{r}V_k^*(x)=\frac{(-1)^{k+r}\sqrt{\pi}(k+1/2)(k+r)!}{(k-r)!} \times\sum_{p=0}^{k-r}{}_4\tilde{F}_3\left(\begin{matrix}r-k, 1, k+r+1, \frac{3}{2}\\ r+\frac{3}{2}, 1-p, p+2\end{matrix}\,\Big|\,1\right)V_p^*(x). \quad (31)$$

With the aid of Lemma 4, the last formula reduces to the following one:

$$D^{r}V_k^*(x)=\frac{(-1)^{k+r}\sqrt{\pi}(k+1/2)(k+r)!}{(k-r)!} \times\sum_{p=0}^{k-r}\frac{(3/2)_p(r-k)_p(k+r+1)_p}{(2p+1)!\Gamma(p+r+3/2)}\,{}_3F_2 \cdot\left(\begin{matrix}-(k-p-r), p+\frac{3}{2}, k+p+r+1\\ 2p+2, p+r+\frac{3}{2}\end{matrix}\,\Big|\,1\right)V_p^*(x). \quad (32)$$

It is clear that the last relation can be written in the following alternative relation:

$$D^r V_k^*(x) = \frac{(-1)^{k+r}\sqrt{\pi}\,(k+1/2)\,(k+r)!}{(k-r)!} \cdot \sum_{\ell=0}^{k-r} \frac{(3/2)_{k-\ell-r}\,(r-k)_{k-\ell-r}\,(k+r+1)_{k-\ell-r}}{\Gamma(k-\ell+3/2)\,(2k-2\ell-2r+1)!} \times {}_3F_2\left(\begin{matrix} -\ell, 2k-\ell+1, k-\ell-r+\frac{3}{2} \\ k-\ell+\frac{3}{2}, 2k-2\ell-2r+2 \end{matrix}\middle|1\right) V_{k-r-\ell}^*(x). \tag{33}$$

In virtue of Lemma 3, the ${}_3F_2(1)$ that appears in (32) can be summed in a closed form, and then, after performing some manipulation, the following formula can be obtained

$$D^r V_k^*(x) = \frac{2^{2r}}{(r-1)!} \sum_{\ell=0}^{\lfloor (k-r)/2 \rfloor} \frac{(k-\ell)!(\ell+r-1)!}{\ell!(k-\ell-r)!} V_{k-2\ell-r}^*(x) + \frac{2^{2r}}{(r-1)!} \sum_{\ell=0}^{\lfloor (k-r-1)/2 \rfloor} \frac{(k-\ell-1)!(\ell+r)!}{\ell!(k-\ell-r)!} V_{k-2\ell-r-1}^*(x). \tag{34}$$

This proves formula (30). □

Remark 8. The result in (30) matches that obtained in [28].

4. Tau Stratagem for Handling a Type of Pantograph Differential Equations with Quadratic Nonlinearity

In this section, we are interested in employing a new expression of the fractional derivatives of the third-kind Chebyshev polynomials along with the application of the spectral tau method to treat the following fractional pantograph differential equation with quadratic nonlinearity ([38, 45, 48]).

$$D^\gamma \nu(t) + \xi_1 \nu'(t) + \xi_2 \nu(t) + \xi_3 \nu\left(\frac{t}{\tau}\right) + \xi_4 \nu^2(t) = g(t), t \in (0,1), \tag{35}$$

governed by the boundary conditions:

$$\nu(0) = \varrho_0, \nu(1) = \varrho_1, \tag{36}$$

or the initial conditions

$$\nu(0) = \tilde{\varrho}_0, \nu'(0) = \tilde{\varrho}_1, \tag{37}$$

where $1 < \gamma \le 2$, $\tau > 1$,$\xi_1, \xi_2, \xi_3, \xi_4, \varrho_0, \varrho_1, \tilde{\varrho}_0$, and $\tilde{\varrho}_1$ are real constants, and $g(t)$ is a known continuous source function.

Before moving further with the implementation of our proposed method, the following two lemmas are needed. The first lemma presents the duplication formula of the shifted Chebyshev polynomials of the third kind, whereas the second lemma exhibits the linearization formula of the shifted third-kind Chebyshev polynomials.

Lemma 9. *Let i be any positive integer, and A is a nonzero real number. We have the following formula:*

$$V_i^*(A\,x) = A^i (2i+1)! \sum_{p=0}^{i} \frac{1}{p!(2i-p+1)!} \, {}_2F_1 \cdot \left(\begin{matrix} -p, -2i+p-1 \\ -2i \end{matrix}\middle|\frac{1}{A}\right) V_{i-p}^*(x). \tag{38}$$

Proof. Formula (37) can be easily obtained by making use of (10) along with (11). □

Lemma 10. *For all nonnegative integers i and j, the following linearization formula is valid ([49])*

$$V_i^*(x)\,V_j^*(x) = \sum_{p=0}^{2\min(i,j)} (-1)^p\,V_{i+j-p}^*(x). \tag{39}$$

The key idea behind solving (35)–(36) is to use the spectral tau approach. We represent the inner product in $L^2(0,1)$, namely, $(\cdot, \cdot)$

$$(\phi(t), \psi(t))_w = \int_0^1 w(t)\,\phi(t)\,\psi(t)\,dt. \tag{40}$$

If we suppose that the right-hand side of (35) may be written as

$$g(t) = \sum_{i=0}^{\infty} g_i\,V_i^*(t), \tag{41}$$

the following approximation of $g(t)$ can therefore be considered:

$$g(t) = \sum_{i=0}^{N} g_i\,V_i^*(t)\,;\, g_i = \frac{2}{\pi}(g, V_i^*(t))_w. \tag{42}$$

We also take into account the approximate solution of (35) as:

$$\nu(t) \approx \nu_n(t) = \sum_{i=0}^{n} u_i\,V_i^*(t) = U.\Phi, \tag{43}$$

where

$$U = (u_0, u_1, \cdots, u_n), \Phi = \begin{pmatrix} V_0^*(t) \\ V_1^*(t) \\ \cdot \\ \cdot \\ \cdot \\ V_n^*(t) \end{pmatrix}. \tag{44}$$

Now, we are going to employ the tau method for solving (35). First, the residual of (35) is given by

$$R_n(t) = D^{\gamma} v_n(t) + \xi_1\, D v_n(t) + \xi_2\, v_n(t) + \xi_3\, v_n\left(\frac{t}{\tau}\right) + \xi_4\, v_n^2(t) - g(t). \tag{45}$$

We approximate each term on the right-hand side of (45) in terms of the shifted third-kind Chebyshev polynomials. First, to approximate the term $D^{\gamma} V_i^*(t)$, we make use of Theorem 5, to get

$$D^{\gamma} V_i^*(t) \approx \sum_{p=0}^{N} d_{p,i,\gamma}\, V_p^*(t), \tag{46}$$

with

$$d_{p,i,\gamma} = \frac{(2i+1)\Gamma(-\gamma+\lceil\gamma\rceil+3/2)(-1)^{\lceil\gamma\rceil+i}(\lceil\gamma\rceil+i)!}{\Gamma(\lceil\gamma\rceil+3/2)(i-\lceil\gamma\rceil)!\Gamma(-p-\gamma+\lceil\gamma\rceil+1)\Gamma(p-\gamma+\lceil\gamma\rceil+2)} \times {}_4F_3\left(\begin{matrix} 1, \lceil\gamma\rceil - i, \lceil\gamma\rceil + i + 1, -\gamma + \lceil\gamma\rceil + \dfrac{3}{2} \\ \lceil\gamma\rceil + \dfrac{3}{2}, -\gamma + \lceil\gamma\rceil - p + 1, -\gamma + \lceil\gamma\rceil + p + 2 \end{matrix} \,\middle|\, 1\right), \tag{47}$$

and accordingly, we have

$$D^{\gamma} v_n(t) = \sum_{i=0}^{n}\sum_{p=0}^{N} u_i\, d_{p,i,\gamma}\, V_p^*(t). \tag{48}$$

Also, Lemma 9 enables one to approximate $V_i^*(t/\tau)$ for any positive integer $\tau > 1$ as

$$V_i^*\left(\frac{t}{\tau}\right) = \sum_{p=0}^{i} \Delta_{i,p}\, V_{i-p}^*, \tag{49}$$

with

$$\Delta_{i,p} = \frac{(2i+1)!}{\tau^i\, p!(2i-p+1)!}\, {}_2F_1\left(\begin{matrix} -p, -2i+p-1 \\ -2i \end{matrix} \,\middle|\, \tau\right). \tag{50}$$

In addition, the linearization formula of the third-kind Chebyshev polynomials leads to the following expression for $v_n^2(t)$:

$$v_n^2(t) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{p=0}^{2\min(i,j)} (-1)^p\, u_i\, u_j V_{i+j-p}^*(t). \tag{51}$$

Now, in virtue of the three formulas (48), (49), and (51), the residual of (35) can be discretized as

$$\begin{aligned} R_n(t) &= \sum_{i=0}^{n}\sum_{p=0}^{N} u_i\, d_{p,i,\gamma}\, V_p^*(t) + \xi_1 \sum_{i=0}^{n}\sum_{p=0}^{i-1} u_i\, d_{p,i,1}\, V_p^*(t) \\ &+ \xi_2 \sum_{i=0}^{n} u_i\, \phi_i + \xi_3 \sum_{i=0}^{n}\sum_{p=0}^{i} u_i\, \Delta_{i,p}, V_{i-p}^*(t) \\ &+ \xi_4 \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{p=0}^{2\min(i,j)} (-1)^p\, u_i\, u_j\, V_{i+j-p}^*(t) - \sum_{i=0}^{n} g_i\, V_i^*(t). \end{aligned} \tag{52}$$

The standard tau technique ([26]) is used in this case to yield

$$\left(R_n(t), V_j^*(t)\right)_w = 0\,;\, 0 \le j \le n-2, \tag{53}$$

which consequently out-turn

$$\begin{aligned} &\sum_{i=0}^{n}\sum_{p=0}^{N} u_i\, d_{p,i,\gamma}\left(V_p^*, V_j^*\right)_w + \xi_1 \sum_{i=0}^{n}\sum_{p=0}^{i-1} u_i\, d_{p,i,1}\left(V_p^*, V_j^*\right)_w \\ &+ \xi_2 \sum_{i=0}^{n} u_i \left(V_i^*, V_j^*\right)_w + \xi_3 \sum_{i=0}^{n}\sum_{p=0}^{i} u_i\, \Delta_{i,p}\left(V_{i-p}^*, V_j^*\right)_w \\ &+ \xi_4 \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{p=0}^{2\min(i,j)} (-1)^p\, u_i\, u_j\left(V_{i+j-p}^*, V_j^*\right)_w - \sum_{i=0}^{n} g_i\left(V_i^*, V_j^*\right)_w = 0. \end{aligned} \tag{54}$$

The benefit of the orthogonality relation leads to the following equations

$$\begin{aligned} &\sum_{i=0}^{n}\sum_{p=0}^{N} u_i\, d_{p,i,\gamma}\, \delta_{p,j} + \xi_1 \sum_{i=0}^{n}\sum_{p=0}^{i-1} u_i\, d_{p,i,1}\, \delta_{p,j} + \xi_2 \sum_{i=0}^{n} u_i\, \delta_{i,j} + \xi_3 \sum_{i=0}^{n}\sum_{p=0}^{i} u_i\, \Delta_{i,p}\, \delta_{i-p,j} \\ &+ \xi_4 \sum_{i=0}^{n}\sum_{k=0}^{n}\sum_{p=0}^{2\min(i,k)} (-1)^p\, u_i\, u_k \delta_{i+k-p,j} - \sum_{i=0}^{n} g_i\, \delta_{i,j} = 0, \end{aligned} \tag{55}$$

or equivalently

$$\begin{aligned} &\sum_{i=0}^{n}\sum_{p=0}^{N} u_i\, d_{p,i,\gamma}\, \delta_{p,j} + \xi_1 \sum_{i=0}^{n}\sum_{p=0}^{i-1} u_i\, d_{p,i,1}\, \delta_{p,j} + \xi_2\, u_j + \xi_3 \sum_{i=0}^{n}\sum_{p=0}^{i} u_i\, \Delta_{i,p}\, \delta_{i-p,j} \\ &+ \xi_4 \sum_{i=0}^{n}\sum_{k=0}^{n}\sum_{p=0}^{2\min(i,k)} (-1)^p\, u_i\, u_k \delta_{i+k-p,j} - g_j \\ &= 0\,;\, j \in \{0, 1, \cdots, n-2\}. \end{aligned} \tag{56}$$

The boundary conditions return

$$\sum_{i=0}^{n} (-1)^i (2i+1) u_i = \varrho_0, \quad \sum_{i=0}^{n} u_i = \varrho_1, \tag{57}$$

while the initial conditions return

$$\sum_{i=0}^{n}(-1)^i(2i+1)u_i=\tilde{\varrho}_0,\ \sum_{i=1}^{N}(-1)^{i-1}2(-1)^{i-1}(i)_2(2i+1)\,u_i=3\,\tilde{\varrho}_1. \tag{58}$$

Equations (56)–(57) or (56)–(58) generate a set of algebraic equations with dimension $(n+1)$ in the unknown expansion coefficients u_j that may be solved via Newton's iterative scheme.

5. Error Estimate

This section examines the proposed polynomial series expansion's convergence and error analysis in depth. As a result, several necessary lemmas are employed in this research. Three theorems will also be stated and proved.

In what follows, by writing $A_n \lesssim B_n$, this implies the existence of a generic constant C, such that $A_n \leq C\,B_n$.

Lemma 11. *Lat $\gamma \in [1,2)$. One has:*

$$|d_{i,p,\gamma}| \lesssim 4^{6(1+\gamma-i)}\,i^{4(1+\gamma)}. \tag{59}$$

Proof. In virtue of: $\Gamma(r+\beta)\approx r!r^{\beta-1}$ (see, [50]), and after some algebraic computations, we get the result. □

Lemma 12. *We have:*

$$|\Delta_{i,p}| \lesssim \frac{(2i+1)!}{\tau^i\,(2i+1-p)!p!}. \tag{60}$$

Proof. Similar to the proof of Lemma 11. □

Lemma 13. *For all $i>0$, we have:*

$$|V_i^*(t)| \leq 2i+1. \tag{61}$$

Theorem 14. *For $k>3$, assume that $v(t)$ is C^k – function, and let $v(t)$ can be approximated as:*

$$v(t)\approx v_n(t)=\sum_{i=0}^{n}u_i\,V_i^*(t), \tag{62}$$

then, the following estimate can be obtained

$$|u_i| \lesssim i^{-k}. \tag{63}$$

Proof. We can get the required result by employing steps similar to those used in [25].

If $v(t)$ obeys the assumptions of Theorem 14, we have the following two theorems. □

Theorem 15. *The following truncation error estimate is valid*

$$|v-v_n| \lesssim n^{2-k}. \tag{64}$$

Proof. By the result of Theorem 14and the help of Lemma 13, we get the result. □

Theorem 16. *If we define*

$$\mathscr{E}_n=\left|D^\gamma v_n(t)+\xi_1\,v'_n(t)+\xi_2\,v_n(t)+\xi_3\,v_n\left(\frac{t}{\tau}\right)+\xi_4\,v_n^2(t)-g(t)\right|, \tag{65}$$

then, for $1<\tau<4$, we have the following global error estimate:

$$|\mathscr{E}_n| \lesssim \max\left\{n^{3-k},\ \frac{m^2\,n^{4+4\gamma-k}}{2^{12n}},\ \frac{n^{1/2-k}\,\tau^n}{4^n}\right\}. \tag{66}$$

Proof. We have

$$\begin{aligned}&D^\gamma v_n(t)+\xi_1\,v'_n(t)+\xi_2\,v_n(t)+\xi_3\,v_n\left(\frac{t}{\tau}\right)+\xi_4\,v_n^2(t)\approx g_n(t),\\ &D^\gamma v(t)+\xi_1\,v'(t)+\xi_2\,v(t)+\xi_3\,v\left(\frac{t}{\tau}\right)+\xi_4\,v^2(t)=g(t).\end{aligned} \tag{67}$$

Substitution by (24) into (23), we get

$$\begin{aligned}|\mathscr{E}_n| \leq{}& |D^\gamma(v-v_n)|+\left|\xi_1\,(v-v_n)'\right|+|\xi_2\,(v-v_n)|\\ &+\left|\xi_3\,(v-v_n)\left(\frac{t}{\tau}\right)\right|+\left|\xi_4\,\left(v^2-v_n^2\right)\right|.\end{aligned} \tag{68}$$

By the boundedness of v and with the help of Theorem 15, we have

$$|\mathscr{E}_n| \lesssim |D^\gamma(v-v_n)|+\left|\xi_1\,(v-v_n)'\right|+\left|\xi_3\,(v-v_n)\left(\frac{t}{\tau}\right)\right|+n^{2-k}. \tag{69}$$

Now, we have:

$$\begin{aligned}D(v-v_n)&=\sum_{i=n+1}^{\infty}u_i\,DV_i^*,\\ D^\gamma(v-v_n)&=\sum_{i=n+1}^{\infty}\sum_{p=0}^{N}u_i\,d_{p,i,\gamma}\,V_p^*,\\ (v-v_n)\left(\frac{t}{\tau}\right)&=\sum_{i=n+1}^{\infty}\sum_{p=0}^{i}u_i\,\Delta_{i,p}\,V_{i-p}^*.\end{aligned} \tag{70}$$

By the application of Theorem 14, Lemmas 11, 12, and 13, respectively, and after some algebraic manipulation, we get

$$\begin{aligned}(v-v_n)' &\lesssim n^{3-k},\\ D^\gamma(v-v_n) &\lesssim \frac{m^2\,n^{4+4\gamma-k}}{2^{12n}},\\ (v-v_n)\left(\frac{t}{\tau}\right) &\lesssim \frac{n^{1/2-k}\,\tau^n}{4^n},\end{aligned} \tag{71}$$

which ends the proof of the theorem. □

6. Numerical Simulations

This section is confined to presenting four test problems to clarify the accuracy and applicability of the Chebyshev third-kind tau method (C3TM) that derived in Section 4.

Problem 1. Consider the linear fractional pantograph differential equation ([38]):

$$D^{\gamma}v(t)-\frac{3}{4}v(t)-v\left(\frac{1}{2}t\right)=2-t^2,\ t\in(0,1)\,;\ 1<\gamma\le 2,$$
$$v(0)=v'(0)=0. \tag{72}$$

In case $\gamma=2$, the exact solution is: $v(t)=t^2$.

First, we discuss the case corresponding to $\gamma=2$. In this case, after applying our algorithm, with $n=2$, the following system of equations can be obtained:

$$\begin{aligned}256\,u_2+12\,u_1-14\,u_0&=11,\\ 5\,u_2-3\,u_1+u_0&=0,\\ 5\,u_2-u_1&=0,\end{aligned} \tag{73}$$

which yields

$$u_0=\frac{5}{8},\ u_1=\frac{5}{16},\ u_2=\frac{1}{16}, \tag{74}$$

and consequently,

$$v(t)=\frac{5}{8}(1)+\frac{5}{16}(-3+4t)+\frac{1}{16}\left(5-20t+16t^2\right)=t^2, \tag{75}$$

which is the exact solution.

Second, when $1<\gamma<2$. Since the exact solution is not available, we define the following error norm

$$\mathscr{E}_n=\max_{0\le t\le 1}\left|D^{\gamma}v_n(t)-\frac{3}{4}v_n(t)-v_n\left(\frac{1}{2}t\right)-2+t^2\right|. \tag{76}$$

We apply our algorithm with $n=2, N=3$. The values of $\mathscr{E}$, for various values of γ, are listed in Table 1.

Note 2. We would like to report here that the authors in [38] obtained an error of order 10^{-17}, when $\gamma=2$, while, we obtained the exact solution.

Problem 2. Consider the following fractional pantograph differential equation: ([38]):

$$D^{\gamma}v(t)+\frac{5}{6}v(t)-4\,v\left(\frac{1}{2}t\right)-9\,v\left(\frac{1}{3}t\right)=t^2-1,\ t\in(0,1)\,;\ 0<\gamma\le 1,$$
$$v(0)=1. \tag{77}$$

In case $\gamma=1$, the exact solution is: $v(t)=12157/1296\,t^3+1675/72\,t^2+67/6\,t+1$.

First, we discuss the case corresponding to $\gamma=1$. In this case, after applying our algorithm, with $n=3$, we get the following system of equations:

$$\begin{aligned}-292\,u_0+672\,u_1-144\,u_2-4\,u_3+9&=0,\\ -200\,u_1+1104\,u_2-648\,u_3-15&=0,\\ -56\,u_2+968\,u_3-3&=0,\\ u_0-3\,u_1+5\,u_2-7\,u_3&=1,\end{aligned} \tag{78}$$

which yields

$$u_0=\frac{2409095}{82944},\ u_1=\frac{363283}{27648},\ u_2=\frac{205699}{82944},\ u_3=\frac{12157}{82944}, \tag{79}$$

and consequently

$$v_3(t)=\frac{2409095}{82944}+\frac{363283}{27648}(4t-3)+\frac{205699}{82944}\left(16t^2-20t+5\right)+\frac{12157}{82944}\left(64t^3-112t^2+56t-7\right), \tag{80}$$

and therefore, we get

$$v_3(t)=\frac{12157t^3}{1296}+\frac{1675t^2}{72}+\frac{67t}{6}+1, \tag{81}$$

which is the exact solution.

Second, when $0<\gamma<1$. Since, the exact solution is not available, we define the following error norm

$$\mathscr{E}_n=\max_{0\le t\le 1}\left|D^{\gamma}v_n(t)+\frac{5}{6}v_n(t)-4\,v_n\left(\frac{1}{2}t\right)-9\,v_n\left(\frac{1}{3}t\right)-t^2+1\right|. \tag{82}$$

We apply our algorithm with $n=3, N=4$. The values of $\mathscr{E}$, for various values of γ are listed in Table 2.

Problem 3. Consider the fractional-delay BVP [25]:

$$v^{(\gamma)}(t)+v'(t)+v\left(\frac{t}{\tau}\right)+v(t)=r(t)\,;\ t\in(0,1), \tag{83}$$

governed by

$$v(0)=1,\ v(1)=\frac{1}{e}, \tag{84}$$

Table 1: Residual error of example 1.

γ	3/2	1.6	1.7	1.8	1.9
$\mathscr{E}$	2.22×10^{-16}	2.22×10^{-16}	2.22×10^{-16}	2.22×10^{-17}	2.22×10^{-18}

Table 2: Residual error of example 2.

γ	0.5	0.6	0.7	0.8	0.9
$\mathscr{E}$	3.56×10^{-13}	4.24×10^{-14}	3.58×10^{-15}	3.94×10^{-15}	2.22×10^{-16}

Table 3: Maximum point-wise error of example 3.

γ	τ	N	4	6	8	10	12	14
	4/3		3.42×10^{-5}	4.76×10^{-7}	5.38×10^{-9}	2.27×10^{-11}	4.68×10^{-14}	2.22×10^{-16}
5/4	2	E	4.51×10^{-5}	5.27×10^{-7}	6.39×10^{-9}	7.95×10^{-11}	6.23×10^{-14}	2.22×10^{-16}
	4		5.37×10^{-5}	8.26×10^{-7}	2.39×10^{-9}	5.95×10^{-11}	7.21×10^{-14}	2.22×10^{-16}
	4/3		$3/27 \times 10^{-5}$	2.25×10^{-7}	5.92×10^{-9}	5.61×10^{-11}	6.34×10^{-14}	2.22×10^{-16}
3/2	2	E	5.13×10^{-5}	7.25×10^{-7}	6.38×10^{-9}	9.34×10^{-11}	2.15×10^{-14}	2.22×10^{-16}
	4		2.87×10^{-5}	2.68×10^{-7}	2.36×10^{-9}	5.27×10^{-11}	2.27×10^{-14}	2.22×10^{-16}
	4/3		4.37×10^{-5}	7.85×10^{-7}	2.96×10^{-9}	3.65×10^{-11}	2.65×10^{-14}	2.22×10^{-16}
7/4	2	E	2.33×10^{-5}	6.64×10^{-7}	2.37×10^{-9}	5.92×10^{-11}	2.84×10^{-14}	2.22×10^{-16}
	4		2.68×10^{-5}	5.61×10^{-7}	3.98×10^{-9}	9.34×10^{-11}	2.38×10^{-14}	2.22×10^{-16}

and $r(t)$ is selected so that $v(t) = \exp(-t)$ is the exact solution. The $C3TM$ is applied for various choices of τ and n. Table 3 lists the maximum point-wise error E that is computed by the following formula:

$$E = \max_{t\in[0,1]} \left| \text{Exact}(t) - \text{Approximate}(t) \right|, \tag{85}$$

for $\gamma = 5/4, 3/2, 7/4$ and $\tau = 4/3, 2, 4$. In Figure 1, the log errors are presented in case of $\tau = 2$.

Problem 4. Consider the fractional-delay initial value problem [48]:

$$v^{(\gamma)}(t) + \eta\, v(t) + \mu\, v\left(\frac{t}{\tau}\right) = (\eta - 1)\sin t + \mu \sin\left(\frac{t}{\tau}\right); t \in [0,1], 1 < \gamma \le 2, \tag{86}$$

governed by the initial conditions:

$$v(0) = v'(0) - 1 = 0, \tag{87}$$

and the exact solution is: $v(t) = \sin t$ for $\gamma = 2$, and η and μ are any real constants. Table 4 presents the errors if $C3TM$ is applied for $n = 15$. We list the maximum absolute residual error defined by:

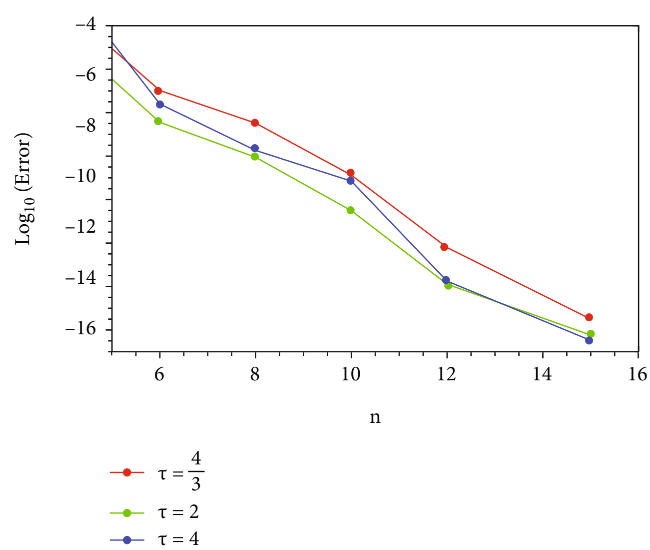

Figure 1: Log errors of example 3.

$$\mathscr{E}_n = \max_{0\le t\le 1} \left| v_n^{(\gamma)}(t) + \eta\, v_n(t) + \mu\, v_n\left(\frac{t}{\tau}\right) - (\eta - 1)\sin t - \mu \sin\left(\frac{t}{\tau}\right) \right|, \tag{88}$$

for the case that corresponds to $\eta = 1, \mu = 1/2, \gamma = 2, 7/4, 3/2, 5/4,$ and $\tau = 2,$ and $\tau = 4$. Figure 2 illustrates the absolute errors for the case $\gamma = 2, n = 15$.

Table 4: The maximum absolute residual error $\mathscr{E}_n$ for various values of γ, for example 4.

n	$\gamma = 2$		$\gamma = 7/4$		$\gamma = 3/2$		$\gamma = 5/4$	
	$\tau = 2$	$\tau = 4$	$\tau = 2$	$\tau = 4$	$\tau = 2$	$\tau = 4$	$\tau = 2$	$\tau = 4$
4	2.2×10^{-3}	4.5×10^{-3}	5.2×10^{-2}	9.8×10^{-3}	7.3×10^{-2}	3.7×10^{-2}	4.4×10^{-2}	2.8×10^{-2}
8	4.6×10^{-7}	5.7×10^{-7}	2.4×10^{-4}	2.9×10^{-4}	3.3×10^{-4}	6.8×10^{-4}	4.2×10^{-4}	1.4×10^{-4}
12	2.8×10^{-10}	4.0×10^{-10}	3.7×10^{-8}	6.2×10^{-8}	5.9×10^{-8}	8.2×10^{-8}	9.2×10^{-8}	6.4×10^{-8}
16	4.4×10^{-16}	4.4×10^{-16}	5.4×10^{-12}	2.4×10^{-12}	5.4×10^{-12}	3.7×10^{-12}	7.5×10^{-12}	8.7×10^{-12}

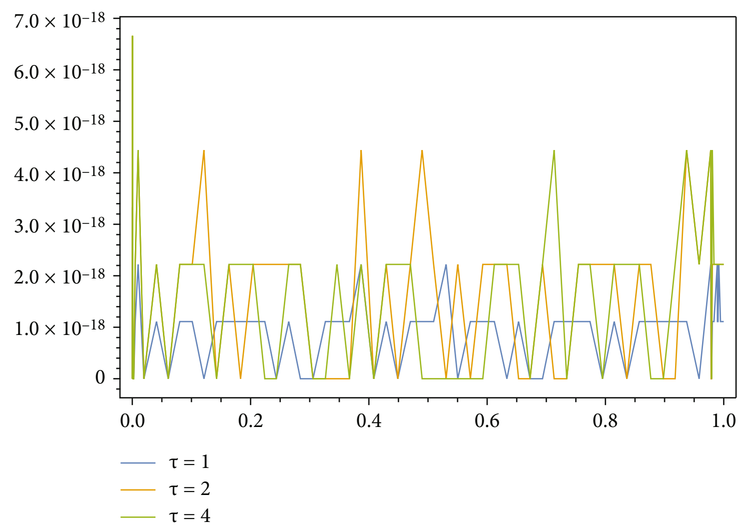

Figure 2: Absolute errors of example 4—$\gamma = 2, N = 16$.

7. Conclusion

Herein, we have established a new formula that gives an approximation of the fractional derivatives of the nonsymmetric polynomials, namely, shifted third-kind Chebyshev polynomials in the Caputo sense. We demonstrated that this formula contains a terminating hypergeometric function of type ${}_4F_3(1)$, which can be simplified in the integer case to match the well-known derivative formula of the Chebyshev polynomials of the third kind. A certain nonlinear fractional pantograph differential equation was treated via the application of the spectral tau method depending on the developed fractional derivatives formula. The convergence analysis of the method was investigated. The algorithm was tested through four examples that show the high accuracy and the efficiency of the presented algorithm. We believe that the theoretical results in this paper can be utilized to treat other types of fractional differential equations.

Data Availability

No data is associated with this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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